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Global weak solution to a generic reaction-diffusion nonlinear parabolic system †

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Summary

We consider a new generic reaction-diffusion system, given as the following form:

$$\begin{cases} \dfrac{\partial u}{\partial t} - div\Big(g\big(|\nabla(u_\sigma)|\big)\nabla u\Big) = f(t,x,u,v,\nabla v), \quad \text{in} \quad Q_T, \\ \\ \dfrac{\partial v}{\partial t} - d_v\Delta v = p(t,x,u,v,\nabla u) \quad \text{in} \quad Q_T, \\ \\ u(0,.) = u_0, v(0,.) = v_0 \quad \text{in} \quad \Omega, \\ \\ \dfrac{\partial u}{\partial \eta} = 0, \dfrac{\partial v}{\partial \eta} = 0 \quad \text{in} \quad \Sigma_T, \end{cases} \tag{1}$$

where $\Omega =]0,1[\times]0,1[$, $Q_T =]0,T[\times\Omega$ and $\Sigma_T =]0,T[\times\partial\Omega$, $(T > 0)$,η is an outward normal to domain Ω and u_0, v_0 is the image to be processed, $G_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp^{-(\frac{|x|^2}{4\sigma})}$, $x \in \mathbb{R}$, $\sigma > 0$ and $\nabla u_\sigma = u * \nabla G_\sigma$.

In this study we are going to proof that there is a global weak solution to the ptoblem (1), we truncate the system and show that it can be solved by using Schauder fixed point theorem in Banach spaces. Finally by making some estimations, we prove that the solution of the truncated system converge to the solution of the problem.

KEYWORDS:
nonlinear parabolic system, reaction-diffusion system, weak solution

1 | INTRODUCION

Image processing is a method to perform some operations on images, in order to get an enhanced image or to extract some useful information from it; recentlly, this topic has become a big challenging problem and a very active field of computer applications and researches[6]. Over the last few years, some wonderful results were observed and has used a reaction-diffusion models, which can describe many natural phenomena in a wide rage of disciplines. One of the most recently and active topic in this field has been the restortion of image, that is a technique in which we enhance the data (raw images) sensors placed on different artifacts of the life for various specified applications; the most common problem posed in this topic is how to eliminate the noise from the data image, the answer of this question is given in 1987[11] by Perona-Malik, in his nonlinear diffusion model; also called anisotropic diffusion model, it is one of the first attempts in which they include local information to reduce noise and enhance constrast while preserving the edge; but the basic model of Perona-Malik is ill-posed in the sense of Hadamard.

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To overcome this problem, in 1992[5], Catté, Lions, Morel and coll suggested introducing the regularization in space and time directly into the continuous equation in order to obtain a related well-posed model. They prove the existence, uniqueness and regularity for the related model and demonstrate exprimentally that the related model gives similar results to the Perona-Malik model[11]. The study of morfu in 2006[10] was based on the contrast enhancement and noise filtering. His model was ruled by the Fisher equation. The model proposed by Morfu is:

$$\begin{aligned} \tfrac{\partial u}{\partial t} - div(g(|\nabla u|)\nabla u) &= f(u) \ \text{ in } Q_T, \\ u(0,x) &= u_0(x) \ \text{ in } \Omega, \\ \tfrac{\partial u}{\partial \upsilon} &= 0 \ \text{ on } \Sigma_T, \end{aligned} \tag{2}$$

where Ω is the domaine of the image, $T > 0$, u_0 is the original image to be processed and $f(s) = s(s-a)(1-s)$ with $0 < a < 1$. The major inconveniences of this model are two, the first is the sensitivity to noise and the second is that no results of existence and consistency is proved. From this, the idea of Alaa and the others in 2014[2] combined the regularization procedure in catté[5] with Morfu[10] model. They suggested to modify the model proposed by Morfu[10] by appling a gaussian filter on the gradient of the noisy image during the calculation of coefficient of anisotropic diffusion; this study is devoted to a generalization of their work in the case of f is nonzero. The proposed model is:

$$\begin{aligned} \tfrac{\partial u}{\partial t} - div(g(|\nabla u_\sigma|)\nabla u) &= f(t,x,u) \text{ in } Q_T, \\ u(0,x) &= u_0(x) \text{ in } \Omega, \\ \tfrac{\partial u}{\partial \upsilon} &= 0 \ \text{ on } \Sigma_T, \end{aligned} \tag{3}$$

where $\Omega =]0,1[\times]0,1[$ denotes picture domain with boundary $\partial\Omega$, with Neumann boundary conditions, $u(t,x)$ is the solution of this PDE (restored image) and u_0 is the original image to be processed, $Q_T =]0,T[\times\Omega$ and $\Sigma_T =]0,T[\times\partial\Omega$ where T is a fixed reel number ($T > 0$), $\sigma > 0$, G_σ is the gaussian filter where:

$$G_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp(-\frac{|x|^2}{4\sigma}), \quad x \in \mathbb{R}^2.$$

They consider the gradient norm of ω as:

$$|\nabla\omega| = \left(\sum_{i=1}^{i=2} (\frac{\partial\omega}{\partial x_i})^2 \right)^{1/2},$$

$\nabla\omega_\sigma$ is the smoothed version of gradient norme where :

$$\nabla\omega_\sigma = \nabla(\omega * G_\sigma) = \omega * \nabla G_\sigma$$

.

The diffusivity g is smooth decreasing function defined by

$$g(0) = 1, \lim_{s\to\infty} g(s) = 0, \tag{4}$$

one of the diffusivities Perona and Malik proposed is:

$$g(s) = \frac{d}{\sqrt{1 + \upsilon\left(\frac{s}{\lambda}\right)^2}},$$

where $\upsilon \geq 0$, $d > 0$ and λ is a threshold (contrast) parameter that separates forward and backward diffusion[16].
The previous study shows how differential equations can be employed to restore a digital image, in which they give a demonstration of the consistency of proposed model. They used a new technique recently introduced by Pierre[13] for study of semi-linear isotropic systems. In 2016[4], Bassam Al-Hamzah and Naji Yabari proposed a new reaction-diffusion model in image processing, which they proved the existence of global solution for the nonlinear reaction-diffusion model. this study deals with the equation:

$$\begin{aligned} \tfrac{\partial u}{\partial t} - div(g(|\nabla u_\sigma|)\nabla u) &= f(t,x,u,\nabla u) \text{ in } Q_T, \\ u(0,x) &= u_0(x) \geq 0 \text{ in } \Omega, \\ \tfrac{\partial u}{\partial \upsilon} &= 0 \ \text{ on } \Sigma_T, \end{aligned} \tag{5}$$

where $\Omega =]0,1[\times]0,1[$, $Q_T =]0,T[\times\Omega$ and $\Sigma_T =]0,T[\times\partial\Omega$, where ($T > 0$), $G_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp^{-(\frac{|x|^2}{4\sigma})}$, $x \in \mathbb{R}$, $\sigma > 0$ and $\nabla u_\sigma = \nabla(u * G_\sigma) = u * \nabla G_\sigma$.
In fact these results $f(t,x,u,\nabla u)$ are a generalization of the work $f = 0$ presented by Catté[5], and the work $f = f(t,x,u)$

presented by Alaa[2]. In the same year[7], Lecheheb and all interested in the study of the existence of weak solutions of boundary value problem for the nonlinear System of Convection-Diffusion-Reaction. In 2018[1] Aaraba, Alaa, and Khalfi provided the existence of global solution to a generic reaction-diffusion system with application to image restoration and anhancement. This study is a generalization of the work presented by[[2],[5],[12]] in the case of reaction-diffusion equations. They gave an example of the application demonstrated on a novel bio-inspired image restoration model[1]. In the same year, Alaa and Zirhem[3], proposed a new model of nonlinear and anisotropic reaction diffusion system applied to image restoration and to contrast enhancement, this Model is based on a system of partial differential equations of type Fitzhugh-Nagumo, where they compared the performance of their alghorithm with that of classical Fizhugh- Nagumo model. In 2020[8], A. Ouaoua, A. Khaldi and M. Maouni give a new study of the stabilization to the solution for a Kirchhoff type reaction-diffusion equation.
The aim of this work is to give a generalisation of the work presented by[[2],[4],[1],[3]], in the case of reaction diffusion equations. We prove the existence of global weak solution to a generic reaction-diffusion nonlinear parabolic system. Our work is given by:

$$\begin{cases} \dfrac{\partial u}{\partial t} - div\big(g\big(|\nabla(u_\sigma)|\big)\nabla u\big) = f(t,x,u,v,\nabla v), \quad \text{in} \quad Q_T, \\ \\ \dfrac{\partial v}{\partial t} - d_v \Delta v = p(t,x,u,v,\nabla u) \quad \text{in} \quad Q_T, \\ \\ u(0,.) = u_0, v(0,.) = v_0 \quad \text{in} \quad \Omega, \\ \\ \dfrac{\partial u}{\partial \eta} = 0, \dfrac{\partial v}{\partial \eta} = 0 \quad \text{in} \quad \Sigma_T, \end{cases} \tag{6}$$

