GLOBAL WELL-POSEDNESS AND OPTIMAL TIME DECAY RATES OF SOLUTIONS TO THE THREE-DIMENSIONAL MAGNETO-MICROPOLAR FLUID EQUATIONS

DONGJUAN NIU AND HAIFENG SHANG

Abstract. This paper deals with the global existence and decay estimates of solutions to the three-dimensional magneto-micropolar fluid equations with only velocity dissipation and magnetic diffusion in the whole space with various Sobolev and Besov spaces. Specifically, we first investigate the global existence and optimal decay estimates of weak solutions. Then we prove the global existence of solutions with small initial data in H^s, $B^s_{2,\infty}$ and critical Besov spaces, respectively. Furthermore, the optimal decay rates of these global solutions are correspondingly established in $\dot{H}^m$ and $\dot{B}^m_{2,\infty}$ spaces with $0 \le m \le s$ and in $\dot{B}^m_{2,1}$ with $0 \le m \le \frac{1}{2}$, when the initial data belongs to $\dot{B}^{-l}_{2,\infty}$ $(0 < l \le \frac{3}{2})$. The main difficulties lie in the presence of linear terms and the lack of micro-rotation velocity dissipation. To overcome them, we make full use of the special structure of the system and employ various techniques involved with the energy methods, the improved Fourier splitting, Fourier analysis and the regularity interpolation methods.

1. Introduction

The three-dimensional (3D) incompressible magneto-micropolar fluid equations can be written as

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - (\mu+\chi)\Delta u = -\nabla P + b\cdot\nabla b + 2\chi\nabla\times w, \\ \partial_t w + (u\cdot\nabla)w + 4\chi w - \kappa\nabla\nabla\cdot w - \gamma\Delta w = 2\chi\nabla\times u, \\ \partial_t b + (u\cdot\nabla)b - \nu\Delta b = b\cdot\nabla u, \\ \nabla\cdot u = \nabla\cdot b = 0, \\ u(x,0) = u_0(x), w(x,0) = w_0(x), b(x,0) = b_0(x), \end{cases} \tag{1.1}$$

where $(x,t) \in \mathbb{R}^3 \times \mathbb{R}^+$. u, w, b and P denote the velocity fields of the fluid, micro-rotational velocity fields, the magnetic field and the pressure, respectively. μ, χ and ν are, respectively, kinematic viscosity, vortex viscosity and magnetic resistivity. κ and γ are angular viscosities.

The magneto-micropolar fluid equations usually describe the motion of aggregates of small solid ferromagnetic particles relative to viscous magnetic fluids under the action of magnetic fields, such as salt water, ester, fluorocarbon [1, 14]. Using spectral Galerkin method, Rojas-Medar [29] proved the existence and uniqueness of local strong solutions to (1.1) in bounded domain. Later, Rojas-Medar and Boldrini [30] established the

2010 *Mathematics Subject Classification.* 35Q35, 35B40, 76D03.
Key words and phrases. 3D magneto-micropolar equations, partial dissipation, global well-posedness, decay estimate.

global existence of weak solutions to system (1.1) in bounded domain. Later, the global existence of classical solutions to (1.1) with small initial data and Dirichlet boundary condition was proved by Ortega-Torres and Rojas-Medar in [27]. Recently, the global existence of weak solutions and global strong solutions with small initial data to (1.1) in the whole space $\mathbb{R}^3$ were established (see, e.g., [25, 40]). For more results related to the global regularity and regularity criteria, one refers to [5, 7, 11, 15, 21, 43, 45, 49, 50, 51] and the references therein.

When ignoring the angular viscosity (see, e.g., [48, 33]), system (1.1) becomes the following 3D magneto-micropolar fluid equations,

$$\begin{cases} \partial_t u + u\cdot\nabla u - (\mu+\chi)\Delta u = -\nabla P + b\cdot\nabla b + 2\chi\nabla\times w, \\ \partial_t w + u\cdot\nabla w + 4\chi w - \kappa\nabla\nabla\cdot w = 2\chi\nabla\times u, \\ \partial_t b + u\cdot\nabla b - \nu\Delta b = b\cdot\nabla u, \\ \nabla\cdot u = 0, \nabla\cdot b = 0, \\ u(x,0) = u_0(x), w(x,0) = w_0(x), b(x,0) = b_0(x). \end{cases} \tag{1.2}$$

So far, to our best knowledge, the previous well-posedness results on (1.2) mainly focus on the two-dimensional system. In particular, Yamazaki [48] proved the global existence of smooth solutions in $\mathbb{R}^2$ by fully exploiting the structure of (1.2) and applying the techniques of Littlewood-Paley decomposition. Recently, system (1.2) with fractional dissipation was studied by Shang and Wu [33]. Moreover, Shang and Gu [32] studied the global regularity of solutions to (1.2) with only partial magnetic diffusion. For more results about the global well-posedness with various partial dissipation, one can refer to [6, 8, 12, 13, 16, 22, 28, 34, 44, 47] and the references therein for interested readers.

Once the global well-posedness is established, one may wonder the large time behavior of global solutions. This is an important issue in the fields of partial differential equations. It is well-known that the L^2 decay problem of weak solutions to the 3D Navier-Stokes equations, i.e., (1.2) with $w = 0$ and $b = 0$, was proposed by the celebrated work of Leray [20]. By introducing the elegant method of Fourier splitting, Schonbek [35, 36] successfully established the optimal time decay rate of weak solutions of the Navier-Stokes equations, see also [19, 46]. Recently, by virtue of the structure to (1.1) and the Fourier splitting method, Li and Shang [25] established the decay estimates for weak solutions of (1.1) and obtain the same rate as those of the 3D Navier-Stokes equations. One can refer to [9, 18] for more results about the decay rate of system (1.1). Very recently, Braz e Silva, Cruz, Freitas and Zingano [4] improved the decay rate of w for the solution of the 3D micropolar equations (1.1) with $b = 0$. In addition, the decay estimates of the higher order derivatives of smooth solutions of 3D Navier-Stokes equations and magneto-micropolar equations with small initial data were obtained in [26, 37] and [25, 40] respectively. We remark that the methods to derive the decay results above are strongly depending on the full Laplacian dissipation in all equations.

In this paper we are interested in the global well-posedness and decay estimates of solutions to (1.2) with only velocity dissipation and magnetic diffusion. We establish the global existence results to system (1.2) in various Sobolev and Besov spaces. Furthermore, the optimal large time decay rates of these global solutions in the corresponding spaces are also established.

We now list the main results of the paper. The first result is the following global existence and optimal decay estimates of weak solutions in L^2 space.

Theorem 1.1. *Let $\mu > 0, \chi > 0, \nu > 0$ and $\kappa \geq 0$. Assume that $(u_0, w_0, b_0) \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Then the following statements hold:*

(i) For any $T > 0$, system (1.2) has a global weak solution (u, w, b) satisfying

$$(u, b) \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)), \quad w \in L^\infty(0, T; L^2(\mathbb{R}^3)). \tag{1.3}$$

(ii) Let $(u_0, w_0, b_0) \in L^p(\mathbb{R}^3)$ with $1 \leq p < 2$. Then the global weak solution (u, w, b) satisfies for all $t > 0$,

$$\|u(t)\|_{L^2(\mathbb{R}^3)} + \|w(t)\|_{L^2(\mathbb{R}^3)} + \|b(t)\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{4}(\frac{2}{p}-1)}, \tag{1.4}$$

where the positive constant C depends on μ, χ, ν and the initial data (u_0, w_0, b_0).

Remark 1.2. *The decay estimate (1.4) for the velocity field u and magnetic field b are optimal, because it agrees with that of the heat kernel. In addition, we believe that the decay rate for micro-rotational field w in (1.4) cannot be improved better. In fact, if we take the L^2-inner product of the second equation in system (1.2) with w, we obtain*

$$\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 + 4\chi\|w\|_{L^2}^2 \leq 2\chi\|\nabla u\|_{L^2}\|w\|_{L^2},$$

which yields

$$\|w(t)\|_{L^2} \leq e^{-4\chi t}\|w_0\|_{L^2} + 2\chi\int_0^t e^{-4\chi(t-\tau)}\|\nabla u(\tau)\|_{L^2} d\tau.$$

This implies that to improve the decay rate of $\|w\|_{L^2}$, we need to get the decay estimate of $\|\nabla u\|_{L^2}$. However, due to the strong coupling structure of system (1.2) and the lack of dissipation in the equation of w, it is extremely difficult to achieve this goal for weak solutions.

The proof of global existence of solutions in Theorem 1.1 can be referred to the Friedrichs method and fully exploiting the structure of (1.2). Classical Fourier splitting methods developed by [19, 46] are not applicable to establish the decay estimate (1.4) because of the lack of the dissipation of w. The main observation is that the low frequency effect dominates in the study of the L^2 decay of weak solutions. Therefore, we overcome the difficulty based on the improved Fourier splitting methods and fully utilizing of the damping term $4\chi w$ in (1.2), then eventually establish the desired decay estimate.

The next three theorems are devoted to the global existence and the time decay estimates of solutions to system (1.2) with small initial data in Sobolev and Besov spaces. In general, the approach to establish the global existence includes two main steps. The first one is related to the local (in time) well-posedness while the second extends the local solution into a global one based on a global (in time) *a priori* bounds. For simplicity, we omit the proof of local well-posedness of solutions and pay attention on the global *a priori* bounds of (u, w, b). We are able to obtain the global existence in three Besov spaces. Furthermore, by using the interpolation method, energy estimates and the technique of Fourier analysis, we obtain the correspondingly optimal time decay rates of these solutions.

More precisely, the first theorem establishes a unique global solution when the initial data (u_0, w_0, b_0) is sufficiently small in $H^s(\mathbb{R}^3)$, and obtain the optimal decay rates of the global solution itself and its higher order derivatives, as stated below.

Theorem 1.3. *Let $\mu > 0, \chi > 0, \nu > 0$ and $\kappa \geq 0$. Assume that $(u_0, w_0, b_0) \in H^s(\mathbb{R}^3)$ with $s > 0$ and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Then the following two statements hold:*

(i) Let $s > \frac{3}{2}$. Then there exists a positive constant C_0 such that for all $0 < \epsilon < C_0$, if

$$\|u_0\|^2_{H^s(\mathbb{R}^3)} + \|w_0\|^2_{H^s(\mathbb{R}^3)} + \|b_0\|^2_{H^s(\mathbb{R}^3)} < \epsilon, \tag{1.5}$$

then system (1.2) has a unique global solution (u, w, b) satisfying, for any $t > 0$,

$$\begin{aligned} &\|u(t)\|^2_{H^s(\mathbb{R}^3)} + \|w(t)\|^2_{H^s(\mathbb{R}^3)} + \|b(t)\|^2_{H^s(\mathbb{R}^3)} \\ &\quad + \int_0^t (\|\nabla u(\tau)\|^2_{H^s(\mathbb{R}^3)} + \|w(\tau)\|^2_{H^s(\mathbb{R}^3)} + \|\nabla b(\tau)\|^2_{H^s(\mathbb{R}^3)}) d\tau \leq C\epsilon, \end{aligned} \tag{1.6}$$

where $C > 0$ is a constant independent of t.

(ii) Let $s > \frac{5}{2}$. Suppose that $(u_0, w_0, b_0) \in \dot{B}^{-l}_{2,\infty}(\mathbb{R}^3)$ with $0 < l \leq \frac{3}{2}$. Then the global solution (u, w, b) satisfies the following decay estimates:

i) For all real number m with $0 \leq m \leq s$,

$$\|D^m u(t)\|_{L^2(\mathbb{R}^3)} + \|D^m w(t)\|_{L^2(\mathbb{R}^3)} + \|D^m b(t)\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{m}{2}-\frac{l}{2}}. \tag{1.7}$$

ii) For $0 \leq m \leq s-1$, the following improved decay estimates for w hold

$$\|D^m w(t)\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{m+1}{2}-\frac{l}{2}}. \tag{1.8}$$

iii) Assume also that there exists a positive constant C_1 such that

$$\|D^m e^{\nu \Delta t} b_0\|_{L^2(\mathbb{R}^3)} \geq C_1(1+t)^{-\frac{m}{2}-\frac{l}{2}}. \tag{1.9}$$

Then the following lower bound decay estimates for b hold

$$\|D^m b\|_{L^2(\mathbb{R}^3)} \geq C(1+t)^{-\frac{m}{2}-\frac{l}{2}}. \tag{1.10}$$

Remark 1.4. *(1) Theorem 1.3 establishes the optimal $L^p(\mathbb{R}^3)$-$L^2(\mathbb{R}^3)$ type of decay rates for $p \in [1, 2)$ for all of the derivatives of order $0 \leq m \leq s$ for (u, b) and $0 \leq m \leq s-1$ for w in the larger space $\dot{B}^{-l}_{2,\infty}(\mathbb{R}^3)$, since $L^p(\mathbb{R}^3) \hookrightarrow \dot{B}^{-l}_{2,\infty}(\mathbb{R}^3)$ with $1 \leq p < 2$ and $l = \frac{3}{p} - \frac{3}{2}$.*

(2) Our global existence part (i) is an obvious generalization of that in [40], where the $H^s(\mathbb{R}^3)$ with $s \geq 3$ smoothness on the initial data is required. In fact, our assumption $s > \frac{3}{2}$ is crucial and necessary to ensure $L^\infty(\mathbb{R}^3) \hookrightarrow H^s(\mathbb{R}^3)$.

(3) The decay estimates of solutions to (1.1) with full dissipation were studied in [40]. Our results of Theorem 1.3 improves that of Theorem 1.2 ([40]) in two aspects: First, the decay estimate (1.7) is only valid for $0 \leq m \leq s-2$ in [40]. Second, the upper bound decay estimate (1.8) of w is improved to (1.8) other than (1.7) in [40].

(4) A sufficient condition of lower bound (1.9) is due to Oliver and Titi [26].

(5) We also remark that it may be difficult to obtain the lower bounds of the decay rates of u and w because of the presence of linear terms in the equation of u and w and lack of the dissipation in the equation of w. Therefore, it would be interesting and challenging to derive the lower bound of decay estimates for u and w in the near future.

The proof of the global existence part of Theorem 1.3 relies on the global *a priori* bound for $\|(u, w, b)\|_{H^s(\mathbb{R}^3)}$. Due to the lack of dissipation in the equation of w, we utilize the special structure of system (1.2) and take advantage of the velocity dissipation and damping term in equation of w to overcome the difficulty from the linear terms $2\chi\nabla \times w$ and $2\chi\nabla \times u$. We remark that Sohinger and Strain [38] first introduced negative homogeneous Besov space $\dot{B}^{-l}_{2,\infty}(\mathbb{R}^3)$ to study the decay estimates of the Boltzmann equation. Later, this space is used to study the decay estimates of other evolutional equations (see, e.g., [10, 39, 40]), whose advantage comes from the fact that the solution can preserve the evolution of time. As we know, it is difficult to verify that the $L^p(\mathbb{R}^3)$ (with $1 \le p < 2$) norm of the solution can be propagated.

The second theorem states that system (1.2) has a unique global solution when the initial data (u_0, w_0, b_0) is sufficiently small in $B^s_{2,\infty}(\mathbb{R}^3)$, and also establish their optimal decay rates.

Theorem 1.5. *Let $\mu > 0, \chi > 0, \nu > 0$ and $\kappa \ge 0$. Assume that $(u_0, w_0, b_0) \in B^s_{2,\infty}(\mathbb{R}^3)$ with $s > 0$ and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Then the following two statements hold:*

(i) Let $s > \frac{3}{2}$. Then there exists a positive constant C_0 such that for all $0 < \epsilon < C_0$, if

$$\|u_0\|^2_{B^s_{2,\infty}(\mathbb{R}^3)} + \|w_0\|^2_{B^s_{2,\infty}(\mathbb{R}^3)} + \|b_0\|^2_{B^s_{2,\infty}(\mathbb{R}^3)} < \epsilon, \tag{1.11}$$

then system (1.2) has a unique global solution (u, w, b) satisfying, for any $t > 0$,

$$\begin{aligned} &\|u(t)\|^2_{B^s_{2,\infty}(\mathbb{R}^3)} + \|w(t)\|^2_{B^s_{2,\infty}(\mathbb{R}^3)} + \|b(t)\|^2_{B^s_{2,\infty}(\mathbb{R}^3)} \\ &\quad + \|\nabla u\|^2_{\tilde{L}^2_t(B^s_{2,\infty}(\mathbb{R}^3))} + \|w\|^2_{\tilde{L}^2_t(B^s_{2,\infty}(\mathbb{R}^3))} + \|\nabla b\|^2_{\tilde{L}^2_t(B^s_{2,\infty}(\mathbb{R}^3))} \le C\epsilon, \end{aligned} \tag{1.12}$$

where $C > 0$ independent of t is a pure constant.

(ii) Let $s > \frac{5}{2}$. Suppose that $(u_0, w_0, b_0) \in \dot{B}^{-l}_{2,\infty}(\mathbb{R}^3)$ with $0 < l \le \frac{3}{2}$. Then the global solution (u, w, b) satisfies the following decay estimates:

i) For all real number m with $0 \le m \le s$,

$$\|u(t)\|_{\dot{B}^m_{2,\infty}(\mathbb{R}^3)} + \|w(t)\|_{\dot{B}^m_{2,\infty}(\mathbb{R}^3)} + \|b(t)\|_{\dot{B}^m_{2,\infty}(\mathbb{R}^3)} \le C(1+t)^{-\frac{m}{2}-\frac{l}{2}}. \tag{1.13}$$

ii) For $0 \le m \le s - 1$, the following improved decay estimates for w hold

$$\|w(t)\|_{\dot{B}^m_{2,\infty}(\mathbb{R}^3)} \le C(1+t)^{-\frac{m+1}{2}-\frac{l}{2}}. \tag{1.14}$$

Remark 1.6. *Since $\dot{H}^s(\mathbb{R}^3) \hookrightarrow \dot{B}^s_{2,\infty}(\mathbb{R}^3)$, the decay estimates of Theorem 1.5 lies in a space which is larger than $\dot{H}^s(\mathbb{R}^3)$.*

To ensure the global existence results of Theorem 1.5, it is important to prove the uniform bounds of $\|(u, w, b)\|_{B^s_{2,\infty}(\mathbb{R}^3)}$. Due to the lack of dissipation in the equation of w, we need to employ the interpolation inequality of Besov spaces and utilize the properties of the heat operators to overcome the difficulties.

Finally, the following theorem shows that system (1.2) has a unique global solution when the initial data (u_0, w_0, b_0) is sufficiently small in critical Besov spaces. Furthermore, the decay rates to the global solution are also established.

Theorem 1.7. *Let $\mu > 0, \chi > 0, \nu > 0$ and $\kappa \geq 0$. Assume that $(u_0, b_0) \in \dot{B}^{\frac{1}{2}}_{2,1}(\mathbb{R}^3)$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$ and $w_0 \in \dot{B}^{\frac{3}{2}}_{2,1}(\mathbb{R}^3)$. Then the following two statements hold:*

(i) If there exist positive constants C_0 and C_1 such that

$$C_0\chi < \min\{\mu + \chi, \nu\}, \tag{1.15}$$

and for any $0 < \epsilon < C_1$,

$$\|u_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}(\mathbb{R}^3)} + \|w_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}(\mathbb{R}^3)} + \|b_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}(\mathbb{R}^3)} < \epsilon, \tag{1.16}$$

then system (1.2) has a unique global solution (u, w, b) satisfying, for any $t > 0$,

$$\begin{aligned}
&\|u(t)\|_{\dot{B}^{\frac{1}{2}}_{2,1}(\mathbb{R}^3)} + \|w(t)\|_{\dot{B}^{\frac{3}{2}}_{2,1}(\mathbb{R}^3)} + \|b(t)\|_{\dot{B}^{\frac{1}{2}}_{2,1}(\mathbb{R}^3)} \\
&\quad + \int_0^t (\|\nabla u(\tau)\|_{\dot{B}^{\frac{3}{2}}_{2,1}(\mathbb{R}^3)} + \|w(\tau)\|_{\dot{B}^{\frac{3}{2}}_{2,1}(\mathbb{R}^3)} + \|\nabla b(\tau)\|_{\dot{B}^{\frac{3}{2}}_{2,1}(\mathbb{R}^3)})d\tau \leq C\epsilon,
\end{aligned} \tag{1.17}$$

where $C > 0$ independent of t is a pure constant.

(ii) Suppose that $(u_0, w_0, b_0) \in \dot{B}^{-l}_{2,\infty}(\mathbb{R}^3)$ with $0 < l \leq \frac{3}{2}$. Then for all real number m with $0 \leq m \leq \frac{1}{2}$, the global solution (u, w, b) established in (i) satisfies the following decay estimates

$$\|u(t)\|_{\dot{B}^{m}_{2,1}(\mathbb{R}^3)} + \|w(t)\|_{\dot{B}^{m+1}_{2,1}(\mathbb{R}^3)} + \|b(t)\|_{\dot{B}^{m}_{2,1}(\mathbb{R}^3)} \leq C(1+t)^{-\frac{m}{2}-\frac{l}{2}}. \tag{1.18}$$

To prove Theorem 1.7, we focus on the uniform bounds of $\|(u,b)\|_{L^2(0,t,B^{\frac{5}{2}}_{2,\infty}(\mathbb{R}^3))}$ and $\|w\|_{L^2(0,t,B^{\frac{3}{2}}_{2,\infty}(\mathbb{R}^3))}$. Due to the lack of dissipation in the equation of w, it seems extremely difficult to deal with the linear terms on the right hand sides of system (1.2) in critical Besov spaces. To overcome these difficulties, a condition of coupling parameters (1.15) is imposed. Therefore, it would be interesting to establish the global well-posedness and decay estimates to system (1.2) without the assumption (1.15), which is left to be investigated in future.

