Existence and nonexistence of global solutions for logarithmic hyperbolic equation *

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Abstract

In this paper, we study the initial-boundary value problem of a class of degenerate quasilinear hyperbolic equation with logarithmic nonlinearity. By applying Galerkin method and the logarithmic Sobolev inequality, we prove the existence of global weak solutions for this problem. Meanwhile,the global nonexistence of solutions is verified by means of the concavity analysis when the initial energy is positive and appropriately bounded.

Keywords: Logarithmic hyperbolic equation; Stable and unstable sets; Existence and nonexistence; Global solutions; Initial-boundary value problem.

2000MR Subject Classification: 35L10; 35L15; 35L20; 35L80.

1. Introduction

In this paper, we study the initial-boundary value problem for logarithmic hyperbolic equation of $p-$Laplacian type

$$u_{tt} + \Delta_p u = |u|^{p-2}u \ln |u|, \quad (x,t) \in \Omega \times R^+, \tag{1.1}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.2}$$

$$u(x,t) = 0,\ (x,t) \in \partial\Omega \times R^+, \tag{1.3}$$

where

$$\Delta_p u = -div(|\nabla u|^{p-2}\nabla u),\ p > 2. \tag{1.4}$$

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$\Omega \subset R^N$ is a bounded domain with smooth boundary $\partial\Omega$, and the parameter p satisfies

$$2 < p < +\infty,\ N \leq p;\quad 2 < p \leq \frac{Np}{N-p},\ N > p. \tag{1.5}$$

When the logarithmic term in (1.1) is replaced by nonlinear source term $|u|^{r-2}u$, the equation (1.1) becomes

$$u_{tt} + \Delta_p u = |u|^{r-2}u,\quad (x,t) \in \Omega \times R^+. \tag{1.6}$$

In the case of $p = 2$ and $r > 2$, J.Ball [1] obtained the finite time blow-up of solutions with negative initial energy. In fact, M.Tsustsumi [2] has obtained stronger result by using the same method. Based on the concavity method, H.A.Levine and L.E.Payne [3] established the nonexistence of global weak solutions of the equation (1.6) under conditions (1.2) and (1.3). By using the potential well theory and the analysis method, Y.J.Ye [4] proved the existence and blow-up result of solutions to the initial-boundary value problem of the equation (1.6) by using the potential well theory and the analysis method. S.Ibrahim and A.Lyaghfouri [5] considered the Cauchy problem of (1.6), under appropriate assumptions on the initial data, they established the finite time blow-up of solutions and, hence, extended a result by V.A.Galaktionov and S.I.Pohozaev [6]. For the equation (1.6) with the dissipative term, Y.J.Ye [7, 8] studied the global solutions by constructing a stable set in $W_0^{1,p}(\Omega)$ and the asymptotic behavior of solution through the use of an important lemma of V.Komornik [9].

S.A.Messaoudi and B.S.Said-Houari [10] considered the nonlinear hyperbolic type equation

$$u_{tt} + \Delta_\alpha u + \Delta_\beta u_t - \Delta u_t + a|u_t|^{m-2}u_t = b|u|^{p-2}u,$$

where $a, b > 0, \alpha, \beta, m, p > 2$ and Ω is a bounded domain in $R^N (N \geq 1)$. Under appropriate conditions on α, β, m, p, they proved a global nonexistence result of solutions with negative initial energy. For the nonlinear wave equation of p-Laplacian type

$$u_{tt} + \Delta_p u - \Delta u_t + q(x,u) = f(x),$$

where $2 \leq p < N$ and f, q are given functions. C.Chen et al.[11] obtained the global existence and uniqueness of solutions and established the long-time behavior of solutions.

L.C.Nhan and T.X.Le [12] studied the existence and nonexistence of global weak solutions for a class of p-Laplacian evolution equations with logarithmic nonlinearity and gave sufficient conditions for the large time decay and blow-up of solutions. Later, Y,Z,Han etc [13] also considered this problem, and they proved the existence of global solutions and established the blow-up solution for arbitrarily high initial energy. For a mixed pseudo-parabolic p-Laplacian type equation with the logarithmic term, under various assumptions about initial values, H.Ding and J,Zhou [14] proved the solution exists globally and blow up in finite time. Moreover, T.Boudjeriou [15] was concerned with the fractional p-Laplacian with logarithmic nonlinearity, by applying the potential well theory and a differential inequality, he proved the existence and decay estimates of global solutions and obtained the blow-up result of solutions.

For the following nonlinear logarithmic wave equations

$$u_{tt} - \Delta u = u \ln |u|. \tag{1.7}$$

T.Cazenave and A.Haraux [16] gave the existence and uniqueness of solutions for the Cauchy problem of equation (1.7). P.Gorka [17] obtained the global weak solutions to the initial-boundary value problem of (1.7) in a one-dimensional case by using some compactness arguments for all $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. K.Bartkowski and P.Gorka [18] proved the existence of classical solutions and weak solutions for the corresponding one dimensional Cauchy problem of equation (1.7). In the case of $0 < \mathcal{E}(0) \leq d$, W.Lian et al. [19] proved the global existence of solution and obtain the blow-up of solution for the equation (1.7) with conditions (1.2) and (1.3) by using potential well theory combined with logarithmic Sobolev inequality. In paper [20], C.C.Liu and Y.H.Ma considered the following fourth order hyperbolic equation with the logarithmic nonlinearity

$$\begin{cases} u_{tt} + u_t + D^4 u = -D(|Du|^{p-2}) Du \ln |Du|, \quad (x,t) \in Q_T, \\ u|_{\partial\Omega} = Du|_{\partial\Omega} = 0, \\ u(x,0) = \varphi(x), \ u_t(x,0) = \psi(x), \ x \in \Omega, \end{cases} \tag{1.8}$$

where $D = \frac{\partial}{\partial x}$, $Q_T = \Omega \times (0,T)$, $\Omega \subset R$ is an open interval and $p > 2$. By means of a first order differential inequality technique, they derived a blow-up result and established the upper bound of the blow-up time for the problem (1.8).