where Ω is smooth bounded domain in $\mathbb{R}^n$ and $T \in [0,\infty[$, $Q_T =]0,T[\times\Omega$ and $\Sigma_T =]0,T[\times\partial\Omega$ where $\partial\Omega$ denotes the boundary of Ω, the initial conditions u_0, v_0 are only assumed to be square integrable, η is an outward normal to domain Ω. Let $\sigma > 0$, ∇u_σ be a regularization by convolution of ∇u. It is defined as $\nabla u_\sigma = \nabla(G_\sigma * u)$ and the diffusivity g check the same properties provided by Alaa[2], which is given in the equation (4) .
The nonlinear functions $f, p : Q_T \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ mesurable for (t,x) and locally lipshitz continous for u and v, $\exists r > 0$ for almost $(t,x) \in Q_T$ such that
$|f(t,x,u,v,q) - f(t,x,\bar{u},v,q_1)| \le k_1(r)[|u-\bar{u}| + \|q-q_1\|]$,
$|p(t,x,u,v,q) - p(t,x,u,\bar{v},q_2)| \le k_2(r)[|v-\bar{v}| + \|q-q_2\|]$,
for all $0 \le |u|, |\bar{u}|, \|q\|, \|q_2\| \le r$ and $0 \le |v|, |\bar{v}|, \|q\|, \|q_2\| \le r$.
In addition the nonlinearities satisfy the following main properties:
(H_1)- the positivity property:
For almost $(t,x) \in Q_T$

$$f(t,x,0,s_1,\nabla s_1) \ge 0, \quad \text{and} \quad p(t,x,s_2,0,\nabla s_2) \ge 0, \quad \forall s_i \ge 0, i = 1,2.$$

$(H_2) - \forall(u,v,l_1) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$, $uf(t,x,u,v,q_1) \le 0$ and $vf(t,x,u,v,q_1) \le 0$,

$- \forall(u,v,q_2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$, $up(t,x,u,v,q_2) \le 0$ and $vp(t,x,u,v,q_2) \le 0$,

$$(u+v)(f+p)(t,x,u,v,q_1,q_2) \le 0.$$

. Furthermore,

$$\sup_{|r|+|s| \le R} \left(|f(t,x,l_1,l_2,\nabla l_2)| + |p(t,x,l_1,l_2,\nabla l_1)|\right) \in L^1(Q_T), \quad \text{for} \quad R = 2r > 0.$$

In this work we give a proof of the global weak solution of the problem (1), we truncate the system ans show that there is a solution by using Schauder fixed point theorem in Banach spaces. Finally by making some estimations we prove that the solution of the truncated system converge to the solution of the proposed problem.
Now we will recall some functional spaces that willbe used throughout this paper. For all $k \in \mathbb{N}$, $H^k(\Omega)$ is the set of functions u defined in Ω such as u and its order $D^m u$ derivatives where $|m| = \sum_n^{j=1} m_j \le k$ are in $L^2(\Omega)$. $H^k(\Omega)$ is Hilbert space for the norm

$$\|u\|_{H^k(\Omega)} = \left(\sum_{|m| \le k} \int_\Omega |D^m u|^2 dx \right)^{\frac{1}{2}}. \tag{7}$$

We denote by $(H^1(\Omega))'$ the dual of $H^1(\Omega)$.
$L^p(0,T,H^k(\Omega))$ is the set of functions u such that, for all every $t \in (0,T)$, $u(t)$ belongs to $H^k(\Omega)$ with the norm

$$\|u\|_{L^p(0,T,H^k(\Omega))} = \left(\int_0^T \|u(t)\|^p_{H^k(\Omega)} dt \right)^{\frac{1}{p}}, \quad 1 < p < \infty, \quad k \in \mathbb{N}. \tag{8}$$

$L^\infty(0,T,L^2(\Omega))$ is the set of functions u such that, for all every $t \in (0,T)$, $u(t)$ belongs to $L^2(\Omega)$ with the norm

$$\|u\|_{L^\infty(0,T,L^2(\Omega))} = \left(\sup_{0<t<T} \|u(t)\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}. \tag{9}$$

$L^\infty(0,T,C^\infty(\Omega))$ is the set of funcions u such that, for all every $t \in (0,T)$, $u(t)$ belongs to $C^\infty(\Omega)$ with the norm

$$\|u\|_{L^\infty(0,T,C^\infty(\Omega))} = \inf\{c, \|u(t)\|_{C^\infty(\Omega)} \leq C \quad in \quad (0,T)\}. \tag{10}$$

We have that

$$\|(u,v)\|_{L^p(0,T,H^k(\Omega))^2} = \|u\|_{L^p(0,T,H^k(\Omega))} + \|v\|_{L^p(0,T,H^k(\Omega))}, \quad 1 < p < \infty, \quad k \in \mathbb{N}. \tag{11}$$

$$\|(u,v)\|_{L^\infty(0,T,L^2(\Omega))^2} = \|u\|_{L^\infty(0,T,L^2(\Omega))} + \|v\|_{L^\infty(0,T,L^2(\Omega))}. \tag{12}$$

$$\|(u,v)\|_{L^\infty(0,T,C^\infty(\Omega))} = \|u\|_{L^\infty(0,T,C^\infty(\Omega))} + \|v\|_{L^\infty(0,T,C^\infty(\Omega))}. \tag{13}$$

2 | THE THEORETICAL STUDY

First, we clearly state our definition of weak solution to the reaction-diffusion system, we define the folowing spaces:

$$X = \{u, v \in L^2(0,T,H^1(\Omega)) \cap C(0,T,L^2(\Omega)), u(0,.) = u_0 \text{ and } v(0,.) = v_0 \}.$$

$$Z = \{\phi, \psi \in C^1(Q_T) \text{ such that } \phi(T,.) = 0 \text{ and } \psi(T,.) = 0\}.$$

$$D = \{u, v \in L^2(0,T,H^1(\Omega)) \cap C(0,T,L^2(\Omega))\}.$$

Definition 1. We call (u,v) a weak solution of system (1) if
• $\forall u, v \in X$ and $\forall \phi, \psi \in Z$ we have

$$\begin{cases} -\int\limits_{Q_T} u\dfrac{\partial \phi}{\partial t} dxdt + \int\limits_{Q_T} g(|\nabla u_\sigma|)\nabla u \nabla \phi dxdt = \int\limits_{Q_T} f(t,x,u,v,\nabla v)\phi dxdt + \int\limits_{\Omega} u_0 \phi(0,x)dx, \\ -\int\limits_{Q_T} v\dfrac{\partial \psi}{\partial t} dxdt + \int\limits_{Q_T} d_v \nabla v \nabla \psi dxdt = \int\limits_{Q_T} p(t,x,u,v,\nabla u)\psi dxdt + \int\limits_{\Omega} v_0 \psi(0,x)dx, \end{cases} \tag{14}$$

where $f(t,x,u,v,\nabla v)$, $p(t,x,u,v,\nabla u) \in L^2(Q_T)$.

Theorem 1. Under the assumption (H_1), (H_2) and for the diffusivity g given in (4). The reaction-diffusion system (1) admits a weak positive solution (u,v) in the sense defined in (14) for all $u_0, v_0 \in L^2(\Omega)$ such that $u_0, v_0 \geq 0$.

Proof. The proof of the Theorem (1) is done in four step:

2.1 | Step1: Existence result for bounded nonlinearities

First, we will show the existence result for bounded source terms f, p.

Lemma 1. Under the obove assumption (H_1) and (H_2) of the nonlinearities, if there exists M_f, M_p such that for almost every $(t,x) \in Q_T$,

$$|f(t,x,s_1,s_2,\nabla s_2)| \le M_f, \quad |p(t,x,s_1,s_2,\nabla s_1)| \le M_p \quad \forall (s_1,s_2) \in \mathbb{R}^2, \tag{15}$$

then for every u_0,v_0 in $L^2(\Omega)$, there exists a weak solution (u,v) to the considered system (14). Moreover there exists $C(M_f, M_p, T, \sigma, \|u_0\|_{L^2(\Omega)}, \|v_0\|_{L^2(\Omega)})$ such that

$$\|(u,v)\|_{L^\infty(0,T,L^2(\Omega))^2} + \|(u,v)\|_{L^2(0,T,H^1(\Omega))^2} \le C. \tag{16}$$

Furthermore if u_0, v_0 are positive and f, p are quasi-postive, then $u(t,x) \ge 0$ and $v(t,x) \ge 0$ for almost every $(t,x) \in Q_T$.