Throughout this manuscript, to simplify the notation, we will write $\int f$ for $\int_{\mathbb{R}^3} f dx$, $\|f\|_{L^p}$ for $\|f\|_{L^p(\mathbb{R}^3)}$, $\|f\|_{\dot{H}^s}$ and $\|f\|_{H^s}$ for $\|f\|_{\dot{H}^s(\mathbb{R}^3)}$ and $\|f\|_{H^s(\mathbb{R}^3)}$ respectively, $\dot{B}^s_{p,r}$ and $B^s_{p,r}$ for $\dot{B}^s_{p,r}(\mathbb{R}^3)$ and $B^s_{p,r}(\mathbb{R}^3)$ respectively, and $L^q_t(\dot{B}^s_{p,r})$ and $\tilde{L}^q_t(\dot{B}^s_{p,r})$ for $L^q_t(\dot{B}^s_{p,r}(\mathbb{R}^3))$ and $\tilde{L}^q_t(\dot{B}^s_{p,r}(\mathbb{R}^3))$ respectively.

The rest of this paper is divided into six sections. In section 2, we introduce some definitions and related results of Besov spaces and give several useful calculus inequalities to be used in the subsequent sections. Sections 3, 4, 5 and 6 state the proofs of Theorem 1.1, Theorem 1.3, Theorem 1.5 and Theorem 1.7, respectively. The proofs of Lemma 2.5 and Lemma 2.6 below are provided in Appendix A.

2. Preliminaries

This section provides several notations of functional spaces and calculus inequalities, which will be helpful in the next sections.

2.1. **Functional spaces.** In this preparatory subsection, we recall the Littlewood-Paley operators and their elementary properties which allow us to define the Besov spaces. Related results can be found in several books and many papers (see, e.g., [2, 3, 24, 31, 42]).

We start with several notations. $\mathcal{S}$ denotes the usual Schwarz class and $\mathcal{S}'$ its dual, the space of tempered distributions. To introduce the Littlewood-Paley decomposition, we write for each $j \in \mathbb{Z}$

$$A_j = \left\{\xi \in \mathbb{R}^d : \ 2^{j-1} \le |\xi| < 2^{j+1}\right\}.$$

The Littlewood-Paley decomposition asserts the existence of a sequence of functions $\{\Phi_j\}_{j\in\mathbb{Z}} \subset \mathcal{S}$ such that

$$\mathrm{supp}\widehat{\Phi}_j \subset A_j, \qquad \widehat{\Phi}_j(\xi) = \widehat{\Phi}_0(2^{-j}\xi) \quad \text{or} \quad \Phi_j(x) = 2^{jd}\Phi_0(2^j x),$$

and

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi) = \begin{cases} 1, & \text{if } \xi \in \mathbb{R}^d \setminus \{0\}, \\ 0, & \text{if } \xi = 0. \end{cases}$$

Therefore, for a general function $\psi \in \mathcal{S}$, we have

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for } \xi \in \mathbb{R}^d \setminus \{0\}.$$

We now choose $\Psi \in \mathcal{S}$ such that

$$\widehat{\Psi}(\xi) = 1 - \sum_{j=0}^{\infty} \widehat{\Phi}_j(\xi), \quad \xi \in \mathbb{R}^d.$$

Then, for any $\psi \in \mathcal{S}$,

$$\Psi * \psi + \sum_{j=0}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\Psi * f + \sum_{j=0}^{\infty} \Phi_j * f = f \tag{2.1}$$

in $\mathcal{S}'$ for any $f \in \mathcal{S}'$. To define the inhomogeneous Besov space, we set

$$\Delta_j f = \begin{cases} 0, & \text{if } j \le -2, \\ \Psi * f, & \text{if } j = -1, \\ \Phi_j * f, & \text{if } j = 0, 1, 2, \cdots. \end{cases} \tag{2.2}$$

To define the homogeneous Besov space, we set

$$\dot{\Delta}_j f = \Phi_j * f, \quad \text{if } j = 0, \pm 1, \pm 2, \cdots. \tag{2.3}$$

Besides the Fourier localization operators Δ_j, the partial sum S_j is also a useful notation. For an integer j,

$$S_j \equiv \sum_{k=-1}^{j-1} \Delta_k.$$

For any $f \in \mathcal{S}'$, the Fourier transform of $S_j f$ is supported on the ball of radius 2^j. It is clear from (2.1) that $S_j \to Id$ as $j \to \infty$ in the distributional sense.

Definition 2.1. *The inhomogeneous and homogeneous Besov spaces* $B^s_{p,q}$ *and* $\dot{B}^s_{p,q}$ *with* $s \in \mathbb{R}$ *and* $p, q \in [1, \infty]$ *consists of* $f \in \mathcal{S}'$ *satisfying*

$$\|f\|_{B^s_{p,q}} \equiv \|2^{js}\|\Delta_j f\|_{L^p}\|_{l^q_j} < \infty,$$

and

$$\|f\|_{\dot{B}^s_{p,q}} \equiv \|2^{js}\|\dot{\Delta}_j f\|_{L^p}\|_{l^q_j} < \infty,$$

respectively.

Several classical function spaces are special cases of Besov spaces. The following proposition lists some useful equivalence and embedding relations.

Lemma 2.2. *For any* $s \in \mathbb{R}$,

$$H^s \sim B^s_{2,2}, \ \ \dot{H}^s \sim \dot{B}^s_{2,2}.$$

For any $s \in \mathbb{R}$ *and* $1 < q < \infty$,

$$B^s_{q,\min\{q,2\}} \hookrightarrow W^s_q \hookrightarrow B^s_{q,\max\{q,2\}}.$$

For any non-integer $s > 0$, *the Hölder space* C^s *is equivalent to* $B^s_{\infty,\infty}$.

We have also used the space-time space defined below.

Definition 2.3. *For* $t > 0$, $s \in \mathbb{R}$ *and* $1 \leq p, q, r \leq \infty$, *the inhomogeneous and homogeneous space-time spaces* $L^r_t B^s_{p,q}$, $L^r_t \dot{B}^s_{p,q}$ *and* $\widetilde{L}^r_t B^s_{p,q}$, $\widetilde{L}^r_t \dot{B}^s_{p,q}$ *are defined through the norms*

$$\|f\|_{L^r_t B^s_{p,q}} \equiv \|\|2^{js}\|\Delta_j f\|_{L^p}\|_{l^q_j}\|_{L^r_t}, \ \ \|f\|_{L^r_t \dot{B}^s_{p,q}} \equiv \|\|2^{js}\|\dot{\Delta}_j f\|_{L^p}\|_{l^q_j}\|_{L^r_t},$$

and

$$\|f\|_{\widetilde{L}^r_t B^s_{p,q}} \equiv \|2^{js}\|\Delta_j f\|_{L^r_t L^p}\|_{l^q_j}, \ \ \|f\|_{\widetilde{L}^r_t \dot{B}^s_{p,q}} \equiv \|2^{js}\|\dot{\Delta}_j f\|_{L^r_t L^p}\|_{l^q_j},$$

respectively.

The inhomogeneous space-time space has the following properties.

$$L^r_t B^s_{p,q} \hookrightarrow \widetilde{L}^r_t B^s_{p,q}, \text{ if } q \geq r, \ \ \widetilde{L}^r_t B^s_{p,q} \hookrightarrow L^r_t B^s_{p,q}, \text{ if } r \geq q.$$

As $q = r$,

$$\|f\|_{L^r_t B^s_{p,q}} \approx \|f\|_{\widetilde{L}^r_t B^s_{p,q}}.$$

The homogeneous space-time space has the similar properties.

Bernstein's inequalities are useful tools in dealing with Fourier localized functions. These inequalities trade integrability for derivatives. The following proposition provides Bernstein type inequalities for fractional derivatives. The upper bounds also hold when the fractional operators are replaced by partial derivatives.

Lemma 2.4. *Let* $\alpha \geq 0$ *and* $1 \leq p \leq q \leq \infty$.

1) *If* f *satisfies*

$$supp\, \widehat{f} \subset \{\xi \in \mathbb{R}^d : \ |\xi| \leq K 2^j\},$$

for some integer j *and a constant* $K > 0$, *then*

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_1\, 2^{2\alpha j + jd(\frac{1}{p} - \frac{1}{q})}\|f\|_{L^p(\mathbb{R}^d)}.$$

2) *If f satisfies*

$$supp\,\widehat{f}\subset\{\xi\in\mathbb{R}^d:\ K_1 2^j\le|\xi|\le K_2 2^j\}$$

for some integer j and constants $0<K_1\le K_2$, then

$$C_1\,2^{2\alpha j}\|f\|_{L^q(\mathbb{R}^d)}\le\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)}\le C_2\,2^{2\alpha j+jd(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R}^d)},$$

where C_1 and C_2 are constants depending on α,p and q.

2.2. **Calculus inequalities.** As preparations we first give two lemmas regarding to commutator estimates and product law, which proofs are put in the Appendix A.

Lemma 2.5. *Let $s>-1$, $(p,r,p_1,p_2,q_1,q_2)\in[1,+\infty]$ with $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{q_1}=\frac{1}{p_2}+\frac{1}{q_2}$ and u be a smooth divergence free vector field. Then for $j\in\mathbb{Z}$,*

$$\|2^{js}\|[\dot\Delta_j,u\cdot\nabla]v\|_{L^p}\|_{l_j^r}\le C(\|\nabla u\|_{L^{p_1}}\|\nabla v\|_{\dot B^{s-1}_{q_1,r}}+\|\nabla v\|_{L^{q_2}}\|\nabla u\|_{\dot B^{s-1}_{p_2,r}}),\tag{2.4}$$

$$\|2^{js}\|[\dot\Delta_j,u\cdot\nabla]v\|_{L^p}\|_{l_j^r}\le C(\|\nabla u\|_{L^{p_1}}\|v\|_{\dot B^{s}_{q_1,r}}+\|v\|_{L^{q_2}}\|\nabla u\|_{\dot B^{s}_{p_2,r}}),\tag{2.5}$$

where $[\dot\Delta_j,u\cdot\nabla]v=\dot\Delta_j(u\cdot\nabla v)-u\cdot\dot\Delta_j(\nabla v)$.

Lemma 2.6. *Suppose that $s>0$ and $(p,r,p_1,p_2,q_1,q_2)\in[1,+\infty]$ with $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{q_1}=\frac{1}{p_2}+\frac{1}{q_2}$. Then the following holds true*

$$\|fg\|_{\dot B^s_{p,r}}\le C(\|f\|_{L^{p_1}}\|g\|_{\dot B^s_{q_1,r}}+\|f\|_{\dot B^s_{p_2,r}}\|g\|_{L^{q_2}}).\tag{2.6}$$

Remark 2.7. *In particular, Lemma 2.5 and Lemma 2.6 with $p_1=q_2=\infty$ and $q_1=p_2$ have previously been obtained in* [2, 24].

Finally, we recall the following three inequalities for homogeneous Besov space.

Lemma 2.8. *(see* [38]*) Let $0\le l\le\frac{3}{2}$ and $1\le p\le 2$ with $\frac{1}{2}+\frac{l}{3}=\frac{1}{p}$. Then*

$$\|f\|_{\dot B^{-l}_{2,\infty}(\mathbb{R}^3)}\le C\|f\|_{L^p(\mathbb{R}^3)}.\tag{2.7}$$

Lemma 2.9. *(see* [38]*) Let $s_1\le s<s_2$, $1\le p_2\le p\le p_1\le\infty$ and $1\le q_2\le q\le q_1\le\infty$. Then*

$$\|f\|_{\dot B^s_{p,q}(\mathbb{R}^3)}\le C\|f\|^{\frac{s_2-s}{s_2-s_1}}_{\dot B^{s_1}_{p_1,q_1}(\mathbb{R}^3)}\|f\|^{\frac{s-s_1}{s_2-s_1}}_{\dot B^{s}_{p_2,q_2}(\mathbb{R}^3)}.\tag{2.8}$$

Lemma 2.10. *(see* [2, 24]*) Let s,s_1 and s_2 be real numbers. Let $s_1<s_2$, $0<\theta<1$, $1\le p\le\infty$ and $1\le r_1\le r_2\le\infty$. Then*

$$\|f\|_{\dot B^s_{p,r_2}(\mathbb{R}^3)}\le C\|f\|_{\dot B^s_{p,r_1}(\mathbb{R}^3)}.\tag{2.9}$$

$$\|f\|_{\dot B^{\theta s_1+(1-\theta)s_2}_{p,1}(\mathbb{R}^3)}\le C\|f\|^{\theta}_{\dot B^{s_1}_{p,\infty}(\mathbb{R}^3)}\|f\|^{1-\theta}_{\dot B^{s_2}_{p,\infty}(\mathbb{R}^3)}.\tag{2.10}$$

3. The proof of Theorem 1.1

In this section, we shall prove Theorem 1.1. The global existence part (i) of Theorem 1.1 can be attained according to Friedrichs method. For simplicity, we sketch the key ideas as follows.

Proof of assertion (i) of Theorem 1.1. Consider the approximation problem

$$\begin{cases} \partial_t u^N + \mathbb{P}J_N(\mathbb{P}J_N u^N \cdot \nabla \mathbb{P}J_N u^N) - (\mu+\chi)\Delta \mathbb{P}J_N u^N = \mathbb{P}J_N(J_N b^N \cdot \nabla J_N b^N) \\ \quad +2\chi\nabla\times J_N w^N, \\ \partial_t w^N + J_N(\mathbb{P}J_N u^N \cdot \nabla J_N w^N) + 4\chi J_N w^N - \kappa\nabla\nabla\cdot J_N w^N = 2\chi\nabla\times\mathbb{P}J_N u^N, \\ \partial_t b^N + J_N(\mathbb{P}J_N u^N\cdot\nabla J_N b^N) - \nu\Delta J_N b^N = J_N(J_N b^N\cdot\nabla\mathbb{P}J_N u^N), \\ u^N(x,0) = J_N u_0,\ w^N(x,0) = J_N w_0,\ b^N(x,0) = J_N b_0, \end{cases} \tag{3.1}$$

where we denote $\mathbb{P}$ the Leray projection onto divergence free vector fields, and the spectral cutoff J_N

$$\widehat{J_N f}(\xi) = \chi_{B(0,N)}(\xi)\hat{f}(\xi),\ N>0,$$

where $B(0,N) = \{\xi\in\mathbb{R}^3 \| \xi| \le N\}$, and $\chi_{B(0,N)}$ is the characteristic function on $B(0,N)$. Moreover, we define

$$L^2_N = \{f\in L^2(\mathbb{R}^3) | \mathrm{supp}\hat{f}\subset B(0,N)\}.$$

Using Picard's theorem in [23], it is easy to prove that system (3.1) has a unique local solution $(u^N, w^N, b^N)\in C^1([0,T_N]; L^2_N)$ with $T_N>0$. Furthermore, we derive that

$$\mathbb{P}u^N = u^N, J_N u^N = u^N, J_N w^N = w^N, J_N b^N = b^N, \nabla\cdot u^N = 0 \text{ and } \nabla\cdot b^N = 0.$$

Therefore, equations (3.1) can be reduced to

$$\begin{cases} \partial_t u^N + \mathbb{P}J_N(u^N\cdot\nabla u^N) = (\mu+\chi)\Delta u^N + \mathbb{P}J_N(b^N\cdot\nabla b^N) + 2\chi\nabla\times w^N, \\ \partial_t w^N + J_N(u^N\cdot\nabla w^N) + 4\chi w^N - \kappa\nabla\nabla\cdot w^N = 2\chi\nabla\times u^N, \\ \partial_t b^N + J_N(u^N\cdot\nabla b^N) = \nu\Delta b^N + J_N(b^N\cdot\nabla u^N) \\ \nabla\cdot u^N = 0, \nabla\cdot b^N = 0, \\ u^N(x,0) = J_N u_0,\ w^N = J_N w_0,\ b^N(x,0) = J_N b_0. \end{cases} \tag{3.2}$$

To extend the local solution to the global one, we just need to show that (u^N, w^N, b^N) remains bounded in L^2_N for all time. Taking the L^2-inner products with (u^N, w^N, b^N) to (3.2), then standard calculations yield

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u^N\|^2_{L^2} + \|w^N\|^2_{L^2} + \|b^N\|^2_{L^2}) + (\mu+\chi)\|\nabla u^N\|^2_{L^2} + 4\chi\|w^N\|^2_{L^2} + \nu\|\nabla b^N\|^2_{L^2} \\ &\le 4\chi\int\nabla\times u^N\cdot w^N. \end{aligned} \tag{3.3}$$

where we used the facts that

$$\int\nabla\times w^N\cdot u^N = \int\nabla\times u^N\cdot w^N,\quad \int b^N\cdot\nabla b^N\cdot u^N + \int b^N\cdot\nabla u^N\cdot b^N = 0.$$

Applying the Young inequality, we derive that

$$\int\nabla\times u^N\cdot w^N \le \frac{\frac{\mu}{2}+\chi}{4\chi}\|\nabla u^N\|^2_{L^2} + \frac{\chi}{\frac{\mu}{2}+\chi}\|w^N\|^2_{L^2}. \tag{3.4}$$

Inserting (3.4) into (3.3), one obtains

$$\frac{1}{2}\frac{d}{dt}(\|u^N\|_{L^2}^2+\|w^N\|_{L^2}^2+\|b^N\|_{L^2}^2)+\frac{\mu}{2}\|\nabla u^N\|_{L^2}^2+\frac{4\chi\mu}{\mu+2\chi}\|w^N\|_{L^2}^2+\nu\|\nabla b^N\|_{L^2}^2\le 0. \tag{3.5}$$

Then it follows from this and Picard's theorem that (u^N,w^N,b^N) exists for all time and satisfies for any $T>0$,

$$\|(u^N,w^N,b^N)\|_{L^\infty(0,T;L^2)}\le C,\ \ \|(\nabla u^N,w^N,\nabla b^N)\|_{L^2(0,T;L^2)}\le C, \tag{3.6}$$

where C is independent of N. These estimates imply that there exists a subsequence of (u^N,w^N,b^N) (still denoted by (u^N,w^N,b^N) for convenience), such that

$$(u^N,w^N,b^N)\rightharpoonup^* (u,w,b) \text{ weakly in } L^\infty(0,T;L^2_{loc}(\mathbb{R}^3)). \tag{3.7}$$

$$(\nabla u^N,w^N,\nabla b^N)\rightharpoonup(\nabla u,w,\nabla b) \text{ weakly in } L^2(0,T;L^2_{loc}(\mathbb{R}^3)). \tag{3.8}$$

Subsequently, for any $T>0$, standard computations show that

$$\partial_t u^N\in L^{\frac{4}{3}}(0,T;H^{-1}),\quad \partial_t b^N\in L^{\frac{4}{3}}(0,T;H^{-1}). \tag{3.9}$$

Thus by virtue of Aubin-Lions Lemma in [41] , there exists a subsequence of (u^N,b^N) such that

$$(u^N,b^N)\to(u,b) \text{ strongly in } L^2(0,T;L^2_{loc}(\mathbb{R}^3)). \tag{3.10}$$

These convergence, together with the method of [20], verify that (u,w,b) is a weak solution of (1.2). We thus complete (i) of Theorem 1.1.

□

Now we start to prove assertion (ii) of Theorem 1.1. We remark that the following proof is formal, which can be rigorously applied to the approximate solution (u^N,w^N,b^N) above.

Similar as (3.3)-(3.5), we obtain the uniformly global *a priori* estimates as follows.

Lemma 3.1. *Let $(u_0,w_0,b_0)\in L^2(\mathbb{R}^3)$. Then for any $t>0$, the solution (u,w,b) of system (1.2) satisfies*

$$\frac{d}{dt}(\|u\|_{L^2}^2+\|w\|_{L^2}^2+\|b\|_{L^2}^2)+c_0(\|\nabla u\|_{L^2}^2+\|w\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\le 0, \tag{3.11}$$

and

$$\begin{aligned}&\|u\|_{L^2}^2+\|w\|_{L^2}^2+\|b\|_{L^2}^2+c_0\int_0^t(\|\nabla u(\tau)\|_{L^2}^2+\|w(\tau)\|_{L^2}^2+\|\nabla b(\tau)\|_{L^2}^2)d\tau\\ &\le\|u_0\|_{L^2}^2+\|w_0\|_{L^2}^2+\|b_0\|_{L^2}^2,\end{aligned} \tag{3.12}$$

where $c_0=\min\{\mu,\frac{8\chi\mu}{\mu+2\chi},2\nu\}$.

Another preparation is the following lemma which will play an important role to derive the decay estimates of Theorem 1.1.

