THE BREZIS-NIRENBERG PROBLEM FOR FRACTIONAL SYSTEMS WITH HARDY POTENTIALS

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Abstract. In this work we study the existence of positive solutions to the following fractional elliptic systems with Hardy-type singular potentials, and coupled by critical homogeneous nonlinearities

$$\begin{cases} (-\Delta)^s u - \mu_1 \frac{u}{|x|^{2s}} = |u|^{2_s^*-2}u + \frac{\eta\alpha}{2_s^*}|u|^{\alpha-2}|v|^{\beta}u + \frac{1}{2}Q_u(u,v) \ \ in\ \Omega, \\ (-\Delta)^s v - \mu_2 \frac{v}{|x|^{2s}} = |v|^{2_s^*-2}v + \frac{\eta\beta}{2_s^*}|u|^{\alpha}|v|^{\beta-2}v + \frac{1}{2}Q_v(u,v) \ \ in\ \Omega, \\ \ u,\ v>0 \quad in\ \ \Omega, \\ u=v=0 \quad in\ \ \mathbb{R}^N\backslash\Omega, \end{cases}$$

where $(-\Delta)^s$ denotes the fractional Laplace operator, $\Omega\subset\mathbb{R}^N$ is a smooth bounded domain such that $0\in\Omega$, $\mu_1,\mu_2\in[0,\Lambda_{N,s})$, $\Lambda_{N,s}=2^{2s}\frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})}$ is the best constant of the fractional Hardy inequality and $2_s^*=\frac{2N}{N-2s}$ is the fractional critical Sobolev exponent. In order to prove the main result, we establish some refined estimates on the extremal functions of the fractional Hardy-Sobolev type inequalities and we get the existence of positive solutions to the systems through variational methods.

1. Introduction and main results

In the last few decades a lot of mathematical efforts have been devoted to elliptic problems involving critical nonlinearities and Hardy potentials and many important results were obtained. In the celebrated paper [33], Terracini showed that for $\mu\in(0,\bar{\mu})$,

$$U_\mu^\varepsilon(x)=\frac{\big(4N(\bar{\mu}-\mu)/(N-2)\big)^{\frac{N-2}{4}}\varepsilon^{\sqrt{\bar{\mu}-\mu}}}{\big(\varepsilon^{\frac{4\sqrt{\bar{\mu}-\mu}}{N-2}}|x|^{\frac{\gamma'}{\sqrt{\bar{\mu}}}}+|x|^{\frac{\gamma}{\sqrt{\bar{\mu}}}}\big)^{\sqrt{\bar{\mu}}}},$$

satisfies equation

$$-\Delta u-\mu\frac{u}{|x|^2}=|u|^{2^*-2}u \quad in\ \mathbb{R}^N, \tag{1.1}$$

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where $\varepsilon > 0$, $\bar{\mu} = (N-2)^2/4$, $\gamma' = \sqrt{\bar{\mu}} - \sqrt{\bar{\mu}-\mu}$, $\gamma = \sqrt{\bar{\mu}} + \sqrt{\bar{\mu}-\mu}$. After that, rely on the results obtained in [33], the research on related perturbative equations with a singular potential and a critical nonlinearity were extensively studied, see e.g. [9, 11, 18] and the references therein. We mention that in the paper [24], Jannelli dealt with the problem

$$-\Delta u - \mu \frac{u}{|x|^2} = \lambda u + |u|^{2^*-2}u, \quad u \in H_0^1(\Omega), \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain such that $0 \in \Omega$, $0 \leq \mu < \bar{\mu}$. The author investigated the relationship between critical dimension and L^2_{loc} integrability of the Green function associated with the Hardy-Schrödinger operator $L_{\mu,2} = -\Delta - \frac{\mu}{|x|^2}$. In particular, Jannelli showed the existence of nontrivial solutions to (1.2) for $\lambda \in (0, \lambda_1(\mu))$ and $\mu \in [0, \bar{\mu}-1]$, where $\lambda_1(\mu)$ is the first eigenvalue of the Hardy-Schrödinger operator $L_{\mu,2}$.

Concerning doubly critical elliptic systems with singular potentials, in 2009, Abdellaoui, Felli and Peral [1] studied the following class of systems

$$\begin{cases} -\Delta u - \lambda_1 \frac{u}{|x|^2} = |u|^{2^*-2}u + \nu h(x)\alpha |u|^{\alpha-2}|v|^{\beta}u, \quad in \ \ \mathbb{R}^N, \\ -\Delta v - \lambda_2 \frac{v}{|x|^2} = |v|^{2^*-2}v + \nu h(x)\beta |u|^{\alpha}|v|^{\beta-2}v, \quad in \ \ \mathbb{R}^N, \end{cases} \tag{1.3}$$

where $\lambda_1, \lambda_2 \in (0, (N-2)^2/4)$, ν is a positive parameter and $\alpha, \beta > 1$ are positive constants such that $\alpha + \beta \leq 2^*$. By variational arguments, the authors obtained the existence of different types of positive solutions to system (1.3). Later, inspired by the work in [1], many results were obtained in this regard and we refer the reader to see, for instance, [12, 13, 34] for the whole space and [16, 25–27] in bounded domains.

Contrary to the local case that has been widely investigated, in recent years, several studies focus on the elliptic problems involving the fractional Laplace operator and singular Hardy potential. Recently, Dipierro, Montoro, Peral and Sciunzi [15] studied the existence and asymptotic behavior of extremals for the fractional Hardy-Sobolev inequality. Based on the results given in [15], Ghoussoub, Robert, Shakerian and Zhao [21] investigated the fractional Hardy-Sobolev inequality with weights and as an application, the authors considered the following boundary value problem involving the fractional Hardy-Schrödinger operator

$$\begin{cases} (-\Delta)^s u - \gamma \frac{u}{|x|^{2s}} = \lambda u + \frac{|u|^{2_s^*(\alpha)-2}u}{|x|^{\alpha}}, \quad u \geq 0 \ \ in \ \ \Omega, \\ u = 0 \quad in \ \ \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.4}$$

where $\gamma \in [0, \Lambda_{N,s})$, $0 \leq \alpha < 2s < N$. By introducing the fractional Hardy singular interior mass of a domain and using very technical and complicated analysis, the authors obtained the existence of least energy solutions to (1.4) under different conditions on Hardy term. We remark that when $\gamma = \alpha = 0$, problem (1.4) reduces to the well-known fractional Brezis-Nirenberg problem that has been well studied by Servadei and Valdinoci in [29, 30]. Moreover, in the paper [17], Faria, Miyagaki, Pereira, Squassina and Zhang investigated the following Brezis-Nirenberg problem for nonlocal systems

$$\begin{cases} (-\Delta)^s u = au + bv + \frac{2p}{p+q} u^{p-1} v^q \quad in\ \Omega, \\ (-\Delta)^s v = bu + cv + \frac{2q}{p+q} u^p v^{q-1} \quad in\ \Omega, \\ \ u,\ v > 0 \quad in\ \ \Omega, \\ u = v = 0 \quad in\ \ \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.5}$$

where $p + q \leq 2_s^*$ and $a, b, c \in \mathbb{R}$ satisfying suitable hypotheses. For more literatures regarding fractional elliptic problems with singular potentials and lower order perturbations, we refer the readers to [2, 3, 5, 19, 31] and the references therein.

Motivated by the aforementioned works, in this paper we focus our attention on fractional elliptic system in bounded domains with Hardy potentials. More precisely, we consider the following class of systems of nonlinear elliptic equations in bounded domains

$$\begin{cases} (-\Delta)^s u - \mu_1 \frac{u}{|x|^{2s}} = |u|^{2_s^*-2} u + \frac{\eta\alpha}{2_s^*} |u|^{\alpha-2} |v|^\beta u + \frac{1}{2} Q_u(u, v) \quad in\ \Omega, \\ (-\Delta)^s v - \mu_2 \frac{v}{|x|^{2s}} = |v|^{2_s^*-2} v + \frac{\eta\beta}{2_s^*} |u|^\alpha |v|^{\beta-2} v + \frac{1}{2} Q_v(u, v) \quad in\ \Omega, \\ \ u,\ v > 0 \quad in\ \ \Omega, \\ u = v = 0 \quad in\ \ \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.6}$$

where $s \in (0, 1)$, $N > 2s$ and $2_s^* = \frac{2N}{N-2s}$, Ω is a smooth bounded domain in $\mathbb{R}^N$ containing the origin and the fractional Laplace operator $(-\Delta)^s$ is defined on smooth functions as

$$(-\Delta)^s u(x) := c_{N,s} p.v. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy,$$

where $c_{N,s} = 2^{2s} \pi^{-\frac{N}{2}} \frac{\Gamma(\frac{N+2s}{2})}{|\Gamma(-s)|}$ and p.v. denotes the principle value of the integral, Q_u, Q_v are partial derivatives of the homogeneous $\mathcal{C}^1$-function $Q(u, v)$:

$$Q(u, v) = au^2 + 2buv + cv^2,$$

and the parameters satisfy

$(\mathcal{H}_1)$ $\eta > 0$, $\mu_1, \mu_2 \in [0, \Lambda_{N,s})$ with $\Lambda_{N,s}$ being the best constant of the fractional Hardy inequality (see (2.4)), $\alpha, \beta > 1$ satisfying $\alpha + \beta = 2_s^*$. Without loss of generality, along the work, we assume that $0 \leq \mu_2 \leq \mu_1 < \Lambda_{N,s}$.