Proof. We show the existence of a weak solution by the classical Schauder fixed point theorem.
We introduce the space

$$W(0,T) : \{u, v \in L^2(0,T,H^1(\Omega)) \cap L^\infty(0,T,L^2(\Omega)), \frac{\partial u}{\partial t}, \frac{\partial v}{\partial t} \in L^2(0,T,(H^1(\Omega))')\}$$

Let $w = (w_1, w_2) \in W(0,T)^2$ and (u,v) be the solution of a linearization of problem (1) given by:
- $\forall u, v \in D$ and $\forall \phi, \psi \in Z$

$$\begin{cases} -\int\limits_{Q_T} u\frac{\partial \phi}{\partial t} dxdt + \int\limits_{Q_T} g(|\nabla (w_1)_\sigma|)\nabla u \nabla \phi dxdt = \int\limits_{Q_T} f(t,x,w_1,w_2,\nabla w_2)\phi dxdt + \int\limits_{\Omega} u_0 \phi(0,x)dx, \\ -\int\limits_{Q_T} v\frac{\partial \psi}{\partial t} dxdt + \int\limits_{Q_T} d_v \nabla v \nabla \psi dxdt = \int\limits_{Q_T} p(t,x,w_1,w_2,\nabla w_1)\psi dxdt + \int\limits_{Q_T} v_0 \psi(0,x)dx. \end{cases} \tag{17}$$

The application $w \in W(0,T) \longrightarrow (u,v) \in W(0,T)$ is clearly well defined. In fact $w_1 \in L^\infty(0,T,L^2(\Omega))$ and g, G_σ are in $C^\infty(\Omega)$ so $g(|\nabla (w_1)_\sigma|) \in L^\infty(0,T,C^\infty(Q_T))$ and since g is non-increasing it satisfied

$$a \le g(|\nabla (w_1)_\sigma|) \le d, \tag{18}$$

where $d > 0$ and a is a positive constant that depends only on σ and g. Let (u,v) the solution of the linearized problem (17).
Now we establish some important estimates to construct the functional setting where Schauder fixed point theory is applicable.
For all $0 < t < T$ and for $\phi = u$ and $\psi = v$

$$\begin{cases} -\int\limits_{Q_t} u\frac{\partial u}{\partial t} dxdt + \int\limits_{Q_t} g(|\nabla (w_1)_\sigma|)\nabla u \nabla u dxdt = \int\limits_{Q_t} f(t,x,w_1,w_2,\nabla w_2) u dxdt + \int\limits_{\Omega} u_0^2 dx, \\ -\int\limits_{Q_t} v\frac{\partial v}{\partial t} dxdt + d_v \nabla v \nabla v dxdt = \int\limits_{Q_t} p(t,x,w_1,w_2,\nabla w_1) v dxdt + \int\limits_{Q_T} v_0^2 dx. \end{cases}$$

$$\begin{cases} -\frac{1}{2}\int\limits_{\Omega} u^2(t)dx + \frac{1}{2}\int\limits_{\Omega} u^2(0)dx + \int\limits_{Q_t} g(|\nabla (w_1)_\sigma|)|\nabla u|^2 dxdt = \int\limits_{Q_t} f(t,x,w_1,w_2,\nabla w_2)u + \int\limits_{\Omega} u_0^2, \\ -\frac{1}{2}\int\limits_{\Omega} v^2(t)dx + \frac{1}{2}\int\limits_{\Omega} v^2(0)dx + d_v \int\limits_{Q_t} |\nabla v|^2 dxdt = \int\limits_{Q_t} p(t,x,w_1,w_2,\nabla w_1) v dxdt + \int\limits_{\Omega} v_0^2 dx, \end{cases}$$

$$\begin{cases} \frac{1}{2}\int\limits_{\Omega} u^2(t)dx + \int\limits_{Q_t} g(|\nabla (w_1)_\sigma|)|\nabla u|^2 dxdt = \int\limits_{Q_t} f(t,x,w_1,w_2,\nabla w_2) u dxdt + \frac{1}{2}\int\limits_{\Omega} u_0^2 dx, \\ \frac{1}{2}\int\limits_{\Omega} v^2(t)dx + d_v \int\limits_{Q_t} |\nabla v|^2 dxdt = \int\limits_{Q_t} p(t,x,w_1,w_2,\nabla w_1) v dxdt + \frac{1}{2}\int\limits_{\Omega} v_0^2 dx, \end{cases} \tag{19}$$

$$\begin{cases} \frac{1}{2}\int\limits_{\Omega} u^2(t)dx + a\int\limits_{Q_t} |\nabla u|^2 dxdt \le \int\limits_{Q_t} |M_f||u| dxdt + \frac{1}{2}\int\limits_{\Omega} u_0^2 dx, \\ \frac{1}{2}\int\limits_{\Omega} v^2(t)dx + d_v \int\limits_{Q_t} |\nabla v|^2 dxdt \le \int\limits_{Q_t} |M_p||v| dxdt + \frac{1}{2}\int\limits_{\Omega} v_0^2 dx, \end{cases} \tag{20}$$

with $a\int_{Q_t}|\nabla u|^2 \geq 0$ and $d_v\int_{Q_t}|\nabla v|^2 \geq 0$.

$$\begin{cases} \frac{1}{2}\int\limits_{\Omega} u^2(t)dx \leq \int\limits_{Q_T} |M_f||u|dxdt + \frac{1}{2}\int\limits_{\Omega} u_0^2 dx, \\ \frac{1}{2}\int\limits_{\Omega} v^2(t)dx \leq \int\limits_{Q_T} |M_p||v|dxdt + \frac{1}{2}\int\limits_{\Omega} v_0^2 dx, \end{cases}$$

with Cauchy Schwartz

$$\begin{cases} \frac{1}{2}\int\limits_{\Omega} u^2(t)dx \leq \left(\int\limits_{Q_T} |M_f|^2 dxdt\right)^{\frac{1}{2}}\left(\int\limits_{Q_T} |u|^2 dxdt\right)^{\frac{1}{2}} + \frac{1}{2}\int\limits_{\Omega} u_0^2 dx, \\ \frac{1}{2}\int\limits_{\Omega} v^2(t)dx \leq \left(\int\limits_{Q_T} |M_p|^2 dxdt\right)^{\frac{1}{2}}\left(\int\limits_{Q_T} |v|^2 dxdt\right)^{\frac{1}{2}} + \frac{1}{2}\int\limits_{\Omega} v_0^2 dx, \end{cases}$$

with Young

$$\begin{cases} \frac{1}{2}\int\limits_{\Omega} u^2(t)dx \leq \frac{\varepsilon}{2}\int\limits_{Q_T} |M_f|^2 dxdt + \frac{1}{2\varepsilon}\int\limits_{Q_T} u^2 dxdt + \frac{1}{2}\int\limits_{\Omega} u_0^2 dx. \\ \frac{1}{2}\int\limits_{\Omega} v^2(t)dx \leq \frac{\varepsilon}{2}\int\limits_{Q_T} |M_p|^2 dxdt + \frac{1}{2\varepsilon}\int\limits_{Q_T} v^2 dxdt + \frac{1}{2}\int\limits_{\Omega} v_0^2 dx. \end{cases}$$

Let $\varepsilon = 1$ and $\int_{Q_T}|M_f|^2 dxdt = M_f$, $\int_{Q_T}|M_p|^2 dxdt = M_p$

$$\begin{cases} \int\limits_{\Omega} u^2(t)dx \leq |M_f| + \int\limits_{Q_T} u^2 dxdt + \int\limits_{\Omega} u_0^2 dx. \\ \int\limits_{\Omega} v^2(t)dx \leq |M_p| + \int\limits_{Q_T} v^2 dxdt + \int\limits_{\Omega} v_0^2 dx. \end{cases} \tag{21}$$

Using Gronwall's inequality we obtain

$$\begin{cases} \int\limits_{Q_T} u^2 dxdt \leq \Big(\exp(T) - 1\Big)\Big(M_f + \int\limits_{\Omega} u_0^2 dx\Big), \\ \int\limits_{Q_T} v^2 dxdt \leq \Big(\exp(T) - 1\Big)\Big(M_p + \int\limits_{\Omega} v_0^2 dx\Big), \end{cases} \tag{22}$$

$$\begin{cases} \sup\limits_{0\leq t\leq T}\int\limits_{\Omega} u^2(t)dx \leq M_f + \Big(\exp(T) - 1\Big)\Big(M_f + \int\limits_{\Omega} u_0^2 dx\Big) + \int\limits_{\Omega} u_0^2 dx := c_1. \\ \sup\limits_{0\leq t\leq T}\int\limits_{\Omega} v^2(t)dx \leq M_p + \Big(\exp(T) - 1\Big)\Big(M_f + \int\limits_{\Omega} v_0^2 dx\Big) + \int\limits_{\Omega} v_0^2 dx := c_2. \end{cases}$$

