Received ; | Revised ; | Accepted

DOI: xxx/xxxx

RESEARCH ARTICLE

Global well-posedness and uniform boundedness of a higher-dimensional crime model with exponential decay

Yani Feng*[1,3,4] | Deqi Wang[2]

[1]School of Information Science and Technology, ShanghaiTech University, Shanghai, China
[2]Department of Statistics, Chengdu University of Information and Technology, Chengdu, China
[3]Shanghai Institute of Microsystem and Information Technology, Chinese Academy of Sciences, Shanghai, China
[4]University of Chinese Academy of Sciences, Beijing, China

Correspondence
*Yani Feng, School of Information Science and Technology, ShanghaiTech University, Shanghai, China. Email: fengyn@shanghaitech.edu.cn

We study a class of reaction–advection–diffusion system

$$\begin{cases} u_t = \nabla\cdot(\nabla u - \chi\frac{\nabla v}{v}u) - uv - u^{2+\beta} + B_1, & x\in\Omega, t>0,\\ v_t = \Delta v - v + uv + B_2, & x\in\Omega, t>0,\end{cases}$$

which is in fact motivated by recent modeling approaches in criminology, for $\chi > 0$ in a smooth bounded domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$. While there are results regarding the existence of global solution of the original crime model[1], the restriction of χ is existed for $n \geq 2$. We prove that, for $n \geq 2$, suppose

$$\beta > \begin{cases} 0, & \text{if } n < 4,\\ \frac{n}{4} - 1, & \text{if } n \geq 4,\end{cases}$$

then the classical solutions of above system are uniformly bounded for any $\chi > 0$. Our result expand χ to be arbitrary.

KEYWORDS:
Crime model, energy functional, integral inequality, global uniform boundedness

1 | INTRODUCTION

The work is considered with the coupled parabolic chemotaxis system with singular sensitivity

$$\begin{cases} u_t = \nabla\cdot(\nabla u - \chi\frac{\nabla v}{v}u) - uv - u^{2+\beta} + B_1, & x\in\Omega, t>0,\\ v_t = \Delta v - v + uv + B_2, & x\in\Omega, t>0,\\ \partial_\nu u = \partial_\nu v = 0, & x\in\partial\Omega, t>0,\\ u(x,0) = u_0, \quad v(x,0) = v_0, & x\in\Omega,\end{cases} \tag{1}$$

for $\chi, \beta > 0$, where $\Omega \in \mathbb{R}^n$ is a bounded domain with smooth boundary. u and v are functions of location and time. The initial functions $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in W^{1,\infty}(\Omega)$ satisfying $u_0 \geq 0$ and $v_0 > 0$.

While crime may occur everywhere, certain regions in space have a disproportionately high level of crime empirically. There exists spatially heterogeneous especially clustered crime data, forming so-call hotpot. Although many social and economic forces contribute to the heterogeneity of spatial distributions in criminal activity, opportunity is the most important factor leading to crime[2,3]. Over the pass few decades, two theories have been popularized in the study of criminal behavior at the social level[4,5,6]. One is repeat and near-repeat victimization effect interpreting that neighborhoods of a burgled house as well as the same house become more likely to be burglarized soon[7,8]. The other theory is established on the fact that burglarization of a house tends to attract more burglars, named as broken-window effect[9].

Taking the two theory into consideration, in 2008, Short *et al.*[1] first proposed a system

$$\begin{cases} u_t = \nabla\cdot(\nabla u - \chi\frac{\nabla v}{v}u) - uv + B_1, \\ v_t = \Delta v - v + uv + B_2, \end{cases} \tag{2}$$

over a 2D lattice to describe the urban criminal agents pattern and spatial-temporal distribution of crime hotspot. u and v denote density of criminal agents and attractiveness of regions, respectively. From the first equation, we see that crime agents enjoy self-diffusion and attractiveness-depending diffusion, interpreting movement toward high concentrations of the attractiveness value. The attraction rate $\chi = 2$ illustrates that there is a complete broken-window effect. The decay term $-uv$ roughly are criminals which we see an abstaining from a second crime, whereas the near-repeat victimization effect is translated by its positive counterpart uv in the second equation. The non-negative function B_1 quantifies the criminal agents at beginning, and B_2 represents some information, whatever those may be, leading certain neighborhoods tend to be more attractive. B_1 and B_2 have a significant affect on the long term behavior of the solutions. For a review of agent-based urban crime modeling, We refer to[10,11].

In view of the sociological meaningful question whether or not criminal agents spontaneously form aggregates, the literature on initial-value problems for (2) is still at quite an early stage and there are only few relative results. For instance, local existence and uniqueness result is achieved for regularized version of (2) in[12]. For one-dimensional case, Rodríguez and Winkler[13] proved the existence and uniqueness of global solutions for $\chi \in (0, \frac{\sqrt{6\sqrt{3}+9}}{2})$, and Qi *et al.*[14] extended this to arbitrary $\chi \in (0,\infty)$. Both of them cover the case $\chi = 2$. For the higher dimension, global solution is achieved by[15] under small χ. Further, there are some modified versions is studied under technical conditions (see the works of[16,17,18]). From the viewpoint of mathematical analysis, it is not hard to recognize that (2) shares essential component with the Keller–Segel chemotaxis model (see[19]). However, the nonlinear kinetics in second equation (2) are quite different from the chemotaxis models and they bring difficulties in that a suitable method for the latter may not apply to the former. We would like to refer to[20,21,22,23,24,25] and references therein for chemotaxis model and[26,27,28,29] for other version of chemotaxis model that may enlighten readers.

It should be emphasized that the coefficient $\chi = 2$ is embedded in the sociological phenomenon described in system (2) and cannot be scaled out by linear or nonlinear transformations. It our goal of this paper to achieve global solution for arbitrary χ, which covers the case $\chi = 2$, through appropriately modification of system (2) in higher dimension. The first equation of system (1) roughly interprets that the criminals enjoy an extreme abstaining from a second crime. Indeed, this may prevent the aggregation of crime agents resulting from large advection rate. Consequently, in this paper we establish the global existence and uniform boundedness of the classical solution of (1), which reads as follows.