$(\mathcal{H}_2)$ $a,\ c > 0$ and there exist constants $\lambda_1, \lambda_2 > 0$ such that

$$\lambda_1(u^2 + v^2) \leq Q(u, v) \leq \lambda_2(u^2 + v^2), \quad \forall\ (u, v) \in H_0^s(\Omega) \times H_0^s(\Omega).$$

To state our main results, we need to introduce some notations. For any $\mu \in [0, \Lambda_{N,s})$ and $\mu_1, \mu_2 \in [0, \Lambda_{N,s})$, we define

$$S_\mu := \inf_{u \in \dot{H}^s(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx - \mu \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2s}} dx}{(\int_{\mathbb{R}^N} |u|^{2^*_s} dx)^{\frac{2}{2^*_s}}}, \tag{1.7}$$

$$S(\mu_1, \mu_2) := \inf_{(u,v) \in \dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N) \setminus \{(0,0)\}} \frac{\int_{\mathbb{R}^N} E(u,v) dx}{(\int_{\mathbb{R}^N} F(u,v) dx)^{\frac{2}{2^*_s}}}, \tag{1.8}$$

where $\dot{H}^s(\mathbb{R}^N)$ denotes the homogeneous fractional Sobolev space, $E(u,v) = |(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2 - \mu_1 \frac{u^2}{|x|^{2s}} - \mu_2 \frac{v^2}{|x|^{2s}}$ and $F(u,v) = |u|^{2^*_s} + |v|^{2^*_s} + \eta |u|^\alpha |v|^\beta$. Moreover, under the assumption $(\mathcal{H}_1)$, we let

$$f(\tau) := \frac{1 + \frac{\Lambda_{N,s} - \mu_2}{\Lambda_{N,s} - \mu_1} \tau^2}{F(1,\tau)^{\frac{2}{2^*_s}}}, \quad \tau \geq 0, \quad f(\tau_{\min}) := \min_{\tau \geq 0} f(\tau) > 0,$$

$$\eta_1 := \frac{N(\Lambda_{N,s} - \mu_2)}{(N-2s)(\Lambda_{N,s} - \mu_1)}, \ \eta_2 := \inf_{\tau > 0} \Big(\tau^{-\beta} \big((1 + \frac{\Lambda_{N,s} - \mu_2}{\Lambda_{N,s} - \mu_1} \tau^2)^{\frac{2^*_s}{2}} - (1 + \tau^{2^*_s}) \big) \Big).$$

By $(\mathcal{H}_1)$ we have $f \in \mathcal{C}^1(\mathbb{R}^+, \mathbb{R}^+)$, $f(0) = 1$ and $\lim_{\tau \to +\infty} f(\tau) = \frac{\Lambda_{N,s} - \mu_2}{\Lambda_{N,s} - \mu_1} \geq 1$, therefore $\min_{\tau \geq 0} f(\tau) \leq 1$ must be achieved at some finite $\tau_{\min} \in [0, \infty)$.

Notice that by testing (1.8) with $(u, 0)$, where u being the extremal for S_{μ_1}, it is easy to obtain that $S(\mu_1, \mu_2) \leq S_{\mu_1}$. For this, the first objective of this paper is to investigate the relationship between S_{μ_1} and $S(\mu_1, \mu_2)$. Following the ideas used in [27], we show that the strict inequality $S(\mu_1, \mu_2) < S_{\mu_1}$ holds under suitable hypotheses, which is crucial to verify the minimization problem (1.8) being achieved.

Theorem 1.1. *Suppose that $(\mathcal{H}_1)$ holds and one of the following conditions is satisfied: (i) $f(\tau_{\min}) < 1$; (ii) $\beta < 2$; (iii) $\beta = 2$, $\eta > \eta_1$; (iv) $\eta > \eta_2$. Then $0 < S(\mu_1, \mu_2) < S_{\mu_1}$.*

Theorem 1.2. *Suppose that $(\mathcal{H}_1)$ holds and $S(\mu_1, \mu_2) < S_{\mu_1}$. Let $u^\varepsilon_\mu(x)$ $(\varepsilon > 0)$ denote the minimizers for (1.7) satisfying (2.5), then there exists constant $\eta^* \geq 0$, $t_i, l_i > 0$ $(i = 1, 2)$, such that $S(\mu_1, \mu_2)$ has positive minimizers of the form $\{(t_i u^\varepsilon_{\mu_1}, l_i u^\varepsilon_{\mu_2})\}$ $(i = 1, 2)$ for all $\eta > \eta^*$.*

By the previous results, we study the existence of positive solutions to (1.6) and the existence result for (1.6) is given by:

Theorem 1.3. *Suppose that $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ hold, $S(\mu_1, \mu_2) < S_{\mu_1}$, $\eta > \eta^*$, $\mu_1 \leq \mu^*$, $\lambda_2 < \Lambda(\mu_1)$, $N \geq 4s$. Then (1.6) has a positive solution $(u, v) \in H^s_0(\Omega) \times H^s_0(\Omega)$.*

As far as we know, there is no work concerned with problem (1.6) in bounded domains. To study problem (1.6), we will follow the arguments used in [8, 30], that is, we will use the Mountain Pass Theorem [4] to prove the existence of positive solutions to (1.6). Comparing to the systems of elliptic equations without Hardy potentials or containing a single Hardy potential (see e.g. [17,23]), the two possible different Hardy terms $\mu_1\frac{1}{|x|^{2s}}$ and $\mu_2\frac{1}{|x|^{2s}}$ appeared in (1.6) make it more difficult to verify the action functional satisfies the $(PS)_c$ condition. To overcome this difficulty, we shall exploit the asymptotic properties of the positive extremals of the fractional Hardy-Sobolev inequality obtained in [15] to establish the desired estimates and we prove that the Mountain Pass critical level of the action functional lies below the threshold of application of the $(PS)_c$ condition.

This paper is organized as follows. In Section 2, we describe the appropriate functional setting for the study of problem (1.6), and we introduce the asymptotic behavior of the extremals of the fractional Hardy-Sobolev inequality. Then we devote Section 3 to the proofs of Theorems 1.1 and 1.2. In Section 4, we establish some refined estimates on the extremal functions of the fractional Hardy-Sobolev type inequalities and the proof of our main result in this paper is completed.

Notation. In this paper, we will use the same C to denote various generic positive constant, and we shall use $o(1)$ to denote quantities that tend to 0 as n tends to ∞.

2. Preliminaries and functional setting

In this section, we recall some known results for readers' convenience and later use. We begin introducing the fractional Sobolev space

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : [u]_s < \infty\},$$

endowed with norm

$$\|u\|_{H^s} = (\|u\|^2_{L^2(\mathbb{R}^N)} + [u]_s^2)^{\frac{1}{2}},$$

where

$$[u]_s = \Big(\frac{c_{N,s}}{2}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\Big)^{\frac{1}{2}} = \Big(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 dx\Big)^{\frac{1}{2}},$$

is the Gagliardo semi-norm of a measurable function $u : \mathbb{R}^N \to \mathbb{R}$. Now, let Ω be a smooth bounded domain in $\mathbb{R}^N$ and we consider the closed subspace

$$H_0^s(\Omega) := \{u \in H^s(\mathbb{R}^N) : u = 0 \ \ a.e.\ in\ \mathbb{R}^N\backslash\Omega\ \}.$$

We refer to [30] for more details on $H_0^s(\Omega)$ and by the fractional Hardy inequality (see (2.4)), we employ the following equivalent norm by setting

$$\|\cdot\|_\mu = \Big(\frac{c_{N,s}}{2}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy - \mu\int_\Omega \frac{u^2}{|x|^{2s}}dx\Big)^{\frac{1}{2}}, \tag{2.1}$$

for $\mu \in [0, \Lambda_{N,s})$. We also recall that $(H_0^s(\Omega), \|\cdot\|_\mu)$ is a Hilbert space and the embedding $H_0^s(\Omega) \hookrightarrow L^r(\Omega)$ is continuous for $r \in [1, 2_s^*]$ and compact for $r \in [1, 2_s^*)$. In this paper, we work in the Hilbert space given by the product

$$W = H_0^s(\Omega) \times H_0^s(\Omega),$$

endowed with the norm

$$\|(u,v)\|_W^2 = \|u\|_{\mu_1}^2 + \|v\|_{\mu_2}^2.$$

Associated with problem (1.6), we consider the energy functional

$$J(u,v) = \frac{1}{2}\int_{\mathbb{R}^N} (E(u,v) - Q(u,v))dx - \frac{1}{2_s^*}\int_\Omega F(u,v)dx,$$

then $J \in \mathcal{C}^1(W,\mathbb{R})$ and its critical points correspond to solutions of (1.6).