Therefore by setting $C_1 = c_1 + c_2$ we get

$$\|(u,v)\|_{L^\infty(0,T,L^2(\Omega))^2} \leq C_1. \tag{23}$$

$$\begin{cases} \frac{1}{2}\int\limits_{\Omega} u^2(t)dx + a\int\limits_{Q_T} |\nabla u|^2 dxdt \leq \int\limits_{Q_T} |M_f||u|dxdt + \frac{1}{2}\int\limits_{\Omega} u_0^2 dx, \\ \frac{1}{2}\int\limits_{\Omega} v^2(t)dx + d_v\int\limits_{Q_T} |\nabla v|^2 dxdt \leq \int\limits_{Q_T} |M_p||v|dxdt + \frac{1}{2}\int\limits_{\Omega} v_0^2 dx, \end{cases}$$

$$\begin{cases} \frac{1}{2}\int\limits_{Q_T} |u|^2 dxdt + a\int\limits_{Q_T} |\nabla u|^2 dxdt \leq \frac{1}{2}M_f + \frac{1}{2}\int\limits_{\Omega} u^2(t)dx + \frac{1}{2}\int\limits_{\Omega} u_0^2 dx, \\ \frac{1}{2}\int\limits_{Q_T} |v|^2 + d_v\int\limits_{Q_T} |\nabla v|^2 dxdt \leq \frac{1}{2}M_p + \frac{1}{2}\int\limits_{\Omega} v^2(t)dx + \frac{1}{2}\int\limits_{\Omega} v_0^2 dx, \end{cases}$$

$$\begin{cases} \frac{1}{2}\int\limits_{Q_T}|u|^2dxdt + a\int\limits_{Q_T}|\nabla u|^2dxdt \le M_f + \int\limits_{\Omega}u^2(t)dx + \int\limits_{\Omega}u_0^2dx, \\ \frac{1}{2}\int\limits_{Q_T}|v|^2dxdt + d_v\int\limits_{Q_T}|\nabla v|^2dxdt \le M_p + \int\limits_{\Omega}v^2(t)dx + \int\limits_{\Omega}v_0^2dx, \end{cases}$$

$$\begin{cases} \min(\frac{1}{2},a)\int\limits_{Q_T}|u|^2dxdt + \int\limits_{Q_T}|\nabla u|^2dxdt \le M_f + \int\limits_{\Omega}u^2(t)dx + \int\limits_{\Omega}u_0^2dx, \\ \min(\frac{1}{2},d_v)\int\limits_{Q_T}|v|^2dxdt + \int\limits_{Q_T}|\nabla v|^2dxdt \le M_p + \int\limits_{\Omega}v^2(t)dx + \int\limits_{\Omega}v_0^2dx, \end{cases}$$

$$\begin{cases} \int\limits_{Q_T}|u|^2dxdt + \int\limits_{Q_T}|\nabla u|^2dxdt \le \dfrac{M_f + \int_\Omega u^2(t)dx + \int_\Omega u_0^2dx}{\min(\frac{1}{2},a)} := c_4, \\ \int\limits_{Q_T}|v|^2dxdt + \int\limits_{Q_T}|\nabla v|^2dxdt \le \dfrac{M_p + \int_\Omega v^2(t)dx + \int_\Omega v_0^2dx}{\min(\frac{1}{2},d_v)} := c_5, \end{cases}$$

setting $C_2 = c_4 + c_5$

$$\|(u,v)\|_{L^2(0,T,H^1(\Omega))^2} \le C_2. \tag{24}$$

Next we estimate the $\frac{\partial u}{\partial t}$ and $\frac{\partial v}{\partial t}$ in $L^2(0,T,(H^1(\Omega))')$ we have

$$\begin{cases} \dfrac{\partial u}{\partial t} = div(g(|\nabla u_\sigma|)\nabla u) + f(t,x,u,v,\nabla v), \\ \dfrac{\partial v}{\partial t} = d_v\Delta v + p(t,x,u,v,\nabla u), \end{cases}$$

$$\|\partial u_t\|_{L^2(0,T,(H^1(\Omega))')} \le c\|\nabla u\|_{L^2(Q_T)} + M_f T := C_3,$$
$$\|\partial v_t\|_{L^2(0,T,(H^1(\Omega))')} \le d_v\|\nabla v\|_{L^2(Q_T)} + M_p T := C_4,$$

setting $C_5 = C_3 + C_4$

$$\|(\partial u_t, \partial v_t)\|_{L^2(0,T,(H^1(\Omega))')^2} \le C_5. \tag{25}$$

Now we are in a position to apply Schauder fixed point in the functional space

$$W_0(0,T) = \{(u,v) \in L^2(0,T,H^1(\Omega))^2 \cap L^\infty(0,T,(H^1(\Omega))')^2, \|(u,v)\|_{L^\infty(0,T,L^2(\Omega))^2} \le C_1, \|(u,v)\|_{L^2(0,T,H^1(\Omega))^2} \le C_2,$$
$$\|(\partial u_t, \partial v_t)\|_{L^2(0,T,(H^1(\Omega))')^2} \le C_5, u(.,0) = u_0, v(.,0) = v_0\}.$$

We can easily verify that $W_0(0,T)$ is a nonemply closed convex in $W(0,T)$ to use Schauder fixed point, we will show that the aplication

$$\begin{array}{rccc} F: & W_0(0,T) & \longrightarrow & W_0(0,T) \\ & w & \mapsto & F(w), \end{array} \tag{26}$$

is a weakly continuous.
We consider a sequence $w_n \in W_0(0,T)$ such that $w_n \rightharpoonup w$ in $W_0(0,T)$ and let $F(w_n) = (u_n, v_n)$ thus

$$\begin{cases} \dfrac{\partial u_n}{\partial t} = div(g(|\nabla w_{1n\sigma}|)\nabla u_n) + f(t,x,w_{1n},w_{2n},\nabla w_{2n}), \\ \dfrac{\partial v_n}{\partial t} = d_v\Delta v_n + p(t,x,w_{1n},w_{2n},\nabla w_{1n}), \end{cases}$$

we have that (u_n, v_n) is bounded in $L^2(0,T,H^1(\Omega))^2$ and $(\partial_t u_n, \partial_t v_n)$ is bounded in $L^2(0,T,(H^1(\Omega))')^2$ then by Simon[14] is relatively compact in $L^2(Q_T)$ which means we can extract a subsequence denoted w_n such that:
•$u_n \rightharpoonup u$ in $L^2(0,T,H^1(\Omega))$.
•$v_n \rightharpoonup v$ in $L^2(0,T,H^1(\Omega))$.
•$f(t,x,w_n,\nabla v_n) \to f(t,x,w,\nabla v)$ in $L^2(Q_T)$.
•$p(t,x,w_n,\nabla u_n) \to p(t,x,w,\nabla u)$ in $L^2(Q_T)$.
•$u_n \to u$ in $L^2(0,T,L^2(\Omega))$ and almost every where in Q_T.
•$u_n \to u$ in $L^2(0,T,L^2(\Omega))$ and almost every where in Q_T.
•$\nabla u_n \rightharpoonup \nabla u$ in $L^2(0,T,L^2(\Omega))$.
•$\nabla v_n \rightharpoonup \nabla v$ in $L^2(0,T,L^2(\Omega))$.
•$w_n \to w$ in $L^2(0,T,L^2(\Omega))$ and almost every where in Q_T.
•$g(|\nabla(w_1)_{n_\sigma}|) \to g(|\nabla(w_1)_\sigma|)$ in $L^2(0,T,H^1(\Omega))$.
•$\partial_t u_n \rightharpoonup \partial_t u$ in $L^2(0,T,(H^1(\Omega))')$.
•$\partial_t v_n \rightharpoonup \partial_t v$ in $L^2(0,T,(H^1(\Omega))')$.
Using these convergence, we can pass to the limit and show that the limit (u,v) are solution of the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = div(g(|\nabla w_{1\sigma}|)\nabla u) + f(t,x,w_1,w_2,\nabla w_2), \\ \\ \dfrac{\partial v}{\partial t} = d_v \Delta v + p(t,x,w_1,w_2,\nabla w_1), \end{cases}$$

thus $F(w) = (u,v)$ then F is weakly continuous, then we deduce the existence of $w = (u,v) \in W_0(0,T)$ such as $w = F(w)$ and thus thee existence of $(u,v) \in W(0,T)$. □

2.2 | Step2: Existence result for truncated nonlinearities

In this case, we truncate f and p using truncation function $\Psi_n \in C_c^\infty(\mathbb{R})$, such that $0 \le \Psi_n \le 1$ and

$$\Psi_n(r) = \begin{cases} 1 & \text{if} \quad |r| \le n, \\ 0 & \text{if} \quad |r| \ge n+1. \end{cases} \tag{27}$$

Thus, we can state that the approximate problem

$$\begin{cases} \dfrac{\partial u_n}{\partial t} = div(g(|\nabla u_{n\sigma}|)\nabla u_n) + f_n(t,x,u_n,v_n,\nabla v_n), \\ \dfrac{\partial v_n}{\partial t} = d_v \Delta v_n + p_n(t,x,u_n,v_n,\nabla u_n), \end{cases} \tag{28}$$

where

$$f_n(t,x,u_n,v_n,\nabla v_n) = \Psi_n(|u|) f(t,x,u,v,\nabla v),$$

and

$$p_n(t,x,u_n,v_n,\nabla u_n) = \Psi_n(|u|) p(t,x,u,v,\nabla u),$$

by the means of theorem (1) the problem (28) admits a weak solution. Now we show that a subsequence (u_n, v_n) converges to the weak solution (u,v) of problem (1), for this we need to prove the following results.