Lemma 3.2. *Let (u,w,b) be a smooth solution to system (1.2) with $\nabla\cdot u_0=\nabla\cdot b_0=0$. Then for any $|\xi|\le 1$, there exists a constant $C>0$ such that*

$$|\hat{u}(\xi,t)|^2+|\hat{w}(\xi,t)|^2+|\hat{b}(\xi,t)|^2\le 2e^{-2c_0|\xi|^2t}(|\hat{u}_0|^2+|\hat{w}_0|^2+|\hat{b}_0|^2)+C|\xi|^{-2}. \tag{3.13}$$

Proof. Applying the Fourier transform to the first three equations of system (1.2), we obtain

$$\begin{cases} \partial_t \hat{u} + (\mu+\chi)|\xi|^2\hat{u} = -\mathcal{F}(\nabla P) + \mathcal{F}(b\cdot\nabla b) + 2\chi i\xi \times \hat{w} - \mathcal{F}(u\cdot\nabla u), \\ \partial_t \hat{w} + 4\chi\hat{w} + \kappa\xi\xi\cdot\hat{w} = 2\chi i\xi\times\hat{u} - \mathcal{F}(u\cdot\nabla w), \\ \partial_t\hat{b} + \nu|\xi|^2\hat{b} = \mathcal{F}(b\cdot\nabla u) - \mathcal{F}(u\cdot\nabla b). \end{cases} \tag{3.14}$$

Taking the dot product to $(3.14)_1$, $(3.14)_2$ and $(3.14)_3$ with $\bar{\hat{u}}$, $\bar{\hat{w}}$ and $\bar{\hat{b}}$ respectively, multiplying $(3.14)_1$, $(3.14)_2$ and $(3.14)_3$ conjugated by $\hat{u}$, $\hat{w}$ and $\hat{b}$ respectively, and adding them up, we obtain

$$\begin{aligned} &\partial_t(|\hat{u}|^2 + |\hat{w}|^2 + |\hat{b}|^2) + 2(\mu+\chi)|\xi|^2|\hat{u}|^2 + 8\chi|\hat{w}|^2 + 2\nu|\xi|^2|\hat{b}|^2 \\ &= -\mathcal{F}(\nabla P)\cdot\bar{\hat{u}} - \overline{\mathcal{F}(\nabla P)}\cdot\hat{u} + \mathcal{F}(b\cdot\nabla b)\cdot\bar{\hat{u}} + \overline{\mathcal{F}(b\cdot\nabla b)}\cdot\hat{u} + 2\chi i\xi\times\hat{w}\cdot\bar{\hat{u}} + 2\chi i\xi\times\bar{\hat{w}}\cdot\hat{u} \\ &\quad - \mathcal{F}(u\cdot\nabla u)\cdot\bar{\hat{u}} - \overline{\mathcal{F}(u\cdot\nabla u)}\cdot\hat{u} + 2\chi i\xi\times\hat{u}\cdot\bar{\hat{w}} + 2\chi i\xi\times\bar{\hat{u}}\cdot\hat{w} - \mathcal{F}(u\cdot\nabla w)\cdot\bar{\hat{w}} \\ &\quad - \overline{\mathcal{F}(u\cdot\nabla w)}\cdot\hat{w} + \mathcal{F}(b\cdot\nabla u)\cdot\bar{\hat{b}} + \overline{\mathcal{F}(b\cdot\nabla u)}\cdot\hat{b} - \mathcal{F}(u\cdot\nabla b)\cdot\bar{\hat{b}} - \overline{\mathcal{F}(u\cdot\nabla b)}\cdot\hat{b}. \end{aligned} \tag{3.15}$$

The terms in (3.15) will be labeled as K_1, K_2, $\cdots$, K_{16} according to the order they appear in (3.15).

Taking advantage of the divergence free condition $\nabla\cdot u = 0$, we derive that

$$K_1 = K_2 = 0.$$

Using the properties of Fourier transform and the divergence free condition $\nabla\cdot b = 0$, we obtain

$$\begin{aligned} |K_3 + K_4| &\le 2|\xi||\mathcal{F}(b\otimes b)||\hat{u}| \\ &\le 2|\xi|\|b\otimes b\|_{L^1}|\hat{u}| \\ &\le 2|\xi|\|b\|_{L^2}^2|\hat{u}|. \end{aligned}$$

Similarly, we have

$$|K_7 + K_8| \le 2|\xi|\|u\|_{L^2}^2|\hat{u}|.$$

$$|K_{11} + K_{12}| \le 2|\xi|(\|u\|_{L^2}^2 + \|w\|_{L^2}^2)|\hat{w}|.$$

$$|K_{13} + K_{14} + K_{15} + K_{16}| \le 4|\xi|(\|u\|_{L^2}^2 + \|b\|_{L^2}^2)|\hat{b}|,$$

Applying the Young inequality,

$$\begin{aligned} |K_5 + K_6 + K_9 + K_{10}| &\le 8\chi|\xi||\hat{w}||\hat{u}| \\ &\le (\mu+2\chi)|\xi|^2|\hat{u}|^2 + \frac{16\chi^2}{\mu+2\chi}|\hat{w}|^2. \end{aligned}$$

Combining the above estimates with (3.15), yield

$$\begin{aligned} &\partial_t(|\hat{u}|^2 + |\hat{w}|^2 + |\hat{b}|^2) + \mu|\xi|^2|\hat{u}|^2 + \frac{8\chi\mu}{\mu+2\chi}|\hat{w}|^2 + 2\nu|\xi|^2|\hat{b}|^2 \\ &\le C|\xi|(\|u\|_{L^2}^2 + \|w\|_{L^2}^2 + \|b\|_{L^2}^2)(|\hat{u}| + |\hat{w}| + |\hat{b}|). \end{aligned}$$

Hence for $|\xi| \le 1$, this inequality implies

$$\begin{aligned}&\partial_t(|\hat{u}|^2+|\hat{w}|^2+|\hat{b}|^2)+c_0|\xi|^2(|\hat{u}|^2+|\hat{w}|^2+|\hat{b}|^2)\\&\le C|\xi|(\|u\|_{L^2}^2+\|w\|_{L^2}^2+\|b\|_{L^2}^2)(|\hat{u}|+|\hat{w}|+|\hat{b}|),\end{aligned}$$

which immediately yields

$$\partial_t\sqrt{|\hat{u}|^2+|\hat{w}|^2+|\hat{b}|^2}+c_0|\xi|^2\sqrt{|\hat{u}|^2+|\hat{w}|^2+|\hat{b}|^2}\le C|\xi|(\|u\|_{L^2}^2+\|w\|_{L^2}^2+\|b\|_{L^2}^2). \tag{3.16}$$

Integrating (3.16) in $[0,t]$, we obtain

$$\begin{aligned}\sqrt{|\hat{u}|^2+|\hat{w}|^2+|\hat{b}|^2}&\le e^{-c_0|\xi|^2t}\sqrt{|\hat{u}_0|^2+|\hat{w}_0|^2+|\hat{b}_0|^2}\\&\quad+C\int_0^t e^{-c_0|\xi|^2(t-\tau)}|\xi|(\|u(\tau)\|_{L^2}^2+\|w(\tau)\|_{L^2}^2+\|b(\tau)\|_{L^2}^2)d\tau.\end{aligned} \tag{3.17}$$

Then this inequality, together with Lemma 3.1, yield the desired result (3.13).

□

With the help of Lemma 3.1 and Lemma 3.2 at our disposal, we are ready to prove assertion (ii) of Theorem 1.1.

Proof of assertion (ii) of Theorem 1.1. Set

$$S(t)=\{\xi\in\mathbb{R}^3||\xi|\le\rho(t)\}$$

with $\rho(t)\le 1$ to be chosen later. By Plancherel's theorem,

$$\begin{aligned}\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2&=\int_{\mathbb{R}^3}|\xi|^2(|\hat{u}(\xi)|^2+|\hat{b}(\xi)|^2)d\xi\\&\ge\int_{|\xi|\ge\rho(t)}|\xi|^2(|\hat{u}(\xi)|^2+|\hat{b}(\xi)|^2)d\xi\\&\ge\rho^2(t)\int_{|\xi|\ge\rho(t)}(|\hat{u}(\xi)|^2+|\hat{b}(\xi)|^2)d\xi\\&=\rho^2(t)(\|u\|_{L^2}^2+\|b\|_{L^2}^2)-\rho^2(t)\int_{|\xi|\le\rho(t)}(|\hat{u}(\xi)|^2+|\hat{b}(\xi)|^2)d\xi.\end{aligned} \tag{3.18}$$

Note that $\rho(t)\le 1$, we obviously obtain

$$\|w\|_{L^2}^2\ge\rho^2(t)\|w\|_{L^2}^2. \tag{3.19}$$

Inserting (3.18) and (3.19) into (3.11), it derives that

$$\begin{aligned}&\frac{d}{dt}(\|u\|_{L^2}^2+\|w\|_{L^2}^2+\|b\|_{L^2}^2)+c_0\rho^2(t)(\|u\|_{L^2}^2+\|w\|_{L^2}^2+\|b\|_{L^2}^2)\\&\le\rho^2(t)\int_{|\xi|\le\rho(t)}(|\hat{u}(\xi)|^2+|\hat{b}(\xi)|^2)d\xi.\end{aligned} \tag{3.20}$$

Combining (3.13) with (3.20), we obtain

$$
\begin{aligned}
&\frac{d}{dt}(\|u\|_{L^2}^2+\|w\|_{L^2}^2+\|b\|_{L^2}^2)+c_0\rho^2(t)(\|u\|_{L^2}^2+\|w\|_{L^2}^2+\|b\|_{L^2}^2)\\
&\le 2\rho^2(t)\int_{|\xi|\le\rho(t)}(|\hat{u}_0|^2+|\hat{w}_0|^2+|\hat{b}_0|^2)d\xi+C\rho^2(t)\int_{|\xi|\le\rho(t)}|\xi|^{-2}d\xi\\
&\le C\rho^{2+3(\frac{2}{p}-1)}(t)(\|u_0\|_{L^p}^2+\|w_0\|_{L^p}^2+\|b_0\|_{L^p}^2)+C\rho^3(t),
\end{aligned}
\tag{3.21}
$$

where we have used Hölder's inequality

$$
\int_{S(t)}|\hat{f}(\xi)|^2d\xi\le\left(\int_{S(t)}|\hat{f}(\xi)|^qd\xi\right)^{\frac{2}{q}}\left(\int_{S(t)}d\xi\right)^{1-\frac{2}{q}},
$$

and the Riesz theorem

$$
\|\hat{f}(\xi)\|_{L^q}\le C\|f(\xi)\|_{L^p}
$$

with $\frac{1}{p}+\frac{1}{q}=1$ and $1\le p\le 2$.

Choosing $T_0>0$ and

$$
\rho(t)=\left(\frac{3}{c_0(1+t)}\right)^{\frac{1}{2}},
\tag{3.22}
$$

such that $\rho(t)\le 1$ for all $t>T_0$.

Inserting (3.22) into (3.21), and multiplying the result by $(1+t)^3$, we obtain

$$
\frac{d}{dt}((1+t)^3(\|u\|_{L^2}^2+\|w\|_{L^2}^2+\|b\|_{L^2}^2))\le C(1+t)^{2-\frac{3}{2}(\frac{2}{p}-1)}+C(1+t)^{\frac{3}{2}}.
$$

Integrating it in $[T_0,t]$, it leads to

$$
\begin{aligned}
\|u\|_{L^2}^2+\|w\|_{L^2}^2+\|b\|_{L^2}^2&\le(1+t)^{-3}(1+T_0)^3(\|u(T_0)\|_{L^2}^2+\|w(T_0)\|_{L^2}^2+\|b(T_0)\|_{L^2}^2)\\
&\quad+C(1+t)^{-\frac{3}{2}(\frac{2}{p}-1)}+C(1+t)^{-\frac{1}{2}}\\
&\le C(1+t)^{-\frac{1}{2}}.
\end{aligned}
\tag{3.23}
$$

When $|\xi|\le\rho(t)$, combining (3.12) and (3.23) with (3.17), we have for $t>T_0$,

$$
\begin{aligned}
\sqrt{|\hat{u}|^2+|\hat{w}|^2+|\hat{b}|^2}&\le e^{-c_0|\xi|^2t}\sqrt{|\hat{u}_0|^2+|\hat{w}_0|^2+|\hat{b}_0|^2}\\
&\quad+C\int_0^{T_0}e^{-c_0|\xi|^2(t-s)}(\|u_0\|_{L^2}^2+\|w_0\|_{L^2}^2+\|b_0\|_{L^2}^2)ds\\
&\quad+C(1+t)^{-\frac{1}{2}}\int_{T_0}^te^{-c_0|\xi|^2(t-s)}(1+s)^{-\frac{1}{2}}ds\\
&\le\sqrt{|\hat{u}_0|^2+|\hat{w}_0|^2+|\hat{b}_0|^2}+C.
\end{aligned}
\tag{3.24}
$$

Inserting (3.24) into (3.22), we eventually obtain

$$
\frac{d}{dt}((1+t)^3(\|u\|_{L^2}^2+\|w\|_{L^2}^2+\|b\|_{L^2}^2))\le C(1+t)^{2-\frac{3}{2}(\frac{2}{p}-1)}+C(1+t)^{\frac{1}{2}}.
$$

This immediately yields

$$
\begin{aligned}
\|u\|_{L^2}^2+\|w\|_{L^2}^2+\|b\|_{L^2}^2 &\le (1+t)^{-3}(1+T_0)^3(\|u(T_0)\|_{L^2}^2+\|w(T_0)\|_{L^2}^2+\|b(T_0)\|_{L^2}^2)\\
&\quad + C(1+t)^{-\frac{3}{2}(\frac{2}{p}-1)}+C(1+t)^{-\frac{3}{2}}\\
&\le C(1+t)^{-\frac{3}{2}(\frac{2}{p}-1)}, \qquad (3.25)
\end{aligned}
$$

which implies (1.4) for $t>T_0$. For the case $t\le T_0$, we have

$$
(1+t)^{-\frac{3}{2}(\frac{2}{p}-1)}(\|u\|_{L^2}^2+\|w\|_{L^2}^2+\|b\|_{L^2}^2)\le (1+T_0)^{-\frac{3}{2}(\frac{2}{p}-1)}(\|u_0\|_{L^2}^2+\|w_0\|_{L^2}^2+\|b_0\|_{L^2}^2)\le C,
$$

where we have used the fact $1\le p<2$. Thus the proof of assertion (ii) of of Theorem 1.1 is completed.

□

4. Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3. We first prove the global well-posedness part (i) of Theorem 1.3. As we know, it suffices to establish the global *a priori* H^s estimates.

Proof of assertion (i) of Theorem 1.3. Applying $\dot{\Delta}_j$ to the first three equations of (1.2), we have

$$
\partial_t\dot{\Delta}_j u+u\cdot\nabla\dot{\Delta}_j u-(\mu+\chi)\dot{\Delta}_j\Delta u=-\dot{\Delta}_j\nabla p-[\dot{\Delta}_j,u\cdot\nabla]u+\dot{\Delta}_j(b\cdot\nabla b)+2\chi\dot{\Delta}_j\nabla\times w, \quad (4.1)
$$

$$
\partial_t\dot{\Delta}_j w+u\cdot\nabla\dot{\Delta}_j w+4\chi\dot{\Delta}_j w-\kappa\dot{\Delta}_j\nabla\nabla\cdot w=-[\dot{\Delta}_j,u\cdot\nabla]w+2\chi\dot{\Delta}_j\nabla\times u, \quad (4.2)
$$

$$
\partial_t\dot{\Delta}_j b+u\cdot\nabla\dot{\Delta}_j b-\nu\dot{\Delta}_j\Delta b=-[\dot{\Delta}_j,u\cdot\nabla]b+\dot{\Delta}_j(b\cdot\nabla u), \quad (4.3)
$$

where $[\dot{\Delta}_j,f\cdot\nabla]g=\dot{\Delta}_j(f\cdot\nabla g)-f\cdot\nabla\dot{\Delta}_j g$ is commutator.

Dotting (4.1)-(4.3) by $\dot{\Delta}_j u$, $\dot{\Delta}_j w$ and $\dot{\Delta}_j b$ respectively, integrating resulting equations in $\mathbb{R}^3$, and adding them together, we easily obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\dot{\Delta}_j u\|_{L^2}^2+\|\dot{\Delta}_j w\|_{L^2}^2+\|\dot{\Delta}_j b\|_{L^2}^2)+(\mu+\chi)\|\dot{\Delta}_j\nabla u\|_{L^2}^2+4\chi\|\dot{\Delta}_j w\|_{L^2}^2+\nu\|\dot{\Delta}_j\nabla b\|_{L^2}^2\\
&\le -\int[\dot{\Delta}_j,u\cdot\nabla]u\cdot\dot{\Delta}_j u+\int[\dot{\Delta}_j,b\cdot\nabla]b\cdot\dot{\Delta}_j u-\int[\dot{\Delta}_j,u\cdot\nabla]w\cdot\dot{\Delta}_j w\\
&\quad +4\chi\int\dot{\Delta}_j\nabla\times u\cdot\dot{\Delta}_j w-\int[\dot{\Delta}_j,u\cdot\nabla]b\cdot\dot{\Delta}_j b+\int[\dot{\Delta}_j,b\cdot\nabla]u\cdot\dot{\Delta}_j b, \qquad (4.4)
\end{aligned}
$$

where we use the divergence free conditions $\nabla\cdot u=0$ and $\nabla\cdot b=0$, and the facts

$$
\int\dot{\Delta}_j\nabla\times w\cdot\dot{\Delta}_j u=\int\dot{\Delta}_j\nabla\times u\cdot\dot{\Delta}_j w,
$$

$$
\int b\cdot\nabla\dot{\Delta}_j b\cdot\dot{\Delta}_j u+\int b\cdot\nabla\dot{\Delta}_j u\cdot\dot{\Delta}_j b=0.
$$

Multiplying (4.4) by 2^{2sj}, taking the l_j^2 over $j\in\mathbb{Z}$, noting that $\dot{B}^s_{2,2}=\dot{H}^s$ and using the Hölder's inequality, we yield

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|u\|^2_{\dot{H}^s}+\|w\|^2_{\dot{H}^s}+\|b\|^2_{\dot{H}^s})+(\mu+\chi)\|\nabla u\|^2_{\dot{H}^s}+4\chi\|w\|^2_{\dot{H}^s}+\nu\|\nabla b\|^2_{\dot{H}^s}\\
&\le \|2^{sj}\|[\dot{\Delta}_j,u\cdot\nabla]u\|_{L^2}\|_{l_j^2}\|u\|_{\dot{H}^s}+\|2^{sj}\|[\dot{\Delta}_j,b\cdot\nabla]b\|_{L^2}\|_{l_j^2}\|u\|_{\dot{H}^s}\\
&\quad+\|2^{sj}\|[\dot{\Delta}_j,u\cdot\nabla]w\|_{L^2}\|_{l_j^2}\|w\|_{\dot{H}^s}+4\chi\|\nabla u\|_{\dot{H}^s}\|w\|_{\dot{H}^s}\\
&\quad+\|2^{sj}\|[\dot{\Delta}_j,u\cdot\nabla]b\|_{L^2}\|_{l_j^2}\|b\|_{\dot{H}^s}+\|2^{sj}\|[\dot{\Delta}_j,b\cdot\nabla]u\|_{L^2}\|_{l_j^2}\|b\|_{\dot{H}^s}. \qquad (4.5)
\end{aligned}$$

With the help of the Young inequality, one yields

$$4\chi\|\nabla u\|_{\dot{H}^s}\|w\|_{\dot{H}^s}\le(\frac{\mu}{2}+\chi)\|\nabla u\|^2_{\dot{H}^s}+\frac{4\chi^2}{\frac{\mu}{2}+\chi}\|w\|^2_{\dot{H}^s}. \qquad (4.6)$$

Inserting (4.6) into (4.5), adding the resulting inequality and (3.11) together, we have

$$\begin{aligned}
&\frac{d}{dt}(\|u\|^2_{H^s}+\|w\|^2_{H^s}+\|b\|^2_{H^s})+c_0(\|\nabla u\|^2_{H^s}+\|w\|^2_{H^s}+\|\nabla b\|^2_{H^s})\\
&\le 2\|2^{sj}\|[\dot{\Delta}_j,u\cdot\nabla]u\|_{L^2}\|_{l_j^2}\|u\|_{H^s}+2\|2^{sj}\|[\dot{\Delta}_j,b\cdot\nabla]b\|_{L^2}\|_{l_j^2}\|u\|_{H^s}\\
&\quad+2\|2^{sj}\|[\dot{\Delta}_j,u\cdot\nabla]w\|_{L^2}\|_{l_j^2}\|w\|_{H^s}+2\|2^{sj}\|[\dot{\Delta}_j,u\cdot\nabla]b\|_{L^2}\|_{l_j^2}\|b\|_{H^s}\\
&\quad+2\|2^{sj}\|[\dot{\Delta}_j,b\cdot\nabla]u\|_{L^2}\|_{l_j^2}\|b\|_{H^s}. \qquad (4.7)
\end{aligned}$$

Using commutator estimate (2.4), and noting that for $s>\frac{3}{2}$,

$$\|f\|_{L^\infty}\le C\|f\|_{H^s},\ \|f\|_{\dot{B}^{s-1}_{2,2}}\le C\|f\|_{B^s_{2,2}}=C\|f\|_{H^s},$$

one obviously derives

$$\|2^{js}\|[\dot{\Delta}_j,u\cdot\nabla]u\|_{L^2}\|_{l_j^2}\le C\|\nabla u\|_{L^\infty}\|\nabla u\|_{\dot{B}^{s-1}_{2,2}}\le C\|\nabla u\|^2_{H^s}.$$

Similarly, we have

$$\|2^{js}\|[\dot{\Delta}_j,b\cdot\nabla]b\|_{L^2}\|_{l_j^2}\le C\|\nabla b\|^2_{H^s},$$

$$\begin{aligned}
\|2^{js}\|[\dot{\Delta}_j,u\cdot\nabla]b\|_{L^2}\|_{l_j^2}&\le C(\|\nabla u\|_{L^\infty}\|\nabla b\|_{\dot{B}^{s-1}_{2,2}}+\|\nabla b\|_{L^\infty}\|\nabla u\|_{\dot{B}^{s-1}_{2,2}})\\
&\le C\|\nabla u\|_{H^s}\|\nabla b\|_{H^s},
\end{aligned}$$

and

$$\|2^{js}\|[\dot{\Delta}_j,b\cdot\nabla]u\|_{L^2}\|_{l_j^2}\le C\|\nabla u\|_{H^s}\|\nabla b\|_{H^s}.$$

Taking advantage of the commutator estimate (2.5), we imply that

$$\begin{aligned}
\|2^{js}\|[\dot{\Delta}_j,u\cdot\nabla]w\|_{L^2}\|_{l_j^2}&\le C(\|\nabla u\|_{L^\infty}\|w\|_{\dot{B}^s_{2,2}}+\|w\|_{L^\infty}\|\nabla u\|_{\dot{B}^s_{2,2}})\\
&\le C\|\nabla u\|_{H^s}\|w\|_{H^s}.
\end{aligned}$$

Combining the above estimates together, we get

$$\begin{aligned}
&\frac{d}{dt}(\|u\|^2_{H^s}+\|w\|^2_{H^s}+\|b\|^2_{H^s})+c_0(\|\nabla u\|^2_{H^s}+\|w\|^2_{H^s}+\|\nabla b\|^2_{H^s})\\
&\le C\|\nabla u\|^2_{H^s}\|u\|_{H^s}+C\|\nabla b\|^2_{H^s}\|u\|_{H^s}+C\|\nabla u\|_{H^s}\|w\|^2_{H^s}\\
&\quad+C\|\nabla u\|_{H^s}\|\nabla b\|_{H^s}\|b\|_{H^s}.
\end{aligned}$$

Then the Young inequality leads to

$$\begin{aligned}&\frac{d}{dt}(\|u\|_{H^s}^2+\|w\|_{H^s}^2+\|b\|_{H^s}^2)+\frac{c_0}{2}(\|\nabla u\|_{H^s}^2+\|w\|_{H^s}^2+\|\nabla b\|_{H^s}^2)\\&\le C(\|u\|_{H^s}^2+\|w\|_{H^s}^2+\|b\|_{H^s}^2)(\|\nabla u\|_{H^s}^2+\|w\|_{H^s}^2+\|\nabla b\|_{H^s}^2).\end{aligned}\tag{4.8}$$

This inequality indicates that, if the initial data (u_0, w_0, b_0) satisfy, for $0<\epsilon<C_0=\frac{c_0}{2C}$,

$$\|u_0\|_{H^s}^2+\|w_0\|_{H^s}^2+\|b_0\|_{H^s}^2<\epsilon,$$

then the corresponding solution remains for all time. Namely,

$$\|u(t)\|_{H^s}^2+\|w(t)\|_{H^s}^2+\|b(t)\|_{H^s}^2<\epsilon.\tag{4.9}$$

In fact, if suppose (4.9) is not true and T_0 is the first time such that (4.9) is violated, i.e.,

$$\|u(T_0)\|_{H^s}^2+\|w(T_0)\|_{H^s}^2+\|b(T_0)\|_{H^s}^2=\epsilon,$$

and (4.9) holds for any $0\le t<T_0$. We can deduce from (4.8) that for any $0\le t\le T_0$,

$$\frac{d}{dt}(\|u\|_{H^s}^2+\|w\|_{H^s}^2+\|b\|_{H^s}^2)+(\frac{c_0}{2}-C\epsilon)(\|\nabla u\|_{H^s}^2+\|w\|_{H^s}^2+\|\nabla b\|_{H^s}^2)\le 0.$$

Therefore,

$$\|u(t)\|_{H^s}^2+\|w(t)\|_{H^s}^2+\|b(t)\|_{H^s}^2\le\|u_0\|_{H^s}^2+\|w_0\|_{H^s}^2+\|b_0\|_{H^s}^2<\epsilon.\tag{4.10}$$

This is a contradiction. Thus, we get the uniform bound of (4.9). In addition,

$$\int_0^t(\|\nabla u(\tau)\|_{H^s}^2+\|w(\tau)\|_{H^s}^2+\|\nabla b(\tau)\|_{H^s}^2)ds\le C\epsilon.\tag{4.11}$$

Therefore, the proof of (i) of Theorem 1.1 is completed.