We say $(u,v) \in W$ is a weak solution of (1.6) if for every $(\varphi,\psi) \in W$, one has

$$\begin{aligned}
&\frac{c_{N,s}}{2}\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dxdy - \mu_1\int_\Omega \frac{u\varphi}{|x|^{2s}}dx \\
&+ \frac{c_{N,s}}{2}\int_{\mathbb{R}^{2N}} \frac{(v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+2s}}dxdy - \mu_2\int_\Omega \frac{v\psi}{|x|^{2s}}dx \\
=&\frac{1}{2}\int_\Omega (Q_u(u,v)\varphi + Q_v(u,v)\psi)dx + \frac{1}{2_s^*}\int_\Omega (F_u(u,v)\varphi + F_v(u,v)\psi)dx.
\end{aligned}$$

To continue, we introduce the asymptotic properties of the positive extremals for the fractional Hardy-Sobolev inequality. We first recall the fractional Sobolev inequality [14], which asserts that for $N > 2s$, there exists a constant $S > 0$ such that

$$S\|u\|_{L^{2_s^*}(\mathbb{R}^N)}^2 \le \frac{c_{N,s}}{2}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy, \quad u \in \dot{H}^s(\mathbb{R}^N). \tag{2.2}$$

The best Sobolev constant S is attained at the function

$$u_\varepsilon(x) = \frac{C\varepsilon^{\frac{N-2s}{2}}}{(\varepsilon^2+|x|^2)^{\frac{N-2s}{2}}}, \quad \varepsilon > 0. \tag{2.3}$$

On the other hand, the classical fractional Hardy inequality (see [20, 22]) states that

$$\Lambda_{N,s}\int_{\mathbb{R}^N} \frac{u^2}{|x|^{2s}}dx \le \frac{c_{N,s}}{2}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy, \quad u \in \dot{H}^s(\mathbb{R}^N), \tag{2.4}$$

and the best Hardy constant $\Lambda_{N,s}$ never achieved in $\dot{H}^s(\mathbb{R}^N)$. Rely on (2.2) and (2.4), for $\mu \in [0,\Lambda_{N,s})$, Dipierro et al. [15] established the existence of positive minimizers for (1.7). As a consequence, up to a constant, the minimizers for (1.7) lead to the variational solutions to the problem

$$(-\Delta)^s u - \mu\frac{u}{|x|^{2s}} = u^{2_s^*-1} \quad in\ \mathbb{R}^N. \tag{2.5}$$

In what follows, we will denote by $u_\mu^\varepsilon(x) = \varepsilon^{-\frac{N-2s}{2}}u_\mu(\frac{x}{\varepsilon})$ $(\varepsilon > 0)$ the positive variational solutions to (2.5), then test (2.5) with $u_\mu^\varepsilon(x)$ and using (1.7) we derive that

$$\frac{c_{N,s}}{2}\int_{\mathbb{R}^{2N}} \frac{|u_\mu^\varepsilon(x)-u_\mu^\varepsilon(y)|^2}{|x-y|^{N+2s}}dxdy - \mu\int_{\mathbb{R}^N} \frac{|u_\mu^\varepsilon(x)|^2}{|x|^{2s}}dx = \int_{\mathbb{R}^N} |u_\mu^\varepsilon(x)|^{2_s^*}dx = S_\mu^{\frac{N}{2s}}. \tag{2.6}$$

Furthermore, it is worthy to point out that the authors [15] gave a control of the extremal for S_μ with $\mu \in (0, \Lambda_{N,s})$. More precisely, there exists positive constants c_1 and c_2, such that

$$\frac{c_1}{\left(|x|^{1-\eta_\mu}(1+|x|^{2\eta_\mu})\right)^{\frac{N-2s}{2}}} \le u_\mu(x) \le \frac{c_2}{\left(|x|^{1-\eta_\mu}(1+|x|^{2\eta_\mu})\right)^{\frac{N-2s}{2}}}, \quad x \in \mathbb{R}^N \backslash \{0\}, \tag{2.7}$$

where $\eta_\mu = 1 - \frac{2\alpha_\mu}{N-2s}$ and $\alpha_\mu \in (0, \frac{N-2s}{2})$ is a suitable parameter whose explicit value will be determined as the unique solution to equation

$$2^{2s}\frac{\Gamma(\frac{\beta+2s}{2})\Gamma(\frac{N-2s}{2})}{\Gamma(\frac{N-\beta-2s}{2})\Gamma(\frac{\beta}{2})} = \Psi_{N,s}(\beta) = \mu, \quad \mu \in (0, \Lambda_{N,s}).$$

Γ is the Gamma function and the function $\beta \mapsto \Psi_{N,s}(\beta)$ satisfies the following known properties.

Lemma 2.1. *(see [20, 21]) The following properties hold:*

1. $\Psi_{N,s}(\beta) > 0$ *for all* $\beta \in (0, N-2s)$.
2. *The graph of* $\Psi_{N,s}$ *in* $(0, N-2s)$ *is symmetric with respect to* $\frac{N-2s}{2}$, *that is,* $\Psi_{N,s}(\beta) = \Psi_{N,s}(N-\beta-2s)$ *for all* $\beta \in (0, N-2s)$.
3. $\Psi_{N,s}$ *is strictly increasing and surjective in* $(0, \frac{N-2s}{2})$, *and strictly decreasing in* $(\frac{N-2s}{2}, N-2s)$.
4. $\Psi_{N,s}(\frac{N-2s}{2}) = \Lambda_{N,s}$.
5. $\lim_{\beta\to 0}\Psi_{N,s}(\beta) = \lim_{\beta\to N-2s}\Psi_{N,s}(\beta) = 0$.

In what follows, for simplicity, we define

$$a(\mu) = (1-\eta_\mu)\delta, \quad b(\mu) = (1+\eta_\mu)\delta, \tag{2.8}$$

where $\mu \in (0, \Lambda_{N,s})$ and $\delta = \frac{N-2s}{2}$. We now extend $a(\mu)$, $b(\mu)$ to the interval $[0, \Lambda_{N,s})$ by defining

$$a(0) = 0, \quad b(0) = N-2s,$$

which is consistent with (2.3) and Lemma 2.1. Therefore, for $\mu \in [0, \Lambda_{N,s})$, we have

$$0 \le a(\mu) < \delta < b(\mu) \le N-2s. \tag{2.9}$$

It is worth noticing that from the definition of $b(\mu)$, we know that $b(\mu)$ is strictly decreasing on $[0, \Lambda_{N,s})$. Thus, if $N \ge 4s$, there exists a unique $\mu^* \in [0, \Lambda_{N,s})$ such that

$$2b(\mu) > N \Leftrightarrow \mu < \mu^*, \quad 2b(\mu) = N \Leftrightarrow \mu = \mu^*. \tag{2.10}$$

By (2.8), we can rewrite (2.7) as the following form

$$\frac{c_1}{\left(|x|^{\frac{a(\mu)}{\delta}} + |x|^{\frac{b(\mu)}{\delta}}\right)^{\delta}} \le u_\mu(x) \le \frac{c_2}{\left(|x|^{\frac{a(\mu)}{\delta}} + |x|^{\frac{b(\mu)}{\delta}}\right)^{\delta}}, \quad x \in \mathbb{R}^N \backslash \{0\}. \tag{2.11}$$

Remark 2.2. For $\mu \in [0, \Lambda_{N,s})$, we define

$$\Lambda(\mu) = \inf_{u \in H_0^s(\Omega)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx - \mu \int_\Omega \frac{u^2}{|x|^{2s}} dx}{\int_\Omega |u|^2 dx}.$$

By applying Hölder's inequality and the fractional Hardy-Sobolev inequality, we get that $\Lambda(\mu)$ is well defined and positive on $H_0^s(\Omega)$.

3. Extremals of $S(\mu_1, \mu_2)$

In this section, we study the existence of extremals for $S(\mu_1, \mu_2)$ under suitable conditions which will be crucially used in the proof of our main result. To do this, we first prove Theorem 1.1, which guarantees the strict inequality $S(\mu_1, \mu_2) < S_{\mu_1}$ holds.

Proof of Theorem 1.1. (i) Since $0 \le \mu_2 \le \mu_1 < \Lambda_{N,s}$ and $f(\tau_{\min}) < 1$, for all $(u, v) \in \dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)$, by the fractional Hardy inequality (2.4) we have that

$$\begin{aligned}
\|u\|_{\mu_1}^2 + \|v\|_{\mu_2}^2 &= \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx - \mu_1 \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2s}} dx + \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2 dx - \mu_2 \int_{\mathbb{R}^N} \frac{v^2}{|x|^{2s}} dx \\
&= \|u\|_{\mu_1}^2 + \|v\|_{\mu_1}^2 + (\mu_1 - \mu_2) \int_{\mathbb{R}^N} \frac{v^2}{|x|^{2s}} dx \\
&\le \|u\|_{\mu_1}^2 + \|v\|_{\mu_1}^2 + \frac{\mu_1 - \mu_2}{\Lambda_{N,s}} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2 dx \\
&\le \|u\|_{\mu_1}^2 + \|v\|_{\mu_1}^2 + \frac{\mu_1 - \mu_2}{\Lambda_{N,s} - \mu_1} \|v\|_{\mu_1}^2 \\
&= \|u\|_{\mu_1}^2 + \frac{\Lambda_{N,s} - \mu_2}{\Lambda_{N,s} - \mu_1} \|v\|_{\mu_1}^2. \qquad (3.1)
\end{aligned}$$

Testing (1.8) with $(u_{\mu_1}, \tau_{\min} u_{\mu_1})$, by (3.1) we have that

$$S(\mu_1, \mu_2) \le f(\tau_{\min}) \frac{\|u_{\mu_1}\|_{\mu_1}^2}{\left(\int_{\mathbb{R}^N} |u_{\mu_1}|^{2_s^*} dx\right)^{\frac{2}{2_s^*}}} = f(\tau_{\min}) S_{\mu_1} < S_{\mu_1}.$$

On the other hand, by (1.8), $(\mathcal{H}_1)$ and arguing as in Theorem 1.1 of [23], we get that $S(\mu_1, \mu_2) \ge S(\mu_1, \mu_1) = S_{\mu_1} \inf_{\tau \ge 0} \frac{1+\tau^2}{F(1,\tau)^{\frac{2}{2_s^*}}} > 0$.

(ii) From the definition of $f(\tau)$, it follows that

$$f'(\tau) = \frac{2\tau^{\beta-1}\left(2_s^* \frac{\Lambda_{N,s}-\mu_2}{\Lambda_{N,s}-\mu_1} \tau^{2-\beta} + \frac{\Lambda_{N,s}-\mu_2}{\Lambda_{N,s}-\mu_1} \eta\alpha\tau^2 - 2_s^* \tau^\alpha - \eta\beta\right)}{2_s^* \big(F(1,\tau)\big)^{\frac{2}{2_s^*}+1}}.$$

If $\beta < 2,\ \eta > 0$, since $f(0) = 1,\ f'(\tau) < 0$ as $\tau \to 0^+$, we have $f(\tau_{\min}) < 1$ and thus $0 < S(\mu_1, \mu_2) < S_{\mu_1}$ by the argument of (i).