Lemma 2. Let (u_n, v_n) be the solution of the approximate problem (28) then
(1) There exists a constant $M\left(\int_\Omega u_{n0}dx, \int_\Omega v_{n0}dx, T\right)$ such that

$$\int_{Q_T} (u_n + v_n) dxdt \le M \quad \forall t \in [0,T].$$

(2) There exists $M_1 > 0$ such that

$$\int_{Q_T} |\nabla u_n|^2 + |\nabla v_n|^2 dxdt \le M_1.$$

(3) There exists $M_2 > 0$ such that
$$\int\limits_{Q_T} |f_n| + |p_n| dxdt \leq M_2.$$

Proof. (1) We have the approximatif problem
$$\begin{cases} \dfrac{\partial u_n}{\partial t} - div(g(|\nabla u_{n\sigma}|)\nabla u_n) = f_n(t,x,u_n,v_n,\nabla v_n), \\ \\ \dfrac{\partial v_n}{\partial t} - d_v \Delta v_n = p_n(t,x,u_n,v_n,\nabla u_n), \end{cases}$$
$$u_n \frac{\partial u_n}{\partial t} - div(g(|\nabla u_{n\sigma}|)\nabla u_n) u_n = u_n f_n(t,x,u_n,v_n,\nabla v_n),$$
$$v_n \frac{\partial u_n}{\partial t} - div(g(|\nabla u_{n\sigma}|)\nabla u_n) v_n = v_n f_n(t,x,u_n,v_n,\nabla v_n),$$
$$(u_n+v_n)\frac{\partial u_n}{\partial t} - div(g(|\nabla u_{n\sigma}|)\nabla u_n)(u_n+v_n) = (u_n+v_n) f_n(t,x,u_n,v_n,\nabla v_n),$$
$$u_n \frac{\partial v_n}{\partial t} - d_v \Delta v_n u_n) u_n = u_n p_n(t,x,u_n,v_n,\nabla u_n),$$
$$v_n \frac{\partial v_n}{\partial t} - d_v \Delta v_n v_n = v_n p_n(t,x,u_n,v_n,\nabla u_n),$$
$$(u_n+v_n)\frac{\partial v_n}{\partial t} - d_v \Delta v_n (u_n+v_n) = (u_n+v_n) p_n(t,x,u_n,v_n,\nabla u_n),$$
$$\begin{cases} (u_n+v_n)\dfrac{\partial u_n}{\partial t} - div(g(|\nabla u_{n\sigma}|)\nabla u_n)(u_n+v_n) = (u_n+v_n) f_n(t,x,u_n,v_n,\nabla v_n), \\ (u_n+v_n)\dfrac{\partial v_n}{\partial t} - d_v \Delta v_n (u_n+v_n) = (u_n+v_n) p_n(t,x,u_n,v_n,\nabla u_n), \end{cases}$$
$$|\int\limits_{Q_T} (u_n+v_n)\frac{\partial(u_n+v_n)}{\partial t} dxdt - \int\limits_{Q_T} \left[div(g(|\nabla u_{n\sigma}|)\nabla u_n) + d_v \Delta v_n\right](u_n+v_n) dxdt| = |\int\limits_{Q_T} (u_n+v_n)\left[(f_n+p_n)(t,x,u_n,v_n,\nabla v_n,\nabla u_n)\right] dxdt|,$$
$$\int\limits_{Q_T} (u_n+v_n)\frac{\partial(u_n+v_n)}{\partial t} dxdt \leq \int\limits_{Q_T} (u_n+v_n)\left[(|f_n|+|p_n|)(t,x,u_n,v_n,\nabla v_n,\nabla u_n)\right] dxdt,$$
$$\frac{1}{2}\int\limits_{\Omega} (u_n(t)+v_n(t)) dx - \frac{1}{2}\int\limits_{\Omega} (u_n(0)+v_n(0)) dx \leq \int\limits_{Q_T} (u_n+v_n)\left[(|f_n|+|p_n|)(t,x,u_n,v_n,\nabla v_n,\nabla u_n)\right] dxdt,$$
$$\begin{aligned} \frac{1}{2}\int\limits_{\Omega} (u_n(t)+v_n(t))^2 dx - \frac{1}{2}\int\limits_{\Omega} (u_{n0}+v_{n0})^2 dx &\leq \int\limits_{Q_T} (u_n+v_n)\left[(|f_n|+|p_n|)(t,x,u_n,v_n,\nabla v_n,\nabla u_n)\right] dxdt \\ &\leq \int\limits_{Q_T} (u_n+v_n)(|M_f|+|M_p|) dxdt, \end{aligned}$$

setting $|M_f| + |M_p| = c$
$$\frac{1}{2}\int\limits_{\Omega} (u_n(t)+v_n(t))^2 dx - \frac{1}{2}\int\limits_{\Omega} (u_{n0}+v_{n0})^2 dx \leq c\int\limits_{Q_T} (u_n+v_n) dxdt,$$
$$\frac{1}{2}\int\limits_{\Omega} (u_n(t)+v_n(t))^2 dx \leq c\int\limits_{Q_T} (u_n+v_n) dxdt \frac{1}{2}\int\limits_{\Omega} (u_{n0}+v_{n0})^2 dx,$$
$$\int\limits_{\Omega} (u_n(t)+v_n(t))^2 dx \leq 2c\int\limits_{Q_T} (u_n+v_n) dxdt \int\limits_{\Omega} (u_{n0}+v_{n0})^2 dx,$$

using a standard Gronwall's argument we get:

$$\int_{Q_T}(u_n+v_n)dxdt \le \left(\int_\Omega (u_{n0}+v_{n0})^2dx\right)\left(\exp(\int_0^T 1ds)\right),$$
$$\le \left(\int_\Omega (u_{n0}+v_{n0})^2dx\right)\left(\exp(T)\right),$$
$$\int_{Q_T}(u_n+v_n)dxdt \le M.$$

(2) We have:

$$\frac{\partial u_n}{\partial t} - div(g(|\nabla u_{n\sigma}|)\nabla u_n) = f_n(t,x,u_n,v_n,\nabla v_n),$$
$$u_n\frac{\partial u_n}{\partial t} - div(g(|\nabla u_{n\sigma}|)\nabla u_n)u_n = u_nf_n(t,x,u_n,v_n,\nabla v_n),$$
$$\int_{Q_t} u_n\frac{\partial u_n}{\partial t}dxdt - \int_{Q_t} div(g(|\nabla u_{n\sigma}|)\nabla u_n)u_ndxdt = \int_{Q_t} u_nf_n(t,x,u_n,v_n,\nabla v_n)dxdt,$$
$$\int_{Q_t} u_n\frac{\partial u_n}{\partial t}dxdt + \int_{Q_t} g(|\nabla u_{n\sigma}|)|\nabla u_n|^2dxdt \le \int_{Q_t}|u_n||f_n(t,x,u_n,v_n,\nabla v_n)|dxdt,$$
$$\le \int_{Q_t}|u_n||M_f|dxdt,$$
$$\frac{1}{2}\int_\Omega u_n(t)^2dx - \frac{1}{2}\int_\Omega u_{n0}^2dx + a\int_{Q_T}|\nabla u_n|^2dxdt \le \int_{Q_T}|u_n||M_f|dxdt, \tag{29}$$

using Cauchy Schwartz and young's we get

$$\frac{1}{2}\int_\Omega u_n(t)^2dx - \frac{1}{2}\int_\Omega u_{n0}^2dx \le \Big(\int_{Q_T}|u_n|^2dxdt\Big)^{\frac{1}{2}}\Big(\int_{Q_T}|M_f|^2dxdt\Big)^{\frac{1}{2}},$$
$$\le \frac{\varepsilon}{2}\int_{Q_T}|u_n|^2dxdt + \frac{M_f}{2\epsilon}mes(Q_T),$$
$$\le \frac{1}{2}\int_{Q_T}u_n^2dxdt + \frac{M_f}{2}mes(Q_T),$$
$$\int_\Omega u_n^2(t)dx - \int_\Omega u_{n0}^2dx \le \int_{Q_T}u_n^2dxdt + M_fmes(Q_T),$$
$$\int_\Omega u_n^2(t)dx \le \int_{Q_T}u_n^2dxdt + \int_\Omega u_{n0}^2dx + M_fmes(Q_T), \tag{30}$$

using Gronwall's we get

$$\int_{Q_T}u_n^2dxdt \le \left(\int_\Omega u_{n0}^2dx + M_fmes(Q_T)\right)\exp(T), \tag{31}$$
$$\int_{Q_T}u_n^2dxdt \le c_1, \tag{32}$$

we have that $\int_{Q_T}u_n^2dxdt$ is bounded.
Let

$$\frac{\partial v_n}{\partial t} - d_v\Delta v_n = p_n(t,x,u_n,v_n,\nabla u_n)$$