Theorem 1. Let Ω be a bounded domain with smooth boundary $\partial\Omega$ on $\mathbb{R}^n$ for $n \geq 2$, initial data $v_0 > 0$ and $u_0 \geq, \not\equiv 0$ in Ω with $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in W^{1,\infty}(\Omega)$. Whenever

$$\beta > \begin{cases} 0, & \text{if } n < 4, \\ \frac{n}{4} - 1, & \text{if } n \geq 4, \end{cases} \tag{3}$$

(1) admits a unique classical solution (u, v) from $C^0(\bar{\Omega}\times[0,\infty)) \bigcap C^{2,1}(\bar{\Omega}\times(0,\infty))$ for all $\chi \in (0,\infty)$. Moreover, there exist constants $\delta, C > 0$ such that $\delta \leq v < C$ and $0 \leq u < C$ for all $t \in (0,\infty)$.

2 | PRELIMINARIES

The starting point of our analysis of (1) is the existence of its local-in-time solutions. The local existence of classical solution is well-established by methods of standard parabolic regularity theory and an appropriate fixed point framework, referring to Theorem 3.1 in[30] or Theorem 5.2 in[31]. We show it in the following.

Lemma 1. Let $\Omega \in \mathbb{R}^n$ be a bounded domain with smooth boundary, $B_1 \in C^1\big(\Omega\times[0,\infty)\big)$ and $B_2 \in C^1\big(\Omega\times[0,\infty)\big)$ are non-negative as well as $0 \leq u_0 \in C^0(\bar{\Omega})$, $0 < v_0 \in W^{1,q}(\Omega)$ for $1 \leq n < q$, then for $\chi > 0$, there exists $T_{\max} \in (0,\infty]$ and a pair of unique non-negative solution satisfying

$$\begin{cases} u \in C^0(\bar{\Omega}\times[0,\infty)) \bigcap C^{2,1}(\bar{\Omega}\times(0,\infty)), \\ v \in C^0(\bar{\Omega}\times[0,\infty)) \bigcap C^{2,1}(\bar{\Omega}\times(0,\infty)) \bigcap L^\infty_{loc}([0,T_{\max}); W^{1,q}(\Omega)), \end{cases}$$

such that $(u, v) > 0$ solves (1) classically on $\Omega\times(0,T_{\max})$ and moreover, if $T < \infty$, then $\lim\limits_{t\to T_{\max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,q}(\Omega)} = \infty$.

Throughout the sequel, without explicit mention we shall assume the requirements of Theorem (1) to be met, and let u, v and $T_{\max}$ be as provided by Lemma 1. Besides, For simplicity, we skip $\mathrm{d}\Omega$ in the following integrals and adjust the values of generic constants C and C_i that may vary from line to line.

With those facts, we can deduce the non-negative lower boundedness of v from the abstract representation formula of v-equation.

Lemma 2. Let (u, v) satisfy Lemma 1, non-negative functions $B_1 \in C^1\big(\Omega \times [0,\infty)\big)$ and $B_2 \in C^1\big(\Omega \times [0,\infty)\big)$, then there exists $T_{\max} > 0$ and a nonnegative constant δ depending on v_0 such that

$$v \geq \delta > 0,\ \forall(\mathbf{x}, t) \in (\bar{\Omega} \times [0, T_{\max})). \tag{4}$$

Proof. Due to $B_2 \in C^1\big(\Omega \times [0,\infty)\big)$ are non-negative, we have

$$\inf_{t>0} \int_\Omega B_2(\mathbf{x}, t) > 0.$$

Then following the process of Lemma 2.1 in the work of[13], v satisfies (4) for $T_{\max} > 0$. □

The following is L^1 norms of (u, v).

Lemma 3. There is a positive constant C such that we have

$$\int_\Omega u + \int_\Omega v \leq C, \quad \forall t > 0. \tag{5}$$

Proof. By Lemma 2, this follows immediately from computing

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u = -\int_\Omega uv - \int_\Omega u^{2+\beta} + \int_\Omega B_1 \leq -\delta \int_\Omega u + \int_\Omega B_1. \tag{6}$$

After solving the differential inequality (6), we have that for $C > 0$,

$$\int_\Omega u \leq C. \tag{7}$$

On the other hand, integrating the v-equation gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega v = -\int_\Omega v + \int_\Omega uv + \int_\Omega B_2.$$

This together with (7) and the fact $(B_1, B_2) \in L^\infty(\bar{\Omega} \times [0, \infty])$ leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega u + \int_\Omega v\right) + \left(\int_\Omega u + \int_\Omega v\right) \leq \int_\Omega u + \int_\Omega \big(B_1 + B_2\big) \leq C.$$

An application of Gronwall's inequality deduces (5). □

For the purpose of using it in the following, we recall two point-wise identities and an inequality.

Lemma 4. Let $\Omega \in \mathbb{R}^n$, $n \geq 1$ be a smooth bounded domain. Any function $f \in C^2(\Omega)$ satisfies

$$i.\ \nabla|\nabla f|^2 = 2\nabla f \cdot D^2 f, \tag{8}$$

$$ii.\ (\Delta f)^2 \leq n|D^2 f|^2, \tag{9}$$

$$iii.\ \nabla f \nabla \Delta f = \frac{1}{2}\Delta|\nabla f|^2 - |D^2 f|^2. \tag{10}$$

All those identities and inequality can be obtained from straightforward calculation. One can see[23,29] and Lemma 3.1 in[32] for their application. We could not find a precise reference in the literature that covers all that is necessary for our purpose; therefore we conclude a short lemma here.

3 | A USER-FRIENDLY INTEGRAL INEQUALITY

Our proof of Theorem 1 is based on generalization and application of an integral inequality by Q.Wang [14], where the inequality is subject to one dimension. We calculate a multi-dimensional form in the following.