(iii) If $\beta = 2,\ \eta > \eta_1$, since $f'(\tau) < 0$ as $\tau \to 0^+$, we conclude that $f(\tau_{\min}) < 1$ and $0 < S(\mu_1, \mu_2) < S_{\mu_1}$.

(iv) If $\eta > \eta_2$, the definition of η_2 implies that there exists $\tau_1 > 0$ such that

$$\eta > \tau_1^{-\beta}\big((1+\frac{\Lambda_{N,s}-\mu_2}{\Lambda_{N,s}-\mu_1}\tau_1^2)^{\frac{2_s^*}{2}} - (1+\tau_1^{2_s^*})\big),$$

then we have $f(\tau_{\min}) \leq f(\tau_1) < 1$ and thus $0 < S(\mu_1,\mu_2) < S_{\mu_1}$. □

With Theorem 1.1, we can verify the existence of extremals for $S(\mu_1,\mu_2)$.

Proof of Theorem 1.2. Since $0 < S(\mu_1,\mu_2) < S_{\mu_1} \leq S_{\mu_2}$, we have that

$$t' := \big(\frac{S(\mu_1,\mu_2)}{S_{\mu_1}}\big)^{\frac{1}{2}\cdot\frac{N}{2s}} < t'' := \big(\frac{S(\mu_1,\mu_2)}{S_{\mu_1}}\big)^{\frac{1}{2_s^*}\cdot\frac{N}{2s}} < 1,$$

$$l' := \big(\frac{S(\mu_1,\mu_2)}{S_{\mu_2}}\big)^{\frac{1}{2}\cdot\frac{N}{2s}} < l'' := \big(\frac{S(\mu_1,\mu_2)}{S_{\mu_2}}\big)^{\frac{1}{2_s^*}\cdot\frac{N}{2s}} < 1.$$

For all $t, l \geq 0$, we consider the equations

$$\begin{cases} t^2 S_{\mu_1}^{\frac{N}{2s}} + l^2 S_{\mu_2}^{\frac{N}{2s}} = S(\mu_1,\mu_2)^{\frac{N}{2s}}, \\ t^{2_s^*} S_{\mu_1}^{\frac{N}{2s}} + l^{2_s^*} S_{\mu_2}^{\frac{N}{2s}} + \eta t^\alpha l^\beta \int_{\mathbb{R}^N} |u_{\mu_1}^\varepsilon|^\alpha |u_{\mu_2}^\varepsilon|^\beta dx = S(\mu_1,\mu_2)^{\frac{N}{2s}}, \end{cases} \tag{3.2}$$

that is,

$$\begin{cases} \frac{t^2}{\left(\frac{S(\mu_1,\mu_2)}{S_{\mu_1}}\right)^{\frac{N}{2s}}} + \frac{l^2}{\left(\frac{S(\mu_1,\mu_2)}{S_{\mu_2}}\right)^{\frac{N}{2s}}} = 1, \\ \frac{t^{2_s^*}}{\left(\frac{S(\mu_1,\mu_2)}{S_{\mu_1}}\right)^{\frac{N}{2s}}} + \frac{l^{2_s^*}}{\left(\frac{S(\mu_1,\mu_2)}{S_{\mu_2}}\right)^{\frac{N}{2s}}} + \frac{\eta t^\alpha l^\beta}{\frac{S(\mu_1,\mu_2)^{\frac{N}{2s}}}{\int_{\mathbb{R}^N}|u_{\mu_1}^\varepsilon|^\alpha |u_{\mu_2}^\varepsilon|^\beta dx}} = 1. \end{cases} \tag{3.3}$$

Take $t^* \in (0,t') \subset (0,1)$ and $l^* \in (0,l') \subset (0,1)$ such that (t^*,l^*) satisfies the first equation in (3.3). Since $1 < 2 < 2_s^*$, there exists $\bar{\eta} > 0$ such that

$$\frac{|t^*|^{2_s^*}}{\left(\frac{S(\mu_1,\mu_2)}{S_{\mu_1}}\right)^{\frac{N}{2s}}} + \frac{|l^*|^{2_s^*}}{\left(\frac{S(\mu_1,\mu_2)}{S_{\mu_2}}\right)^{\frac{N}{2s}}} + \frac{\bar{\eta}|t^*|^\alpha |l^*|^\beta}{\frac{S(\mu_1,\mu_2)^{\frac{N}{2s}}}{\int_{\mathbb{R}^N}|u_{\mu_1}^\varepsilon|^\alpha |u_{\mu_2}^\varepsilon|^\beta dx}} = 1, \tag{3.4}$$

which implies that for all $\eta > \bar{\eta}$, there exists $l^{**} \in (0, l^*)$ such that

$$\frac{|t^*|^{2_s^*}}{\left(\frac{S(\mu_1,\mu_2)}{S_{\mu_1}}\right)^{\frac{N}{2s}}} + \frac{|l^{**}|^{2_s^*}}{\left(\frac{S(\mu_1,\mu_2)}{S_{\mu_2}}\right)^{\frac{N}{2s}}} + \frac{\eta|t^*|^\alpha |l^{**}|^\beta}{\frac{S(\mu_1,\mu_2)^{\frac{N}{2s}}}{\int_{\mathbb{R}^N}|u_{\mu_1}^\varepsilon|^\alpha |u_{\mu_2}^\varepsilon|^\beta dx}} = 1. \tag{3.5}$$

Since $(t',0)$, $(0,l')$, (t^*,l^*) satisfy the first equation in (3.3), $(t'',0)$, $(0,l'')$, (t^*,l^{**}) satisfy the second one in (3.3) and $t'' > t'$, $l'' > l'$, $l^{**} < l^*$, (3.3)-(3.5) yield that for all $\eta > \bar{\eta}$, the two continuous curves in (3.2) and (3.3) must have at least two intersection points $(t_i,l_i) \in (0,t') \times (0,l')$, $i = 1,2$. Define

$$\eta^* = \inf\{\bar{\eta} \mid \bar{\eta} > 0 \ is \ \ defined \ as \ in \ (3.4)\} \geq 0.$$

Then for all $\eta > \eta^*$, by (1.7), (1.8), (2.6) and (3.2), there exist $(t_i, l_i) \in (0, t') \times (0, l')$, $i = 1, 2$, such that $\{(t_i u^{\varepsilon}_{\mu_1}, l_i u^{\varepsilon}_{\mu_2})\}$ are minimizers of $S(\mu_1, \mu_2)$. $\square$

4. Proof of Theorem 1.3

The purpose of this section is to prove our main result, for this we need the following two auxiliary estimates, which will be crucially used in the sequel. We first establish the general estimates for $\varphi u^{\varepsilon}_{\mu}$, where we will borrow some ideas from [21].

Let us fix $\rho > 0$ small enough such that $B_{4\rho} \subset \Omega$ and let $\varphi \in C_0^{\infty}(\mathbb{R}^N)$ be a cutting-off function such that $0 \le \varphi \le 1$ in $\mathbb{R}^N$, $\varphi = 1$ in B_ρ and $\varphi = 0$ in $B^c_{2\rho} = \mathbb{R}^N \backslash B_{2\rho}$, where $B_\rho = B_\rho(0)$ is the ball centered at origin with radius ρ. Set $v^{\varepsilon}_{\mu}(x) = \varphi(x) u^{\varepsilon}_{\mu}(x)$, $\mu \in [0, \Lambda_{N,s})$. Then we get the following results.

Lemma 4.1. *Assume that $s \in (0,1)$ and $N > 2s$, $\mu \in [0, \Lambda_{N,s})$. Then the following estimates hold true:*

$$\|v^{\varepsilon}_{\mu}\|^2_{\mu} = S_{\mu}^{\frac{N}{2s}} + O(\varepsilon^{2(b(\mu)-\delta)}), \tag{4.1}$$

$$\int_{\Omega} |v^{\varepsilon}_{\mu}|^{2^*_s} dx = S_{\mu}^{\frac{N}{2s}} + O\big(\varepsilon^{2^*_s \cdot (b(\mu)-\delta)}\big), \tag{4.2}$$

and

$$\|v^{\varepsilon}_{\mu}\|^2_{L^2(\Omega)} \ge \begin{cases} C\varepsilon^{2s}, & if \quad \frac{N}{b(\mu)} < 2, \\ C\varepsilon^{2s} |\ln \varepsilon|, & if \quad \frac{N}{b(\mu)} = 2, \\ C\varepsilon^{2(b(\mu)-\delta)}, & if \quad 2 < \frac{N}{b(\mu)}, \end{cases} \tag{4.3}$$

as $\varepsilon \to 0$, for some positive constant C independent of ε.