,

$$v_n \frac{\partial v_n}{\partial t} - d_v \Delta v_n v_n = v_n p_n(t, x, u_n, v_n, \nabla u_n),$$

$$\int_{Q_t} v_n \frac{\partial v_n}{\partial t} dxdt - \int_{Q_t} d_v \Delta v_n v_n dxdt = \int_{Q_t} v_n p_n(t, x, u_n, v_n, \nabla u_n) dxdt,$$

$$\int_{Q_t} v_n \frac{\partial v_n}{\partial t} dxdt + d_v \int_{Q_t} \nabla v_n \nabla v_n dxdt = \int_{Q_t} v_n p_n(t, x, u_n, v_n, \nabla u_n) dxdt,$$

$$\int_{Q_t} v_n \frac{\partial v_n}{\partial t} dxdt + d_v \int_{Q_t} |\nabla v_n|^2 dxdt \leq \int_{Q_t} |v_n||p_n(t, x, u_n, v_n, \nabla u_n)| dxdt, \tag{33}$$

using Cauchy Schwartz and Young's:

$$\begin{aligned}
\int_{Q_t} v_n \frac{\partial v_n}{\partial t} dxdt &\leq \int_{Q_t} |v_n||M_p| dxdt, \\
&\leq M_p \left(\int_{Q_T} |1|^2 dxdt \right)^{\frac{1}{2}} \left(\int_{Q_t} |v_n|^2 dxdt \right)^{\frac{1}{2}}, \\
&\leq \frac{M_p \varepsilon}{2} mes(Q_T) + \frac{1}{2\varepsilon} \int_{Q_t} v_n^2 dxdt
\end{aligned}$$

$$\int_{\Omega} v_n^2(t) dx - \int_{\Omega} v_{n0}^2 dx \leq M_p mes(Q_T) + \int_{Q_T} v_n^2 dxdt, \tag{34}$$

using Gronwall's

$$\int_{Q_T} v_n^2 dxdt \leq \left(M_p mes(Q_T) + \int_{\Omega} v_{n0}^2 dx \right) exp(T), \tag{35}$$

$$\int_{Q_T} v_n^2 dxdt \leq c_2 \tag{36}$$

we have that $\int_{Q_T} v_n^2 dxdt$ is bounded.
From (33) we have:

$$\begin{aligned}
\int_{Q_t} v_n \frac{\partial v_n}{\partial t} dxdt + d_v \int_{Q_t} |\nabla v_n|^2 dxdt &\leq \int_{Q_t} |v_n||p_n(t, x, u_n, v_n, \nabla u_n)| dxdt, \\
&\leq \int_{Q_t} |v_n||M_p| dxdt, \\
&\leq \frac{M_p \varepsilon}{2} mes(Q_t) + \frac{1}{2\varepsilon} \int_{Q_t} |v_n|^2 dxdt,
\end{aligned}$$

$$\frac{1}{2} \int_{\Omega} v_n^2(t) dx - \frac{1}{2} \int_{\Omega} v_{n0}^2 dx + d_v \int_{Q_T} |\nabla v_n|^2 dxdt \leq \frac{M_p \varepsilon}{2} mes(Q_T) + \frac{1}{2\varepsilon} \int_{Q_T} |v_n|^2 dxdt,$$

$$\int_{\Omega} v_n^2(t) dx - \int_{\Omega} v_{n0}^2 dx + 2d_v \int_{Q_T} |\nabla v_n|^2 dxdt \leq M_p mes(Q_T) + \int_{Q_T} |v_n|^2 dxdt,$$

$$\int_{\Omega} v_n^2(t) dx + 2d_v \int_{Q_T} |\nabla v_n|^2 dxdt \leq M_p mes(Q_T) + \int_{Q_T} |v_n|^2 dxdt + \int_{\Omega} v_{n0}^2 dx,$$

from the previous result, we have that $\int_{Q_T} |v_n|^2 dxdt$ is bounded with $c_2 > 0$

$$\int_{\Omega} v_n^2(t) dx + 2d_v \int_{Q_T} |\nabla v_n|^2 dxdt \leq M_p mes(Q_T) + c_2 + \int_{\Omega} v_{n0}^2 dx, \tag{37}$$

using Gronwall's:

$$\int_{Q_T} |\nabla v_n|^2 dxdt \leq \left(M_p mes(Q_T) + c_2 + \int_{\Omega} v_{n0}^2 dx \right) \exp(T), \tag{38}$$

$$\leq C_1, \tag{39}$$

we have that $\int_{Q_T} |\nabla v_n|^2 dxdt$ is bounded, now let us show that $\int_{Q_T} |\nabla u_n| dxdt$ is bounded.
From (29), we have:

$$\frac{1}{2}\int_{\Omega} u_n(t)^2 dx - \frac{1}{2}\int_{\Omega} u_{n0}^2 dx + a \int_{Q_T} |\nabla u_n|^2 dxdt \leq \int_{Q_T} |u_n||M_f| dxdt,$$

$$\leq \frac{M_f \varepsilon}{2} mes(Q_T) + \frac{1}{2\varepsilon}\int_{Q_T} |u_n|^2 dxdt,$$

$$\int_{\Omega} u_n^2(t) dx + 2a \int_{Q_T} |\nabla u_n|^2 dxdt \leq M_f mes(Q_T) + \int_{Q_T} |u_n|^2 dxdt + \int_{\Omega} u_{n0}^2 dx,$$

we have that $\int_{Q_T} |u_n|^2 dxdt$ is bounded with $c_1 > 0$:

$$\int_{\Omega} u_n^2(t) dx + 2a \int_{Q_T} |\nabla u_n|^2 dxdt \leq M_f mes(Q_T) + c_1 + \int_{\Omega} u_{n0}^2 dx, \tag{40}$$

using Gronwall's:

$$\int_{Q_T} |\nabla u_n|^2 dxdt \leq \left(M_f mes(Q_T) + c_1 + \int_{\Omega} u_{n0}^2 dx \right) exp(T), \tag{41}$$

$$\leq C_2, \tag{42}$$

we have $\int_{Q_T} |\nabla u_n|^2 dxdt$ is bounded.

$$\int_{Q_T} |\nabla u_n|^2 + |\nabla v_n|^2 dxdt \leq M_1, \quad \text{where} \quad M_1 = C_1 + C_2 > 0. \tag{43}$$

(3) We have:

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - div(g(|\nabla u_{n\sigma}|)\nabla u_n) = f_n(t, x, u_n, v_n, \nabla v_n), \\ \dfrac{\partial v_n}{\partial t} - d_v \Delta v_n = p_n(t, x, u_n, v_n, \nabla u_n), \end{cases}$$

$$\begin{cases} u_n \dfrac{\partial u_n}{\partial t} - div(g(|\nabla u_{n\sigma}|)\nabla u_n) u_n = f_n(t, x, u_n, v_n, \nabla v_n), \\ v_n \dfrac{\partial v_n}{\partial t} - d_v \Delta v_n v_n = v_n p_n(t, x, u_n, v_n, \nabla u_n), \end{cases}$$

$$\begin{cases} \displaystyle\int_{Q_t} u_n \frac{\partial u_n}{\partial t} dxdt - \int_{Q_t} div(g(|\nabla u_{n\sigma}|)\nabla u_n) u_n dxdt = \int_{Q_t} f_n(t, x, u_n, v_n, \nabla v_n) dxdt, \\ \displaystyle\int_{Q_t} v_n \frac{\partial v_n}{\partial t} dxdt - \int_{Q_t} d_v \Delta v_n v_n dxdt = \int_{Q_t} v_n p_n(t, x, u_n, v_n, \nabla u_n) dxdt, \end{cases}$$

$$\begin{cases} \displaystyle\frac{1}{2}\int_{\Omega} u_n^2(t) dx + a \int_{Q_T} |\nabla u_n|^2 dxdt \leq \int_{Q_T} |u_n||f_n(t, x, u_n, v_n, \nabla v_n)| dxdt + \frac{1}{2}\int_{\Omega} u_{n0}^2 dx, \\ \displaystyle\frac{1}{2}\int_{\Omega} v_n^2(t) dx + d_v \int_{Q_T} |\nabla v_n|^2 dxdt \leq \int_{Q_T} |v_n||p_n(t, x, u_n, v_n, \nabla u_n)| dxdt + \frac{1}{2}\int_{\Omega} v_{n0}^2 dx, \end{cases}$$

$$\begin{cases} a\int\limits_{Q_T} |\nabla u_n|^2 dxdt \leq \int\limits_{Q_T} |u_n||f_n(t,x,u_n,v_n,\nabla v_n)|dxdt + \frac{1}{2}\int\limits_{\Omega} u_{n0}^2 dx, \\ d_v\int\limits_{Q_T} |\nabla v_n|^2 dxdt \leq \int\limits_{Q_T} |v_n||p_n(t,x,u_n,v_n,\nabla u_n)|dxdt + \frac{1}{2}\int\limits_{\Omega} v_{n0}^2 dx, \end{cases}$$

using Cauchy Schwartz and Young's:

$$\begin{cases} a\int\limits_{Q_T} |\nabla u_n|^2 dxdt \leq \left(\int\limits_{Q_T} |u_n|^2 dxdx\right)^{\frac{1}{2}} \left(\int\limits_{Q_T} |f_n(t,x,u_n,v_n,\nabla v_n)|^2 dxdt\right)^{\frac{1}{2}} + \frac{1}{2}\int\limits_{\Omega} u_{n0}^2 dx, \\ d_v\int\limits_{Q_T} |\nabla v_n|^2 dxdt \leq \left(\int\limits_{Q_T} |v_n|^2 dxdt\right)^{\frac{1}{2}} \left(\int\limits_{Q_T} |p_n(t,x,u_n,v_n,\nabla u_n)|^2 dxdt\right)^{\frac{1}{2}} + \frac{1}{2}\int\limits_{\Omega} v_{n0}^2 dx, \end{cases}$$