It makes the problem (1.1) different from the one (1.6) that a logarithmic nonlinear term $u \ln |u|$ is introduced. For this reason, fewer results are, at the present time, known for the logarithmic hyperbolic equation of p-Laplacian type. In this paper, we derive the global existence and nonexistence of solution for the problem (1.1)-(1.3) by means of the potential well theory ([21, 22, 23, 24]) and the concavity analysis method ([25]).

For simplicity, hereafter we denote the Lebesgue space $L^p(\Omega)$ norm by $\|\cdot\|_p$ and $L^2(\Omega)$ norm by $\|\cdot\|$. We write equivalent norm $\|\nabla \cdot\|_p$ instead of $W_0^{1,p}(\Omega)$ norm $\|\cdot\|_{W_0^{1,p}(\Omega)}$.

2. Preliminaries

At first, we give the definition of weak solutions to the problem (1.1)-(1.3) and list up several useful lemmas.

Definition 2.1 *If*

$$u \in C([0,T], W_0^{1,p}(\Omega)), \ u_t \in C([0,T], L^2(\Omega)), \ u_{tt} \in C([0,T], W^{-1,p'}(\Omega))$$

and satisfies

$$\int_\Omega u_{tt}\varphi dx + \int_\Omega \Delta_p u\varphi dx = \int_\Omega |u|^{p-2}u\ln|u|\varphi dx,$$

then the function $u(t)$ *is said to be a weak solution of (1.1)-(1.3) on* $[0,T]$, *where* $\varphi \in W_0^{1,p}(\Omega)$.

Lemma 2.1 *Let* q *be a real number with* $2 \le q < +\infty$ *if* $2 \le n \le p$ *and* $2 \le q \le \frac{np}{n-p}$ *if* $2 < p < n$. *Then there exists a positive constant* C *depending on* Ω, p *and* q *such that* $\|u\|_q \le C\|\nabla u\|_p$.

Lemma 2.2[26] *Let* B_0, B, B_1 *be Banach spaces with* $B_0 \subseteq B \subseteq B_1$ *and*

$$X = \{u:\ u \in L^p[0,T];B_0),\ u_t \in L^q[0,T];B_1)\},\ 1 \le p,q \le +\infty.$$

Suppose that B_0 *is compactly embedded in* B *and that* B *is continuously embedded in* B_1, *then (i) the embedding of* X *into* $L^p(0,T;B)$ *is compact if* $p < +\infty$. *(ii) the embedding of* X *into* $C([0,T];B)$ *is compact if* $p = +\infty$ *and* $q > 1$.

Lemma 2.3[27] *Assume that* $u_n(x)$ *is a bounded sequence in* $L^q(\Omega), 1 \le q < +\infty$ *such that* $u_n(x) \to u(x)$ *almost everywhere. Then* $u(x) \in L^q(\Omega)$ *and* $u_n(x) \to u(x)$ *weakly converges in* $L^q(\Omega)$.

Lemma 2.4[28,29,30] *(*L^2-*logarithmic Sobolev inequality) If* $v \in H_0^1(\Omega)$, *then for each* $a > 0$, *one has the inequality*

$$\int_\Omega |v|^2\ln|v|dx \le \|v\|^2\ln\|v\| + \frac{a^2}{2\pi}\|\nabla v\|^2 - \frac{n}{2}(1+\ln a)\|v\|^2. \quad (2.1)$$

In order to deal with the logarithmic term $|u|^{p-2}u\ln|u|$ in equation (1.1), we introduce the following L^p-logarithmic Sobolev inequality.

Lemma 2.5[31] *(*L^p-*logarithmic Sobolev inequality) Let* $u \in W_0^{1,p}(\Omega)$, *then one has the inequality*

$$\int_\Omega |u|^p\ln|u|dx \le \|u\|_p^p\ln\|u\|_p + \frac{(p-2)a^2}{4\pi}\|u\|_p^p + \frac{a^2}{2\pi}\|\nabla u\|_p^p - \frac{n}{p}(1+\ln a)\|u\|_p^p, \quad (2.2)$$

where $a > 0$ *is a constant.*

For convenience, in the following we are going to give the proof of Lemma 2.5.

Proof By (2.1) in Lemma 2.4, we have

$$\begin{aligned}\int_\Omega |v|^2\ln|v|dx &\le \|v\|^2\ln\|v\| + \frac{a^2}{2\pi}\|\nabla v\|^2 - \frac{n}{2}(1+\ln a)\|v\|^2\\ &= \int_\Omega |v|^2dx\cdot\ln\Big(\int_\Omega |v|^2dx\Big)^{\frac{1}{2}} + \frac{a^2}{2\pi}\int_\Omega |\nabla v|^2dx - \frac{n}{2}(1+\ln a)\int_\Omega |v|^2dx.\end{aligned} \quad (2.3)$$

Let $v = u^{\frac{p}{2}}$ in (2.3), then we obtain

$$\begin{aligned}\frac{1}{2}\int_\Omega |u|^p \ln |u|^p dx &\leq \int_\Omega |u|^p dx \cdot \ln\Big(\int_\Omega |u|^p dx\Big)^{\frac{1}{2}} \\ &+\frac{a^2}{2\pi}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx - \frac{n}{2}(1+\ln a)\int_\Omega |u|^p dx \\ &= \frac{1}{2}\|u\|_p^p \ln \|u\|_p^p + \frac{a^2}{2\pi}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx - \frac{n}{2}(1+\ln a)\|u\|_p^p.\end{aligned} \tag{2.4}$$