□

Now we turn to prove the decay estimates assertion (ii) of Theorem 1.3. As a tool, we first verify the following lemma in negative Besov space $\dot{B}_{2,\infty}^{-l}$ with $0\le l\le\frac{3}{2}$.

Lemma 4.1. *Let $c_0=\min\{\mu,\frac{8\chi\mu}{\mu+2\chi},2\nu\}$. Then we have*

(1) For $0\le l\le\frac{1}{2}$,

$$\begin{aligned}&\frac{d}{dt}(\|u\|_{\dot{B}_{2,\infty}^{-l}}^2+\|w\|_{\dot{B}_{2,\infty}^{-l}}^2+\|b\|_{\dot{B}_{2,\infty}^{-l}}^2)+c_0(\|\nabla u\|_{\dot{B}_{2,\infty}^{-l}}^2+\|w\|_{\dot{B}_{2,\infty}^{-l}}^2+\|\nabla b\|_{\dot{B}_{2,\infty}^{-l}}^2)\\&\le C(\|\nabla u\|_{L^2}^{\frac{1}{2}+l}+\|\nabla b\|_{L^2}^{\frac{1}{2}+l})(\|\Delta u\|_{L^2}^{\frac{1}{2}-l}+\|\Delta b\|_{L^2}^{\frac{1}{2}-l})(\|\nabla u\|_{L^2}+\|\nabla w\|_{L^2}+\|\nabla b\|_{L^2})\\&\quad\times(\|u\|_{\dot{B}_{2,\infty}^{-l}}+\|w\|_{\dot{B}_{2,\infty}^{-l}}+\|b\|_{\dot{B}_{2,\infty}^{-l}}).\end{aligned}\tag{4.12}$$

(2) For $\frac{1}{2}<l\le\frac{3}{2}$,

$$\begin{aligned}&\frac{d}{dt}(\|u\|_{\dot{B}_{2,\infty}^{-l}}^2+\|w\|_{\dot{B}_{2,\infty}^{-l}}^2+\|b\|_{\dot{B}_{2,\infty}^{-l}}^2)+c_0(\|\nabla u\|_{\dot{B}_{2,\infty}^{-l}}^2+\|w\|_{\dot{B}_{2,\infty}^{-l}}^2+\|\nabla b\|_{\dot{B}_{2,\infty}^{-l}}^2)\\&\le C(\|u\|_{L^2}^{\frac{2l-1}{2}}+\|w\|_{L^2}^{\frac{2l-1}{2}}+\|b\|_{L^2}^{\frac{2l-1}{2}})(\|\nabla u\|_{L^2}^{\frac{5-2l}{2}}+\|\nabla w\|_{L^2}^{\frac{5-2l}{2}}+\|\nabla b\|_{L^2}^{\frac{5-2l}{2}})\\&\quad\times(\|u\|_{\dot{B}_{2,\infty}^{-l}}+\|w\|_{\dot{B}_{2,\infty}^{-l}}+\|b\|_{\dot{B}_{2,\infty}^{-l}}).\end{aligned}\tag{4.13}$$

Proof. From (4.4), together with divergence free conditions $\nabla\cdot u=\nabla\cdot b=0$, we derive that

$$\frac{1}{2}\frac{d}{dt}(\|\dot\Delta_j u\|_{L^2}^2+\|\dot\Delta_j w\|_{L^2}^2+\|\dot\Delta_j b\|_{L^2}^2)+(\mu+\chi)\|\dot\Delta_j\nabla u\|_{L^2}^2+4\chi\|\dot\Delta_j w\|_{L^2}^2+\nu\|\dot\Delta_j\nabla b\|_{L^2}^2$$
$$\le-\int\dot\Delta_j(u\cdot\nabla u)\cdot\dot\Delta_j u+\int\dot\Delta_j(b\cdot\nabla b)\cdot\dot\Delta_j u-\int\dot\Delta_j(u\cdot\nabla w)\cdot\dot\Delta_j w$$
$$+4\chi\int\dot\Delta_j\nabla\times u\cdot\dot\Delta_j w-\int\dot\Delta_j(u\cdot\nabla b)\cdot\dot\Delta_j b+\int\dot\Delta_j(b\cdot\nabla u)\cdot\dot\Delta_j b.$$

Multiplying this inequality by 2^{-2lj} and taking the l_j^∞ over $j\in\mathbb{Z}$, we obtain

$$\begin{aligned}&\frac{d}{dt}(\|u\|_{\dot B_{2,\infty}^{-l}}^2+\|w\|_{\dot B_{2,\infty}^{-l}}^2+\|b\|_{\dot B_{2,\infty}^{-l}}^2)+2(\mu+\chi)\|\nabla u\|_{\dot B_{2,\infty}^{-l}}^2+8\chi\|w\|_{\dot B_{2,\infty}^{-l}}^2+2\nu\|\nabla b\|_{\dot B_{2,\infty}^{-l}}^2\\
&\le 2\|2^{-2lj}\|\dot\Delta_j(u\cdot\nabla u)\|_{L^2}\|\dot\Delta_j u\|_{L^2}\|_{l_j^\infty}+2\|2^{-2lj}\|\dot\Delta_j(b\cdot\nabla b)\|_{L^2}\|\dot\Delta_j u\|_{L^2}\|_{l_j^\infty}\\
&\quad+2\|2^{-2lj}\|\dot\Delta_j(u\cdot\nabla w)\|_{L^2}\|\dot\Delta_j w\|_{L^2}\|_{l_j^\infty}+8\chi\|2^{-2lj}\|\dot\Delta_j(\nabla\times u)\|_{L^2}\|\dot\Delta_j w\|_{L^2}\|_{l_j^\infty}\\
&\quad+2\|2^{-2lj}\|\dot\Delta_j(u\cdot\nabla b)\|_{L^2}\|\dot\Delta_j b\|_{L^2}\|_{l_j^\infty}+2\|2^{-2lj}\|\dot\Delta_j(b\cdot\nabla u)\|_{L^2}\|\dot\Delta_j b\|_{L^2}\|_{l_j^\infty}\\
&\le 2\|u\cdot\nabla u\|_{\dot B_{2,\infty}^{-l}}\|u\|_{\dot B_{2,\infty}^{-l}}+2\|b\cdot\nabla b\|_{\dot B_{2,\infty}^{-l}}\|u\|_{\dot B_{2,\infty}^{-l}}+2\|u\cdot\nabla w\|_{\dot B_{2,\infty}^{-l}}\|w\|_{\dot B_{2,\infty}^{-l}}\\
&\quad+8\chi\|\nabla u\|_{\dot B_{2,\infty}^{-l}}\|w\|_{\dot B_{2,\infty}^{-l}}+2\|u\cdot\nabla b\|_{\dot B_{2,\infty}^{-l}}\|b\|_{\dot B_{2,\infty}^{-l}}+2\|b\cdot\nabla u\|_{\dot B_{2,\infty}^{-l}}\|b\|_{\dot B_{2,\infty}^{-l}}.\end{aligned}\tag{4.14}$$

Applying the Young inequality, we derive that

$$8\chi\|\nabla u\|_{\dot B_{2,\infty}^{-l}}\|w\|_{\dot B_{2,\infty}^{-l}}\le(\mu+2\chi)\|\nabla u\|_{\dot B_{2,\infty}^{-l}}+\frac{8\chi^2}{\frac{\mu}{2}+\chi}\|w\|_{\dot B_{2,\infty}^{-l}}.$$

Therefore, it follows from this, (4.14) and Lemma 2.8 that

$$\begin{aligned}&\frac{d}{dt}(\|u\|_{\dot B_{2,\infty}^{-l}}^2+\|w\|_{\dot B_{2,\infty}^{-l}}^2+\|b\|_{\dot B_{2,\infty}^{-l}}^2)+c_0(\|\nabla u\|_{\dot B_{2,\infty}^{-l}}^2+\|w\|_{\dot B_{2,\infty}^{-l}}^2+\|\nabla b\|_{\dot B_{2,\infty}^{-l}}^2)\\
&\le 2\|u\cdot\nabla u\|_{L^{\frac{6}{3+2l}}}\|u\|_{\dot B_{2,\infty}^{-l}}+2\|b\cdot\nabla b\|_{L^{\frac{6}{3+2l}}}\|u\|_{\dot B_{2,\infty}^{-l}}+2\|u\cdot\nabla w\|_{L^{\frac{6}{3+2l}}}\|w\|_{\dot B_{2,\infty}^{-l}}\\
&\quad+2\|u\cdot\nabla b\|_{L^{\frac{6}{3+2l}}}\|b\|_{\dot B_{2,\infty}^{-l}}+2\|b\cdot\nabla u\|_{L^{\frac{6}{3+2l}}}\|b\|_{\dot B_{2,\infty}^{-l}}\\
&:=J_1+J_2+J_3+J_4+J_5.\end{aligned}\tag{4.15}$$

Next we divide the proof into two cases.

1) If $0\le l\le\frac{1}{2}$, by Hölder's inequality and Gagliardo-Nirenberg inequality, we have

$$\|u\cdot\nabla u\|_{L^{\frac{6}{3+2l}}}\le\|u\|_{L^{\frac{3}{l}}}\|\nabla u\|_{L^2},\tag{4.16}$$

and

$$\|u\|_{L^{\frac{3}{l}}}\le C\|\nabla u\|_{L^2}^{\frac{1+2l}{2}}\|\Delta u\|_{L^2}^{\frac{1-2l}{2}}.$$

Thus

$$J_1\le C\|\nabla u\|_{L^2}^{\frac{3+2l}{2}}\|\Delta u\|_{L^2}^{\frac{1-2l}{2}}\|u\|_{\dot B_{2,\infty}^{-l}}.$$

Similarly, we obtain

$$J_2\le C\|\nabla b\|_{L^2}^{\frac{3+2l}{2}}\|\Delta b\|_{L^2}^{\frac{1-2l}{2}}\|u\|_{\dot B_{2,\infty}^{-l}}.$$

$$J_3\le C\|\nabla u\|_{L^2}^{\frac{1+2l}{2}}\|\Delta u\|_{L^2}^{\frac{1-2l}{2}}\|\nabla w\|_{L^2}\|w\|_{\dot B_{2,\infty}^{-l}}.$$

$$J_4\le C\|\nabla u\|_{L^2}^{\frac{1+2l}{2}}\|\Delta u\|_{L^2}^{\frac{1-2l}{2}}\|\nabla b\|_{L^2}\|b\|_{\dot B_{2,\infty}^{-l}}.$$

$$J_5 \leq C\|\nabla b\|_{L^2}^{\frac{1+2l}{2}}\|\Delta b\|_{L^2}^{\frac{1-2l}{2}}\|\nabla u\|_{L^2}\|b\|_{\dot{B}^{-l}_{2,\infty}}.$$

Inserting the above bounds into (4.15), we get the desired estimate (4.12).

2) If $\frac{1}{2} < l \leq \frac{3}{2}$, applying the Gagliardo-Nirenberg inequality, we have

$$\|u\|_{L^{\frac{3}{l}}} \leq C\|u\|_{L^2}^{\frac{2l-1}{2}}\|\nabla u\|_{L^2}^{\frac{3-2l}{2}}.$$

Together with (4.16), we yield

$$J_1 \leq C\|u\|_{L^2}^{\frac{2l-1}{2}}\|\nabla u\|_{L^2}^{\frac{5-2l}{2}}\|u\|_{\dot{B}^{-l}_{2,\infty}}.$$

Similarly, we have

$$J_2 \leq C\|b\|_{L^2}^{\frac{2l-1}{2}}\|\nabla b\|_{L^2}^{\frac{5-2l}{2}}\|u\|_{\dot{B}^{-l}_{2,\infty}}.$$

$$J_3 \leq C\|u\|_{L^2}^{\frac{2l-1}{2}}\|\nabla u\|_{L^2}^{\frac{3-2l}{2}}\|\nabla w\|_{L^2}\|w\|_{\dot{B}^{-l}_{2,\infty}}.$$

$$J_4 \leq C\|u\|_{L^2}^{\frac{2l-1}{2}}\|\nabla u\|_{L^2}^{\frac{3-2l}{2}}\|\nabla b\|_{L^2}\|b\|_{\dot{B}^{-l}_{2,\infty}}.$$

$$J_5 \leq C\|b\|_{L^2}^{\frac{2l-1}{2}}\|\nabla b\|_{L^2}^{\frac{3-2l}{2}}\|\nabla u\|_{L^2}\|b\|_{\dot{B}^{-l}_{2,\infty}}.$$

Inserting the above estimates into (4.15), we obtain (4.13).

□

By means of Lemma 4.1, we now start to prove assertion (ii) of Theorem 1.3.

Proof of assertion (ii) of Theorem 1.3. The proof is slightly long. For the sake of clarity, we divide it into three parts.

(1) We first prove (1.7). Resorting to (4.4), together with Hölder's, Bernstein's and Sobolev's inequalities and the divergence free conditions $\nabla \cdot u = \nabla \cdot b = 0$, we yield

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\dot{\Delta}_j u\|_{L^2}^2 + \|\dot{\Delta}_j w\|_{L^2}^2 + \|\dot{\Delta}_j b\|_{L^2}^2) + (\mu+\chi)\|\dot{\Delta}_j \nabla u\|_{L^2}^2 + 4\chi\|\dot{\Delta}_j w\|_{L^2}^2 + \nu\|\dot{\Delta}_j \nabla b\|_{L^2}^2 \\
&\leq -\int \dot{\Delta}_j(u\cdot\nabla u)\cdot\dot{\Delta}_j u + \int \dot{\Delta}_j(b\cdot\nabla b)\cdot\dot{\Delta}_j u - \int \dot{\Delta}_j(u\cdot\nabla b)\cdot\dot{\Delta}_j b \\
&\quad + \int \dot{\Delta}_j(b\cdot\nabla u)\cdot\dot{\Delta}_j b - \int [\dot{\Delta}_j, u\cdot\nabla]w\cdot\dot{\Delta}_j w + 4\chi\int \dot{\Delta}_j\nabla\times u\cdot\dot{\Delta}_j w \\
&\leq \|\dot{\Delta}_j(u\cdot\nabla u)\|_{L^{\frac{6}{5}}}\|\dot{\Delta}_j u\|_{L^6} + \|\dot{\Delta}_j(b\cdot\nabla b)\|_{L^{\frac{6}{5}}}\|\dot{\Delta}_j u\|_{L^6} + \|\dot{\Delta}_j(u\cdot\nabla b)\|_{L^{\frac{6}{5}}}\|\dot{\Delta}_j b\|_{L^6} \\
&\quad + \|\dot{\Delta}_j(b\cdot\nabla u)\|_{L^{\frac{6}{5}}}\|\dot{\Delta}_j b\|_{L^6} + \|[\dot{\Delta}_j, u\cdot\nabla]w\|_{L^2}\|\dot{\Delta}_j w\|_{L^2} + 4\chi\|\dot{\Delta}_j\nabla\times u\|_{L^2}\|\dot{\Delta}_j w\|_{L^2} \\
&\leq C2^j\|\dot{\Delta}_j(u\otimes u)\|_{L^{\frac{6}{5}}}\|\dot{\Delta}_j\nabla u\|_{L^2} + C2^j\|\dot{\Delta}_j(b\otimes b)\|_{L^{\frac{6}{5}}}\|\dot{\Delta}_j\nabla u\|_{L^2} \\
&\quad + C2^j\|\dot{\Delta}_j(u\otimes b)\|_{L^{\frac{6}{5}}}\|\dot{\Delta}_j\nabla b\|_{L^2} + \|[\dot{\Delta}_j, u\cdot\nabla]w\|_{L^2}\|\dot{\Delta}_j w\|_{L^2} \\
&\quad + 4\chi\|\dot{\Delta}_j\nabla\times u\|_{L^2}\|\dot{\Delta}_j w\|_{L^2}.
\end{aligned} \tag{4.17}$$

Applying the Young inequality,

$$4\chi\|\dot{\Delta}_j\nabla\times u\|_{L^2}\|\dot{\Delta}_j w\|_{L^2} \leq (\frac{\mu}{2}+\chi)\|\dot{\Delta}_j\nabla u\|_{L^2}^2 + \frac{4\chi^2}{\frac{\mu}{2}+\chi}\|\dot{\Delta}_j w\|_{L^2}^2. \tag{4.18}$$

Inserting (4.18) into (4.17), multiplying the resulting inequality by 2^{-2mj} and taking the l^2_j over $j \in \mathbb{Z}$ together with Lemma 2.5 and Lemma 2.6, we obtain

$$\begin{aligned}
&\frac{d}{dt}(\|u\|^2_{\dot{B}^m_{2,2}} + \|w\|^2_{\dot{B}^m_{2,2}} + \|b\|^2_{\dot{B}^m_{2,2}}) + c_0(\|\nabla u\|^2_{\dot{B}^m_{2,2}} + \|w\|^2_{\dot{B}^m_{2,2}} + \|\nabla b\|^2_{\dot{B}^m_{2,2}})\\
&\le C\|2^{(m+1)j}\|\dot{\Delta}_j(u\otimes u)\|_{L^{\frac{6}{5}}}\|_{l^2_j}\|\nabla u\|_{\dot{B}^m_{2,2}} + C\|2^{(m+1)j}\|\dot{\Delta}_j(b\otimes b)\|_{L^{\frac{6}{5}}}\|_{l^2_j}\|\nabla u\|_{\dot{B}^m_{2,2}}\\
&\quad + C\|2^{(m+1)j}\|\dot{\Delta}_j(u\otimes b)\|_{L^{\frac{6}{5}}}\|_{l^2_j}\|\nabla b\|_{\dot{B}^m_{2,2}} + C\|2^{mj}\|[\dot{\Delta}_j, u\cdot\nabla]w\|_{L^2}\|_{l^2_j}\|w\|_{\dot{B}^m_{2,2}}\\
&\le C\|u\|_{L^3}\|\nabla u\|^2_{\dot{B}^m_{2,2}} + C\|b\|_{L^3}\|\nabla b\|_{\dot{B}^m_{2,2}}\|\nabla u\|_{\dot{B}^m_{2,2}}\\
&\quad + C(\|u\|_{L^3}\|\nabla b\|_{\dot{B}^m_{2,2}} + \|b\|_{L^3}\|\nabla u\|_{\dot{B}^m_{2,2}})\|\nabla b\|_{\dot{B}^m_{2,2}}\\
&\quad + C(\|\nabla u\|_{L^\infty}\|w\|_{\dot{H}^m} + \|u\|_{\dot{H}^{m+1}}\|w\|_{L^\infty})\|w\|_{\dot{H}^m}\\
&\le C(\|u\|^{\frac{1}{2}}_{L^2}\|\nabla u\|^{\frac{1}{2}}_{L^2} + \|b\|^{\frac{1}{2}}_{L^2}\|\nabla b\|^{\frac{1}{2}}_{L^2} + \|\nabla u\|_{L^\infty} + \|w\|_{L^\infty})\\
&\quad \times (\|\nabla u\|^2_{\dot{B}^m_{2,2}} + \|w\|^2_{\dot{B}^m_{2,2}} + \|\nabla b\|^2_{\dot{B}^m_{2,2}}).
\end{aligned} \tag{4.19}$$

Choosing the ϵ in (1.6) sufficiently small, and noting that $\dot{B}^m_{2,2} = \dot{H}^m$, we get

$$\frac{d}{dt}(\|u\|^2_{\dot{H}^m} + \|w\|^2_{\dot{H}^m} + \|b\|^2_{\dot{H}^m}) + \frac{c_0}{2}(\|\nabla u\|^2_{\dot{H}^m} + \|w\|^2_{\dot{H}^m} + \|\nabla b\|^2_{\dot{H}^m}) \le 0. \tag{4.20}$$

Applying Lemma 2.9, we derive that

$$\|u\|_{\dot{H}^m} \le C\|u\|^{\frac{1}{m+l+1}}_{\dot{B}^{-l}_{2,\infty}}\|\nabla u\|^{\frac{m+l}{m+l+1}}_{\dot{H}^m}. \tag{4.21}$$

$$\|b\|_{\dot{H}^m} \le C\|b\|^{\frac{1}{m+l+1}}_{\dot{B}^{-l}_{2,\infty}}\|\nabla b\|^{\frac{m+l}{m+l+1}}_{\dot{H}^m}. \tag{4.22}$$

Therefore, if

$$\|u\|_{\dot{B}^{-l}_{2,\infty}} + \|w\|_{\dot{B}^{-l}_{2,\infty}} + \|b\|_{\dot{B}^{-l}_{2,\infty}} \le C, \tag{4.23}$$

then inserting (4.21) and (4.22) into (4.20), we obtain

$$\frac{d}{dt}(\|u\|^2_{\dot{H}^m} + \|w\|^2_{\dot{H}^m} + \|b\|^2_{\dot{H}^m}) + C(\|u\|^2_{\dot{H}^m} + \|w\|^2_{\dot{H}^m} + \|b\|^2_{\dot{H}^m})^{\frac{m+l+1}{m+l}} \le 0. \tag{4.24}$$

It follows that

$$\|u\|^2_{\dot{H}^m} + \|w\|^2_{\dot{H}^m} + \|b\|^2_{\dot{H}^m} \le C(1+t)^{-m-l}, \tag{4.25}$$

which immediately yields (1.7).