Proof. From now on, for simplicity, we will omit the constant $\frac{c_{N,s}}{2}$. By the definition of v^{ε}_{μ} together with (2.5) and (2.6), we get

$$\begin{aligned}
\|v^{\varepsilon}_{\mu}\|^2_{\mu} &= \int_{\mathbb{R}^{2N}} \frac{|v^{\varepsilon}_{\mu}(x) - v^{\varepsilon}_{\mu}(y)|^2}{|x-y|^{N+2s}} dxdy - \mu \int_{\Omega} \frac{|v^{\varepsilon}_{\mu}(x)|^2}{|x|^{2s}} dx \\
&= \int_{\mathbb{R}^{2N}} \frac{(\varphi(x) u^{\varepsilon}_{\mu}(x) - \varphi(y) u^{\varepsilon}_{\mu}(y))^2}{|x-y|^{N+2s}} dxdy - \mu \int_{\Omega} \frac{|\varphi(x) u^{\varepsilon}_{\mu}(x)|^2}{|x|^{2s}} dx \\
&= \int_{\mathbb{R}^{2N}} \frac{\varphi^2(x)(u^{\varepsilon}_{\mu}(x))^2 - \varphi^2(x) u^{\varepsilon}_{\mu}(x) u^{\varepsilon}_{\mu}(y) + \varphi^2(y)(u^{\varepsilon}_{\mu}(y))^2 - \varphi^2(y) u^{\varepsilon}_{\mu}(x) u^{\varepsilon}_{\mu}(y)}{|x-y|^{N+2s}} dxdy \\
&\quad + \int_{\mathbb{R}^{2N}} \frac{\varphi^2(x) u^{\varepsilon}_{\mu}(x) u^{\varepsilon}_{\mu}(y) + \varphi^2(y) u^{\varepsilon}_{\mu}(x) u^{\varepsilon}_{\mu}(y) - 2\varphi(x)\varphi(y) u^{\varepsilon}_{\mu}(x) u^{\varepsilon}_{\mu}(y)}{|x-y|^{N+2s}} dxdy \\
&\quad - \mu \int_{\Omega} \frac{|\varphi(x) u^{\varepsilon}_{\mu}(x)|^2}{|x|^{2s}} dx
\end{aligned}$$

$$
\begin{aligned}
&=\int_{\mathbb{R}^{2N}}\frac{(u_\mu^\varepsilon(x)-u_\mu^\varepsilon(y))(\varphi^2(x)u_\mu^\varepsilon(x)-\varphi^2(y)u_\mu^\varepsilon(y))}{|x-y|^{N+2s}}dxdy\\
&\quad+\int_{\mathbb{R}^{2N}}\frac{(\varphi(x)-\varphi(y))^2u_\mu^\varepsilon(x)u_\mu^\varepsilon(y)}{|x-y|^{N+2s}}dxdy-\mu\int_\Omega\frac{\varphi^2(x)(u_\mu^\varepsilon(x))^2}{|x|^{2s}}dx\\
&=\mu\int_\Omega\frac{\varphi^2(x)(u_\mu^\varepsilon(x))^2}{|x|^{2s}}dx+\int_\Omega\varphi^2(x)|u_\mu^\varepsilon(x)|^{2_s^*}dx\\
&\quad+\int_{\mathbb{R}^{2N}}\frac{(\varphi(x)-\varphi(y))^2u_\mu^\varepsilon(x)u_\mu^\varepsilon(y)}{|x-y|^{N+2s}}dxdy-\mu\int_\Omega\frac{\varphi^2(x)(u_\mu^\varepsilon(x))^2}{|x|^{2s}}dx\\
&=\int_{\mathbb{R}^{2N}}\frac{(\varphi(x)-\varphi(y))^2u_\mu^\varepsilon(x)u_\mu^\varepsilon(y)}{|x-y|^{N+2s}}dxdy+\int_\Omega\varphi^2(x)|u_\mu^\varepsilon(x)|^{2_s^*}dx\\
&=\int_{\mathbb{R}^N}|u_\mu^\varepsilon(x)|^{2_s^*}dx+I_1+I_2\\
&=S_\mu^{\frac{N}{2s}}+I_1+I_2,
\end{aligned}\tag{4.4}
$$

where

$$
I_1=\int_{\mathbb{R}^{2N}}\frac{(\varphi(x)-\varphi(y))^2u_\mu^\varepsilon(x)u_\mu^\varepsilon(y)}{|x-y|^{N+2s}}dxdy,
$$

$$
I_2=\int_{\mathbb{R}^N}(\varphi^2(x)-1)|u_\mu^\varepsilon(x)|^{2_s^*}dx.
$$

To estimate I_1, we use the decomposition

$$
\mathbb{R}^N\times\mathbb{R}^N=(B_{2\rho}\times B_{2\rho})\cup(B_{2\rho}\times B_{2\rho}^c)\cup(B_{2\rho}^c\times B_{2\rho})\cup(B_{2\rho}^c\times B_{2\rho}^c).
$$

By the definition of φ, it is easy to see that

$$
\int_{B_{2\rho}^c\times B_{2\rho}^c}\frac{(\varphi(x)-\varphi(y))^2u_\mu^\varepsilon(x)u_\mu^\varepsilon(y)}{|x-y|^{N+2s}}dxdy=0.
$$

Note that $\int_{\mathbb{R}^N}\frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{N+2s}}dy\in L^\infty(\mathbb{R}^N)$, then by (2.11) and the fact that $b(\mu)\in(\frac{N-2s}{2},N-2s]$, for $(x,y)\in B_{2\rho}\times B_{2\rho}^c$, we obtain

$$
\begin{aligned}
&\int_{B_{2\rho}}\int_{B_{2\rho}^c}\frac{(\varphi(x)-\varphi(y))^2u_\mu^\varepsilon(x)u_\mu^\varepsilon(y)}{|x-y|^{N+2s}}dxdy\\
\le&C\varepsilon^{2b(\mu)-2\delta}\rho^{-b(\mu)}\int_{|x|\le2\rho}\frac{1}{|x|^{b(\mu)}}dx\int_{\mathbb{R}^N}\frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{N+2s}}dy\\
\le&C\varepsilon^{2(b(\mu)-\delta)}.
\end{aligned}
$$

Similarly, for $(x,y)\in B_{2\rho}^c\times B_{2\rho}$, one can prove that

$$\int_{B_{2\rho}^c}\int_{B_{2\rho}}\frac{(\varphi(x)-\varphi(y))^2u_\mu^\varepsilon(x)u_\mu^\varepsilon(y)}{|x-y|^{N+2s}}dxdy\le C\varepsilon^{2(b(\mu)-\delta)}.$$

For $(x,y)\in B_{2\rho}\times B_{2\rho}$, by the definition of φ and (2.11), we have

$$\begin{aligned}&\int_{B_{2\rho}}\int_{B_{2\rho}}\frac{(\varphi(x)-\varphi(y))^2u_\mu^\varepsilon(x)u_\mu^\varepsilon(y)}{|x-y|^{N+2s}}dxdy\\ =&\int_{|x|\le\rho}\int_{\rho\le|y|\le2\rho}\frac{(\varphi(x)-\varphi(y))^2u_\mu^\varepsilon(x)u_\mu^\varepsilon(y)}{|x-y|^{N+2s}}dxdy\\ &+\int_{\rho\le|x|\le2\rho}\int_{|y|\le2\rho}\frac{(\varphi(x)-\varphi(y))^2u_\mu^\varepsilon(x)u_\mu^\varepsilon(y)}{|x-y|^{N+2s}}dxdy\\ \le& C\varepsilon^{2b(\mu)-2\delta}\int_{|x|\le\rho}\frac{1}{|x|^{b(\mu)}}dx+C\varepsilon^{2b(\mu)-2\delta}\int_{|y|\le2\rho}\frac{1}{|y|^{b(\mu)}}dy\\ \le& C\varepsilon^{2(b(\mu)-\delta)}.\end{aligned}$$

Collecting the estimates above, we obtain that

$$I_1=\int_{\mathbb{R}^{2N}}\frac{(\varphi(x)-\varphi(y))^2u_\mu^\varepsilon(x)u_\mu^\varepsilon(y)}{|x-y|^{N+2s}}dxdy\le C\varepsilon^{2(b(\mu)-\delta)}. \tag{4.5}$$

Now, we estimate I_2. By the definition of φ and (2.11), we deduce that

$$\begin{aligned}|I_2|&=|\int_{\mathbb{R}^N}(\varphi^2(x)-1)|u_\mu^\varepsilon(x)|^{2_s^*}dx|\\ &\le 2\int_{|x|\ge\rho}|u_\mu^\varepsilon(x)|^{2_s^*}dx\\ &\le 2\varepsilon^{2_s^*(b(\mu)-\delta)}\int_{|x|\ge\rho}\frac{1}{|x|^{2_s^*\cdot b(\mu)}}dx\\ &\le C\varepsilon^{2_s^*(b(\mu)-\delta)}\\ &\le C\varepsilon^{2(b(\mu)-\delta)}.\end{aligned} \tag{4.6}$$

Therefore, combining (4.4), (4.5) and (4.6), we get (4.1). Then, we estimate $\int_\Omega|v_\mu^\varepsilon|^{2_s^*}dx$. By the definition of φ and (2.11), we have

$$\begin{aligned}|\int_\Omega|v_\mu^\varepsilon|^{2_s^*}dx-\int_{\mathbb{R}^N}|u_\mu^\varepsilon|^{2_s^*}dx|&\le\int_{\mathbb{R}^N}|\varphi^{2_s^*}-1||u_\mu^\varepsilon|^{2_s^*}dx\\ &\le\int_{|x|\ge\rho}|u_\mu^\varepsilon|^{2_s^*}dx\end{aligned}$$

$$\leq C\varepsilon^{2_s^*\cdot(b(\mu)-\delta)},$$

which implies that

$$\int_\Omega |v_\mu^\varepsilon|^{2_s^*}dx = S_\mu^{\frac{N}{2s}} + O(\varepsilon^{2_s^*\cdot(b(\mu)-\delta)}).$$