By direct computation, we get

$$\nabla u^{\frac{p}{2}} = \frac{p}{2}u^{\frac{p}{2}-1}\cdot \nabla u = \frac{p}{2}u^{\frac{p-2}{2}}\cdot \nabla u. \tag{2.5}$$

From (2.5) and Hölder inequality, we receive

$$\begin{aligned}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx &= \frac{p^2}{4}\int_\Omega |u|^{p-2}|\nabla u|^2 dx \\ &\leq \frac{p^2}{4}\Big(\int_\Omega |u|^p dx\Big)^{\frac{p-2}{p}}\Big(\int_\Omega |\nabla u|^p dx\Big)^{\frac{2}{p}} = \frac{p^2}{4}\|u\|_p^{p-2}\|\nabla u\|_p^2.\end{aligned} \tag{2.6}$$

By Young inequality $XY \leq \frac{X^\alpha}{\alpha} + \frac{Y^\beta}{\beta}$ with $\alpha = \frac{p}{p-2},\ \beta = \frac{p}{2}$, we conclude that

$$\frac{p^2}{4}\|u\|_p^{p-2}\|\nabla u\|_p^2 \leq \frac{p^2}{4}\Big(\frac{p-2}{p}\|u\|_p^p + \frac{2}{p}\|\nabla u\|_p^p\Big) = \frac{p(p-2)}{4}\|u\|_p^p + \frac{p}{2}\|\nabla u\|_p^p. \tag{2.7}$$

It follows from (2.4), (2.6) and (2.7) that

$$\frac{1}{2}\int_\Omega |u|^p \ln |u|^p dx \leq \frac{1}{2}\|u\|_p^p \ln \|u\|_p^p + \frac{a^2}{2\pi}\Big(\frac{p(p-2)}{4}\|u\|_p^p + \frac{p}{2}\|\nabla u\|_p^p\Big) - \frac{n}{2}(1+\ln a)\|u\|_p^p,$$

which implies

$$\int_\Omega |u|^p \ln |u| dx \leq \|u\|_p^p \ln \|u\|_p + \frac{(p-2)a^2}{4\pi}\|u\|_p^p + \frac{a^2}{2\pi}\|\nabla u\|_p^p - \frac{n}{p}(1+\ln a)\|u\|_p^p.$$

This completes the proof of Lemma 2.5.

Next, for the problem (1.1)-(1.3), we define the functionals

$$\mathcal{J}(u) = \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{p}\int_\Omega |u|^p \ln |u| dx + \frac{1}{p^2}\|u\|_p^p \tag{2.8}$$

and

$$\mathcal{K}(u) = \|\nabla u\|_p^p - \frac{1}{p}\int_\Omega |u|^p \ln |u| dx, \tag{2.9}$$

for $u \in W_0^{1,p}(\Omega)$. By (2.8) and (2.9), we have

$$\mathcal{J}(u) = \frac{1}{p^2}\|u\|_p^p + \frac{1}{p}\mathcal{K}(u). \tag{2.10}$$

We denote the energy functional by

$$\mathcal{E}(t)=\frac{1}{2}\|u_t\|^2+\frac{1}{p}\|\nabla u\|_p^p-\frac{1}{p}\int_\Omega |u|^p\ln|u|dx+\frac{1}{p^2}\|u\|_p^p=\frac{1}{2}\|u_t\|^2+\mathcal{J}(u), \tag{2.11}$$

for $u\in W_0^{1,p}(\Omega),\ t\geq 0.$

$$\mathcal{E}(0)=\frac{1}{2}\|u_1\|^2+\frac{1}{p}\|\nabla u_0\|_p^p-\frac{1}{p}\int_\Omega |u_0|^p\ln|u_0|dx+\frac{1}{p^2}\|u_0\|_p^p=\frac{1}{2}\|u_1\|^2+\mathcal{J}(u_0) \tag{2.12}$$

is the initial total energy.

Moreover, we define the Nehari manifold

$$\mathcal{N}=\{u\in W_0^{1,p}(\Omega)/\{0\}:\ \mathcal{K}(u)=0,\ \|\nabla u\|_p\neq 0\},$$

the stable set

$$\mathcal{W}=\{u\in W_0^{1,p}(\Omega):\ \mathcal{K}(u)>0,\ \mathcal{J}(u)<d\}\cup\{0\}$$

and the unstable set

$$\mathcal{U}=\{u\in W_0^{1,p}(\Omega):\ \mathcal{K}(u)<0\ \mathcal{J}(u)<d\},$$

where

$$d=\inf\{\sup_{\theta\geq 0}\mathcal{J}(\theta u):\ u\in W_0^{1,p}(\Omega),\ \|\nabla u\|_p\neq 0\}. \tag{2.13}$$

It is readily seen that the potential well depth d defined in (2.13) can also be characterized as

$$d=\inf_{u\in\mathcal{N}}\mathcal{J}(u). \tag{2.14}$$

Lemma 2.6 *If $u\in W_0^{1,p}(\Omega)$ and $\|u\|_p\neq 0$, then we have*

$$(a)\quad \lim_{\theta\to 0^+}\mathcal{J}(\theta u)=0,\ \lim_{\theta\to+\infty}\mathcal{J}(\theta u)=-\infty;$$

$$(b)\quad \mathcal{K}(\theta u)=\theta\mathcal{J}'(\theta u)\begin{cases}>0, & 0<\theta<\theta_*,\\ =0, & \theta=\theta_*,\\ <0, & \theta_*<\theta<+\infty.\end{cases} \tag{2.15}$$

Proof (a) For $u\in W_0^{1,p}(\Omega)$,

$$\mathcal{J}(\theta u)=\frac{\theta^p}{p}\|\nabla u\|_p^p-\frac{\theta^p}{p}\int_\Omega |u|^p\ln|u|dx-\frac{\theta^p}{p}\|u\|_p^p\ln\theta+\frac{\theta^p}{p^2}\|u\|_p^p.$$

It is easy to get from $\|u\|_p\neq 0$ that (a) is valid.