Therefore, it is left to verify that (4.23) holds for $0 < l \le \frac{3}{2}$. To this end, we divide the proof into four cases.

Case 1. ($0 < l \le \frac{1}{2}$) By Lemma 4.1, we obtain

$$\begin{aligned}
&\frac{d}{dt}(\|u\|^2_{\dot{B}^{-l}_{2,\infty}} + \|w\|^2_{\dot{B}^{-l}_{2,\infty}} + \|b\|^2_{\dot{B}^{-l}_{2,\infty}})\\
&\le C(\|\nabla u\|^2_{H^s} + \|w\|^2_{H^s} + \|\nabla b\|^2_{H^s})(\|u\|_{\dot{B}^{-l}_{2,\infty}} + \|w\|_{\dot{B}^{-l}_{2,\infty}} + \|b\|_{\dot{B}^{-l}_{2,\infty}}).
\end{aligned} \tag{4.26}$$

Integrating this inequality in $[0,t]$, one yields

$$
\begin{aligned}
&\|u(t)\|^2_{\dot{B}^{-l}_{2,\infty}} + \|w(t)\|^2_{\dot{B}^{-l}_{2,\infty}} + \|b(t)\|^2_{\dot{B}^{-l}_{2,\infty}} \\
&\le \|u_0\|^2_{\dot{B}^{-l}_{2,\infty}} + \|w_0\|^2_{\dot{B}^{-l}_{2,\infty}} + \|b_0\|^2_{\dot{B}^{-l}_{2,\infty}} + C\sup_{0\le\tau\le t}(\|u(\tau)\|_{\dot{B}^{-l}_{2,\infty}} + \|w(\tau)\|_{\dot{B}^{-l}_{2,\infty}} + \|b(\tau)\|_{\dot{B}^{-l}_{2,\infty}}) \\
&\quad \times \int_0^t (\|\nabla u(\tau)\|^2_{H^s} + \|w(\tau)\|^2_{H^s} + \|\nabla b(\tau)\|^2_{H^s}) d\tau \\
&\le C + C\sup_{0\le\tau\le t}(\|u(\tau)\|_{\dot{B}^{-l}_{2,\infty}} + \|w(\tau)\|_{\dot{B}^{-l}_{2,\infty}} + \|b(\tau)\|_{\dot{B}^{-l}_{2,\infty}}),
\end{aligned}
\tag{4.27}
$$

where we have used (1.6) in the last inequality. Then utilizing Young inequality to this above inequality, it implies (4.23).

Case 2. ($\frac{1}{2} < l < 1$) Since $\|f\|_{\dot{B}^{-l}_{2,\infty}} \cap \|f\|_{L^2} \hookrightarrow \|f\|_{\dot{B}^{-\frac{1}{2}}_{2,\infty}}$, then (4.25) with $l = \frac{1}{2}$ yields

$$
\|u(t)\|_{L^2} + \|w(t)\|_{L^2} + \|b(t)\|_{L^2} \le C(1+t)^{-\frac{1}{4}}. \tag{4.28}
$$

$$
\|u(t)\|_{\dot{H}^s} + \|w(t)\|_{\dot{H}^s} + \|b(t)\|_{H^s} \le C(1+t)^{-\frac{2s+1}{4}}. \tag{4.29}
$$

Therefore, the Gagliardo-Nirenberg inequality, together with (4.28) and (4.29), leads to

$$
\begin{aligned}
\|\nabla u\|_{L^2} &\le C\|u\|_{L^2}^{\frac{s-1}{s}} \|u\|_{\dot{H}^s}^{\frac{1}{s}} \\
&\le C(1+t)^{-\frac{3}{4}}.
\end{aligned}
\tag{4.30}
$$

Similarly, we have

$$
\|\nabla w\|_{L^2} + \|\nabla b\|_{L^2} \le C(1+t)^{-\frac{3}{4}}. \tag{4.31}
$$

Applying the Young inequality to (4.13), together with (4.28)-(4.31), we obtain

$$
\begin{aligned}
&\frac{d}{dt}(\|u\|^2_{\dot{B}^{-l}_{2,\infty}} + \|w\|^2_{\dot{B}^{-l}_{2,\infty}} + \|b\|^2_{\dot{B}^{-l}_{2,\infty}}) \\
&\le C[(\|u\|^{2l-1}_{L^2} + \|w\|^{2l-1}_{L^2} + \|b\|^{2l-1}_{L^2})(\|\nabla u\|^{3-2l}_{L^2} + \|\nabla w\|^{3-2l}_{L^2} + \|\nabla b\|^{3-2l}_{L^2}) \\
&\quad + \|\nabla u\|^2_{L^2} + \|\nabla w\|^2_{L^2} + \|\nabla b\|^2_{L^2}](\|u\|_{\dot{B}^{-l}_{2,\infty}} + \|w\|_{\dot{B}^{-l}_{2,\infty}} + \|b\|_{\dot{B}^{-l}_{2,\infty}}) \\
&\le C((1+t)^{-(2-l)} + \|\nabla u\|^2_{H^s} + \|w\|^2_{H^s} + \|\nabla b\|^2_{H^s}) \\
&\quad \times (\|u\|_{\dot{B}^{-l}_{2,\infty}} + \|w\|_{\dot{B}^{-l}_{2,\infty}} + \|b\|_{\dot{B}^{-l}_{2,\infty}}).
\end{aligned}
\tag{4.32}
$$

Integrating this inequality in $[0,t]$, and noting that (1.6), we have

$$
\begin{aligned}
&\|u\|^2_{\dot{B}^{-l}_{2,\infty}} + \|w\|^2_{\dot{B}^{-l}_{2,\infty}} + \|b\|^2_{\dot{B}^{-l}_{2,\infty}} \\
&\le \|u_0\|^2_{\dot{B}^{-l}_{2,\infty}} + \|w_0\|^2_{\dot{B}^{-l}_{2,\infty}} + \|b_0\|^2_{\dot{B}^{-l}_{2,\infty}} + C\sup_{0\le\tau\le t}(\|u(\tau)\|_{\dot{B}^{-l}_{2,\infty}} + \|w(\tau)\|_{\dot{B}^{-l}_{2,\infty}} + \|b(\tau)\|_{\dot{B}^{-l}_{2,\infty}}) \\
&\quad \times \left(\int_0^t (1+\tau)^{-(2-l)} d\tau + 1\right) \\
&\le C + C\sup_{0\le\tau\le t}(\|u(\tau)\|_{\dot{B}^{-l}_{2,\infty}} + \|w(\tau)\|_{\dot{B}^{-l}_{2,\infty}} + \|b(\tau)\|_{\dot{B}^{-l}_{2,\infty}}),
\end{aligned}
\tag{4.33}
$$

Then by virtue of the Young inequality, we complete (4.23).

Case 3. $(1 \le l < \frac{3}{2})$ Note that $\|f\|_{\dot{B}^{-l}_{2,\infty}} \cap \|f\|_{L^2} \hookrightarrow \|f\|_{\dot{B}^{-l_0}_{2,\infty}}$ with $0 < l_0 < 1$, then taking $l = l_0$ in (4.25), we have derived that

$$\|u(t)\|_{L^2} + \|w(t)\|_{L^2} + \|b(t)\|_{L^2} \le C(1+t)^{-\frac{l_0}{2}}. \tag{4.34}$$

$$\|u(t)\|_{\dot{H}^s} + \|w(t)\|_{\dot{H}^s} + \|b(t)\|_{\dot{H}^s} \le C(1+t)^{-\frac{s+l_0}{2}}. \tag{4.35}$$

Then using the Gagliardo-Nirenberg inequality, it deduces that

$$\begin{aligned}\|\nabla u\|_{L^2} &\le C\|u\|_{L^2}^{\frac{s-1}{s}}\|u\|_{\dot{H}^s}^{\frac{1}{s}}\\ &\le C(1+t)^{-\frac{l_0+1}{2}}.\end{aligned} \tag{4.36}$$

Similarly, we have

$$\|\nabla w\|_{L^2} + \|\nabla b\|_{L^2} \le C(1+t)^{-\frac{l_0+1}{2}}. \tag{4.37}$$

Therefore, it follows from (4.13) and (4.34)-(4.37) that

$$\begin{aligned}&\frac{d}{dt}(\|u\|^2_{\dot{B}^{-l}_{2,\infty}} + \|w\|^2_{\dot{B}^{-l}_{2,\infty}} + \|b\|^2_{\dot{B}^{-l}_{2,\infty}})\\ &\le C((1+t)^{-\frac{2l_0+3-2l}{2}} + \|\nabla u\|^2_{H^s} + \|w\|^2_{H^s} + \|\nabla b\|^2_{H^s})\\ &\quad \times (\|u\|_{\dot{B}^{-l}_{2,\infty}} + \|w\|_{\dot{B}^{-l}_{2,\infty}} + \|b\|_{\dot{B}^{-l}_{2,\infty}}).\end{aligned} \tag{4.38}$$

Integrating this inequality in $[0,t]$, we obtain

$$\begin{aligned}&\|u\|^2_{\dot{B}^{-l}_{2,\infty}} + \|w\|^2_{\dot{B}^{-l}_{2,\infty}} + \|b\|^2_{\dot{B}^{-l}_{2,\infty}}\\ &\le \|u_0\|^2_{\dot{B}^{-l}_{2,\infty}} + \|w_0\|^2_{\dot{B}^{-l}_{2,\infty}} + \|b_0\|^2_{\dot{B}^{-l}_{2,\infty}} + C\sup_{0\le\tau\le t}(\|u(\tau)\|_{\dot{B}^{-l}_{2,\infty}} + \|w(\tau)\|_{\dot{B}^{-l}_{2,\infty}} + \|b(\tau)\|_{\dot{B}^{-l}_{2,\infty}})\\ &\quad \times \left(\int_0^t (1+\tau)^{-\frac{2l_0+3-2l}{2}}d\tau + 1\right)\\ &\le C + C\sup_{0\le\tau\le t}(\|u(\tau)\|_{\dot{B}^{-l}_{2,\infty}} + \|w(\tau)\|_{\dot{B}^{-l}_{2,\infty}} + \|b(\tau)\|_{\dot{B}^{-l}_{2,\infty}}),\end{aligned} \tag{4.39}$$

where we have used the fact that for any $1 < l < \frac{3}{2}$, we can choose $l_0 < 1$ sufficiently near 1 such that $1 - \frac{2l_0+3-2l}{2} < 0$. Then it eventually yields (4.23).

Case 4. $(l = \frac{3}{2})$ Note that $\|f\|_{\dot{B}^{-\frac{3}{2}}_{2,\infty}} \cap \|f\|_{L^2} \hookrightarrow \|f\|_{\dot{B}^{-l_0}_{2,\infty}}$ with $0 < l_0 < \frac{3}{2}$, then by choosing l_0 near $\frac{3}{2}$, following the idea in Case 3, we complete the proof of this case.

(2) Now we are at a stage to verify (1.8). Applying D^m to the second equation of (1.2), dotting it by $D^m w$, integrating in space domain, then standard calculations yield

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|D^m w\|^2_{L^2} + 4\chi\|D^m w\|^2_{L^2}\\ &\le 2\chi\int D^m\nabla\times u\cdot D^m w - \int D^m(u\cdot\nabla w)\cdot D^m w\\ &\le C(\|D^{m+1}u\|_{L^2} + \|D^{m+1}u\|_{L^2}\|w\|_{L^\infty} + \|u\|_{L^\infty}\|D^{m+1}w\|_{L^2})\|D^m w\|_{L^2}.\end{aligned}$$

Then this inequality, together with (1.6), leads to

$$\frac{d}{dt}\|D^m w\|_{L^2} + 4\chi\|D^m w\|_{L^2} \le C(\|D^{m+1}u\|_{L^2} + \|D^{m+1}w\|_{L^2}).$$

Multiplying this inequality by $e^{4\chi t}$, integrating the resulting inequality in $[0,t]$, together with (1.6) and (1.7), we have

$$\begin{aligned}
\|D^m w\|_{L^2} &\le e^{-4\chi t}\|D^m w_0\|_{L^2} + C\int_0^t e^{-4\chi(t-\tau)}(\|D^{m+1}u(\tau)\|_{L^2} + \|D^{m+1}w(\tau)\|_{L^2})d\tau \\
&\le Ce^{-4\chi t} + C\int_0^{\frac{t}{2}} e^{-4\chi(t-\tau)}(\|D^{m+1}u(\tau)\|_{L^2} + \|D^{m+1}w(\tau)\|_{L^2})d\tau \\
&\quad + C\int_{\frac{t}{2}}^t e^{-4\chi(t-\tau)}(\|D^{m+1}u(\tau)\|_{L^2} + \|D^{m+1}w(\tau)\|_{L^2})d\tau \\
&\le Ce^{-4\chi t} + Ce^{-2\chi t}\left(\int_0^{\frac{t}{2}}(\|D^{m+1}u(\tau)\|_{L^2}^2 + \|D^{m+1}w(\tau)\|_{L^2}^2)d\tau\right)^{\frac{1}{2}}\left(\int_0^{\frac{t}{2}} d\tau\right)^{\frac{1}{2}} \\
&\quad + C\int_{\frac{t}{2}}^t e^{-4\chi(t-\tau)}(1+\tau)^{-\frac{m+1}{2}-\frac{l}{2}}d\tau \\
&\le Ce^{-4\chi t} + Ce^{-2\chi t}t^{\frac{1}{2}} + C(1+t)^{-\frac{m+1}{2}-\frac{l}{2}} \\
&\le C(1+t)^{-\frac{m+1}{2}-\frac{l}{2}}.
\end{aligned}$$

Thus the proof of (1.8) is completed.

(3) Finally, we prove (1.10). We write the third equation of (1.2) into integral form,

$$b = e^{\nu\Delta t}b_0 + \int_0^t e^{\nu\Delta(t-\tau)}(b\cdot\nabla u - u\cdot\nabla b)(\tau)d\tau.$$

Applying D^m to this equation and taking the L^2 norm, together with (1.7), we obtain

$$\begin{aligned}
\|D^m b - D^m e^{\nu\Delta t}b_0\|_{L^2} &\le \int_0^{\frac{t}{2}} \|D^m\nabla e^{\nu\Delta(t-\tau)}(b\otimes u - u\otimes b)(\tau)\|_{L^2}d\tau \\
&\quad + \int_{\frac{t}{2}}^t \|\nabla e^{\nu\Delta(t-\tau)}D^m(b\otimes u - u\otimes b)(\tau)\|_{L^2}d\tau \\
&\le C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{m+1}{2}-\frac{3}{4}}\|(b\otimes u - u\otimes b)(\tau)\|_{L^1}d\tau \\
&\quad + C\int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{2}}\|D^m(b\otimes u - u\otimes b)(\tau)\|_{L^2}d\tau \\
&\le C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{m+1}{2}-\frac{3}{4}}\|u(\tau)\|_{L^2}\|b(\tau)\|_{L^2}d\tau \\
&\quad + C\int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{2}}(\|u(\tau)\|_{L^\infty}\|D^m b(\tau)\|_{L^2} + \|b(\tau)\|_{L^\infty}\|D^m u(\tau)\|_{L^2})d\tau \\
&\le C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{m+1}{2}-\frac{3}{4}}(1+\tau)^{-l}d\tau \\
&\quad + C\int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{3}{2}(\frac{1}{2}+\frac{m}{3})-\frac{m}{2}-l}d\tau \\
&\le CM(t)(1+t)^{-\frac{m}{2}-\frac{l}{2}},
\end{aligned} \tag{4.40}$$

where

$$M(t)=\begin{cases}(1+t)^{-(\frac{1}{4}+\frac{l}{2})}+(1+t)^{-\frac{1}{4}-\frac{m}{2}-\frac{l}{2}}, & if\quad l<1,\\ (1+t)^{-(\frac{5}{4}-\frac{l}{2})}\ln(1+t)+(1+t)^{-\frac{1}{4}-\frac{m}{2}-\frac{l}{2}}, & if\quad l=1,\\ (1+t)^{-(\frac{5}{4}-\frac{l}{2})}+(1+t)^{-\frac{1}{4}-\frac{m}{2}-\frac{l}{2}}, & if\quad l>1.\end{cases}$$

Therefore, combining (4.40) and (1.9), and choosing t sufficiently large, yield

$$\begin{aligned}\|D^m b\|_{L^2}&\geq \|D^m e^{\nu\Delta t}b_0\|_{L^2}-\|D^m b-D^m e^{\nu\Delta t}b_0\|_{L^2}\\ &\geq C_1(1+t)^{-\frac{m}{2}-\frac{l}{2}}-CM(t)(1+t)^{-\frac{m}{2}-\frac{l}{2}}\\ &\geq C(1+t)^{-\frac{m}{2}-\frac{l}{2}},\end{aligned}\tag{4.41}$$

since $M(t)\to 0$ as $t\to\infty$. We thus obtain (1.10). Therefore, the proof of assertion (ii) of Theorem 1.3 is completed.

□

5. Proof of Theorem 1.5

This section is devoted to proving Theorem 1.5. We first show the global existence part (i). The key step is to establish the global *a priori* $B^s_{2,\infty}$ estimates of the solution.