Finally, by the definition of φ and (2.11), we have

$$\begin{aligned}
\int_\Omega |v_\mu^\varepsilon|^2 dx &= \int_{|x|\leq 2\rho} |\varphi(x)u_\mu^\varepsilon(x)|^2 dx \\
&\geq \int_{|x|\leq\rho} |u_\mu^\varepsilon(x)|^2 dx \\
&\geq C\varepsilon^{N-2\delta}\int_{|x|\leq\frac{\rho}{\varepsilon}} \frac{1}{\left(|x|^{\frac{a(\mu)}{\delta}}+|x|^{\frac{b(\mu)}{\delta}}\right)^{2\delta}}dx \\
&\geq C\varepsilon^{N-2\delta}\int_{1\leq|x|\leq\frac{\rho}{\varepsilon}} \frac{1}{\left(|x|^{\frac{a(\mu)}{\delta}}+|x|^{\frac{b(\mu)}{\delta}}\right)^{2\delta}}dx \\
&\geq C\varepsilon^{N-2\delta}\int_{1\leq|x|\leq\frac{\rho}{\varepsilon}} \frac{1}{|x|^{2b(\mu)}}dx \\
&= C\varepsilon^{N-2\delta}\int_1^{\frac{\rho}{\varepsilon}} r^{N-2b(\mu)-1}dr \\
&= \begin{cases} C\varepsilon^{N-2\delta} - C\varepsilon^{2(b(\mu)-\delta)}, & if \quad N<2b(\mu), \\ C\varepsilon^{N-2\delta}|\ln\varepsilon|, & if \quad N=2b(\mu), \\ C\varepsilon^{2(b(\mu)-\delta)} - C\varepsilon^{N-2\delta}, & if \quad N>2b(\mu), \end{cases} \\
&\geq \begin{cases} C\varepsilon^{2s}, & if \quad \frac{N}{b(\mu)}<2, \\ C\varepsilon^{2s}|\ln\varepsilon|, & if \quad \frac{N}{b(\mu)}=2, \\ C\varepsilon^{2(b(\mu)-\delta)}, & if \quad \frac{N}{b(\mu)}>2. \end{cases}
\end{aligned}$$

□

Lemma 4.2. *Assume that $(\mathcal{H}_1)$ holds and $\eta>\eta^*$, $\alpha_1,\beta_1>0$, $\sigma:=\alpha_1+\beta_1<2_s^*$. Let $(t_0u_{\mu_1}^\varepsilon,\ l_0u_{\mu_2}^\varepsilon)$ be a positive minimizer of $S(\mu_1,\mu_2)$ obtained as in Theorem 1.2. Then the following estimates hold as $\varepsilon\to 0$:*

$$\|(t_0v_{\mu_1}^\varepsilon,\ l_0v_{\mu_2}^\varepsilon)\|_W^2 \leq S(\mu_1,\mu_2)^{\frac{N}{2s}} + C\varepsilon^{2(b(\mu_1)-\delta)}, \tag{4.7}$$

$$\int_\Omega F(t_0v_{\mu_1}^\varepsilon, l_0v_{\mu_2}^\varepsilon)dx \geq S(\mu_1,\mu_2)^{\frac{N}{2s}} - C\varepsilon^{2_s^*\cdot(b(\mu_1)-\delta)}, \tag{4.8}$$

$$\int_{\Omega}|v_{\mu_1}^{\varepsilon}|^{\alpha_1}|v_{\mu_2}^{\varepsilon}|^{\beta_1}dx \geq \begin{cases} C\varepsilon^{N-\sigma\delta},\ if\ \alpha_1 b(\mu_1)+\beta_1 b(\mu_2)>N, \\ C\varepsilon^{N-\sigma\delta}|\ln\varepsilon|,\ if\ \alpha_1 b(\mu_1)+\beta_1 b(\mu_2)=N, \\ C\varepsilon^{\alpha_1 b(\mu_1)+\beta_1 b(\mu_2)-\sigma\delta},\ if\ \alpha_1 b(\mu_1)+\beta_1 b(\mu_2)<N. \end{cases} \tag{4.9}$$

Proof. (i) Since $b(\mu)$ is decreasing on $[0,\Lambda_{N,s})$ and (t_0,l_0) satisfies (3.2), by (3.2) and (4.1), we have

$$\begin{aligned}\|(t_0 v_{\mu_1}^{\varepsilon},\ l_0 v_{\mu_2}^{\varepsilon})\|_W^2 &= t_0^2\|v_{\mu_1}^{\varepsilon}\|_{\mu_1}^2 + l_0^2\|v_{\mu_2}^{\varepsilon}\|_{\mu_2}^2 \\ &\leq t_0^2 S_{\mu_1}^{\frac{N}{2s}} + l_0^2 S_{\mu_2}^{\frac{N}{2s}} + C\varepsilon^{2(b(\mu_1)-\delta)} + C\varepsilon^{2(b(\mu_2)-\delta)} \\ &\leq S(\mu_1,\mu_2)^{\frac{N}{2s}} + C\varepsilon^{2(b(\mu_1)-\delta)}.\end{aligned}$$

(ii) By (3.2) and (4.2), we get

$$\begin{aligned}\int_{\Omega}F(t_0 v_{\mu_1}^{\varepsilon}, l_0 v_{\mu_2}^{\varepsilon})dx =& t_0^{2_s^*}\int_{\Omega}|v_{\mu_1}^{\varepsilon}|^{2_s^*}dx + l_0^{2_s^*}\int_{\Omega}|v_{\mu_2}^{\varepsilon}|^{2_s^*}dx + \eta t_0^{\alpha} l_0^{\beta}\int_{\Omega}|v_{\mu_1}^{\varepsilon}|^{\alpha}|v_{\mu_2}^{\varepsilon}|^{\beta}dx \\ \geq& t_0^{2_s^*}S_{\mu_1}^{\frac{N}{2s}} + l_0^{2_s^*}S_{\mu_2}^{\frac{N}{2s}} + \eta t_0^{\alpha} l_0^{\beta}\int_{\Omega}|v_{\mu_1}^{\varepsilon}|^{\alpha}|v_{\mu_2}^{\varepsilon}|^{\beta}dx \\ &- C\varepsilon^{2_s^*\cdot(b(\mu_1)-\delta)} - C\varepsilon^{2_s^*\cdot(b(\mu_2)-\delta)} \\ =& t_0^{2_s^*}S_{\mu_1}^{\frac{N}{2s}} + l_0^{2_s^*}S_{\mu_2}^{\frac{N}{2s}} + \eta t_0^{\alpha} l_0^{\beta}\int_{\mathbb{R}^N}|u_{\mu_1}^{\varepsilon}|^{\alpha}|u_{\mu_2}^{\varepsilon}|^{\beta}dx \\ &- \eta t_0^{\alpha} l_0^{\beta}\int_{\mathbb{R}^N}(1-\varphi^{2_s^*})|u_{\mu_1}^{\varepsilon}|^{\alpha}|u_{\mu_2}^{\varepsilon}|^{\beta}dx - C\varepsilon^{2_s^*\cdot(b(\mu_1)-\delta)} - C\varepsilon^{2_s^*\cdot(b(\mu_2)-\delta)} \\ \geq& S(\mu_1,\mu_2)^{\frac{N}{2s}} - C\int_{|x|\geq\rho}|u_{\mu_1}^{\varepsilon}|^{\alpha}|u_{\mu_2}^{\varepsilon}|^{\beta}dx - C\varepsilon^{2_s^*\cdot(b(\mu_1)-\delta)}.\end{aligned}$$

On the other hand, by (2.11), it follows that

$$\begin{aligned}\int_{|x|\geq\rho}|u_{\mu_1}^{\varepsilon}|^{\alpha}|u_{\mu_2}^{\varepsilon}|^{\beta}dx \leq& C\varepsilon^{\alpha(b(\mu_1)-\delta)}\cdot\varepsilon^{\beta(b(\mu_2)-\delta)}\int_{|x|\geq\rho}|x|^{-(\alpha b(\mu_1)+\beta b(\mu_2))}dx \\ \leq& C\varepsilon^{2_s^*\cdot(b(\mu_1)-\delta)}\int_{|x|\geq\rho}|x|^{-(\alpha b(\mu_1)+\beta b(\mu_2))}dx \\ \leq& C\varepsilon^{2_s^*\cdot(b(\mu_1)-\delta)},\end{aligned}$$

which yields the inequality

$$\int_{\Omega}F(t_0 v_{\mu_1}^{\varepsilon}, l_0 v_{\mu_2}^{\varepsilon})dx \geq S(\mu_1,\mu_2)^{\frac{N}{2s}} - C\varepsilon^{2_s^*\cdot(b(\mu_1)-\delta)}.$$

(iii) Using the similar argument as obtaining the estimate (4.3), we conclude that

$$\begin{aligned}
\int_\Omega |v^\varepsilon_{\mu_1}|^{\alpha_1}|v^\varepsilon_{\mu_2}|^{\beta_1}dx &\geq \int_{|x|\leq\rho} |u^\varepsilon_{\mu_1}(x)|^{\alpha_1}|u^\varepsilon_{\mu_2}(x)|^{\beta_1}dx \\
&= \int_{|x|\leq\frac{\rho}{\varepsilon}} \varepsilon^{N-\sigma\delta}|u_{\mu_1}(x)|^{\alpha_1}|u_{\mu_2}(x)|^{\beta_1}dx \\
&\geq C\varepsilon^{N-\sigma\delta}\int_{1\leq|x|\leq\frac{\rho}{\varepsilon}} |u_{\mu_1}(x)|^{\alpha_1}|u_{\mu_2}(x)|^{\beta_1}dx \\
&\geq C\varepsilon^{N-\sigma\delta}\int_{1\leq|x|\leq\frac{\rho}{\varepsilon}} |x|^{-(\alpha_1 b(\mu_1)+\beta_1 b(\mu_2))}dx \\
&\geq C\varepsilon^{N-\sigma\delta}\int_1^{\frac{\rho}{\varepsilon}} r^{N-1}\cdot r^{-(\alpha_1 b(\mu_1)+\beta_1 b(\mu_2))}dr \\
&\geq \begin{cases} C\varepsilon^{N-\sigma\delta}, \quad if \quad \alpha_1 b(\mu_1)+\beta_1 b(\mu_2) > N, \\ C\varepsilon^{N-\sigma\delta}|\ln\varepsilon|, \quad if \quad \alpha_1 b(\mu_1)+\beta_1 b(\mu_2) = N, \\ C\varepsilon^{\alpha_1 b(\mu_1)+\beta_1 b(\mu_2)-\sigma\delta}, \quad if \quad \alpha_1 b(\mu_1)+\beta_1 b(\mu_2) < N. \end{cases}
\end{aligned}$$

□

Using the previous lemmas we are able to prove the main result of this paper, that is, Theorem 1.3. The method we use here is the Mountain Pass Theorem and we point out that under the assumptions in Theorem 1.3 and Remark 2.2, it is easy to check that the functional J satisfies the geometric features required by the Mountain Pass Theorem and we omit it here. Now, we show the energy functional J verifies the Palais-Smale condition at any level c, provided $c < c^*$. The technique is the concentration compactness result for the fractional Laplacian obtained in e.g. [6, 35], see also [28, Remark I.6] for more details.