Theorem 2. Let $\Omega \in \mathbb{R}^N$ be a smooth bounded domain and $w > 0$ satisfying $w \in C^2(\bar{\Omega})$ and $\frac{\partial w}{\partial v} = 0$ on $\partial\Omega$. Then for all $p \geq 1$, $q \geq -\frac{1}{2}$ and $\epsilon > \frac{p}{2q+1} > 0$ we have that

$$\int_\Omega \frac{|\nabla w|^{2p+2}}{w^{q+2}} \leq \frac{N+4p\epsilon}{2q+1-\frac{p}{\epsilon}} \int_\Omega \frac{|D^2 w|^2 |\nabla w|^{2p-2}}{w^q}. \tag{11}$$

Proof. We introduce $J := \int_\Omega |\Delta \log w|^2 \frac{|\nabla w|^{2p-2}}{w^{q-2}} > 0$ for $p > 1$. Directly calculate $|\Delta \log w|^2$ to get

$$J = \int_\Omega \frac{|\Delta w|^2 |\nabla w|^{2p-2}}{w^q} \overbrace{-2\int_\Omega \frac{|\nabla w|^{2p}\Delta w}{w^{q+1}}}^{J_0} + \int_\Omega \frac{|\nabla w|^{2p+2}}{w^{q+2}}. \tag{12}$$

Since $\frac{\partial w}{\partial v} = 0$ on $\partial\Omega$, employing integration by parts gives

$$\begin{aligned} J_0 =& 2\int_\Omega \frac{\nabla|\nabla w|^{2p}\cdot\nabla w}{w^{q+1}} - 2(q+1)\int_\Omega \frac{|\nabla w|^{2p+2}}{w^{q+2}} \\ =& 2p\int_\Omega \frac{|\nabla w|^{2p-2}\nabla|\nabla w|^2\cdot\nabla w}{w^{q+1}} - 2(q+1)\int_\Omega \frac{|\nabla w|^{2p+2}}{w^{q+2}}. \end{aligned}$$

By (8) and Young's inequality, we have for $\epsilon > 0$ that

$$\begin{aligned} J_0 =& 4p\int_\Omega \frac{|\nabla w|^{2p}\cdot D^2 w}{w^{q+1}} - 2(q+1)\int_\Omega \frac{|\nabla w|^{2p+2}}{w^{q+2}} \\ \leq& 4p\epsilon\int_\Omega \frac{|\nabla w|^{2p-2}|D^2 w|^2}{w^q} - \big(2(q+1) - \frac{p}{\epsilon}\big)\int_\Omega \frac{|\nabla w|^{2p+2}}{w^{q+2}}. \end{aligned} \tag{13}$$

Subscribing (13) into (12), and in the view of the pointwise inequality (9), we have

$$J \leq (N+4p\epsilon)\int_\Omega \frac{|\nabla w|^{2p-2}|D^2 w|^2}{w^q} - \big((2q+1) - \frac{p}{\epsilon}\big)\int_\Omega \frac{|\nabla w|^{2p+2}}{w^{q+2}}. \tag{14}$$

Because $\epsilon > \frac{p}{2q+1} > 0$ and $J > 0$, we have $(2q+1) - \frac{p}{\epsilon} > 0$ and then deduce (11). □

Remark 1. Let $\Omega \in \mathbb{R}^n$ and take $p = 2, \epsilon > \frac{2}{2q+1}$, we have $\frac{N+8\epsilon}{2q+1-\frac{p}{\epsilon}} = \frac{2+8\epsilon}{2q+1-\frac{2}{\epsilon}}$. Note that $\frac{2+8\epsilon}{2q+1-\frac{2}{\epsilon}}$ can achieve its global minimum over certain ϵ and we denote it by $C^\sharp$. Thus we obtain

$$\int_\Omega \frac{|\nabla w|^6}{w^{q+2}} \leq C^\sharp \int_\Omega \frac{|D^2 w|^2 |\nabla w|^2}{w^q}. \tag{15}$$

4 | SOME USEFUL *A PRIORI* ESTIMATES

In preparation for construction and estimate of energy-type functional, some important *a priori* estimates are provided and collected into two lemmas in this section.

Lemma 5. Let $2 \leq q < 3$, for $\epsilon_1, \epsilon_2 > 0$ we have

$$\frac{d}{dt}\int_\Omega \frac{|\nabla v|^4}{v^q} \leq -(12 - \frac{16q}{q+1} - (4\epsilon_1 + 4\epsilon_2)C^\sharp)\int_\Omega \frac{|\nabla v|^2|D^2 v|^2}{v^q} + C_{\epsilon_1}\int_\Omega u^3 v^{4-q} + C_{\epsilon_2}\int_\Omega |\nabla u|^2 v^{4-q} - (4-q)\int_\Omega \frac{|\nabla v|^4}{v^q} - q\int_\Omega \frac{|\nabla v|^4 u}{v^q}$$

$$
-q\int_\Omega \frac{|\nabla v|^4}{v^{q+1}}B_2. \tag{16}
$$

Proof. Through straightforward calculation we can show

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega \frac{|\nabla v|^4}{v^q} &= 4\int_\Omega \frac{|\nabla v|^2\nabla v\cdot\nabla v_t}{v^q} - q\int_\Omega \frac{|\nabla v|^4 v_t}{v^{q+1}} \\
&= \overbrace{4\int_\Omega \frac{|\nabla v|^2\nabla v\cdot\nabla\Delta v}{v^q}}^{I_1} - 4\int_\Omega \frac{|\nabla v|^4}{v^q} + \overbrace{4\int_\Omega \frac{|\nabla v|^2\nabla v\nabla(uv)}{v^q}}^{I_c}\overbrace{-q\int_\Omega \frac{|\nabla v|^4\Delta v}{v^{q+1}}}^{I_2} + q\int_\Omega \frac{|\nabla v|^4}{v^q} - q\int_\Omega \frac{|\nabla v|^4 u}{v^q} \\
&\quad - q\int_\Omega \frac{|\nabla v|^4}{v^{q+1}}B_2.
\end{aligned} \tag{17}
$$