Proof of assertion (i) of Theorem 1.5. Applying Hölder's inequality to (4.4) and integrating the resulting inequality in $[0,t]$, we obtain

$$\begin{aligned}&\|\dot\Delta_j u\|^2_{L^2}+\|\dot\Delta_j w\|^2_{L^2}+\|\dot\Delta_j b\|^2_{L^2}+2(\mu+\chi)\int_0^t\|\dot\Delta_j\nabla u(\tau)\|^2_{L^2}d\tau+8\chi\int_0^t\|\dot\Delta_j w(\tau)\|^2_{L^2}d\tau\\ &\quad+2\nu\int_0^t\|\dot\Delta_j\nabla b(\tau)\|^2_{L^2}d\tau\\ &\leq\|\dot\Delta_j u_0\|^2_{L^2}+\|\dot\Delta_j w_0\|^2_{L^2}+\|\dot\Delta_j b_0\|^2_{L^2}\\ &\quad+2\int_0^t\|[\dot\Delta_j,u\cdot\nabla]u(\tau)\|_{L^2}\|\dot\Delta_j u(\tau)\|_{L^2}d\tau+2\int_0^t\|[\dot\Delta_j,b\cdot\nabla]b(\tau)\|_{L^2}\|\dot\Delta_j u(\tau)\|_{L^2}d\tau\\ &\quad+2\int_0^t\|[\dot\Delta_j,u\cdot\nabla]w(\tau)\|_{L^2}\|\dot\Delta_j w(\tau)\|_{L^2}d\tau+8\chi\int_0^t\|\dot\Delta_j\nabla\times u(\tau)\|_{L^2}\|\dot\Delta_j w(\tau)\|_{L^2}d\tau\\ &\quad+2\int_0^t\|[\dot\Delta_j,u\cdot\nabla]b(\tau)\|_{L^2}\|\dot\Delta_j b(\tau)\|_{L^2}d\tau+2\int_0^t\|[\dot\Delta_j,b\cdot\nabla]u(\tau)\|_{L^2}\|\dot\Delta_j b(\tau)\|_{L^2}d\tau.\end{aligned}$$

Multiplying it by 2^{2sj} and taking the supremum over $j\in\mathbb{Z}$, one yields

$$\begin{aligned}&\|u\|^2_{\tilde L^\infty_t(\dot B^s_{2,\infty})}+\|w\|^2_{\tilde L^\infty_t(\dot B^s_{2,\infty})}+\|b\|^2_{\tilde L^\infty_t(\dot B^s_{2,\infty})}+2(\mu+\chi)\|\nabla u\|^2_{\tilde L^2_t(\dot B^s_{2,\infty})}+8\chi\|w\|^2_{\tilde L^2_t(\dot B^s_{2,\infty})}\\ &\qquad+2\nu\|\nabla b\|^2_{\tilde L^2_t(\dot B^s_{2,\infty})}\\ &\leq\|u_0\|^2_{\dot B^s_{2,\infty}}+\|w_0\|^2_{\dot B^s_{2,\infty}}+\|b_0\|^2_{\dot B^s_{2,\infty}}+L_1+L_2+L_3+L_4+L_5+L_6\end{aligned}\tag{5.1}$$

with

$$L_1=2\sup_{j\in\mathbb{Z}}2^{2sj}\int_0^t\|[\dot\Delta_j,u\cdot\nabla]u(\tau)\|_{L^2}\|\dot\Delta_j u(\tau)\|_{L^2}d\tau,$$

$$L_2 = 2\sup_{j\in\mathbb{Z}} 2^{2sj}\int_0^t \|[\dot\Delta_j, b\cdot\nabla]b(\tau)\|_{L^2}\|\dot\Delta_j u(\tau)\|_{L^2}d\tau,$$
$$L_3 = 2\sup_{j\in\mathbb{Z}} 2^{2sj}\int_0^t \|[\dot\Delta_j, u\cdot\nabla]w(\tau)\|_{L^2}\|\dot\Delta_j w(\tau)\|_{L^2}d\tau,$$
$$L_4 = 8\chi\sup_{j\in\mathbb{Z}} 2^{2sj}\int_0^t \|\dot\Delta_j \nabla\times u(\tau)\|_{L^2}\|\dot\Delta_j w(\tau)\|_{L^2}d\tau,$$
$$L_5 = 2\sup_{j\in\mathbb{Z}} 2^{2sj}\int_0^T \|[\dot\Delta_j, u\cdot\nabla]b(\tau)\|_{L^2}\|\dot\Delta_j b(\tau)\|_{L^2}d\tau,$$
$$L_6 = 2\sup_{j\in\mathbb{Z}} 2^{2sj}\int_0^T \|[\dot\Delta_j, b\cdot\nabla]u(\tau)\|_{L^2}\|\dot\Delta_j b(\tau)\|_{L^2}d\tau.$$

Applying Hölder's inequality and (2.4), we have

$$\begin{aligned}L_1 &\le 2\|u\|_{\tilde L_t^\infty(\dot B^s_{2,\infty})}\sup_{j\in\mathbb{Z}}(2^{sj}\|[\dot\Delta_j, u\cdot\nabla]u\|_{L^1_tL^2})\\ &\le C\|u\|_{\tilde L_t^\infty(\dot B^s_{2,\infty})}\|\nabla u\|_{L^2_tL^\infty}\|\nabla u\|_{\tilde L^2_t(\dot B^{s-1}_{2,\infty})}.\end{aligned}$$

Similarly,

$$L_2 \le C\|u\|_{\tilde L_t^\infty(\dot B^s_{2,\infty})}\|\nabla b\|_{L^2_tL^\infty}\|\nabla b\|_{\tilde L^2_t(\dot B^{s-1}_{2,\infty})},$$

$$L_5+L_6 \le C\|b\|_{\tilde L_t^\infty(\dot B^s_{2,\infty})}(\|\nabla u\|_{L^2_tL^\infty}\|\nabla b\|_{\tilde L^2_t(\dot B^{s-1}_{2,\infty})}+\|\nabla b\|_{L^2_tL^\infty}\|\nabla u\|_{\tilde L^2_t(\dot B^{s-1}_{2,\infty})}).$$

Again applying Hölder's inequality, together with (2.5), we obtain

$$\begin{aligned}L_3 &\le \|w\|_{\tilde L_t^\infty(\dot B^s_{2,\infty})}\sup_{j\in\mathbb{Z}}(2^{sj}\|[\dot\Delta_j, u\cdot\nabla]w\|_{L^1_tL^2})\\ &\le C\|w\|_{\tilde L_t^\infty(\dot B^s_{2,\infty})}(\|\nabla u\|_{L^2_tL^\infty}\|w\|_{\tilde L^2_t(\dot B^s_{2,\infty})}+\|w\|_{L^2_tL^\infty}\|\nabla u\|_{\tilde L^2_t(\dot B^s_{2,\infty})})\end{aligned}$$

Using Hölder's inequality and the Young inequality, it holds that

$$L_4 \le (\mu+2\chi)\|\nabla u\|^2_{\tilde L^2_t(\dot B^s_{2,\infty})}+\frac{16\chi^2}{\mu+2\chi}\|w\|^2_{\tilde L^2_t(\dot B^s_{2,\infty})}.$$

Inserting the above estimates into (5.1), one leads to

$$\begin{aligned}&\|u\|^2_{\tilde L_t^\infty(\dot B^s_{2,\infty})}+\|w\|^2_{\tilde L_t^\infty(\dot B^s_{2,\infty})}+\|b\|^2_{\tilde L_t^\infty(\dot B^s_{2,\infty})}+\mu\|\nabla u\|^2_{\tilde L^2_t(\dot B^s_{2,\infty})}+\frac{8\chi\mu}{\mu+2\chi}\|w\|^2_{\tilde L^2_t(\dot B^s_{2,\infty})}\\ &\quad+2\nu\|\nabla b\|^2_{\tilde L^2_t(\dot B^s_{2,\infty})}\\ &\le \|u_0\|^2_{\dot B^s_{2,\infty}}+\|w_0\|^2_{\dot B^s_{2,\infty}}+\|b_0\|^2_{\dot B^s_{2,\infty}}+C\|\nabla u\|_{L^2_tL^\infty}(\|u\|_{\tilde L_t^\infty(\dot B^s_{2,\infty})}\|\nabla u\|_{\tilde L^2_t(\dot B^{s-1}_{2,\infty})}\\ &\quad+\|b\|_{\tilde L_t^\infty(\dot B^s_{2,\infty})}\|\nabla b\|_{\tilde L^2_t(\dot B^{s-1}_{2,\infty})}+\|w\|_{\tilde L_t^\infty(\dot B^s_{2,\infty})}\|w\|_{\tilde L^2_t(\dot B^s_{2,\infty})})\\ &\quad+C\|\nabla b\|_{L^2_tL^\infty}(\|u\|_{\tilde L_t^\infty(\dot B^s_{2,\infty})}\|\nabla b\|_{\tilde L^2_t(\dot B^{s-1}_{2,\infty})}+\|b\|_{\tilde L_t^\infty(\dot B^s_{2,\infty})}\|\nabla u\|_{\tilde L^2_t(\dot B^{s-1}_{2,\infty})})\\ &\quad+C\|w\|_{\tilde L_t^\infty(\dot B^s_{2,\infty})}\|w\|_{L^2_tL^\infty}\|\nabla u\|_{\tilde L^2_t(\dot B^s_{2,\infty})}.\end{aligned}\tag{5.2}$$

Adding (3.11) and (5.2) up, using the Young inequality, and noting that for $s>\frac{3}{2}$,

$$\|f\|_{L^\infty}\le C\|f\|_{B^s_{2,\infty}},\ \|f\|_{\dot B^{s-1}_{2,\infty}}\le C\|f\|_{B^s_{2,\infty}},$$

we have

$$
\begin{aligned}
&\|u\|^2_{\tilde{L}^\infty_t(B^s_{2,\infty})} + \|w\|^2_{\tilde{L}^\infty_t(B^s_{2,\infty})} + \|b\|^2_{\tilde{L}^\infty_t(B^s_{2,\infty})} + \frac{\mu}{2}\|\nabla u\|^2_{\tilde{L}^2_t(B^s_{2,\infty})} + \frac{4\chi\mu}{\mu+2\chi}\|w\|^2_{\tilde{L}^2_t(B^s_{2,\infty})} \\
&\quad + \nu\|\nabla b\|^2_{\tilde{L}^2_t(B^s_{2,\infty})} \\
&\le \|u_0\|^2_{B^s_{2,\infty}} + \|w_0\|^2_{B^s_{2,\infty}} + \|b_0\|^2_{B^s_{2,\infty}} \\
&\quad + C(\|u\|^2_{\tilde{L}^\infty_t(B^s_{2,\infty})} + \|w\|^2_{\tilde{L}^\infty_t(B^s_{2,\infty})} + \|b\|^2_{\tilde{L}^\infty_t(B^s_{2,\infty})}) \\
&\quad \times (\|\nabla u\|^2_{\tilde{L}^2_t(B^s_{2,\infty})} + \|w\|^2_{\tilde{L}^2_t(B^s_{2,\infty})} + \|\nabla b\|^2_{\tilde{L}^2_t(B^s_{2,\infty})}). \qquad (5.3)
\end{aligned}
$$

This inequality indicates that, if we set $C_0 = \frac{\min\{\frac{\mu}{2}, \frac{4\chi\mu}{\mu+2\chi}, \nu\}}{C}$, and the initial data (u_0, w_0, b_0) satisfy, for $\epsilon < C_0$,

$$
\|u_0\|^2_{B^s_{2,\infty}} + \|w_0\|^2_{B^s_{2,\infty}} + \|b_0\|^2_{B^s_{2,\infty}} < \epsilon,
$$

then we have

$$
\begin{aligned}
&\|u\|^2_{\tilde{L}^\infty_t(B^s_{2,\infty})} + \|w\|^2_{\tilde{L}^\infty_t(B^s_{2,\infty})} + \|b\|^2_{\tilde{L}^\infty_t(B^s_{2,\infty})} \\
&\quad + \|\nabla u\|^2_{\tilde{L}^2_t(B^s_{2,\infty})} + \|w\|^2_{\tilde{L}^2_t(B^s_{2,\infty})} + \|\nabla b\|^2_{\tilde{L}^2_t(B^s_{2,\infty})} \le C\epsilon,
\end{aligned}
$$

which implies the global existence part (i) of Theorem 1.5.

□

We now start to prove the decay estimates assertion (ii) of Theorem 1.5.

Proof of assertion (ii) of Theorem 1.5. For the sake of clarity, we divided it into two parts.

(1) First, we deal with (1.13). Applying divergence free conditions $\nabla \cdot u = \nabla \cdot b = 0$ to (4.4), together with Hölder's, Bernstein's and Sobolev's inequalities, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\dot{\Delta}_j u\|^2_{L^2} + \|\dot{\Delta}_j w\|^2_{L^2} + \|\dot{\Delta}_j b\|^2_{L^2}) + (\mu+\chi)\|\dot{\Delta}_j \nabla u\|^2_{L^2} + 4\chi\|\dot{\Delta}_j w\|^2_{L^2} + \nu\|\dot{\Delta}_j \nabla b\|^2_{L^2} \\
&\le -\int \dot{\Delta}_j(u\cdot\nabla u)\cdot\dot{\Delta}_j u + \int \dot{\Delta}_j(b\cdot\nabla b)\cdot\dot{\Delta}_j u - \int \dot{\Delta}_j(u\cdot\nabla b)\cdot\dot{\Delta}_j b \\
&\quad + \int \dot{\Delta}_j(b\cdot\nabla u)\cdot\dot{\Delta}_j b - \int [\dot{\Delta}_j, u\cdot\nabla]w\cdot\dot{\Delta}_j w + 4\chi\int \dot{\Delta}_j \nabla\times u\cdot\dot{\Delta}_j w \\
&\le \|\dot{\Delta}_j(u\cdot\nabla u)\|_{L^{\frac{6}{5}}}\|\dot{\Delta}_j u\|_{L^6} + \|\dot{\Delta}_j(b\cdot\nabla b)\|_{L^{\frac{6}{5}}}\|\dot{\Delta}_j u\|_{L^6} + \|\dot{\Delta}_j(u\cdot\nabla b)\|_{L^{\frac{6}{5}}}\|\dot{\Delta}_j b\|_{L^6} \\
&\quad + \|\dot{\Delta}_j(b\cdot\nabla u)\|_{L^{\frac{6}{5}}}\|\dot{\Delta}_j b\|_{L^6} + \|[\dot{\Delta}_j, u\cdot\nabla]w\|_{L^2}\|\dot{\Delta}_j w\|_{L^2} + 4\chi\|\dot{\Delta}_j \nabla\times u\|_{L^2}\|\dot{\Delta}_j w\|_{L^2} \\
&\le C2^j\|\dot{\Delta}_j(u\otimes u)\|_{L^{\frac{6}{5}}}\|\dot{\Delta}_j \nabla u\|_{L^2} + C2^j\|\dot{\Delta}_j(b\otimes b)\|_{L^{\frac{6}{5}}}\|\dot{\Delta}_j \nabla u\|_{L^2} \\
&\quad + C2^j\|\dot{\Delta}_j(u\otimes b)\|_{L^{\frac{6}{5}}}\|\dot{\Delta}_j \nabla b\|_{L^2} + \|[\dot{\Delta}_j, u\cdot\nabla]w\|_{L^2}\|\dot{\Delta}_j w\|_{L^2} \\
&\quad + 4\chi\|\dot{\Delta}_j \nabla\times u\|_{L^2}\|\dot{\Delta}_j w\|_{L^2}. \qquad (5.4)
\end{aligned}
$$

Multiplying this inequality by 2^{-2mj} and taking the supermum over $j \in \mathbb{Z}$, together with (4.18), Lemma 2.5 and Lemma 2.6, we derive that

$$\begin{aligned}
&\frac{d}{dt}(\|u\|^2_{\dot{B}^m_{2,\infty}} + \|w\|^2_{\dot{B}^m_{2,\infty}} + \|b\|^2_{\dot{B}^m_{2,\infty}}) + c_0(\|\nabla u\|^2_{\dot{B}^m_{2,\infty}} + \|w\|^2_{\dot{B}^m_{2,\infty}} + \|\nabla b\|^2_{\dot{B}^m_{2,\infty}}) \\
&\le C\|2^{(m+1)j}\|\dot{\Delta}_j(u\otimes u)\|_{L^{\frac{6}{5}}}\|_{l^\infty_j}\|\nabla u\|_{\dot{B}^m_{2,\infty}} + C\|2^{(m+1)j}\|\dot{\Delta}_j(b\otimes b)\|_{L^{\frac{6}{5}}}\|_{l^\infty_j}\|\nabla u\|_{\dot{B}^m_{2,\infty}} \\
&\quad + C\|2^{(m+1)j}\|\dot{\Delta}_j(u\otimes b)\|_{L^{\frac{6}{5}}}\|_{l^\infty_j}\|\nabla b\|_{\dot{B}^m_{2,\infty}} + C\|2^{mj}\|[\dot{\Delta}_j, u\cdot\nabla]w\|_{L^2}\|_{l^\infty_j}\|w\|_{\dot{B}^m_{2,\infty}} \\
&\le C\|u\|_{L^3}\|\nabla u\|^2_{\dot{B}^m_{2,\infty}} + C\|b\|_{L^3}\|\nabla b\|_{\dot{B}^m_{2,\infty}}\|\nabla u\|_{\dot{B}^m_{2,\infty}} \\
&\quad + C(\|u\|_{L^3}\|\nabla b\|_{\dot{B}^m_{2,\infty}} + \|b\|_{L^3}\|\nabla u\|_{\dot{B}^m_{2,\infty}})\|\nabla b\|_{\dot{B}^m_{2,\infty}} \\
&\quad + C(\|\nabla u\|_{L^\infty}\|w\|_{\dot{H}^m} + \|u\|_{\dot{H}^{m+1}}\|w\|_{L^\infty})\|w\|_{\dot{H}^m} \\
&\le C(\|u\|^{\frac{1}{2}}_{L^2}\|\nabla u\|^{\frac{1}{2}}_{L^2} + \|b\|^{\frac{1}{2}}_{L^2}\|\nabla b\|^{\frac{1}{2}}_{L^2} + \|\nabla u\|_{L^\infty} + \|w\|_{L^\infty}) \\
&\quad \times (\|\nabla u\|^2_{\dot{B}^m_{2,\infty}} + \|w\|^2_{\dot{B}^m_{2,\infty}} + \|\nabla b\|^2_{\dot{B}^m_{2,\infty}})
\end{aligned} \tag{5.5}$$

Choosing the suitable ϵ in (1.12) small enough, we obtain

$$\frac{d}{dt}(\|u\|^2_{\dot{B}^m_{2,\infty}} + \|w\|^2_{\dot{B}^m_{2,\infty}} + \|b\|^2_{\dot{B}^m_{2,\infty}}) + \frac{c_0}{2}(\|\nabla u\|^2_{\dot{B}^m_{2,\infty}} + \|w\|^2_{\dot{B}^m_{2,\infty}} + \|\nabla b\|^2_{\dot{B}^m_{2,\infty}}) \le 0. \tag{5.6}$$

Applying Lemma 2.9 to obtain

$$\|u\|_{\dot{B}^m_{2,\infty}} \le C\|u\|^{\frac{1}{m+l+1}}_{\dot{B}^{-l}_{2,\infty}}\|\nabla u\|^{\frac{m+l}{m+l+1}}_{\dot{B}^m_{2,\infty}}. \tag{5.7}$$

$$\|b\|_{\dot{B}^m_{2,\infty}} \le C\|b\|^{\frac{1}{m+l+1}}_{\dot{B}^{-l}_{2,\infty}}\|\nabla b\|^{\frac{m+l}{m+l+1}}_{\dot{B}^m_{2,\infty}}. \tag{5.8}$$

Therefore, if

$$\|u\|_{\dot{B}^{-l}_{2,\infty}} + \|w\|_{\dot{B}^{-l}_{2,\infty}} + \|b\|_{\dot{B}^{-l}_{2,\infty}} \le C, \tag{5.9}$$

then (5.6)-(5.8) imply

$$\frac{d}{dt}(\|u\|^2_{\dot{B}^m_{2,\infty}} + \|w\|^2_{\dot{B}^m_{2,\infty}} + \|b\|^2_{\dot{B}^m_{2,\infty}}) + C(\|u\|^2_{\dot{B}^m_{2,\infty}} + \|w\|^2_{\dot{B}^m_{2,\infty}} + \|b\|^2_{\dot{B}^m_{2,\infty}})^{\frac{m+l+1}{m+l}} \le 0. \tag{5.10}$$

It follows from this that

$$\|u\|^2_{\dot{B}^m_{2,\infty}} + \|w\|^2_{\dot{B}^m_{2,\infty}} + \|b\|^2_{\dot{B}^m_{2,\infty}} \le C(1+t)^{-m-l}. \tag{5.11}$$

This immediately yields (1.13).

To complete the proof, it is enough to verify (5.9). Since the key idea is similar as that of assertion (ii) in Theorem 1.3, we only sketch the proof and point out the main differences for simplicity. We again divide the proof into four cases.

Case 1. $(0 < l \le \frac{1}{2})$ By Lemma 4.1, we have

$$\begin{aligned}
&\frac{d}{dt}(\|u\|^2_{\dot{B}^{-l}_{2,\infty}} + \|w\|^2_{\dot{B}^{-l}_{2,\infty}} + \|b\|^2_{\dot{B}^{-l}_{2,\infty}}) \\
&\le C(\|\nabla u\|^2_{B^s_{2,\infty}} + \|w\|^2_{B^s_{2,\infty}} + \|\nabla b\|^2_{B^s_{2,\infty}})(\|u\|_{\dot{B}^{-l}_{2,\infty}} + \|w\|_{\dot{B}^{-l}_{2,\infty}} + \|b\|_{\dot{B}^{-l}_{2,\infty}}).
\end{aligned} \tag{5.12}$$

Integrating this inequality in $[0,t]$, together with (1.12) and the Young inequality, we eventually obtain (5.9).

Case 2. ($\frac{1}{2} < l < 1$) Note that $\|f\|_{\dot{B}^{-l}_{2,\infty}} \cap \|f\|_{\dot{B}^{0}_{2,\infty}} \hookrightarrow \|f\|_{\dot{B}^{-\frac{1}{2}}_{2,\infty}}$, then with the help of (5.11) we know that

$$\|u(t)\|_{\dot{B}^{s}_{2,\infty}} + \|w(t)\|_{\dot{B}^{s}_{2,\infty}} + \|b(t)\|_{\dot{B}^{s}_{2,\infty}} \leq C(1+t)^{-\frac{2s+1}{4}}. \tag{5.13}$$

Applying Lemma 2.10, we obtain

$$\begin{aligned}\|u\|_{L^2} &\leq C\|u\|^{\frac{2s}{1+2s}}_{\dot{B}^{-\frac{1}{2}}_{2,\infty}} \|u\|^{\frac{1}{1+2s}}_{\dot{B}^{s}_{2,\infty}} \\ &\leq C(1+t)^{-\frac{1}{4}}.\end{aligned} \tag{5.14}$$

Furthermore,

$$\begin{aligned}\|\nabla u\|_{L^2} &\leq C\|u\|^{\frac{s-1}{s}}_{L^2} \|u\|^{\frac{1}{s}}_{\dot{B}^{s}_{2,\infty}} \\ &\leq C(1+t)^{-\frac{3}{4}}.\end{aligned} \tag{5.15}$$

Similarly, we have

$$\|w\|_{L^2} + \|b\|_{L^2} \leq C(1+t)^{-\frac{1}{4}}. \tag{5.16}$$

$$\|\nabla w\|_{L^2} + \|\nabla b\|_{L^2} \leq C(1+t)^{-\frac{3}{4}}. \tag{5.17}$$

Using Lemma 4.1, together with (5.14)-(5.17), we derive that

$$\begin{aligned}&\frac{d}{dt}(\|u\|^2_{\dot{B}^{-l}_{2,\infty}} + \|w\|^2_{\dot{B}^{-l}_{2,\infty}} + \|b\|^2_{\dot{B}^{-l}_{2,\infty}}) \\ &\leq C((1+t)^{-(2-l)} + \|\nabla u\|^2_{B^{s}_{2,\infty}} + \|w\|^2_{B^{s}_{2,\infty}} + \|\nabla b\|^2_{B^{s}_{2,\infty}}) \\ &\quad \times (\|u\|_{\dot{B}^{-l}_{2,\infty}} + \|w\|_{\dot{B}^{-l}_{2,\infty}} + \|b\|_{\dot{B}^{-l}_{2,\infty}}).\end{aligned} \tag{5.18}$$

Integrating this inequality in $[0,t]$, together with (1.12) and the Young inequality, we eventually yield (5.9).