$$\begin{cases} a\int\limits_{Q_T} |\nabla u_n|^2 dxdt \leq \frac{\varepsilon}{2}\int\limits_{Q_T} |u_n|^2 dxdx + \frac{1}{2\varepsilon}\int\limits_{Q_T} |f_n(t,x,u_n,v_n,\nabla v_n)|^2 dxdt + \frac{1}{2}\int\limits_{\Omega} u_{n0}^2 dx, \\ d_v\int\limits_{Q_T} |\nabla v_n|^2 dxdt \leq \frac{\varepsilon}{2}\int\limits_{Q_T} |v_n|^2 dxdt + \frac{1}{2\varepsilon}\int\limits_{Q_T} |p_n(t,x,u_n,v_n,\nabla u_n)|^2 dxdt + \frac{1}{2}\int\limits_{\Omega} v_{n0}^2 dx, \end{cases}$$

$$\begin{cases} 2a\int\limits_{Q_T} |\nabla u_n|^2 dxdt \leq \int\limits_{Q_T} |u_n|^2 dxdx + \int\limits_{Q_T} |f_n(t,x,u_n,v_n,\nabla v_n)|^2 dxdt + \int\limits_{\Omega} u_{n0}^2 dx, \\ 2d_v\int\limits_{Q_T} |\nabla v_n|^2 dxdt \leq \int\limits_{Q_T} |v_n|^2 dxdt + \int\limits_{Q_T} |p_n(t,x,u_n,v_n,\nabla u_n)|^2 dxdt + \int\limits_{\Omega} v_{n0}^2 dx, \end{cases}$$

we have from the previous result that $\int_{Q_T} |\nabla u_n|^2 dxdt$ and $\int_{Q_T} |\nabla v_n|^2 dxdt$ are bounded, moreover $\int_{\Omega} |u_n|^2 dxdt$ and $\int_{\Omega} |v_n|^2 dxdt$ are bounded. Hence

$$\int\limits_{Q_T} |f_n| + |p_n| dxdt \leq M_2, \quad M_2 > 0. \tag{44}$$

□

2.3 | Step3: Convergence

According to the lemma (2) we have (u_n, v_n) is bounded in $L^2(0,T,H^1(\Omega))^2$ and $(\frac{\partial u_n}{\partial t}, \frac{\partial v_n}{\partial t})$ is bounded in $(L^2(0,T,(H^1(\Omega))') + L^1(Q_T))^2$, then with Simon [14] (u_n, v_n) is relatively compact in $(L^2(Q_T))^2$ then we can extract a subsequence (u_n, v_n) in $(L^2(Q_T))^2$ such that:

•$u_n \rightharpoonup u$ in $L^2(Q_T)$ and for almost every where in Q_T.
•$v_n \rightharpoonup v$ in $L^2(Q_T)$ and for almost every where in Q_T.
•$\nabla G_\sigma * u_n \rightharpoonup \nabla G_\sigma * u$ in $L^2(Q_T)$ and for almost every where in Q_T.
•$g(|\nabla G_\sigma * u_n|) \longrightarrow g(|\nabla G_\sigma * u|)$ in $L^2(Q_T)$.
•$f_n(t,x,u_n,v_n,\nabla v_n) \longrightarrow f(t,x,u,v,\nabla v)$ for almost every where in Q_T.
•$p_n(t,x,u_n,v_n,\nabla v_n) \longrightarrow p(t,x,u,v,\nabla v)$ for almost every where in Q_T.

To prove that (u,v) is a weak solution of system (1), we actually need to prove that $f_n(t,x,u_n,v_n,\nabla v_n)$ converges strongly toward $f(t,x,u,v,\nabla v)$ in $L^1(Q_T)$ and this convergence is given by the following lemma.

Lemma 3. Under the additional assumption that, for $R > 0$:

$$\sup_{|r|+|s|\leq R} \Big(|f(t,x,r,s,\nabla s)| + |p(t,x,r,s,\nabla r)| \Big) \in L^1(Q_T).$$

(1) There exists $C > 0$ such that:

$$\int\limits_{Q_T} (u_n + 2v_n)(|f_n| + |p_n|) dxdt \leq M_3.$$

(2) f_n and p_n converges strongly toward f and p in $L^1(Q_T)$.

Proof. (1) we have:

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - div(g(|\nabla u_{n\sigma}|)\nabla u_n) = f_n(t,x,u_n,v_n,\nabla v_n), \\ \\ \dfrac{\partial v_n}{\partial t} - d_v \Delta v_n = p_n(t,x,u_n,v_n,\nabla u_n), \end{cases}$$

$$u_n \frac{\partial u_n}{\partial t} - div(g(|\nabla u_{n\sigma}|)\nabla u_n)u_n = u_n f_n(t,x,u_n,v_n,\nabla v_n),$$

$$2v_n \frac{\partial u_n}{\partial t} - div(g(|\nabla u_{n\sigma}|)\nabla u_n)(2v_n) = (2v_n) f_n(t,x,u_n,v_n,\nabla v_n),$$

$$(u_n+2v_n)\frac{\partial u_n}{\partial t} - div(g(|\nabla u_{n\sigma}|)\nabla u_n)(u_n+2v_n) = (u_n+2v_n) f_n(t,x,u_n,v_n,\nabla v_n), \tag{45}$$

$$u_n \frac{\partial (2v_n)}{\partial t} - 2d_v \Delta v_n u_n = 2u_n p_n(t,x,u_n,v_n,\nabla u_n),$$

$$\frac{\partial (2v_n)}{\partial t} - 2d_v \Delta v_n = 2p_n(t,x,u_n,v_n,\nabla u_n),$$

$$2v_n \frac{\partial (2v_n)}{\partial t} - 2d_v \Delta v_n (2v_n) = 2(2v_n) p_n(t,x,u_n,v_n,\nabla u_n),$$

$$(u_n+2v_n)\frac{\partial (2v_n)}{\partial t} - 2d_v \Delta v_n (u_n+2v_n) = 2(u_n+2v_n) p_n(t,x,u_n,v_n,\nabla u_n), \tag{46}$$

from (45) and (46) we have:

$$\int_{Q_t} (u_n+2v_n)\frac{\partial (u_n+2v_n)}{\partial t}dxdt - \int_{Q_T} \left[div(g(|\nabla u_{n\sigma}|)\nabla u_n) + 2d_v\Delta v_n\right](u_n+2v_n)dxdt = \int_{Q_t}(u_n+2v_n)\big(f_n+p_n\big)dxdt, \tag{47}$$

$$|\int_{Q_t} (u_n+2v_n)\frac{\partial (u_n+2v_n)}{\partial t}dxdt - \int_{Q_t} \left[div(g(|\nabla u_{n\sigma}|)\nabla u_n) + 2d_v\Delta v_n\right](u_n+2v_n)dxdt| = |\int_{Q_t}(u_n+2v_n)\big(f_n+2p_n\big)dxdt|,$$

$$\int_{Q_t} (u_n+2v_n)\frac{\partial (u_n+2v_n)}{\partial t}dxdt + \int_{Q_t} \left[div(g(|\nabla u_{n\sigma}|)\nabla u_n) + 2d_v\Delta v_n\right](u_n+2v_n)dxdt = \int_{Q_t}(u_n+2v_n)\big(|f_n|+2|p_n|\big)dxdt, \tag{48}$$

$$\int_{Q_t} (u_n+2v_n)\frac{\partial (u_n+2v_n)}{\partial t}dxdt + \int_{Q_t} \left[div(g(|\nabla u_{n\sigma}|)\nabla u_n) + 2d_v\Delta v_n\right](u_n+2v_n)dxdt \le \int_{Q_t}(u_n+2v_n)\big(|f_n|+|p_n|\big)dxdt + \int_{Q_t}(u_n+v_n)|p_n|dxdt,$$

$$\begin{aligned} \int_{Q_t} (u_n+2v_n)\frac{\partial (u_n+2v_n)}{\partial t}dxdt &\le \int_{Q_t}(u_n+2v_n)\big(|f_n|+|p_n|\big)dxdt + \int_{Q_t}(u_n+v_n)|p_n|dxdt, \\ &\le \int_{Q_t}(u_n+2v_n)\big(|f_n|+|p_n|\big)dxdt + \int_{Q_t}|u_n||p_n|dxdt + 2\int_{Q_t}|v_n||p_n|dxdt, \\ &\le \int_{Q_t}(u_n+2v_n)\big(|f_n|+|p_n|\big)dxdt + M_p\int_{Q_t}|u_n|dxdt + 2M_p\int_{Q_t}|v_n|dxdt, \end{aligned}$$

$$\frac{1}{2}\int_{\Omega} \big(u_n(t)+2v_n(t)\big)^2 dx - \frac{1}{2}\int_{\Omega}\big(u_{n0}+2v_{n0}\big)^2dx \le \int_{Q_T}(u_n+2v_n)\big(|f_n|+|p_n|\big)dxdt + M_p\int_{Q_T}|u_n|dxdt + 2M_p\int_{Q_T}|v_n|dxdt, \tag{49}$$