(b) An elementary calculation shows that

$$\frac{d}{d\theta}\mathcal{J}(\theta u)=\theta^{p-1}\Big(\|\nabla u\|_p^p-\int_\Omega |u|^p\ln|u|dx-\|u\|_p^p\ln\theta\Big). \tag{2.16}$$

Let $\frac{d}{d\theta}\mathcal{J}(\theta u)=0$, then we have

$$\theta_*=\exp\Big(\frac{\|\nabla u\|_p^p-\int_\Omega |u|^p\ln|u|dx}{\|u\|_p^p}\Big). \tag{2.17}$$

It follows from (2.9) that

$$\mathcal{K}(\theta u)=\theta^p\Big(\|\nabla u\|_p^p-\int_\Omega |u|^p\ln|u|dx-\|u\|_p^p\ln\theta\Big). \tag{2.18}$$

From (2.16), (2.17) and (2.18), the formula (2.15) holds.

Lemma 2.7 *Suppose that $u\in W_0^{1,p}(\Omega)$ and $\|\nabla u\|_p\neq 0$. Then $d\geq M$, where $M=\frac{1}{p^2}(2\pi)^{\frac{n}{2}}e^{\frac{2(n+p)-p^2}{2}}$.*

Proof From Lemma 2.6 and (2.10), we obtain

$$\sup_{\theta\geq 0}\mathcal{J}(\theta u)=\mathcal{J}(\theta_* u)=\frac{1}{p^2}\|\theta_* u\|_p^p+\frac{1}{p}\mathcal{K}(\theta_* u)=\frac{1}{p^2}\|\theta_* u\|_p^p. \tag{2.19}$$

We get from Lemma 2.5 that

$$\begin{aligned}\mathcal{K}(u)&=\|\nabla u\|_p^p-\frac{1}{p}\int_\Omega |u|^p\ln|u|dx\\ &\geq\Big(1-\frac{a^2}{2\pi}\Big)\|\nabla u\|_p^p+\Big(\frac{n}{p}\ln(ae)-\frac{(p-2)a^2}{4\pi}-\ln\|u\|_p\Big)\|u\|_p^p.\end{aligned}$$

Choosing $a=\sqrt{2\pi}$, we have

$$\begin{aligned}\mathcal{K}(u)&\geq\Big(\frac{n}{p}\ln(\sqrt{2\pi}\ e)-\frac{p-2}{2}-\ln\|u\|_p\Big)\|u\|_p^p\\ &=\Big[\ln\Big((2\pi)^{\frac{n}{2p}}e^{\frac{2(n+p)-p^2}{2p}}\Big)-\ln\|u\|_p\Big]\|u\|_p^p.\end{aligned} \tag{2.20}$$

It follows from $\mathcal{K}(\theta_* u)=0$ and (2.20) that

$$\ln\Big((2\pi)^{\frac{n}{2}}e^{\frac{2(n+p)-p^2}{2}}\Big)-\ln\|\theta_* u\|_p^p\leq 0,$$

which implies

$$\|\theta_* u\|_p^p\geq(2\pi)^{\frac{n}{2}}e^{\frac{2(n+p)-p^2}{2}}. \tag{2.21}$$

Thus, we obtain from (2.19) and (2.21) that

$$\sup_{\theta\geq 0}\mathcal{J}(\theta u)\geq\frac{1}{p^2}(2\pi)^{\frac{n}{2}}e^{\frac{2(n+p)-p^2}{2}}. \tag{2.22}$$

Thus, by (2.13) and (2.22), we conclude that $d \geq M > 0$.

3. Existence of global Solutions

In this part, we prove the existence of global solutions for the problem (1.1)-(1.3) by applying the compactness principle and monotone mapping method.

Theorem 3.1 *Assume that p satisfies (1.5), there exists a global solution $u(x,t)$ of the problem (1.1)-(1.3), if $u_0 \in W_0^{1,p}(\Omega)$, $u_1 \in L^2(\Omega)$ and $0 < \mathcal{E}(0) < M$, $\mathcal{K}(u_0) \geq 0$.*

Proof Assume that $\{\omega_j\}_{j=1}^{\infty}$ is a basis of space $W_0^{1,p}(\Omega)$ and that V_k is the subspace of $W_0^{1,p}(\Omega)$ generated by $\{\omega_1, \omega_2, \cdots, \omega_m\}$, $m = 1, 2, \cdots$. We shall look for the approximate solutions $u_m(t) = \sum_{j=1}^{m} g_{jm}(t)\omega_j$ with $g_{jm}(t) \in C^2[0,T]$, $\forall T > 0$. Here the functions $g_{jm}(t)$ fulfil the following system of equations

$$(u_{mtt}, \omega_j) + (\Delta_p u_m, \omega_j) = (|u_m|^{p-2}u_m \ln |u_m|, \omega_j), \ j = 1, 2, \cdots, m \tag{3.1}$$

with initial data

$$u_m(0) = u_{0m}, \ u_{mt}(0) = u_{1m}. \tag{3.2}$$

Because $W_0^{1,p}(\Omega)$ is dense in $L^2(\Omega)$, so there exist α_{jm} and β_{jm} such that

$$u_{0m} = \sum_{j=1}^{m} g_{jm}(0)\omega_j = \sum_{j=1}^{m} \alpha_{jm}\omega_j \to u_0(x) \ \textit{strongly in} \ W_0^{1,p}(\Omega), \ m \to \infty, \tag{3.3}$$

$$u_{1m} = \sum_{j=1}^{m} g'_{jm}(0)\omega_j = \sum_{j=1}^{m} \beta_{jm}\omega_j \to u_1(x) \ \textit{strongly in} \ L^2(\Omega), \ m \to \infty. \tag{3.4}$$