In light of the third identity in Lemma 4, we have

$$
\begin{aligned}
I_1 =& 2\int_\Omega \frac{|\nabla v|^2\Delta|\nabla v|^2}{v^q} - 4\int_\Omega \frac{|\nabla v|^2|D^2v|^2}{v^q} \\
=& -2\int_\Omega \nabla\Big(\frac{|\nabla v|^2}{v^q}\Big)\nabla|\nabla v|^2 - 4\int_\Omega \frac{|\nabla v|^2|D^2v|^2}{v^q} \\
=& -2\int_\Omega \frac{(\nabla|\nabla v|^2)^2}{v^q} + 2q\int_\Omega \frac{\nabla v|\nabla v|^2\nabla|\nabla v|^2}{v^{q+1}} - 4\int_\Omega \frac{|\nabla v|^2|D^2v|^2}{v^q} \\
=& -12\int_\Omega \frac{|\nabla v|^2|D^2v|^2}{v^q} + \overbrace{2q\int_\Omega \frac{\nabla v|\nabla v|^2\nabla|\nabla v|^2}{v^{q+1}}}^{I_{11}},
\end{aligned} \tag{18}
$$

where the second line is from application of integration by parts and fourth line is from (8). Similarly, we calculate that

$$
I_2 = q\int_\Omega \nabla\Big(\frac{|\nabla v|^4}{v^{q+1}}\Big)\nabla v = 2q\int_\Omega \frac{\nabla v|\nabla v|^2\nabla|\nabla v|^2}{v^{q+1}} - q(q+1)\int_\Omega \frac{|\nabla v|^6}{v^{q+2}}. \tag{19}
$$

Combining I_2 and I_{11}, for $\epsilon > 0$, we have from Young's inequality that

$$
\begin{aligned}
I_2 + I_{11} =& 4q\int_\Omega \frac{\nabla v|\nabla v|^2\nabla|\nabla v|^2}{v^{q+1}} - q(q+1)\int_\Omega \frac{|\nabla v|^6}{v^{q+2}} \\
\leq& 4q\epsilon\int_\Omega \frac{(\nabla|\nabla v|^2)^2}{v^q} + \Big(\frac{q}{\epsilon} - q(q+1)\Big)\int_\Omega \frac{|\nabla v|^6}{v^{q+2}}.
\end{aligned}
$$

Here, we take $\epsilon = \frac{4}{(q+1)}$ and apply (8), the inequality goes

$$
I_2 + I_{11} \leq \frac{16q}{q+1}\int_\Omega \frac{|\nabla v|^2|D^2v|^2}{v^q}. \tag{20}
$$

Similarly, employing Young's inequality we have

$$
\begin{aligned}
I_c =& 4\int_\Omega \frac{|\nabla v|^2\nabla v\nabla u}{v^{q-1}} + 4\int_\Omega \frac{|\nabla v|^4 u}{v^q} \\
\leq& (4\epsilon_1 + 4\epsilon_2)\int_\Omega \frac{|\nabla v|^6}{v^{q+2}} + C_{\epsilon_1}\int_\Omega u^3v^{4-q} + C_{\epsilon_2}\int_\Omega |\nabla u|^2v^{4-q}.
\end{aligned} \tag{21}
$$

Substituting (18)-(21) into (17) gives

$$\frac{d}{dt}\int\limits_{\Omega}\frac{|\nabla v|^4}{v^q} \leq -(12-\frac{16q}{q+1})\int\limits_{\Omega}\frac{|\nabla v|^2|D^2v|^2}{v^q}+(4\epsilon_1+4\epsilon_2)\int\limits_{\Omega}\frac{|\nabla v|^6}{v^{q+2}}+C_{\epsilon_1}\int\limits_{\Omega}u^3v^{4-q}+C_{\epsilon_2}\int\limits_{\Omega}|\nabla u|^2v^{4-q}$$
$$-(4-q)\int\limits_{\Omega}\frac{|\nabla v|^4}{v^q}-q\int\limits_{\Omega}\frac{|\nabla v|^4u}{v^q}-q\int\limits_{\Omega}\frac{|\nabla v|^4}{v^{q+1}}B_2. \tag{22}$$

From Remark 1 we can show that (5) satisfies. □

Lemma 6. Let $\kappa_1=(2\chi-4(4-q))$ and $\kappa_2=(2(4-q)\chi-(4-q)(3-q))$. For $2\geq q<3$, there exist small $\epsilon_3>0$ and $\delta>0$ such that

$$\frac{d}{dt}\int\limits_{\Omega}u^2v^{4-q} \leq -(2-\kappa_1\epsilon_3)\int\limits_{\Omega}|\nabla u|^2v^{4-q}+(\kappa_2+\frac{\kappa_1}{4\epsilon_3})\int\limits_{\Omega}u^2|\nabla v|^2v^{2-q}-2\int\limits_{\Omega}u^{3+\beta}v^{4-q}-(4-q)\int\limits_{\Omega}u^2v^{4-q}+(4-q)\int\limits_{\Omega}u^3v^{4-q}$$
$$+2\int\limits_{\Omega}uv^{4-q}B_1+(4-q)\int\limits_{\Omega}u^2v^{3-q}B_2. \tag{23}$$

Proof. In light of (1) and integration by parts we can show that

$$\begin{aligned}\frac{d}{dt}\int\limits_{\Omega}u^2v^{4-q} &= 2\int\limits_{\Omega}uv^{4-q}u_t+(4-q)\int\limits_{\Omega}u^2v^{3-q}v_t\\
&= -2\int\limits_{\Omega}\nabla(uv^{4-q})\cdot(\nabla u-\chi\frac{\nabla v}{v}u)-(4-q)\int\limits_{\Omega}\nabla(u^2v^{3-q})\nabla v-2\int\limits_{\Omega}u^{3+\beta}v^{4-q}-(4-q)\int\limits_{\Omega}u^2v^{4-q}+(4-q)\int\limits_{\Omega}u^3v^{4-q}\\
&\quad +2\mu\int\limits_{\Omega}uv^{4-q}+(4-q)\lambda\int\limits_{\Omega}u^2v^{3-q}-2\int\limits_{\Omega}u^2v^{5-q}\\
&= -2\int\limits_{\Omega}|\nabla u|^2v^{4-q}+\overbrace{\kappa_1\int\limits_{\Omega}u\nabla u\nabla vv^{3-q}}^{I_3}+\kappa_2\int\limits_{\Omega}u^2|\nabla v|^2v^{2-q}-2\int\limits_{\Omega}u^{3+\beta}v^{4-q}-(4-q)\int\limits_{\Omega}u^2v^{4-q}+(4-q)\int\limits_{\Omega}u^3v^{4-q}\\
&\quad -2\int\limits_{\Omega}u^2v^{5-q}+2\mu\int\limits_{\Omega}uv^{4-q}+(4-q)\int\limits_{\Omega}u^2v^{3-q}B_2.\end{aligned} \tag{24}$$

Employing general Young's inequality, for $\epsilon_3>0$, we have that

$$I_3 \leq \kappa_1\epsilon_3\int\limits_{\Omega}|\nabla u|^2v^{4-q}+\frac{\kappa_1}{4\epsilon_3}\int\limits_{\Omega}u^2|\nabla v|^2v^{2-q}. \tag{25}$$

Noting that v enjoys its lower bound for any $t>0$, we denote $\delta^*=\delta^{-\frac{2}{3}}$ the upper bound of $v^{-\frac{2}{3}}$. Substituting (25) into (24), we obtain (23). □

5 | LAYPUNOV FUNCTIONAL

In this section, we shall finish the proof of the Theorem 1. First, we construct the energy functional and prove that each item of functional is uniformly bounded.