Case 3. ($1 \leq l < \frac{3}{2}$) Since $\|f\|_{\dot{B}^{-l}_{2,\infty}} \cap \|f\|_{\dot{B}^{0}_{2,\infty}} \hookrightarrow \|f\|_{\dot{B}^{-l_0}_{2,\infty}}$ with $0 < l_0 < 1$, then we get from (5.11)

$$\|u(t)\|_{\dot{B}^{s}_{2,\infty}} + \|w(t)\|_{\dot{B}^{s}_{2,\infty}} + \|b(t)\|_{\dot{B}^{s}_{2,\infty}} \leq C(1+t)^{-\frac{s+l_0}{2}}. \tag{5.19}$$

Again applying (2.10), we infer that

$$\begin{aligned}\|u\|_{L^2} &\leq C\|u\|^{\frac{s}{s+l_0}}_{\dot{B}^{-l_0}_{2,\infty}} \|u\|^{\frac{l_0}{s+l_0}}_{\dot{B}^{s}_{2,\infty}} \\ &\leq C(1+t)^{-\frac{l_0}{2}}.\end{aligned} \tag{5.20}$$

Furthermore,

$$\begin{aligned}\|\nabla u\|_{L^2} &\leq C\|u\|^{\frac{s-1}{s}}_{L^2} \|u\|^{\frac{1}{s}}_{\dot{B}^{s}_{2,\infty}} \\ &\leq C(1+t)^{-\frac{l_0+1}{2}}.\end{aligned} \tag{5.21}$$

Similarly, we have

$$\|w\|_{L^2} + \|b\|_{L^2} \leq C(1+t)^{-\frac{l_0}{2}}. \tag{5.22}$$

$$\|\nabla w\|_{L^2} + \|\nabla b\|_{L^2} \leq C(1+t)^{-\frac{l_0+1}{2}}. \tag{5.23}$$

Combining (5.20)-(5.20) with Lemma 4.1 together, we derive that

$$\begin{aligned}
&\frac{d}{dt}(\|u\|^2_{\dot{B}^{-l}_{2,\infty}} + \|w\|^2_{\dot{B}^{-l}_{2,\infty}} + \|b\|^2_{\dot{B}^{-l}_{2,\infty}}) \\
&\leq C((1+t)^{-\frac{2l_0+3-2l}{2}} + \|\nabla u\|^2_{B^s_{2,\infty}} + \|w\|^2_{B^s_{2,\infty}} + \|\nabla b\|^2_{B^s_{2,\infty}}) \\
&\quad \times (\|u\|_{\dot{B}^{-l}_{2,\infty}} + \|w\|_{\dot{B}^{-l}_{2,\infty}} + \|b\|_{\dot{B}^{-l}_{2,\infty}}).
\end{aligned} \tag{5.24}$$

This implies (5.9) as explained above.

Case 4. ($l = \frac{3}{2}$) Note that $\|f\|_{\dot{B}^{-\frac{3}{2}}_{2,\infty}} \cap \|f\|_{\dot{B}^{0}_{2,\infty}} \hookrightarrow \|f\|_{\dot{B}^{-l_0}_{2,\infty}}$ with $0 < l_0 < \frac{3}{2}$, then by choosing l_0 near $\frac{3}{2}$, following the step of Case 3, we complete the proof.

(2) Now we prove (1.14). Taking the L^2-inner product to (4.2) with $\dot{\Delta}_j w$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\dot{\Delta}_j w\|^2_{L^2} + 4\chi\|\dot{\Delta}_j w\|^2_{L^2} \\
&\leq 2\chi \int \dot{\Delta}_j(\nabla \times u)\cdot \dot{\Delta}_j w - \int \dot{\Delta}_j(u\cdot\nabla w)\cdot\dot{\Delta}_j w \\
&\leq C(\|\dot{\Delta}_j(\nabla u)\|_{L^2} + \|\dot{\Delta}_j(u\cdot\nabla w)\|_{L^2})\|\dot{\Delta}_j w\|_{L^2}.
\end{aligned}$$

Multiplying this inequality by 2^{2mj}, taking the supremum over $j \in \mathbb{Z}$ and utilizing Bernstein's inequality and Lemma 2.6, we yield

$$\begin{aligned}
\frac{d}{dt}\|w\|_{\dot{B}^{m}_{2,\infty}} + 4\chi\|w\|_{\dot{B}^{m}_{2,\infty}} &\leq C(\|u\|_{\dot{B}^{m+1}_{2,\infty}} + \|u\|_{L^\infty}\|w\|_{\dot{B}^{m+1}_{2,\infty}} + \|w\|_{L^\infty}\|u\|_{\dot{B}^{m+1}_{2,\infty}}) \\
&\leq C(\|u\|_{\dot{B}^{m+1}_{2,\infty}} + \|w\|_{\dot{B}^{m+1}_{2,\infty}}),
\end{aligned}$$

where we used (1.12) in the last inequality. Integrating this inequality in $[0,t]$ and using (1.12) and (1.13), we obtain

$$\begin{aligned}
\|w\|_{\dot{B}^{m}_{2,\infty}} &\leq e^{-4\chi t}\|w_0\|_{\dot{B}^{m}_{2,\infty}} + C\int_0^t e^{-4\chi(t-\tau)}(\|u(\tau)\|_{\dot{B}^{m+1}_{2,\infty}} + \|w(\tau)\|_{\dot{B}^{m+1}_{2,\infty}})d\tau \\
&\leq Ce^{-4\chi t} + C\int_0^{\frac{t}{2}} e^{-4\chi(t-\tau)}(\|u(\tau)\|_{\dot{B}^{m+1}_{2,\infty}} + \|w(\tau)\|_{\dot{B}^{m+1}_{2,\infty}})d\tau \\
&\quad + C\int_{\frac{t}{2}}^t e^{-4\chi(t-\tau)}(\|u(\tau)\|_{\dot{B}^{m+1}_{2,\infty}} + \|w(\tau)\|_{\dot{B}^{m+1}_{2,\infty}})d\tau \\
&\leq Ce^{-4\chi t} + Ce^{-2\chi t}\left(\int_0^{\frac{t}{2}}(\|u(\tau)\|^2_{\dot{B}^{m+1}_{2,\infty}} + \|w(\tau)\|^2_{\dot{B}^{m+1}_{2,\infty}})d\tau\right)^{\frac{1}{2}}\left(\int_0^{\frac{t}{2}} d\tau\right)^{\frac{1}{2}} \\
&\quad + C\int_{\frac{t}{2}}^t e^{-4\chi(t-\tau)}(1+\tau)^{-\frac{m+1}{2}-\frac{l}{2}}d\tau \\
&\leq Ce^{-4\chi t} + Ce^{-2\chi t}t^{\frac{1}{2}} + C(1+t)^{-\frac{m+1}{2}-\frac{l}{2}} \\
&\leq C(1+t)^{-\frac{m+1}{2}-\frac{l}{2}}.
\end{aligned}$$

Thus the proof of (1.14) is completed.

$\square$

6. Proof of Theorem 1.7

This section is devoted to the proof of Theorem 1.7. As above we establish the global *a priori* bounds of the solution in the critical Besov spaces to prove the global existence part (i) of Theorem 1.7,

Proof of assertion (i) of Theorem 1.7. Dotting (4.1) and (4.3) by $\dot\Delta_j u$ and $\dot\Delta_j b$ respectively, and integrating it in space domain, with the aspects of Bernstein's inequality and Hölder's inequality, we yield

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\dot\Delta_j u\|_{L^2}^2+\|\dot\Delta_j b\|_{L^2}^2)+(\mu+\chi)c_1 2^{2j}\|\dot\Delta_j u\|_{L^2}^2+\nu c_1 2^{2j}\|\dot\Delta_j b\|_{L^2}^2\\
&\le -\int[\dot\Delta_j,u\cdot\nabla]u\cdot\dot\Delta_j u+\int[\dot\Delta_j,b\cdot\nabla]b\cdot\dot\Delta_j u+2\chi\int\dot\Delta_j\nabla\times w\cdot\dot\Delta_j u\\
&\quad-\int[\dot\Delta_j,u\cdot\nabla]b\cdot\dot\Delta_j b+\int[\dot\Delta_j,b\cdot\nabla]u\cdot\dot\Delta_j b\\
&\le\|[\dot\Delta_j,u\cdot\nabla]u\|_{L^2}\|\dot\Delta_j u\|_{L^2}+\|[\dot\Delta_j,b\cdot\nabla]b\|_{L^2}\|\dot\Delta_j u\|_{L^2}+2\chi c_2 2^j\|\dot\Delta_j w\|_{L^2}\|\dot\Delta_j u\|_{L^2}\\
&\quad+\|[\dot\Delta_j,u\cdot\nabla]b\|_{L^2}\|\dot\Delta_j b\|_{L^2}+\|[\dot\Delta_j,b\cdot\nabla]u\|_{L^2}\|\dot\Delta_j b\|_{L^2}.
\end{aligned}\tag{6.1}$$

Simple calculations lead to

$$\begin{aligned}
&\frac{d}{dt}\sqrt{\|\dot\Delta_j u\|_{L^2}^2+\|\dot\Delta_j b\|_{L^2}^2}+c_3 2^{2j}\sqrt{\|\dot\Delta_j u\|_{L^2}^2+\|\dot\Delta_j b\|_{L^2}^2}\\
&\le\|[\dot\Delta_j,u\cdot\nabla]u\|_{L^2}+\|[\dot\Delta_j,b\cdot\nabla]b\|_{L^2}+2\chi c_2 2^j\|\dot\Delta_j w\|_{L^2}\\
&\quad+\|[\dot\Delta_j,u\cdot\nabla]b\|_{L^2}+\|[\dot\Delta_j,b\cdot\nabla]u\|_{L^2}
\end{aligned}\tag{6.2}$$

with $c_3=c_1\min\{(\mu+\chi),\nu\}$. Integrating it in $[0,t]$, we obtain

$$\begin{aligned}
&\|\dot\Delta_j u\|_{L^2}+\|\dot\Delta_j b\|_{L^2}+c_3 2^{2j}\|\dot\Delta_j u\|_{L^1_tL^2}+c_4 2^{2j}\|\dot\Delta_j b\|_{L^1_tL^2}\\
&\le 2\|\dot\Delta_j u_0\|_{L^2}+2\|\dot\Delta_j b_0\|_{L^2}+2\|[\dot\Delta_j,u\cdot\nabla]u\|_{L^1_tL^2}+2\|[\dot\Delta_j,b\cdot\nabla]b\|_{L^1_tL^2}+4\chi c_2 2^j\|\dot\Delta_j w\|_{L^1_tL^2}\\
&\quad+2\|[\dot\Delta_j,u\cdot\nabla]b\|_{L^1_tL^2}+2\|[\dot\Delta_j,b\cdot\nabla]u\|_{L^1_tL^2}.
\end{aligned}$$

Multiplying it by $2^{\frac{1}{2}j}$ and taking the l^1_j over $j\in\mathbb{Z}$, one yields

$$\begin{aligned}
&\|u\|_{\dot B^{\frac12}_{2,1}}+\|b\|_{\dot B^{\frac12}_{2,1}}+c_3\|u\|_{\tilde L^1_t(\dot B^{\frac52}_{2,1})}+c_4\|b\|_{\tilde L^1_t(\dot B^{\frac52}_{2,1})}\\
&\le 2\|u_0\|_{\dot B^{\frac12}_{2,1}}+2\|b_0\|_{\dot B^{\frac12}_{2,1}}+2\|\|2^{\frac12 j}[\dot\Delta_j,u\cdot\nabla]u\|_{L^1_tL^2}\|_{l^1_j}+2\|\|2^{\frac12 j}[\dot\Delta_j,b\cdot\nabla]b\|_{L^1_tL^2}\|_{l^1_j}\\
&\quad+4\chi c_2\|w\|_{\tilde L^1_t(\dot B^{\frac32}_{2,1})}+2\|\|2^{\frac12 j}[\dot\Delta_j,u\cdot\nabla]b\|_{L^1_tL^2}\|_{l^1_j}+2\|\|2^{\frac12 j}[\dot\Delta_j,b\cdot\nabla]u\|_{L^1_tL^2}\|_{l^1_j}.
\end{aligned}\tag{6.3}$$

Similarly,

$$\frac{d}{dt}\|\dot\Delta_j w\|_{L^2}+4\chi\|\dot\Delta_j w\|_{L^2}\le\|[\dot\Delta_j,u\cdot\nabla]w\|_{L^2}+2\chi 2^j\|\dot\Delta_j u\|_{L^2},\tag{6.4}$$

and

$$\|w\|_{\dot B^{\frac32}_{2,1}}+4\chi\|w\|_{\tilde L^1_t(\dot B^{\frac32}_{2,1})}\le\|w_0\|_{\dot B^{\frac32}_{2,1}}+\|\|2^{\frac32 j}[\dot\Delta_j,u\cdot\nabla]w\|_{L^1_tL^2}\|_{l^1_j}+2\chi\|u\|_{\tilde L^1_t(\dot B^{\frac52}_{2,1})}.\tag{6.5}$$

Multiplying (6.4) by $e^{4\chi t}$ on both sides, and integrating the resulting inequality in $[0,t]$, we obtain

$$\begin{aligned}\|\dot{\Delta}_j w\|_{L^2} \le e^{-4\chi t}\|\dot{\Delta}_j w_0\|_{L^2} + \int_0^t e^{-4\chi(t-\tau)}\|[\dot{\Delta}_j, u\cdot\nabla]w(\tau)\|_{L^2}d\tau \\ + 2\chi 2^j \int_0^t e^{-4\chi(t-\tau)}\|\dot{\Delta}_j u(\tau)\|_{L^2}d\tau.\end{aligned}$$

Taking the L^1_t norm, and using the Young inequality, we have

$$\|\dot{\Delta}_j w\|_{L^1_t L^2} \le \frac{1}{4\chi}\|\dot{\Delta}_j w_0\|_{L^2} + \frac{1}{4\chi}\|[\dot{\Delta}_j, u\cdot\nabla]w\|_{L^1_t L^2} + \frac{1}{2}2^j\|\dot{\Delta}_j u\|_{L^1_t L^2}.$$

Therefore, multiplying it by $2^{\frac{3}{2}j}$, and taking the l^1_j over $j\in\mathbb{Z}$, it leads to

$$\|w\|_{\tilde{L}^1_t(\dot{B}^{\frac{3}{2}}_{2,1})} \le \frac{1}{4\chi}\|w_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \frac{1}{4\chi}\|2^{\frac{3}{2}j}\|[\dot{\Delta}_j, u\cdot\nabla]w\|_{L^1_t L^2}\|_{l^1_j} + \frac{1}{2}\|u\|_{\tilde{L}^1_t(\dot{B}^{\frac{5}{2}}_{2,1})}. \tag{6.6}$$

Adding (6.3) and (6.5) together, and inserting (6.6) into the resulting inequality, we get

$$\begin{aligned}&\|u\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|b\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|w\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + (c_3 - 2\chi(c_2+1))\|u\|_{\tilde{L}^1_t(\dot{B}^{\frac{5}{2}}_{2,1})} + c_3\|b\|_{\tilde{L}^1_t(\dot{B}^{\frac{5}{2}}_{2,1})} + 4\chi\|w\|_{\tilde{L}^1_t(\dot{B}^{\frac{3}{2}}_{2,1})} \\ &\le 2\|u_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + 2\|b_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + (c_2+1)\|w_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + 2\|\|2^{\frac{1}{2}j}[\dot{\Delta}_j, u\cdot\nabla]u\|_{L^1_t L^2}\|_{l^1_j} \\ &\quad + 2\|\|2^{\frac{1}{2}j}[\dot{\Delta}_j, b\cdot\nabla]b\|_{L^1_t L^2}\|_{l^1_j} + 2\|\|2^{\frac{1}{2}j}[\dot{\Delta}_j, u\cdot\nabla]b\|_{L^1_t L^2}\|_{l^1_j} \\ &\quad + 2\|\|2^{\frac{1}{2}j}[\dot{\Delta}_j, b\cdot\nabla]u\|_{L^1_t L^2}\|_{l^1_j} + c_2\|2^{\frac{3}{2}j}\|[\dot{\Delta}_j, u\cdot\nabla]w\|_{L^1_t L^2}\|_{l^1_j}.\end{aligned} \tag{6.7}$$

Using Lemma 2.5, and noting that $\|f\|_{\tilde{L}^1_t(\dot{B}^s_{2,1})} \approx \|f\|_{L^1_t(\dot{B}^s_{2,1})}$ and $\|f\|_{L^\infty} \le C\|f\|_{\dot{B}^{\frac{3}{2}}_{2,1}}$, we obtain

$$\begin{aligned}&\|u\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|b\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|w\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + (c_3 - 2\chi(c_2+1))\|u\|_{L^1_t(\dot{B}^{\frac{5}{2}}_{2,1})} + c_3\|b\|_{L^1_t(\dot{B}^{\frac{5}{2}}_{2,1})} \\ &\le 2\|u_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + 2\|b_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + (c_2+1)\|w_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \\ &\quad + C\int_0^t (\|u(s)\|_{\dot{B}^{\frac{5}{2}}_{2,1}} + \|b(s)\|_{\dot{B}^{\frac{5}{2}}_{2,1}})(\|u(s)\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|b(s)\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|w(s)\|_{\dot{B}^{\frac{3}{2}}_{2,1}})ds.\end{aligned} \tag{6.8}$$

This inequality indicates that, if

$$2\chi(c_2+1) < c_3 = c_1\min\{(\mu+\chi), \nu\},$$

and for any $0 < \epsilon < C_1 = \frac{c_3 - 2\chi(c_2+1)}{C}$,

$$\|u_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|w_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|b_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}} < \epsilon,$$

then bootstrap argument yields

$$\|u\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|b\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|w\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|u\|_{L^1_t(\dot{B}^{\frac{5}{2}}_{2,1})} + \|b\|_{L^1_t(\dot{B}^{\frac{5}{2}}_{2,1})} \le C\epsilon.$$

Thus the proof of (i) of Theorem 1.7 is completed.

□

We now turn to prove the decay part (ii) of Theorem 1.7.

Proof of assertion (ii) of Theorem 1.7. Using the divergence free condition $\nabla\cdot u=\nabla\cdot b=0$, together with (6.1), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\dot\Delta_j u\|_{L^2}^2+\|\dot\Delta_j b\|_{L^2}^2)+(\mu+\chi)c_1 2^{2j}\|\dot\Delta_j u\|_{L^2}^2+\nu c_1 2^{2j}\|\dot\Delta_j b\|_{L^2}^2\\
&\le -\int\dot\Delta_j(u\cdot\nabla u)\cdot\dot\Delta_j u+\int\dot\Delta_j(b\cdot\nabla b)\cdot\dot\Delta_j u+2\chi\int\dot\Delta_j\nabla\times w\cdot\dot\Delta_j u\\
&\quad-\int\dot\Delta_j(u\cdot\nabla b)\cdot\dot\Delta_j b+\int\dot\Delta_j(b\cdot\nabla u)\cdot\dot\Delta_j b\\
&\le\|\dot\Delta_j(u\cdot\nabla u)\|_{L^2}\|\dot\Delta_j u\|_{L^2}+\|\dot\Delta_j(b\cdot\nabla b)\|_{L^2}\|\dot\Delta_j u\|_{L^2}+2\chi c_2 2^j\|\dot\Delta_j w\|_{L^2}\|\dot\Delta_j u\|_{L^2}\\
&\quad+\|\dot\Delta_j(u\cdot\nabla b)\|_{L^2}\|\dot\Delta_j b\|_{L^2}+\|\dot\Delta_j(b\cdot\nabla u)\|_{L^2}\|\dot\Delta_j b\|_{L^2}. \qquad (6.9)
\end{aligned}$$

It follows from Bernstein's inequality that

$$\begin{aligned}
&\frac{d}{dt}\sqrt{\|\dot\Delta_j u\|_{L^2}^2+\|\dot\Delta_j b\|_{L^2}^2}+c_3 2^{2j}\sqrt{\|\dot\Delta_j u\|_{L^2}^2+\|\dot\Delta_j b\|_{L^2}^2}\\
&\le C2^j\|\dot\Delta_j(u\otimes u)\|_{L^2}+C2^j\|\dot\Delta_j(b\otimes b)\|_{L^2}+2\chi c_2 2^j\|\dot\Delta_j w\|_{L^2}\\
&\quad+C2^j\|\dot\Delta_j(u\otimes b)\|_{L^2}. \qquad (6.10)
\end{aligned}$$