By applying Picard's iteration method, the solutions $g_{jm}(t)$ of the Cauchy problem (3.1)-(3.2) for $t \in [0, t_m)$, $t_m \leq T$. By using the uniformly boundedness of function $g_{jm}(t)$ as well as the extension theorem, these solutions can be extended to the whole interval $[0,T]$ for any given $T > 0$.

Multiplying both sides of (3.1) by $g'_{jm}(t)$, summing on j from 1 to m and then integrating over $[0,t]$, we obtain

$$\mathcal{E}_m(t) = \frac{1}{2}\|u_{mt}(t)\|^2 + \mathcal{J}(u_m(t)) = \frac{1}{2}\|u_{mt}(0)\|^2 + \mathcal{J}(u_m(0)) = \mathcal{E}_m(0) < M \leq d. \tag{3.5}$$

From (3.5), it is easy to verify

$$u_m(t) \in \mathcal{W}, \ \forall t \in [0,T]. \tag{3.6}$$

Suppose that there exists a time $t_1 \in (0,T)$ such that $u_m(t_1) \notin \mathcal{W}$, then according to the continuity of $u_m(t)$ on t, we get $u_m(t_1) \in \partial\mathcal{W}$. By the continuity of $\mathcal{J}(u(t))$ and $\mathcal{K}(u(t))$ with respect to t, we receive either

$$\mathcal{J}(u_m(t_1)) = d, \tag{3.7}$$

or

$$\mathcal{K}(u_m(t_1)) = 0,\ \|\nabla u_m\|_p \neq 0. \tag{3.8}$$

From (3.5), we have $J(u_m(t_1)) < d$. Therefore, the case (3.7) is impossible.

Assume that (3.8) holds, then $u_m(t_1) \in \mathcal{N}$. By (2.7), we get $J(u_m(t_1)) \geq d$ which is contradictive with (3.5). Therefore, the case (3.8) is also impossible as well.

We deduce from (2.3), (3.5) and (3.6) that

$$M > \mathcal{J}(u_m) = \frac{1}{p^2}\|u_m\|_p^p + \frac{1}{p}\mathcal{K}(u_m) > \frac{1}{p^2}\|u_m\|_p^p, \tag{3.9}$$

which implies that

$$\|u_m\|_p^p < Mp^2. \tag{3.10}$$

Taking $a = \sqrt{\pi}$ in (2.2), we have from Lemma 2.5, (2.8) and (2.9) that

$$\begin{aligned}
\|\nabla u_m\|_p^p &= 2\mathcal{K}(u) + 2\int_\Omega |u_m|^p \ln |u_m| dx - \|\nabla u_m\|_p^p \\
&= 2p\mathcal{J}(u_m) - \frac{2}{p}\|u_m\|_p^p - \|\nabla u_m\|_p^p + 2\int_\Omega |u_m|^p \ln |u_m| dx \\
&\leq 2p\mathcal{J}(u_m) - \frac{2}{p}\|u_m\|_p^p + \frac{p-2}{2}\|u_m\|_p^p \\
&-\frac{n}{p}\ln(\pi e^2)\|u_m\|_p^p + 2\|u_m\|_p^p \ln \|u_m\|_p \\
&\leq 2p\mathcal{J}(u_m) + \frac{p-2}{2}\|u_m\|_p^p + 2\|u_m\|_p^p \ln \|u_m\|_p < C_M.
\end{aligned} \tag{3.11}$$

Here $C_M = 2pM + \frac{(p-2)p^2}{2}M + 2pM\ln(p^2 M)$. From (3.5), we have

$$\|u_{mt}\|^2 < 2M. \tag{3.12}$$

For $u, v \in W_0^{1,p}(\Omega)$, by (1.4), we have $(\Delta_p u, v) = \int_\Omega |\nabla u|^{p-2}|\nabla u| \cdot |\nabla v| dx)$. Hence, from Hölder inequality and (3.11), we obtain

$$\|\Delta_p u\|_{W^{-1,p'}(\Omega)} \leq \|\nabla u\|_p^{p-1} < C_M^{\frac{p-1}{p}}. \tag{3.13}$$

From (3.10) to (3.13), it can be concluded that there is a function $u(t)$ and a convergent subsequence of u_k, which are still expressed by u_m. As $m \to \infty$, we get

$$u_m \to u \textit{ weakly star in } L^\infty(0, T; W_0^{1,p}(\Omega)), \tag{3.14}$$

$$u_m \to u \textit{ weakly star in } L^\infty(0, T; L^p(\Omega)), \tag{3.15}$$

$$u_{mt} \to u_t \textit{ weakly star in } L^\infty(0, T; L^2(\Omega)), \tag{3.16}$$

$$\Delta_p u_m \to \chi \textit{ weakly star in } L^\infty(0, T; W^{-1,p'}(\Omega)). \tag{3.17}$$

Combining (3.15), (3.16) with Lemma 2.2, we have

$$u_m \to u \text{ strongly in } C([0,T]; L^2(\Omega)), \tag{3.18}$$

which implies

$$|u_m|^{p-2}u_m \ln |u_m| \to |u|^{p-2}u \ln |u| \text{ almost everywhere } (x,t) \in \Omega \times (0,T). \tag{3.19}$$