Lemma 7. Let $\eta>0$, $2\leq q<3$, and $\mathcal{F}_\eta(u,v)$ be the following form

$$\mathcal{F}_\eta(u,v)=\eta\int\limits_{\Omega}u^2v^{4-q}+\int\limits_{\Omega}\frac{|v|^4}{v^q},$$

then there exists a constant $C>0$ such that

$$\frac{d}{dt}\mathcal{F}_\eta(u,v)+(4-q)\mathcal{F}_\eta(u,v)<C. \tag{26}$$

Moreover, we have that

$$\|u\|_{L^2(\Omega)} < C. \tag{27}$$

Proof. Denoting $\kappa_3 = 12 - \frac{16q}{q+1} - (4\epsilon_1 + 4\epsilon_2)C^\sharp$ and combining Lemma 5 and Lemma 6, we achieve

$$\begin{aligned}
&\frac{d}{dt}\big(\eta \int_\Omega u^2 v^{4-q} + \int_\Omega \frac{|\nabla v|^4}{v^q}\big) \\
&\leq -((2-\kappa_1\epsilon_3)\eta + C_{\epsilon_2}) \int_\Omega |\nabla u|^2 v^{4-q} - \kappa_3 \int_\Omega \frac{|\nabla v|^2 |D^2 v|^2}{v^q} - (4-q)\int_\Omega \frac{|\nabla v|^4}{v^q} + (\kappa_2 + \frac{\kappa_1}{4\epsilon_3})\eta \overbrace{\int_\Omega u^2 |\nabla v|^2 v^{2-q}}^{I_4} \\
&\quad + ((4-q)\eta + C_{\epsilon_1}) \overbrace{\int_\Omega u^3 v^{4-q}}^{I_5} - (4-q)\eta \int_\Omega u^2 v^{4-q} + 2\eta \overbrace{\int_\Omega u v^{4-q} B_1}^{I_6} + (4-q)\eta \overbrace{\int_\Omega u^2 v^{3-q} B_2}^{I_7} - 2\eta \int_\Omega u^{3+\beta} v^{4-q} \\
&\quad - 2\eta \int_\Omega u^2 v^{5-q}.
\end{aligned} \tag{28}$$

From Young's inequality, we have

$$I_4 \leq \epsilon_4 \int_\Omega \frac{|v|^6}{v^{q+2}} + C_{\epsilon_4} \int_\Omega u^3 v^{4-q}. \tag{29}$$

Combining the second term of I_4 with I_5 and employ Young's inequality again to show for $\epsilon_5 > 0$ that

$$\int_\Omega u^3 v^{4-q} = \int_\Omega u^3 v^{\frac{3(4-q)}{3+\beta}} v^{\frac{\beta(4-q)}{3+\beta}} \leq \epsilon_5 \int_\Omega u^{3+\beta} v^{4-q} + C_{\epsilon_5} \int_\Omega v^{4-q}. \tag{30}$$

Similarly, since $\|B_1\|_{L^\infty(\Omega)} < C$ and $\|B_2\|_{L^\infty(\Omega)} < C$ we have from Holder inequality for $\epsilon_6, \epsilon_7 > 0$ that

$$I_6 \leq C \int_\Omega u v^{\frac{5-q}{2}} v^{\frac{3-q}{2}} \leq \epsilon_6 \int_\Omega u^2 v^{5-q} + C(\epsilon_6) \tag{31}$$

and

$$I_7 \leq C \int_\Omega u^2 v^{\frac{2(4-q)}{3+\beta}} v^{\frac{(3-q)(1+\beta)-2}{3+\beta}} \leq \epsilon_7 \int_\Omega u^{3+\beta} v^{4-q} + C(\epsilon_7), \tag{32}$$

where, we have $\|v\|_{L^{3-q}(\Omega)} < C$ and $\|v\|_{L^{3-q-\frac{2}{1+\beta}}(\Omega)} < C$ since $2 \leq q < 3$ and the boundedness of $\|v\|_{L^1(\Omega)}$. Now, to estimate $\|v^{4-q}\|_{L^1(\Omega)}$, we properly rearrange it and apply Gagliardo-Nirenberg inequality to obtain for $C_1, C_2 > 0$ that

$$\|v^{\frac{4-q}{6}}\|_{L^6(\Omega)} \leq C_1 \|v^{\frac{4-q}{6}}\|^{\frac{6}{(3-q)n+6}}_{L^{\frac{6}{4-q}}(\Omega)} \|\nabla(v^{\frac{4-q}{6}})\|^{\frac{(3-q)n}{(3-q)n+6}}_{L^6(\Omega)} + C_2 \|v^{\frac{4-q}{6}}\|_{L^{\frac{6}{4-q}}(\Omega)}. \tag{33}$$

For $\epsilon_8, \epsilon_9 > 0$ and due to the boundedness of $\|v\|_{L^1(\Omega)}$, we apply Young's inequality and from (33) and our integral inequality to have

$$\begin{aligned}
C_{\epsilon_5} \int_\Omega v^{4-q} &\leq C_3\Big(\big(\int_\Omega \frac{|\nabla v|^6}{v^{q+2}}\big)^{\frac{(3-q)n}{(3-q)n+6}} + 1\Big) \\
&\leq \epsilon_8 \int_\Omega \frac{|\nabla v|^6}{v^{q+2}} + C_4 \leq \epsilon_9 \int_\Omega \frac{|\nabla v|^2 |D^2 v|^2}{v^q} + C_5.
\end{aligned} \tag{34}$$