For this, we define the fractional gradient of a function $u \in H^s_0(\Omega)$ as

$$|D^s u(x)|^2 = \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dy.$$

In the sequel, we shall use the notation $|D^s u|$ denotes the fractional gradient of a function u, then $|D^s u| \in L^2(\mathbb{R}^N)$ and it is well defined a.e. in $\mathbb{R}^N$.

Lemma 4.3. *Suppose that* $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ *hold. Then the functional* J *satisfies the* $(PS)_c$ *condition for all* $c < c^* := \frac{s}{N}S(\mu_1,\mu_2)^{\frac{N}{2s}}$.

Proof. Suppose the sequence $\{(u_n, v_n)\} \subset W$ satisfies $J(u_n, v_n) \to c < c^*$ and $J'(u_n, v_n) \to 0$. Standard argument shows that $\{(u_n, v_n)\}$ is bounded in W. Up to a subsequence if necessary, for some $(u, v) \in W$, we have

$$(u_n, v_n) \rightharpoonup (u, v) \quad weakly\ in \quad W,$$

$$(u_n,v_n)\rightharpoonup(u,v)\quad weakly\ in\ (L^2(\Omega),|x|^{-2s})^2,$$
$$(u_n,v_n)\rightharpoonup(u,v)\quad weakly\ in\ (L^{2_s^*}(\Omega))^2,$$
$$(u_n,v_n)\to(u,v)\quad a.e.\ in\ \Omega,$$
$$(u_n,v_n)\to(u,v)\quad strongly\ in\ (L^2(\Omega))^2.$$

By the concentration compactness principle (see e.g. [6, 10, 35]) and up to a subsequence, there exists an at most countable set $\mathcal{T}$, a set of points $x_j\in\Omega\backslash\{0\}$, nonnegative real numbers $\sigma_j,\nu_j,j\in\mathcal{T}$, and σ_0,ν_0,μ_0, such that the following convergences hold in sense of measures:

$$|D^su_n|^2+|D^sv_n|^2\to d\sigma\geq|D^su|^2+|D^sv|^2+\sigma_0\delta_{x_0}+\sum_{j\in\mathcal{T}}\sigma_j\delta_{x_j},$$
$$\frac{\mu_1|u_n|^2+\mu_2|v_n|^2}{|x|^{2s}}\to d\mu=\frac{\mu_1|u|^2+\mu_2|v|^2}{|x|^{2s}}+\mu_0\delta_{x_0},$$
$$F(u_n,v_n)\to d\nu=F(u,v)+\nu_0\delta_{x_0}+\sum_{j\in\mathcal{T}}\nu_j\delta_{x_j},$$

where δ_x is the Dirac mass at x. By (1.8) and following a similar way as in [6] (see also [10, Lemma 4.5]), we have

$$\sigma_0-\mu_0\geq S(\mu_1,\mu_2)\nu_0^{\frac{2}{2_s^*}},\tag{4.10}$$
$$\sigma_j\geq S(0,0)\nu_j^{\frac{2}{2_s^*}},\quad j\in\mathcal{T}.\tag{4.11}$$

Now we consider the possibility of concentration at the origin x_0. For $\varepsilon>0$, let $\phi\in\mathcal{C}_0^\infty(\mathbb{R}^N)$ be a nonincreasing cut-off function verifying $\phi=1$ in $B_1(0)$, $\phi=0$ in $B_2^c(0)$. Let $\phi_\varepsilon(x)=\phi(\frac{x}{\varepsilon})$, clearly $|\nabla\phi_\varepsilon|\leq\frac{C}{\varepsilon}$. Since $(\phi_\varepsilon u_n,\phi_\varepsilon v_n)$ is bounded in W, we have that

$$\langle J'(u_n,v_n),(\phi_\varepsilon u_n,\phi_\varepsilon v_n)\rangle\to 0\quad as\ n\to\infty.$$

Then

$$\begin{aligned}
0=&\int_{\mathbb{R}^{2N}}\frac{(u_n(x)-u_n(y))(\phi_\varepsilon(x)u_n(x)-\phi_\varepsilon(y)u_n(y))}{|x-y|^{N+2s}}dxdy-\mu_1\int_{\mathbb{R}^N}\frac{|u_n|^2}{|x|^{2s}}\phi_\varepsilon dx\\
&+\int_{\mathbb{R}^{2N}}\frac{(v_n(x)-v_n(y))(\phi_\varepsilon(x)v_n(x)-\phi_\varepsilon(y)v_n(y))}{|x-y|^{N+2s}}dxdy-\mu_2\int_{\mathbb{R}^N}\frac{|v_n|^2}{|x|^{2s}}\phi_\varepsilon dx\\
&-\int_\Omega Q(u_n,v_n)\phi_\varepsilon dx-\int_\Omega F(u_n,v_n)\phi_\varepsilon dx+o(1)\\
=&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2\phi_\varepsilon(x)}{|x-y|^{N+2s}}dxdy-\mu_1\int_{\mathbb{R}^N}\frac{|u_n|^2}{|x|^{2s}}\phi_\varepsilon dx\\
&+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u_n(x)-u_n(y))(\phi_\varepsilon(x)-\phi_\varepsilon(y))u_n(y)}{|x-y|^{N+2s}}dxdy
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x)-v_n(y)|^2\phi_\varepsilon(x)}{|x-y|^{N+2s}}dxdy-\mu_2\int_{\mathbb{R}^N}\frac{|v_n|^2}{|x|^{2s}}\phi_\varepsilon dx\\
&+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(v_n(x)-v_n(y))(\phi_\varepsilon(x)-\phi_\varepsilon(y))v_n(y)}{|x-y|^{N+2s}}dxdy\\
&-\int_\Omega Q(u_n,v_n)\phi_\varepsilon dx-\int_\Omega F(u_n,v_n)\phi_\varepsilon dx+o(1), \qquad (4.12)
\end{aligned}
$$

as $n\to\infty$. Using Hölder's inequality and the fact that u_n, v_n are bounded in $H_0^s(\Omega)$, we get

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u_n(y)(u_n(x)-u_n(y))(\phi_\varepsilon(x)-\phi_\varepsilon(y))}{|x-y|^{N+2s}}dxdy\\
&\qquad\le C\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^2|\phi_\varepsilon(x)-\phi_\varepsilon(y)|^2}{|x-y|^{N+2s}}dxdy\Big)^{\frac12},\\
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v_n(y)(v_n(x)-v_n(y))(\phi_\varepsilon(x)-\phi_\varepsilon(y))}{|x-y|^{N+2s}}dxdy\\
&\qquad\le C\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(y)|^2|\phi_\varepsilon(x)-\phi_\varepsilon(y)|^2}{|x-y|^{N+2s}}dxdy\Big)^{\frac12}.
\end{aligned}
$$

From [6,35], it holds

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^2|\phi_\varepsilon(x)-\phi_\varepsilon(y)|^2}{|x-y|^{N+2s}}dxdy=0, \qquad (4.13)$$

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(y)|^2|\phi_\varepsilon(x)-\phi_\varepsilon(y)|^2}{|x-y|^{N+2s}}dxdy=0. \qquad (4.14)$$

On the other hand, by the definition of ϕ_ε, it is easy to prove that

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_\Omega\frac{\mu_1|u_n|^2+\mu_2|v_n|^2}{|x|^{2s}}\phi_\varepsilon dx=\lim_{\varepsilon\to0}\int_\Omega\phi_\varepsilon d\mu=\mu_0, \qquad (4.15)$$

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_\Omega Q(u_n,v_n)\phi_\varepsilon dx=0, \qquad (4.16)$$

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_\Omega F(u_n,v_n)\phi_\varepsilon dx=\lim_{\varepsilon\to0}\int_\Omega\phi_\varepsilon d\nu=\nu_0. \qquad (4.17)$$

Then, by (4.12)-(4.17), we obtain

$$0=\lim_{\varepsilon\to0}\lim_{n\to\infty}\langle J'(u_n,v_n),(\phi_\varepsilon u_n,\phi_\varepsilon v_n)\rangle\ge\sigma_0-\mu_0-\nu_0. \qquad (4.18)$$

Combining (4.10) and (4.18), it follows that $S(\mu_1,\mu_2)\nu_0^{\frac{2}{2_s^*}}\le\nu_0$, which implies that

$$\nu_0=0,\quad or\quad \nu_0\ge S(\mu_1,\mu_2)^{\frac{N}{2s}}. \qquad (4.19)$$

Next we consider the possibility of concentration at $x_j, j \in \mathcal{T}$. Take $\varepsilon > 0$ small enough and $\psi_\varepsilon \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ such that $\psi_\varepsilon = 1$ in $B_\varepsilon(x_j)$, $\psi_\varepsilon = 0$ in $B^c_{2\varepsilon}(x_j)$ and $|\nabla\psi_\varepsilon| \le \frac{C}{\varepsilon}$. Note that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega \frac{\mu_1|u_n|^2 + \mu_2|v_n|^2}{|x|^{2s}}\psi_\varepsilon dx = \lim_{\varepsilon\to 0}\int_\Omega \psi_\varepsilon d\mu = 0.$$

Arguing as in (4.18), we get

$$0 = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\langle J'(u_n, v_n), (\psi_\varepsilon u_n, \psi_\varepsilon v_n)\rangle \ge \sigma_j - \nu_j. \tag{4.20}$$

From (4.11) and (4.20) we have that $S(0,0)\nu_j^{\frac{2}{2^*_s}} \le \nu_j$, which implies that

$$\nu_j = 0, \quad or \quad \nu_j \ge S(0,0)^{\frac{N}{2s}} \ge S(\mu_1,\mu_2)^{\frac{N}{2s}}, \tag{4.21}$$

therefore the set $\mathcal{T}$ is finite.