Let $\Omega_1 = \{x \in \Omega : \ |u_m(x,t) \le 1\}$ and $\Omega_2 = \{x \in \Omega : \ |u_m(x,t) \ge 1\}$, then by means of direct calculation, we know from Lemma 2.1 and (3.11)

$$\begin{aligned}
&\int_\Omega \left| |u_m|^{p-2}u_m \ln |u_m| dx \right|^{p'} \\
&= \int_{\Omega_1} \left| |u_m|^{p-2}u_m \ln |u_m| dx \right|^{p'} + \int_{\Omega_2} \left| |u_m|^{p-2}u_m \ln |u_m| dx \right|^{p'} \\
&\le [(p-1)e]^{-p'}|\Omega| + \left(\frac{n-p}{p(p-1)}\right)^{p'} \int_{\Omega_2} |u_m|^{\frac{np}{n-p}} \\
&\le [(p-1)e]^{-p'}|\Omega| + \left(\frac{n-p}{p(p-1)}\right)^{p'} C^{\frac{np}{n-p}} \|\nabla u_m\|_p^{\frac{np}{n-p}} \le L_M,
\end{aligned} \tag{3.20}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $L_M = [(p-1)e]^{-p'}|\Omega| + \left(\frac{n-p}{p(p-1)}\right)^{p'} C^{\frac{np}{n-p}} C_M^{\frac{n}{n-p}}$. From Lemma 2.3, (3.19) and (3.20), we receive

$$|u_m|^{p-2}u_m \ln |u_m| \to |u|^{p-2}u \ln |u| \text{ weakly in } L^\infty(0,T; L^{p'}(\Omega)). \tag{3.21}$$

Now, we need to prove $\chi = \Delta_p u$. For this purpose, multiplying the two sides of (3.1) by an arbitrary smooth function $\varphi(t) \in C^2[0,T]$ and integrating over $[0,T]$, we have

$$\begin{aligned}
&(u_{mt}(T), \varphi(T)\omega_j) + \int_0^T (\Delta_p u_m, \varphi(t)\omega_j)dt = (u_{mt}(0), \varphi(0)\omega_j) \\
&+ \int_0^T (u_{mt}, \varphi'(t)\omega_j)dt + \int_0^T (|u_m|^{p-2}u_m \ln |u_m|, \varphi(t)\omega_j)dt.
\end{aligned} \tag{3.22}$$

Taking the limitation of both sides of equality (3.22) with j fixed and $m \to \infty$, we get

$$\begin{aligned}
&(u_t(T), \varphi(T)\omega_j) + \int_0^T (\chi, \varphi(t)\omega_j)dt \\
&= (u_t(0), \varphi(0)\omega_j) + \int_0^T (u_t, \varphi'(t)\omega_j)dt + \int_0^T (|u|^{p-2}u \ln |u|, \varphi(t)\omega_j)dt.
\end{aligned} \tag{3.23}$$

By (3.23),we have

$$\begin{aligned}
&(u_t(T), \psi(T)) + \int_0^T (\chi, \psi(t))dt \\
&= (u_t(0), \psi(0)) + \int_0^T (u_t, \psi'(t))dt + \int_0^T (|u|^{p-2}u \ln |u|, \psi(t))dt,
\end{aligned} \tag{3.24}$$

for every $\psi \in L^2(0,T;W_0^{1,p}(\Omega)), \quad \psi' \in L^2(0,T;L^2(\Omega))$. In particular, Setting $\psi = u$ in (3.24) , we obtain

$$\begin{aligned}&(u_t(T),u(T)) + \int_0^T (\chi,u)dt\\&= (u_t(0),u(0)) + \int_0^T \|u_t(t)\|^2dt + \int_0^T (|u|^{p-2}u\ln|u|,u)dt.\end{aligned} \tag{3.25}$$

On the other hand, multiplying the two sides of (3.1) by $g_{jm}(t)$, summing on j from 1 to m and integrating over $[0,T]$, we get

$$\begin{aligned}&(u_{mt}(T),u_m(T)) + \int_0^T (\Delta_p u_m,u_m)dt = (u_{mt}(0),u_m(0)\\&+ \int_0^T \|u_{mt}\|^2dt + \int_0^T (|u_m|^{p-2}u_m\ln|u_m|,u_m)dt.\end{aligned} \tag{3.26}$$

Taking the inferior limitation on both sides of (3.26) as $m \to \infty$, we have

$$\begin{aligned}&(u_t(T),u(T)) + \lim_{m\to\infty}\inf \int_0^T (\Delta_p u_m,u_m)dt\\&\le (u_t(0),u(0)) + \int_0^T \|u_t\|^2dt + \int_0^T (|u|^{p-2}u\ln|u|,u)dt.\end{aligned} \tag{3.27}$$

We conclude from (3.25) and (3.27) that

$$\lim_{m\to\infty}\inf \int_0^T (\Delta_p u_m,u_m)dt \le \int_0^T (\chi,u)dt. \tag{3.28}$$

By the monotonicity of operator Δ_p, we have

$$\int_0^T (\Delta_p u_m - \Delta_p v, u_m - v)dt \ge 0, \ \forall v \in L^\infty(0,T;W_0^{1,p}(\Omega)). \tag{3.29}$$

We get from (3.28) and (3.29) that

$$\begin{aligned}&\lim_{m\to\infty}\inf \int_0^T (\Delta_p u_m - \Delta_p v, u_m - v)dt\\&\le \int_0^T (\chi,u-v)dt - \int_0^T (\Delta_p v,u-v)dt = \int_0^T (\chi - \Delta_p v, u-v)dt.\end{aligned} \tag{3.30}$$

Combining (3.29) with (3.30), we have

$$\int_0^T (\chi - \Delta_p v, u - v)dt \ge 0. \tag{3.31}$$

Let $v = u - \lambda\omega$, then, by (3.31), we obtain

$$\lambda \int_0^T (\chi - \Delta_p(u-\lambda\omega),\omega)dt \ge 0, \tag{3.32}$$

for any $\omega \in L^p(0,T;W_0^{1,p}(\Omega))$ and any real number λ.