Substitute (29)-(32) and (34) to (28), there exists $C > 0$ such that

$$\begin{aligned}
&\frac{d}{dt}\big(\eta \int_\Omega u^2 v^{4-q} + \int_\Omega \frac{|\nabla v|^4}{v^q}\big) \\
&\leq -((2-\kappa_1\epsilon_3)\eta + C_{\epsilon_2}) \int_\Omega |\nabla u|^2 v^{4-q} - \kappa_5 \int_\Omega \frac{|\nabla v|^2 |D^2 v|^2}{v^q} - (2\eta - (4-q)\eta\epsilon_7 - \kappa_4\epsilon_5) \int_\Omega u^{3+\beta} v^{4-q} - (2\eta - 2\eta\epsilon_6) \int_\Omega u^2 v^{5-q}
\end{aligned}$$

$$-(4-q)\int_\Omega \frac{|\nabla v|^4}{v^q}-(4-q)\eta\int_\Omega u^2v^{4-q}+C, \tag{35}$$

where, we denote $\kappa_4=(\kappa_2+\frac{\kappa_1}{4\epsilon_3})\eta C_{\epsilon_4}+(4-q)\eta+C_{\epsilon_1}$ and $\kappa_5=(\kappa_3-(\kappa_2+\frac{\kappa_1}{4\epsilon_3})\eta\epsilon_4C^\sharp-\epsilon_9)$. Take $\epsilon_2=\frac{1}{4\eta}$ and $\epsilon_3=\frac{1}{\kappa_1}$, we have $(2-\kappa_1\epsilon_3)\eta+C_{\epsilon_2}=0$. Then take $\epsilon_5=\frac{\eta}{\kappa_4}$, $\epsilon_7=\frac{1}{(4-q)}$ to obtain $2\eta-(4-q)\eta\epsilon_7-\kappa_4\epsilon_5=0$ and take $\epsilon_6=1$ to show $2\eta-2\eta\epsilon_6=0$. Finally, we take ϵ_1,ϵ_4 and ϵ_9 small such that $\kappa_5\geq 0$. Thus, this simplifies (34) and obtains (27). Then (28) can be achieved due to the lower boundedness of v. □

To prove Theorem 1.1, one shall first prove the boundedness of L^p norm of (u,v). We show this in two cases.

6 | L^P ESTIMATES

6.1 | In the case of $n<4$

By now, we know the boundedness of $\|u\|_{L^2(\Omega)}$ under the condition $\beta>0$. To show Theorem 1, the boundedness of $\|v\|_{W^{1,q}(\Omega)}$ is needed for some $q\in(1,\infty]$. A easy way is an application of heat semi-group estimate on representation formula of v-equation. We present it in the following. As for the details of proof, we refer to[15].

Lemma 8. Suppose that $p>\frac{n}{2}$ and $q\in(n,\infty]$ and suppose $q<\frac{np}{n-p}$ if $p\leq n$. Then there exists a constant $C>0$ depending on q,T and $\|B_2\|_{L^\infty(\Omega)}$ as well as $\|v_0\|_{W^{1,\infty}(\Omega)}$ and an exponent $\gamma>0$ such that

$$\|v(\cdot,t)\|_{W^{1,q}(\Omega)}\leq C(1+\|u\|^\gamma_{L^\infty((0,T);L^p(\Omega))}). \tag{36}$$

With this theorem and (28), in the case $n<4$, we extend q such that $\|v(\cdot,t)\|_{W^{1,q}(\Omega)}$ is bounded, which reads as follows.

Corollary 1. Let $n<4$ and $\frac{n}{2}<q<\frac{2n}{n-2}$. Then we can find a constant $C>0$ depending on q,T and $\|B_2\|_{L^\infty(\Omega)}$ as well as $\|v_0\|_{W^{1,\infty}(\Omega)}$ such that

$$\|v(\cdot,t)\|_{W^{1,q}(\Omega)}\leq C. \tag{37}$$

With this and appropriate choices for the parameters, we can gain the following estimate

Lemma 9. Let $p>1$ and $\beta>\max\{\frac{p}{2}(n-2)-1,0\}$ for $n<4$, there exists a positive constant C depending on p,T and $\|B_1\|_{L^\infty(\Omega)}$ such that

$$\|u\|_{L^p(\Omega)}<C. \tag{38}$$

Proof. Let $2<\theta<\frac{2n}{n-2}$. By a direct calculation and integration by parts, we have for $\chi,\beta>0$ and small $\epsilon_1,\epsilon_2>0$ that

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega u^p=&-(p-1)\int_\Omega|\nabla u|^2u^{p-2}+\chi(p-1)\int_\Omega\nabla u u^{p-1}\frac{\nabla v}{v}\\&-\int_\Omega u^pv-\int_\Omega u^{p+1+\beta}+\int_\Omega u^{p-1}B_1\\ \leq&-(p-1-\epsilon_1)\int_\Omega|\nabla u|^2u^{p-2}+\epsilon_2\int_\Omega u^{\frac{p\theta}{\theta-2}}+C(\epsilon_1,\epsilon_2,\chi,p)\int_\Omega\frac{|\nabla v|^\theta}{v^\theta}\\&-\int_\Omega u^pv-\int_\Omega u^{p+1+\beta}+\int_\Omega u^{p-1}B_1.\end{aligned} \tag{39}$$

Employing Young's inequality and Holder inequality, since $\|B_1\|_{L^\infty(\Omega)}<C$ we have

$$\int_\Omega u^{p-1}B_1\leq\epsilon\int_\Omega u^{p+1+\beta}+C(\epsilon), \tag{40}$$

for small $\epsilon>0$. Similarly, let θ close to $\frac{2n}{n-2}$ and $\beta>\max\{\frac{p}{2}(n-2)-1,0\}$, which implies $\frac{p\theta}{\theta-2}\leq p+1+\beta$, then we obtain

$$\int_\Omega u^{\frac{p\theta}{\theta-2}}\leq C\int_\Omega u^{p+1+\beta}+C, \tag{41}$$

for $C > 0$. Subscribing (40) and (41) into (39), it is concluded from Corollary 1 that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \int_\Omega u^p \le C, \tag{42}$$

for some $C > 0$. The Lemma 9 is satisfied though Gronwall's inequality. □

6.2 | In the case of $n \geq 4$

Lemma 10. Let $p > 1$ and $\beta > \max\{\frac{p}{2} - 1, 0\}$ for $n \geq 4$, there exists a positive constant C depending on p, T and $\|B_1\|_{L^\infty(\Omega)}$ such that

$$\|u\|_{L^p(\Omega)} < C. \tag{43}$$

Proof. To show this, we follow the calculation of (39) and rearrange the estimates to have for $\chi, \beta > 0$ that