We have from the previous result of lemma (2) that $\int_{Q_T}|u_n|dxdt$ and $\int_{Q_T}|v_n|dxdt$ are bounded.

$$\int_{Q_T}(u_n+2v_n)\big(|f_n|+|p_n|\big)dxdt \le \frac{1}{2}\int_{\Omega}\big(u_{n0}+2v_{n0}\big)^2dx + M_p\int_{Q_T}|u_n|dxdt + 2M_p\int_{Q_T}|v_n|dxdt, \tag{50}$$

$$\int_{Q_T} (u_n + 2v_n)\big(|f_n| + |p_n|\big)dxdt \le M_3, \quad M_3 > 0. \tag{51}$$

(2) We have f_n, p_n converge almost every where toward f, p. We will show that f_n and p_n are equi-integrable in $L^1(Q_T)$. The proof will be given for f_n however the same result for p_n.
Let $\varepsilon > 0$ and prove that there exists $\delta > 0$ such that $|E| < \delta$ with $E \subset Q_T$ implies that

$$\int_E |f_n(t, x, u_n, v_n, \nabla v_n)|dxdt < \varepsilon.$$

For all $K \ge 0$:

$$\int_E |f_n(t, x, u_n, v_n, \nabla v_n)|dxdt \le \int_{[E\cap|u_n+2v_n|\le K]} |f_n(t, x, u_n, v_n, \nabla v_n)|dxdt + \int_{[E\cap|u_n+2v_n|>K]} |f_n(t, x, u_n, v_n, \nabla v_n)|dxdt,$$

with

$$\int_{[E\cap|u_n+2v_n|\le K]} |f_n(t, x, u_n, v_n, \nabla v_n)|dxdt \le \int_E \sup_{|u_n+2v_n|\le K} |f_n(t, x, u_n, v_n, \nabla v_n)|dxdt,$$

and $\sup_{|u_n+2v_n|\le K} |f_n(t, x, u_n, v_n, \nabla v_n)| \le \sup_{|u_n+v_n|\le K} |f_n(t, x, u_n, v_n, \nabla v_n)|$. For all $\varepsilon > 0$, there exists $\delta > 0$ such that $|E| < \delta$ we obtain

$$\int_E \sup_{|u_n+v_n|\le K} |f_n(t, x, u_n, v_n, \nabla v_n)|dxdt \le \frac{\varepsilon}{2}, \tag{52}$$

we have $|u_n + 2v_n| > K \Rightarrow |u_n + v_n| > K \Rightarrow \frac{1}{K} < \frac{1}{|u_n+v_n|}$ then we have:

$$\int_{[E\cap|u_n+2v_n|>K]} |f_n(t, x, u_n, v_n, \nabla v_n)|dxdt \le \frac{1}{K}\int_{Q_T} (u_n + 2v_n)|f_n(t, x, u_n, v_n, \nabla v_n)|dxdt,$$

with $E \subset Q_T$

$$\int_{[E\cap|u_n+2v_n|>K]} |f_n(t, x, u_n, v_n, \nabla v_n)|dxdt \le \frac{1}{K}\int_E (u_n + 2v_n)|f_n(t, x, u_n, v_n, \nabla v_n)|dxdt, \tag{53}$$

and since (51) ensures that $\int_E (u_n + 2v_n)|f_n|dxdt$ is bounded, we obtain:

$$\int_E |f_n(t, x, u_n, v_n, \nabla v_n)|dxdt \le \varepsilon. \tag{54}$$

The same thing holds for p_n as well. □

2.4 | Step 4: The positivity of the solution

Consider the function:

$$sign^-(s) = \begin{cases} -1 & \text{if} \quad s < 0, \\ 0 & \text{if} \quad s \ge 0. \end{cases} \tag{55}$$

Let $\varepsilon > 0$ we build a sequence of regular convex functions $j_\varepsilon(s)$ such as $j'_\varepsilon(s)$ is bounded and for all $s \in \mathbb{R}$ where $j'_\varepsilon(r) \to sing^-(s)$ when $\varepsilon \to 0$.
To prove the positivity of the solution (u, v), we proved that the bilinear form is definite and positive, the definite is relies on the quasi-positivity of nonlinearities given in the hypothese (H_1), so it remains to show the positivity, for this we take $u = v$.

$$2\frac{\partial u}{\partial t} - [div(g(|\nabla u_\sigma|)\nabla u) + d_v \Delta u] = (f + p)(t, x, u, u, \nabla u). \tag{56}$$

Let u be a solution of (56), we multiply both of the equation by $j'_\varepsilon(u)$, and integrating on $Q_t =]0, t[\times\Omega$ for $t \in [0, T[$

$$2\int_{Q_t} \frac{\partial u}{\partial t} j'_\varepsilon(u)dxdt - \int_{Q_t} div(g(|\nabla u_\sigma|)\nabla u)j'_\varepsilon(u)dxdt - \int_{Q_t} d_v \Delta u j'_\varepsilon(v)dxdt = \int_{Q_t} (f + p)(s, x, u, u, \nabla u)j'_\varepsilon(u)dxds, \tag{57}$$

$$2\int\limits_{Q_t}\frac{\partial u}{\partial t}j'_\varepsilon(u)dxdt+\int\limits_{Q_t}g(|\nabla u_\sigma|)\nabla u\nabla j'_\varepsilon(u)dxdt+d_v\int\limits_{Q_t}\nabla u\nabla j'_\varepsilon(v)dxdt=\int\limits_{Q_t}(f+p)(s,x,u,u,\nabla u)j'_\varepsilon(u)dxds.$$

From (18) and with

$$\int\limits_{Q_T}\frac{\partial j_\varepsilon(u(t,x))}{\partial t}dxdt=\int\limits_{Q_T}j'_\varepsilon(u)\frac{\partial u}{\partial t}dxdt$$

we have

$$2\int\limits_{Q_t}\frac{\partial j_\varepsilon(u(t,x))}{\partial t}dxdt+a\int\limits_{Q_t}|\nabla u|^2j''_\varepsilon(u)dxdt+d_v\int\limits_{Q_t}|\nabla u|^2j''_\varepsilon(v)dxdt\le\int\limits_{Q_t}(f+p)(s,x,u,u,\nabla u)j'_\varepsilon(u)dxds.$$

$$2\int\limits_{Q_t}\frac{\partial j_\varepsilon(u(t,x))}{\partial t}dxdt+(a+d_v)\int\limits_{Q_t}|\nabla u|^2j''_\varepsilon(u)dxdt\le\int\limits_{Q_t}(f+p)(s,x,u,u,\nabla u)j'_\varepsilon(u)dxds.$$

$$2\int\limits_{\Omega}[j_\varepsilon(u)(t)-j_\varepsilon(u)(0)]dx+(a+d_v)\int\limits_{Q_t}|\nabla u|^2j''_\varepsilon(u)dxdt\le\int\limits_{Q_t}(f+p)(s,x,u,u,\nabla u)j'_\varepsilon(u)dxds.$$

Since $u(0,x)=u_0$, then $j_\varepsilon(u_0(x))=j_\varepsilon(u)(0)=0$, $\int\limits_\Omega j_\varepsilon(u)(0)dx=0$ and $\int\limits_{Q_t}|\nabla u|^2j''_\varepsilon(u)dxdt\ge 0$, then we have

$$2\int\limits_{\Omega}j_\varepsilon(u)(t)dx\le\int\limits_{Q_t}(f+p)(s,x,u,u,\nabla u)j'_\varepsilon(u)dxds.$$

$$\begin{aligned}\int\limits_{\Omega}j_\varepsilon(u)(t)dx&\le\frac{1}{2}\int\limits_{Q_t}(f+p)(s,x,u,u,\nabla u)j'_\varepsilon(u)dxds.\\&\le\frac{1}{2}\int\limits_{(0,t)\times[u<0]}(f+p)(s,x,u,u,\nabla u)j'_\varepsilon(u)dxds+\frac{1}{2}\int\limits_{(0,t)\times[u\le 0]}(f+p)(s,x,u,u,\nabla u)j'_\varepsilon(u)dxds.\end{aligned}$$

Where $u\le 0$ we have $j'_\varepsilon(u)=0$, so $\int\limits_{(0,t)\times[u\le 0]}(f+p)(s,x,u,u,\nabla u)j'_\varepsilon(u)dxds=0$, therefore

$$\int\limits_{\Omega}j_\varepsilon(u)(t)dx\le\frac{1}{2}\int\limits_{(0,t)\times[u<0]}(f+p)(s,x,u,u,\nabla u)j'_\varepsilon(u)dxds,$$

pass to the limit when $\varepsilon\longrightarrow 0$ we obtain

$$\int\limits_{\Omega}(u)^-(t)dx\le-\frac{1}{2}\int\limits_{(0,t)\times[u<0]}(f+p)(s,x,u,u,\nabla u)dxds\le 0,\tag{58}$$

by the hypothese (H_2) of nonlinearities, we obtain $(u)^-(t)dx\ge 0$, then $(u)^-(t)=0$ on Ω, therefore $u\ge 0$ and $v\ge 0$ in Q_t.

□

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