For $\lambda > 0, \lambda \to 0$, we infer from (3.32) and the hemicontinuity of operator Δ_p that

$$\int_0^T (\chi - \Delta_p u, \omega)dt \geq 0. \tag{3.33}$$

For $\lambda < 0, \lambda \to 0$, the same argument yields

$$\int_0^T (\chi - \Delta_p u, \omega)dt \leq 0. \tag{3.34}$$

Thus, for all $\omega \in L^p(0,T;W_0^{1,p}(\Omega))$, we deduce from (3.33) and (3.34) that

$$\int_0^T (\chi - \Delta_p u, \omega)dt = 0, \tag{3.35}$$

which implies that $\chi = \Delta_p u$.

Next, we prove above solution $u(x,t)$ satisfies the initial data conditions (1.2), i.e. $u(x,0) = u_0(x),\ u_t(x,0) = u_1(x)$.

We draw a conclusion from (3.15), (3.16) and Lemma 1.2 that the mapping $u(t)$: $[0,T] \to L^2(\Omega)$ is continuous. Hence, we have $u_m(0) \to u(0)$ weakly in $L^2(\Omega)$. according to (3.3), we obtain $u(0) = u_0$.

To prove $u_t(0) = u_1$, let $\xi(t)$ be a smooth function with $\xi(0) = 1, \xi(T) = 0$. Noting

$$\int_0^T (u_{mtt}, \xi\omega_j)dt = -\int_0^T (u_{mt}, \xi_t\omega_j)dt - (u_{mt}(0), \xi(0)\omega_j).$$

For given j, as $m \to \infty$, we get in the distribution sense

$$\int_0^T (u_{tt}, \xi\omega_j)dt = -\int_0^T (u_t, \xi_t\omega_j)dt - (u_t(0), \xi(0)\omega_j) \tag{3.36}$$

in $\mathcal{D}'([0,T])$. On the other hand,

$$\int_0^T (u_{mtt}, \xi\omega_j)dt = \int_0^T [(-\Delta_p u_m, \xi\omega_j) + (|u_m|^{p-2}u_m \ln|u_m|, \xi\omega_j)]dt$$

converges to

$$\int_0^T [(-\Delta_p u, \xi\omega_j) + (|u|^{p-2}u \ln|u|, \xi\omega_j)]dt = \int_0^T (u_{tt}, \xi\omega_j)dt$$

as $m \to \infty$. Therefore,

$$\int_0^T (u_{tt}, \xi\omega_j)dt = -\int_0^T (u_t, \xi_t\omega_j)dt - (u_1, \xi(0)\omega_j). \tag{3.37}$$

From (3.36) and (3.37), we have $(u_t(0), \omega_j) = (u_1, \omega_j)$. By the density of$\{\omega_j\}_{j=1}^m$ in $L^2(\Omega)$, we get $u_t(0) = u_1$. So far, the proof of Theorem 3.1 is completed.

For the case of $\mathcal{K}(u_0) \geq 0$ and $\mathcal{E}(0) = M \leq d$, the global existence result of solutions to the problem (1.1)-(1.3) reads as follows:

Theorem 3.2 *Assume that p fulfils (1.5), there exists a global solution $u(x,t)$ for the problem (1.1)-(1.3), if initial datum $u_0 \in W_0^{1,p}(\Omega)$, $u_1 \in L^2(\Omega)$ satisfy $\mathcal{E}(0) = M \leq d$ and $\mathcal{K}(u_0) \geq 0$, .*

Proof For the case $\|\nabla u_0\|_p \neq 0$, let us suppose that $\rho_k = 1 - \frac{1}{k}$ and $u_{0k} = \rho_k u_0$, $k \geq 2$. We consider the following problem

$$\begin{cases} u_{tt} + \Delta_p u = |u|^{p-2} u \ln |u|, \quad (x,t) \in \Omega \times R^+, \\ u(x,0) = u_{0k}(x), \; u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u(x,t) = 0, \; (x,t) \in \partial\Omega \times R^+. \end{cases} \tag{3.38}$$

From $\mathcal{K}(u_0) \geq 0$ and Lemma 2.6, we have $\theta^* = \theta^*(u_0) \geq 1$. Accordingly, we get $\mathcal{K}(u_{0k}) > 0$. By (2.3), we obtain

$$0 < \mathcal{J}(u_{0k}) = \frac{1}{p^2}\|u_{0k}\|_p^p + \frac{1}{p}\mathcal{K}(u_{0k}) < \mathcal{J}(u_0). \tag{3.39}$$

Therefore, we receive

$$0 < \mathcal{E}_k(0) = \frac{1}{2}\|u_1\|^2 + \mathcal{J}(u_{0k}) < \frac{1}{2}\|u_1\|^2 + \mathcal{J}(u_0) = \mathcal{E}(0) = M \leq d,$$

which implies that $u_{0k} \in \mathcal{W}$.