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega u^p =& -(p-1)\int_\Omega |\nabla u|^2 u^{p-2} + \chi(p-1)\int_\Omega \nabla u u^{p-1}\frac{\nabla v}{v} - \int_\Omega u^p v - \int_\Omega u^{p+1+\beta} + \int_\Omega u^{p-1}B_1\\ \le& -(p-1-\epsilon_1)\int_\Omega |\nabla u|^2 u^{p-2} + \epsilon_2\int_\Omega u^{\frac{3p}{2}} + C(\epsilon_1,\epsilon_2,\chi,p)\int_\Omega \frac{|\nabla v|^6}{v^6} - \int_\Omega u^p v - \int_\Omega u^{p+1+\beta} + \int_\Omega u^{p-1}B_1.\end{aligned}$$

Due to Holder inequality and $\beta \geq \frac{p}{2} - 1$, for $C > 0$, we have

$$\int_\Omega u^{\frac{3p}{2}} \le C\int_\Omega u^{p+1+\beta} \tag{44}$$

and conclude from Theorem 2 and (40) that

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^p + \int_\Omega u^p \le& C(\epsilon_1,\epsilon_2,\chi,p)\int_\Omega \frac{|\nabla v|^6}{v^6}\\ \le& \tilde{C}\int_\Omega \frac{|\nabla v|^2|D^2 v|^2}{v^q},\end{aligned} \tag{45}$$

where, $\tilde{C} > 0$ depends on $\epsilon_1, \epsilon_2, \chi, p$ and $\|\frac{1}{v}\|_{L^\infty(\Omega)}$. Combining this with (35), there exists a $\tilde{\epsilon} > 0$ such that

$$\begin{aligned}&\frac{d}{dt}(\mathcal{F}_\eta(u,v) + \tilde{\epsilon}\int_\Omega u^p) + (4-q)(\mathcal{F}_\eta(u,v) + \tilde{\epsilon}\int_\Omega u^p)\\ &\le -((2-\kappa_1\epsilon_3)\eta + C_{\epsilon_2})\int_\Omega |\nabla u|^2 v^{4-q} - \kappa_6\int_\Omega \frac{|\nabla v|^2|D^2 v|^2}{v^q}\\ &\quad - (2\eta - (4-q)\eta\epsilon_7 - \kappa_4\epsilon_5)\int_\Omega u^{3+\beta}v^{4-q} - (2\eta - 2\eta\epsilon_6)\int_\Omega u^2 v^{5-q} + C.\end{aligned}$$

Taking $\tilde{\epsilon}$ small enough such that $\kappa_6 = \kappa_3 - (\kappa_2 + \frac{\kappa_1}{4\epsilon_3})\eta\epsilon_4 C^\sharp - \epsilon_9 - \tilde{\epsilon}\tilde{C} > 0$, we readily achieve

$$\frac{d}{dt}(\mathcal{F}_\eta(u,v) + \tilde{\epsilon}\int_\Omega u^p) + (4-q)(\mathcal{F}_\eta(u,v) + \tilde{\epsilon}\int_\Omega u^p) \le C,$$

for $C > 0$. Then (10) satisfies by an application of Gronwall's inequality. □

7 | UNIFORM BOUNDEDNESS

In this section, we shall prove Theorem 1. As in the prevous section, by Lemma 1 we are given here some $T > 0$ and a classical solution (u, v) to (1) in $\Omega \times (0, T)$. L^p estimate of u and Lemma 8 will help us in deriving higher regularity for u by its representation formula

$$\begin{aligned} u(\cdot,t) =& e^{t\Delta}u_0 - \chi \int_0^t e^{(t-s)\Delta}\nabla \cdot \left(\frac{u(\cdot,s)}{v(\cdot,s)}\nabla v(\cdot,s)\right) \mathrm{d}s - \int_0^t e^{(t-s)\Delta}(u(\cdot,s)v(\cdot,s) + u(\cdot,s)^{2+\beta})\mathrm{d}s + \int_0^t e^{(t-s)\Delta}B_1(s)\mathrm{d}s \\ \leq& e^{t\Delta}u_0 - \chi \int_0^t e^{(t-s)\Delta}\nabla \cdot \left(\frac{u(\cdot,s)}{v(\cdot,s)}\nabla v(\cdot,s)\right) \mathrm{d}s + \int_0^t e^{(t-s)\Delta}B_1(s)\mathrm{d}s, \end{aligned} \tag{46}$$

which holds for all $t \in (0,T)$. The idea of its proof resembles the lemma 3.4 in the work of Winkler[25].

Lemma 11. Suppose

$$\beta > \begin{cases} 0, & \text{if } n < 4, \\ \frac{n}{4} - 1, & \text{if } n \geq 4, \end{cases} \tag{47}$$

and there is $p > \frac{n}{2}$ such that

$$\sup_{t\in(0,T)} \|u(\cdot,t)\|_{L^p(\Omega)} < \infty, \tag{48}$$

then there is a $C > 0$ with

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq C,$$

for all $t \in (0,T)$

Proof. Noting that with the condition of β in Lemma 9 and Lemma 10, (11) satisfies for $p > \frac{n}{2}$. Therefore, we can show the Lemma by directly following the proof of Lemma 11 in[15]. □

Now, all the ingredient necessary to verify Theorem 1.