Multiplying (6.10) and (6.4) by 2^{mj} and $c_2 2^{(m+1)j}$ respectively, adding them up, and taking the l_j^1 over $j\in\mathbb{Z}$, we obtain

$$\begin{aligned}
&\frac{d}{dt}(y(t)+c_2\|w\|_{\dot B^{m+1}_{2,1}})+\frac{c_3-2\chi c_2}{2}(\|u\|_{\dot B^{m+2}_{2,1}}+\|b\|_{\dot B^{m+2}_{2,1}})+2\chi c_2\|w\|_{\dot B^{m+1}_{2,1}}\\
&\le C\|2^{(m+1)j}\|\dot\Delta_j(u\otimes u)\|_{L^2}\|_{l_j^1}+C\|2^{(m+1)j}\|\dot\Delta_j(b\otimes b)\|_{L^2}\|_{l_j^1}+C\|2^{(m+1)j}\|\dot\Delta_j(u\otimes b)\|_{L^2}\|_{l_j^1}\\
&\quad+c_2\|2^{(m+1)j}\|[\dot\Delta_j,u\cdot\nabla]w\|_{L^2}\|_{l_j^1}. \qquad (6.11)
\end{aligned}$$

where $y(t)=\|2^{mj}\sqrt{\|\dot\Delta_j u\|_{L^2}^2+\|\dot\Delta_j b\|_{L^2}^2}\|_{l_j^1}$. Using Lemma 2.5 and Lemma 2.6, together with $\|f\|_{L^3}\le C\|f\|_{\dot B^{\frac12}_{2,1}}$, $\|f\|_{\dot B^{m+1}_{6,1}}\le C\|f\|_{\dot B^{m+2}_{2,1}}$, $\|f\|_{L^\infty}\le C\|f\|_{\dot B^{\frac32}_{2,1}}$, $\|\nabla f\|_{L^{\frac{6}{1-2m}}}\le C\|f\|_{\dot B^{m+2}_{2,1}}$ and $\|f\|_{\dot B^{m+1}_{\frac{3}{1+m},1}}\le C\|w\|_{\dot B^{\frac32}_{2,1}}$, we obtain

$$\begin{aligned}
&\frac{d}{dt}(y(t)+c_2\|w\|_{\dot B^{m+1}_{2,1}})+\frac{c_3-2\chi c_2}{2}(\|u\|_{\dot B^{m+2}_{2,1}}+\|b\|_{\dot B^{m+2}_{2,1}})+2\chi c_2\|w\|_{\dot B^{m+1}_{2,1}}\\
&\le C\|u\|_{L^3}\|u\|_{\dot B^{m+1}_{6,1}}+C\|b\|_{L^3}\|b\|_{\dot B^{m+1}_{6,1}}+C(\|u\|_{L^3}\|b\|_{\dot B^{m+1}_{6,1}}+\|b\|_{L^3}\|u\|_{\dot B^{m+1}_{6,1}})\\
&\quad+C(\|\nabla u\|_{L^{\frac{6}{1-2m}}}\|w\|_{\dot B^{m+1}_{\frac{3}{1+m},1}}+\|w\|_{L^\infty}\|u\|_{\dot B^{m+2}_{2,1}})\\
&\le C(\|u\|_{\dot B^{\frac12}_{2,1}}+\|w\|_{\dot B^{\frac32}_{2,1}}+\|b\|_{\dot B^{\frac12}_{2,1}})(\|u\|_{\dot B^{m+2}_{2,1}}+\|b\|_{\dot B^{m+2}_{2,1}}). \qquad (6.12)
\end{aligned}$$

Then this inequality, together with (1.17) with ϵ sufficiently small, we obtain

$$\frac{d}{dt}(y(t)+c_2\|w\|_{\dot B^{m+1}_{2,1}})+\frac{c_3-2\chi c_2}{4}(\|u\|_{\dot B^{m+2}_{2,1}}+\|b\|_{\dot B^{m+2}_{2,1}})+2\chi c_2\|w\|_{\dot B^{m+1}_{2,1}}\le 0. \qquad (6.13)$$

Applying Lemma 2.9, we infer that

$$\|u\|_{\dot B^m_{2,1}}\le C\|u\|^{\frac{2}{m+l+2}}_{\dot B^{-l}_{2,\infty}}\|u\|^{\frac{m+l}{m+l+2}}_{\dot B^{m+2}_{2,1}}. \qquad (6.14)$$

$$\|b\|_{\dot{B}^m_{2,1}} \le C\|b\|^{\frac{2}{m+l+2}}_{\dot{B}^{-l}_{2,\infty}}\|b\|^{\frac{m+l}{m+l+2}}_{\dot{B}^{m+2}_{2,1}}. \tag{6.15}$$

Therefore, if

$$\|u\|_{\dot{B}^{-l}_{2,\infty}} + \|w\|_{\dot{B}^{-l}_{2,\infty}} + \|b\|_{\dot{B}^{-l}_{2,\infty}} \le C, \tag{6.16}$$

then there exists a constant $a_0 > 0$ such that we obtain from (6.13)-(6.15)

$$\frac{d}{dt}(y(t) + c_2\|w\|_{\dot{B}^{m+1}_{2,1}}) + a_0(y(t) + c_2\|w\|_{\dot{B}^{m+1}_{2,1}})^{\frac{m+l+2}{m+l}} \le 0. \tag{6.17}$$

It follows that

$$y(t) + \|w\|_{\dot{B}^{m+1}_{2,1}} \le C(1+t)^{-\frac{m}{2}-\frac{l}{2}}, \tag{6.18}$$

which completes (1.18). At the end, it is left to prove (6.16). Since the idea is very similar to the proof of assertion (ii) of Theorem 1.5, we omit the details for simplicity. □

Appendix A. Proofs of Lemma 2.5 and Lemma 2.6

This section proves Lemma 2.5 and Lemma 2.6.

Proof of Lemma 2.5. We only prove (2.4), since (2.5) can be proved with minor changes. Using Littlewood-Paley decomposition, we obtain

$$\begin{aligned}2^{js}\|[\dot{\Delta}_j, u\cdot\nabla]v\|_{L^p} &\le 2^{js}\sum_{|k-j|\le 2}\|[\dot{\Delta}_j, S_{k-1}u\cdot\nabla]\dot{\Delta}_k v\|_{L^p} + 2^{js}\sum_{|k-j|\le 2}\|[\dot{\Delta}_j, \dot{\Delta}_k u\cdot\nabla]S_{k-1}v\|_{L^p}\\ &\quad + 2^{js}\sum_{k\ge j-1}\|[\dot{\Delta}_j, \tilde{\dot{\Delta}}_k u\cdot\nabla]\dot{\Delta}_k v\|_{L^p}\\ &:= I_1 + I_2 + I_3,\end{aligned} \tag{A.1}$$

where $\tilde{\dot{\Delta}}_k = \dot{\Delta}_{k-1} + \dot{\Delta}_k + \dot{\Delta}_{k+1}$. Since for fixed j, the summation over $|j-k| \le 2$ involves only a finite number of k's. For the sake of brevity, we shall replace the summations by their representative term with $k = j$ in I_1 and I_2. This practice does not change the estimates. To estimate I_1, we make use of the commutator structure to write

$$\begin{aligned}&[\dot{\Delta}_j, S_{j-1}u\cdot\nabla]\dot{\Delta}_j v(x)\\ &\quad = \int \Phi_j(x-y)(S_{j-1}u(y) - S_{j-1}u(x))\cdot\nabla\dot{\Delta}_j v(y)dy\\ &\quad = \int \Phi_j(x-y)\int_0^1 \nabla S_{j-1}u(x+\lambda(y-x))\cdot(y-x)d\lambda\cdot\nabla\dot{\Delta}_j v(y)dy\\ &\quad = \int_0^1\int \Phi_j(\lambda^{-1}z)\nabla S_{j-1}u(x-\lambda z)\cdot(\lambda^{-1}z)\cdot\nabla\dot{\Delta}_j v(x-\lambda^{-1}z)\lambda^{-3}dzd\lambda,\end{aligned}$$

where we set $z = \lambda(x-y)$ in the last equality. Then Minkowski's inequality and Hölder's inequality yield

$$\begin{aligned}
I_1 &\le C2^{js}\int_0^1\int |\Phi_j(\lambda^{-1}z)\lambda^{-1}z| \|\nabla S_{j-1}u(x-\lambda z)\|_{L_x^{p_1}}\|\nabla\dot\Delta_j v(x-\lambda^{-1}z)\|_{L_x^{q_1}}\lambda^{-3}dzd\lambda\\
&\le C2^{js}\|\Phi_j(x)x\|_{L_x^1}\|\nabla u\|_{L^{p_1}}\|\nabla\dot\Delta_j v\|_{L^{q_1}}\\
&\le C2^{(s-1)j}\|\nabla u\|_{L^{p_1}}\|\nabla\dot\Delta_j v\|_{L^{q_1}}.
\end{aligned}$$

Again applying Hölder's inequality and Bernstein's inequality

$$\begin{aligned}
I_2 &\le C2^{js}\|\dot\Delta_j u\|_{L^{p_1}}\|\nabla S_{j-1}v\|_{L^{q_1}}\\
&\le C2^{(s-1)j}\|\dot\Delta_j\nabla u\|_{L^{p_1}}\|\nabla v\|_{L^{q_1}}.
\end{aligned}$$

By divergence free condition $\nabla\cdot u = 0$, we obtain

$$\begin{aligned}
I_3 &\le C2^{(s+1)j}\sum_{k\ge j-1}\|\tilde{\dot\Delta}_k u\|_{L^{p_1}}\|\dot\Delta_k v\|_{L^{q_1}}\\
&\le C\sum_{k\ge j-1}2^{(s+1)(j-k)}\|\dot\Delta_k\nabla u\|_{L^{p_1}}2^{(s-1)k}\|\dot\Delta_k\nabla v\|_{L^{q_1}}\\
&\le C\|\nabla u\|_{L^{p_1}}\sum_{k\ge j-1}2^{(s+1)(j-k)}2^{(s-1)k}\|\dot\Delta_k\nabla v\|_{L^{q_1}}.
\end{aligned}$$

Inserting the above estimates into (A.1), and taking the l_j^r over $j\in\mathbb{Z}$, yield

$$\begin{aligned}
\|2^{js}\|[\dot\Delta_j, u\cdot\nabla]v\|_{L^p}\|_{l_j^r} &\le C\|\nabla u\|_{L^{p_1}}\|\nabla v\|_{\dot B^{s-1}_{q_1,r}} + C\|\nabla v\|_{L^{q_1}}\|\nabla u\|_{\dot B^{s-1}_{p_1,r}}\\
&\quad + C\|\nabla u\|_{L^{p_1}}\|\sum_{k\ge j-1}2^{(s+1)(j-k)}2^{(s-1)k}\|\dot\Delta_k\nabla v\|_{L^{q_1}}\|_{l_j^r}\\
&\le C\|\nabla u\|_{L^{p_1}}\|\nabla v\|_{\dot B^{s-1}_{q_1,r}} + C\|\nabla v\|_{L^{q_1}}\|\nabla u\|_{\dot B^{s-1}_{p_1,r}},
\end{aligned}$$

where the discrete Young inequality has been applied in the last inequality. Note that the above estimates also hold for p_1 and q_1 replaced by p_2 and q_2 respectively, so we have proved (2.4). Thus the proof of Lemma 2.5 is completed.

□

Proof of Lemma 2.6. Using Littlewood-Paley decomposition, we obtain

$$\begin{aligned}
2^{js}\|\dot\Delta_j(fg)\|_{L^p} &= 2^{js}\sum_{|k-j|\le 2}\|\dot\Delta_j(S_{k-1}f\dot\Delta_k g)\|_{L^p} + 2^{js}\sum_{|k-j|\le 2}\|\dot\Delta_j(\dot\Delta_k f S_{k-1}g)\|_{L^p}\\
&\quad + 2^{js}\sum_{k\ge j-1}\|\dot\Delta_j(\tilde{\dot\Delta}_k f\dot\Delta_k g)\|_{L^p}\\
&:= M_1 + M_2 + M_3, \qquad\qquad (A.2)
\end{aligned}$$

where $\tilde{\dot\Delta}_k = \dot\Delta_{k-1} + \dot\Delta_k + \dot\Delta_{k+1}$. Since for fixed j, the summation over $|j-k|\le 2$ involves only a finite number of k's. For the sake of brevity, we shall replace the summations by their representative term with $k = j$ in M_1 and M_2. This practice does not change the estimates. By Hölder's inequality

$$\begin{aligned} M_1 &\le C2^{js}\|S_{j-1}f\|_{L^{p_1}}\|\dot{\Delta}_j g\|_{L^{q_1}} \\ &\le C2^{js}\|f\|_{L^{p_1}}\|\dot{\Delta}_j g\|_{L^{q_1}}. \end{aligned}$$

Similarly, we have

$$M_2 \le C2^{js}\|\dot{\Delta}_j f\|_{L^{p_1}}\|g\|_{L^{q_1}}.$$

Again applying Hölder's inequality yields

$$\begin{aligned} M_3 &\le C2^{sj}\sum_{k\ge j-1}\|\tilde{\dot{\Delta}}_k f\|_{L^{p_1}}\|\dot{\Delta}_k g\|_{L^{q_1}} \\ &\le C\|f\|_{L^{p_1}}\sum_{k\ge j-1}2^{-s(k-j)}2^{sk}\|\dot{\Delta}_k g\|_{L^{q_1}}. \end{aligned}$$

Inserting the above estimates into (A.2), and taking the l^r_j over $j \in \mathbb{Z}$, together with the discrete Young inequality, yield

$$\|fg\|_{\dot{B}^s_{p,r}} \le C(\|f\|_{L^{p_1}}\|g\|_{\dot{B}^s_{q_1,r}} + \|f\|_{\dot{B}^s_{p_1,r}}\|g\|_{L^{q_1}}).$$

Note that the above estimates also hold for p_1 and q_1 replaced by p_2 and q_2 respectively, so it follows from this that the desired estimate (2.6) is derived. We thus complete the proof of Lemma 2.6.

□

Acknowledgements

Niu's research is supported by the National Natural Science Foundation of China (No.11471220 and No.11871046) and the key research project of National Natural Science Foundation (No. 11931010). Shang is supported by the Key Research Projects of University in Henan Province (No. 2015GGJS-070).

References

[1] G. Ahmadi, M. Shahinpoor, Universal stability of magneto-micropolar fuid motions, Int. J. Engng. Sci. 12 (1974) 657-663.
[2] H. Bahouri, J.-Y. Chemin, R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, Springer, 2011.
[3] J. Bergh, J. Löfström, Interpolation Spaces, An Introduction, Springer-Verlag, Berlin-Heidelberg-New York, 1976.
[4] P. Braz e Silva, F.W. Cruz, L.B.S. Freitas, P.R. Zingano, On the L^2 decay of weak solutions for the 3D asymmetric fluids equations, J. Differential Equations 267 (2019), 3578-3609.
[5] P. Braz e Silva, L. Friz, M.A. Rojas-Medar, Exponential stability for magneto-micropolar fluids, Nonlinear Anal. 143 (2016), 211-223.
[6] M. Chen, Global well-posedness of the 2D incompressible micropolar fluid flows with partial viscosity and angular viscosity, Acta Math. Sci. Ser. B Engl. Ed. 33 (2013), 929-935.
[7] Q. Chen, C. Miao, Global well-posedness for the micropolar fluid system in critical Besov spaces, J. Differential Equations 252 (2012), 2698-2724.
[8] J. Cheng, Y. Liu, Global regularity of the 2D magnetic micropolar fluid flows with mixed partial viscosity, Comput. Math. Appl. 70 (2015), 66-72.

[9] F.W. Cruz, M.M. Novais, Optimal L^2 decay of the magneto-micropolar system in $\mathbb{R}^3$, Z. Angew. Math. Phys. 71 (2020), Paper No. 91, 26pp.

[10] R. Danchin, J. Xu, Optimal time-decay estimates for the compressible Navier-Stokes equations in the critical L^p framework, Arch. Rational Mech. Anal. 224 (2017), 53-90

[11] B. Dong, Z. Chen, Asymptotic profiles of solutions to the 2D viscous incompressible micropolar fluid flows, Discrete Contin. Dyn. Syst. 23 (2009), 765-784.

[12] B. Dong, J. Li, J. Wu, Global well-posedness and large-time decay for the 2D micropolar equations, J. Differential Equations 262 (2017), 3488-3523.

[13] B. Dong, Z. Zhang, Global regularity of the 2D micropolar fluid flows with zero angular viscosity, J. Differential Equations 249 (2010), 200-213.

[14] A. Eringen, Theory of micropolar fluids, J. Math. Mech. 16 (1966), 1-18.

[15] S. Gala, Regularity criteria for the 3D magneto-micropolar fluid equations in the Morrey-Campanato space, NoDEA Nonlinear Differential Equations Appl. 17 (2010) 181-194.

[16] Y. Guo, H. Shang, Global well-posedness of two-dimensional magneto-micropolar equations with partial dissipation, Appl. Math. Comput. 313 (2017), 392-407.

[17] Y. Guo, Y. Wang, Decay of dissipative equations and negative Sobolev spaces, Comm. Partial Differential Equations 37 (2012), 2165-2208.

[18] R.H. Guterres, J.R. Nunes, C.F. Perusato, Decay rates for the magneto-micropolar system in $L^2(\mathbb{R}^n)$, Arch. Math. (Basel) 111 (2018), 431-442.

[19] R. Kajikiya, T. Miyakawa, On L^2 decay of weak solutions of the Navier-Stokes equations in $\mathbb{R}^n$, Math. Z. 192 (1986), 135-148.

[20] J. Leray, Sur le mouvement d'un liquide visqueux emplissant l'espace, Acta Math. 63 (1934), 193-248.

[21] G. Łukaszewicz, Micropolar fluids. Theory and applications. Modeling and Simulation in Science, Engineering and Technology. Birkhäuser Boston, Inc., Boston, MA, 1999.

[22] L. Ma, On two-dimensional incompressible magneto-micropolar system with mixed partial viscosity, Nonlinear Anal. Real World Appl. 40 (2018), 95-129.

[23] A.J. Majda, A.L. Bertozzi, Vorticity and Incompressible Flow, Cambridge University Press, Cambridge, 2002.

[24] C. Miao, J. Wu, Z. Zhang, Littlewood-Paley Theory and its Applications in Partial Differential Equations of Fluid Dynamics, Science Press, Beijing, China, 2012 (in Chinese).

[25] M. Li, H. Shang, Large time decay of solutions for the 3D magneto-micropolar equations, Nonlinear Anal. Real World Appl. 44 (2018), 479-496.

[26] M. Oliver, E.S. Titi, Remark on the rate of decay of higher order derivatives for solutions to the Navier-Stokes equations in $\mathbb{R}^n$, J. Funct. Anal. 172 (2000), 1-18.

[27] E. Ortega-Torres, M. Rojas-Medar, Magneto-micropolar fluid motion: global existence of strong solutions, Abstr. Appl. Anal. 4 (1999), 109-125.

[28] D. Regmi, J. Wu, Global regularity for the 2D magneto-micropolar equations with partial dissipation, J. Math. Study 49 (2016), 169-194.

[29] M. Rojas-Medar, Magneto-micropolar fluid motion: existence and uniqueness of strong solution, Math. Nachr. 188 (1997), 301-319.

[30] M. Rojas-Medar, J. Boldrini, Magneto-micropolar fluid motion: existence of weak solutions, Rev. Mat. Complut. 11 (1998), 443-460.

[31] T. Runst, W. Sickel, Sobolev Spaces of fractional order, Nemytskij operators and Nonlinear Partial Differential Equations, Walter de Gruyter, Berlin, New York, 1996.

[32] H. Shang, C. Gu, Global regularity and decay estimates for 2D magneto-micropolar equations with partial dissipation, Z. Angew. Math. Phys. 70 (2019), Art.85, 22pp.

[33] H. Shang, J. Wu, Global regularity for 2D fractional magneto-micropolar equation, Math. Z., accepted for publication.

[34] H. Shang, J. Zhao, Global regularity for 2D magneto-micropolar equations with only micro-rotational velocity dissipation and magnetic diffusion, Nonlinear Anal. 150 (2017), 194-209.

[35] M. Schonbek, L^2 decay for weak solutions of the Navier-Stokes equations, Arch. Rational Mech. Anal. 88 (1985), 209-222.

[36] M. Schonbek, Large time behaviour of solutions to the Navier-Stokes equations, Comm. Partial Differential Equations 11 (1986), 733-763.
[37] M. Schonbek, M. Wiegner, On the decay of higher-order norms of the solutions of Navier-Stokes equations, Proc. Roy. Soc. Edinburgh Sect. A 126 (1996), 677-685.
[38] V. Sohinger, R.M. Strain, The Boltzmann equation, Besov spaces, and optimal time decay rates in $\mathbb{R}_x^n$, Adv. Math. 261 (2014), 274-332.
[39] Z. Tan, Y. Wang, Y. Wang, Decay estimates of solutions to the compressible Euler-Maxwell system in $\mathbb{R}^3$, J. Differential Equations 257 (2014), 2846-2873.
[40] Z. Tan, W. Wu, J. Zhou, Global existence and decay estimate of solutions to magneto-micropolar fluid equations, J. Differential Equations 266 (2019), 4137-4169.
[41] R. Temam, Navier-Stokes equations: Theory and numerical analysis. Reprint of the 1984 edition. AMS Chelsea Publishing, Providence, RI, 2001.
[42] H. Triebel, Theory of Function Spaces II, Birkhauser Verlag, 1992.
[43] Y. Wang, L. Gu, Global regularity of 3D magneto-micropolar fluid equations, Appl. Math. Lett. 99 (2020), 105980, 9pp.
[44] Y. Wang, K. Wang, Global well-posedness of 3D magneto-micropolar fluid equations with mixed partial viscosity, Nonlinear Anal. Real World Appl. 33 (2017) 348-362.
[45] R. Wei, B. Guo, Y. Li, Global existence and optimal convergence rates of solutions for 3D compressible magneto-micropolar fluid equations, J. Differential Equations 263 (2017), 2457-2480.
[46] M. Wiegner, Decay results for weak solutions of the Navier-Stokes equations in $\mathbb{R}^n$, J. London Math. Soc. 35 (1987), 303-313.
[47] L. Xue, Well posedness and zero microrotation viscosity limit of the 2D micropolar fluid equations, Math. Methods Appl. Sci. 34 (2011), 1760-1777.
[48] K. Yamazaki, Global regularity of the two-dimensional magneto-micropolar fluid system with zero angular viscosity, Discrete Contin. Dyn. Syst. 35 (2015), 2193-2207.
[49] B. Yuan, Regularity of weak solutions to magneto-micropolar fluid equations, Acta Math. Sci. Ser. B Engl. Ed. 30 (2010), 1469-1480.
[50] J. Yuan, Existence theorem and blow-up criterion of the strong solutions to the magneto-micropolar fluid equations, Math. Methods Appl. Sci. 31 (2008), 1113-1130.
[51] Z. Zhang, Z. Yao, X. Wang, A regularity criterion for the 3D magneto-micropolar fluid equations in Triebel-Lizorkin spaces, Nonlinear Anal. 74 (2011), 2220-2225.

School of Mathematical Sciences, Capital Normal University, Beijing 100048, People's Republic of China

Email address: djniu@cnu.edu.cn

School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo, Henan 454000, People's Republic of China

Email address: hfshang@163.com