Suppose that there exists $\nu_k, k \in \{0\} \cup \mathcal{T}$, such that $\nu_k \neq 0$. It follows that

$$\begin{aligned} c &= J(u_n, v_n) - \frac{1}{2}\langle J'(u_n, v_n), (u_n, v_n)\rangle + o(1) \\ &= \frac{s}{N}\int_\Omega F(u_n, v_n)dx + o(1) \\ &= \frac{s}{N}\Big(\int_\Omega F(u, v)dx + \nu_0 + \sum_{j\in\mathcal{T}}\nu_j\Big) \end{aligned}$$

By (4.19)-(4.21) and the assumption $c < c^*$, we get a contradiction and we deduce that $\nu_0 = \nu_j = 0, \forall j \in \mathcal{T}$. Up to a subsequence, $(u_n, v_n) \to (u, v)$ strongly in $(L^{2^*_s}(\Omega))^2$. By the weak-to-weak continuous property of the fractional Laplace operator [10, Lemma 2.2] and the Brezis-Lieb lemma [7], it is now enough to get that

$$\begin{aligned} &\lim_{n\to\infty}(\|u_n - u\|^2_{\mu_1} + \|v_n - v\|^2_{\mu_2}) \\ &= \lim_{n\to\infty}\langle J'(u_n, v_n) - J'(u, v), (u_n - u, v_n - v)\rangle \\ &= \lim_{n\to\infty}\langle J'(u_n, v_n), (u_n - u, v_n - v)\rangle = 0, \end{aligned}$$

thus $(u_n, v_n) \to (u, v)$ strongly in W. □

Lemma 4.4. *Under the assumptions of Theorem 1.3, there exists $\varepsilon > 0$ small enough such that*

$$\sup_{\tau\ge 0} J(\tau t_0 v^\varepsilon_{\mu_1}, \tau l_0 v^\varepsilon_{\mu_2}) < c^* = \frac{s}{N}S(\mu_1,\mu_2)^{\frac{N}{2s}}.$$

Proof. For all $\tau \ge 0$, by $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, we have that

$$J(\tau t_0 v^\varepsilon_{\mu_1}, \tau l_0 v^\varepsilon_{\mu_2}) = \frac{\tau^2}{2}\Big\{\|(t_0 v^\varepsilon_{\mu_1}, l_0 v^\varepsilon_{\mu_2})\|^2_W - \int_\Omega Q(t_0 v^\varepsilon_{\mu_1}, l_0 v^\varepsilon_{\mu_2})dx\Big\}$$

$$- \frac{\tau^{2_s^*}}{2_s^*} \int_\Omega F(t_0 v_{\mu_1}^\varepsilon, l_0 v_{\mu_2}^\varepsilon) dx.$$

For all $B_1, B_2 > 0$, there holds that

$$\max_{\tau \geq 0} (\frac{\tau^2}{2} B_1 - \frac{\tau^{2_s^*}}{2_s^*} B_2) = \frac{s}{N} (B_1 B_2^{-\frac{2}{2_s^*}})^{\frac{2_s^*}{2_s^*-2}}. \tag{4.22}$$

By the hypothesis $0 \leq \mu_2 \leq \mu_1 \leq \mu^*$, it follows from (2.10) that

$$N \leq 2b(\mu_1) \leq 2b(\mu_2) \tag{4.23}$$

and

$$2(b(\mu_1) - \delta) = 2b(\mu_1) - 2\delta \geq N - (N - 2s) = 2s. \tag{4.24}$$

Therefore, from (4.22)-(4.24) and Lemmas 4.1 and 4.2, for $\varepsilon > 0$ small enough, we get that

$$\begin{aligned}
\sup_{\tau \geq 0} J(\tau t_0 v_{\mu_1}^\varepsilon, \tau l_0 v_{\mu_2}^\varepsilon) &\leq \frac{s}{N} \Big(\frac{\|(t_0 v_{\mu_1}^\varepsilon, l_0 v_{\mu_2}^\varepsilon)\|_W^2 - \int_\Omega Q(t_0 v_{\mu_1}^\varepsilon, l_0 v_{\mu_2}^\varepsilon) dx}{\big(\int_\Omega F(t_0 v_{\mu_1}^\varepsilon, l_0 v_{\mu_2}^\varepsilon) dx \big)^{\frac{2}{2_s^*}}} \Big)^{\frac{N}{2s}} \\
&\leq \frac{s}{N} \Big(\frac{S(\mu_1, \mu_2)^{\frac{N}{2s}} + C\varepsilon^{2(b(\mu_1)-\delta)} - \int_\Omega Q(t_0 v_{\mu_1}^\varepsilon, l_0 v_{\mu_2}^\varepsilon) dx}{\big(S(\mu_1, \mu_2)^{\frac{N}{2s}} - C\varepsilon^{2_s^* \cdot (b(\mu_1)-\delta)} \big)^{\frac{2}{2_s^*}}} \Big)^{\frac{N}{2s}} \\
&\leq \frac{s}{N} \Big(\frac{S(\mu_1, \mu_2)^{\frac{N}{2s}} + C\varepsilon^{2(b(\mu_1)-\delta)} - C \int_\Omega |v_{\mu_1}^\varepsilon|^2 dx}{S(\mu_1, \mu_2)^{\frac{N}{2s} \cdot \frac{2}{2_s^*}} - C\varepsilon^{2(b(\mu_1)-\delta)}} \Big)^{\frac{N}{2s}} \\
&\leq \frac{s}{N} S(\mu_1, \mu_2)^{\frac{N}{2s}} \Big(\frac{1 + C\varepsilon^{2(b(\mu_1)-\delta)} - C \int_\Omega |v_{\mu_1}^\varepsilon|^2 dx}{1 - C\varepsilon^{2(b(\mu_1)-\delta)}} \Big)^{\frac{N}{2s}} \\
&= \frac{s}{N} S(\mu_1, \mu_2)^{\frac{N}{2s}} \Big(1 + \frac{C\varepsilon^{2(b(\mu_1)-\delta)} - C \int_\Omega |v_{\mu_1}^\varepsilon|^2 dx}{1 - C\varepsilon^{2(b(\mu_1)-\delta)}} \Big)^{\frac{N}{2s}} \\
&\leq \frac{s}{N} S(\mu_1, \mu_2)^{\frac{N}{2s}} \big(1 + C\varepsilon^{2(b(\mu_1)-\delta)} - C \int_\Omega |v_{\mu_1}^\varepsilon|^2 dx \big) \\
&= \frac{s}{N} S(\mu_1, \mu_2)^{\frac{N}{2s}} + C\varepsilon^{2(b(\mu_1)-\delta)} - C \int_\Omega |v_{\mu_1}^\varepsilon|^2 dx \\
&< \frac{s}{N} S(\mu_1, \mu_2)^{\frac{N}{2s}},
\end{aligned}$$

this concludes the proof. □

Proof of Theorem 1.3. Under the assumptions of Theorem 1.3, a standard argument of the Mountain Pass Theorem [4] shows that there exists a sequence $\{(u_n, v_n)\} \subset W$ such that $J(u_n, v_n) \to c$ and $J'(u_n, v_n) \to 0$ as $n \to \infty$, where

$$0 < c \leq \sup_{\tau \in [0,1]} J(\tau t_0 v_{\mu_1}^\varepsilon, \tau l_0 v_{\mu_2}^\varepsilon).$$

Lemma 4.4 implies that $c < c^*$ and by Lemma 4.3, there exists a subsequence of $\{(u_n, v_n)\}$ (still denoted by $\{(u_n, v_n)\}$) such that $(u_n, v_n) \to (u, v)$ strongly in W. Therefore (u, v) is a critical point of J corresponds to a nontrivial solution to problem (1.6). Set $u^+ = \max\{u, 0\}, v^+ = \max\{v, 0\}$, replacing u and v in the terms on the right hand side of equations in (1.6) by u^+ and v^+ respectively and repeating the above process, we obtain the existence of a nontrivial weak solution $(u, v) \neq (0, 0)$ to (1.6). Then, using Lemma 2.5 in [10], we have that $(u, v) \neq (0, 0)$ is nonnegative. From (1.6), $(\mathcal{H}_2)$ and definition of Q, we deduce that $u, v \neq 0$ and by the Maximum Principle [32], we get that $u, v > 0$ in Ω. This finishes the proof of Theorem 1.3. □

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