Proof of Theorem 1.1. Theorem 1 provides us with the solution (u, v) to (1) in $\Omega \times \left(0, T_{\max}\right)$ for some $T_{\max} \in (0,\infty]$ and, for any $T \in (0,\infty)$ such that the solution exists in $\Omega \times (0,T)$ by Lemma 8 and 11 we find $C > 0$ with

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq C,$$

and

$$\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq C,$$

for all $t \in (0,T)$. Considering the alternative in Lemma 1, $T_{\max} < \infty$ can therefore be ruled out. Further, since the nonnegativity of u and v can be deduced from the maximum principle, the additional fact $v \in L^\infty_{\mathrm{loc}}\left(\left[0,\infty\right); W^{1,\infty}(\Omega)\right)$ ensures uniqueness of (1). □

References

1. Short Martin B, D'orsogna Maria R, Pasour Virginia B, et al. A statistical model of criminal behavior. *Mathematical Models and Methods in Applied Sciences.* 2008;18(supp01):1249–1267.

2. Cohen Lawrence E, Felson Marcus. Social change and crime rate trends: A routine activity approach. *American sociological review.* 1979;:588–608.

3. Felson Marcus. Routine activities and crime prevention in the developingmetropolis. *Criminology.* 1987;25(4):911–932.

4. Anselin Luc, Cohen Jacqueline, Cook David, Gorr Wilpen, Tita George. Spatial analyses of crime. *Criminal justice.* 2000;4(2):213–262.

5. Bowers Kate J, Johnson Shane D. Domestic burglary repeats and space-time clusters: The dimensions of risk. *European Journal of Criminology.* 2005;2(1):67–92.

6. Chaturapruek Sorathan, Breslau Jonah, Yazdi Daniel, Kolokolnikov Theodore, McCalla Scott G. Crime modeling with Lévy flights. *SIAM Journal on Applied Mathematics.* 2013;73(4):1703–1720.

7. Johnson Shane D, Bowers Kate, Hirschfield Alex. New insights into the spatial and temporal distribution of repeat victimization. *The British Journal of Criminology.* 1997;37(2):224–241.

8. Short Martin B, D'orsogna Maria R, Brantingham Patricia J, Tita George E. Measuring and modeling repeat and near-repeat burglary effects. *Journal of Quantitative Criminology.* 2009;25(3):325–339.

9. Kelling George L, Wilson James Q, others . Broken windows. *Atlantic monthly.* 1982;249(3):29–38.

10. D'Orsogna Maria R, Perc Matjaž. Statistical physics of crime: A review. *Physics of life reviews.* 2015;12:1–21.

11. Groff Elizabeth R, Johnson Shane D, Thornton Amy. State of the art in agent-based modeling of urban crime: An overview. *Journal of Quantitative Criminology.* 2019;35(1):155–193.

12. Rodriguez Nancy, Bertozzi Andrea. Local existence and uniqueness of solutions to a PDE model for criminal behavior. *Mathematical Models and Methods in Applied Sciences.* 2010;20(supp01):1425–1457.

13. Rodriguez Nancy, Winkler Michael. On the global existence and qualitative behavior of one-dimensional solutions to a model for urban crime. *arXiv preprint arXiv:1903.06331.* 2019;.

14. Wang Qi, Wang Deqi, Feng Yani. Global well-posedness and uniform boundedness of urban crime models: One-dimensional case. *Journal of Differential Equations.* 2020;.

15. Freitag Marcel. Global solutions to a higher-dimensional system related to crime modeling. *Mathematical Methods in the Applied Sciences.* 2018;41(16):6326–6335.

16. Manasevich Raul, Phan Quoc Hung, Souplet Philippe. Global existence of solutions for a chemotaxis-type system arising in crime modeling. *arXiv preprint arXiv:1206.3724.* 2012;.

17. Rodríguez N. On the global well-posedness theory for a class of PDE models for criminal activity. *Physica D: Nonlinear Phenomena.* 2013;260:191–200.

18. Heihoff Frederic. Generalized solutions for a system of partial differential equations arising from urban crime modeling with a logistic source term. *Zeitschrift für angewandte Mathematik und Physik.* 2020;71(3):1–23.

19. Keller Evelyn F, Segel Lee A. Initiation of slime mold aggregation viewed as an instability. *Journal of theoretical biology.* 1970;26(3):399–415.

20. Winkler Michael. Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model. *Journal of Differential Equations.* 2010;248(12):2889–2905.

21. Fujie Kentarou, Yokota Tomomi. Boundedness in a fully parabolic chemotaxis system with strongly singular sensitivity. *Journal of Mathematical Analysis and Applications.* 2014;38:140–143.

22. Fujie Kentarou. Boundedness in a fully parabolic chemotaxis system with singular sensitivity. *Journal of Mathematical Analysis and Applications.* 2015;424(1):675–684.

23. Lankeit Johannes. A new approach toward boundedness in a two-dimensional parabolic chemotaxis system with singular sensitivity. *Mathematical Methods in the Applied Sciences.* 2015;39(3):394–404.

24. Winkler Michael. Absence of collapse in a parabolic chemotaxis system with signal-dependent sensitivity. *Mathematische Nachrichten.* 2010;283(11):1664–1673.

25. Winkler Michael. Global solutions in a fully parabolic chemotaxis system with singular sensitivity. *Mathematical methods in the applied sciences.* 2011;34(2):176–190.

26. Biler Piotr. Global solutions to some parabolic-elliptic systems of chemotaxis. *Advances in Mathematical Sciences and Applications.* 1999;9:347–359.

27. Fujie Kentarou, Winkler Michael, Yokota Tomomi. Blow-up prevention by logistic sources in a parabolic–elliptic Keller-Segel system with singular sensitivity. *Nonlinear Analysis: Theory, Methods and Applications.* 2014;109:56–71.

28. Koichi Osaki, Atsushi Yagi. Finite dimensional attractor for one-dimensional Keller-Segel equations. *Funkcialaj Ekvacioj.* 2014;44(3):441–469.

29. Winkler Michael. Global Large-Data Solutions in a Chemotaxis-(Navier–)Stokes System Modeling Cellular Swimming in Fluid Drops. *Communications in Partial Differential Equations.* 2012;37(2):319–351.

30. Horstmann Dirk, Winkler Michael. Boundedness vs. blow-up in a chemotaxis system. *Journal of Differential Equations.* 2005;215(1):52–107.

31. Amann Herbert. Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems. In: Springer 1993 (pp. 9–126).

32. Marras Monica, Viglialoro Giuseppe. Boundedness in a fully parabolic chemotaxis-consumption system with nonlinear diffusion and sensitivity, and logistic source. *Mathematische Nachrichten.* 2018;291(14-15):2318–2333.

$\square$