According to Theorem 3.1, for each k, there exists a global weak solution $u_k(t)$ of the problem (3.38) such that $u_k(t) \in L^\infty([0,+\infty); W_0^{1,p}(\Omega))$, $u_{kt}(t) \in L^\infty([0,+\infty); L^2(\Omega))$ and

$$(u_{kt}, v) + \int_0^t (\Delta_p u_k, v) ds = (u_1, v) + \int_0^t (|u_k|^{p-2} u_k \ln |u_k|, v) ds \tag{3.40}$$

for any $v \in W_0^{1,p}(\Omega)$.

In addition,

$$\mathcal{E}_k(t) = \frac{1}{2}\|u_{kt}\|^2 + \mathcal{J}(u_k) = \frac{1}{2}\|u_1\|^2 + J(u_{0k}) = \mathcal{E}_k(0) < M \leq d. \tag{3.41}$$

By using (3.41) and combining with the same argument as (3.6), we can prove $u_k(t) \in \mathcal{W}$.

For the case $\|\nabla u_0\|_p = 0$, we get $\mathcal{J}(u_0) = 0$ by $\mathcal{K}(u_0) \geq 0$. So, we have $\mathcal{E}(0) = \frac{1}{2}\|u_1\|^2 = M \leq d$. Let $\rho_k = 1 - \frac{1}{k}$, $u_{1k} = \rho_k u_1(x)$, $k \geq 2$, we concern will the following problem

$$\begin{cases} u_{tt} + \Delta_p u = |u|^{p-2} u \ln |u|, \quad (x,t) \in \Omega \times R^+, \\ u(x,0) = u_0(x), \; u_t(x,0) = u_{1k}(x), \quad x \in \Omega, \\ u(x,t) = 0, \; (x,t) \in \partial\Omega \times R^+. \end{cases} \tag{3.42}$$

Noting

$$0<\mathcal{E}_k(0)=\frac{1}{2}\|u_{1k}\|^2+\mathcal{J}(u_0)=\frac{1}{2}\|\rho_k u_1\|^2<\frac{1}{2}\|u_1\|^2=M\le d. \tag{3.43}$$

According to (3.43) and Theorem 3.1, the problem (3.42) has a global weak solution $u_k(t)$. In which $u_k(t)\in L^\infty(0,+\infty;W_0^{1,p}(\Omega))$, $u_{kt}(t)\in L^\infty(0,+\infty;L^2(\Omega))$ and $u_k(t)\in \mathcal{W}$ for each k.

The rest of the proof of Theorem 3.2 is the same as the process of Theorem 3.1. Here, we have omitted it.

4. Nonexistence of global solutions

We study the nonexistence of global solutions for the problem (1.1)-(1.3) and give the lifespan of solutions in this section.

Lemma 4.1*[32, 33] If nonnegative function $\Phi(t)\in C^2$ satisfies*

$$\Phi(t)\Phi''(t)-(1+\rho)\Phi'(t)^2\ge 0,$$

for $\Phi(0)>0$, $\Phi'(0)>0$ and $\rho>0$, then there exists a time T_* such that $0<T_*\le\frac{\Phi(0)}{\rho\Phi'(0)}$ and $\lim\limits_{t\to T_*^-}\Phi(t)=+\infty$.

Lemma 4.2 *Suppose that $u(t)$ is a solution of (1.1)-(1.3). If $u_0\in\mathcal{U}$ and $\mathcal{E}(0)<d$, then $u(t)\in\mathcal{U}$ and $\mathcal{E}(t)<d$, $\forall t\ge 0$.*

Proof By the conservation of energy, it is proved that we have $\mathcal{E}(t)=\mathcal{E}(0)<d$, $\forall t\ge 0$. From (2.11), we get

$$\mathcal{J}(u)\le\mathcal{E}(t)<d,\ \forall t\ge 0. \tag{4.1}$$

Assume that there is a $t^*\in[0,+\infty)$ such that $u(t^*)\notin\mathcal{U}$, then by continuity of $\mathcal{K}(u(t))$ on t, we obtain $\mathcal{K}(u(t^*))=0$. This means $u(t^*)\in\mathcal{N}$. From (2.14), we have $\mathcal{J}(u(t^*))\ge d$, which is contradiction with (4.1). Therefore, the conclusion in Lemma 4.2 holds.

Theorem 4.1 *If the initial datum $u_0\in\mathcal{U}$, $u_1\in L^2(\Omega)$ satisfy $0<\mathcal{E}(0)<d$ and $\int_\Omega u_0u_1dx>0$, then the solution $u(t)$ of the problem (1.1)-(1.3) does not exist globally in time. Namely, there exists a time T_* such that $\lim\limits_{t\to T_*^-}\|u(t)\|^2=+\infty$, where the lifespan T_* is estimated by $0<T_*<\frac{4\Psi(0)}{(p-2)\Psi'(0)}$, $\Psi(t)$ is given in (4.19).*

Proof By $u_0\in\mathcal{U}$, $\mathcal{E}(0)<d$ and Lemma 4.2, we obtain $u\in\mathcal{U}$, which implies that

$$\mathcal{K}(u)=\|\nabla u\|_p^p-\int_\Omega|u|^p\ln|u|dx<0. \tag{4.2}$$

From (2.13) and (2.18), we have

$$d \leq \sup_{\theta \geq 0} \mathcal{J}(\theta u) = \frac{1}{p^2} \|\theta_* u\|_p^p. \tag{4.3}$$

We deduce from (2.17), (4.2) and (4.3) that

$$d \leq \frac{1}{p^2} \|u\|_p^p. \tag{4.4}$$

Let

$$\Psi(t) = \|u(t)\|^2 = \int_\Omega u^2 dx. \tag{4.5}$$

According to the continuity of $\Psi(t)$ with respect to t, there exists a real number $\alpha > 0$ such that

$$\Psi(t) \geq \alpha > 0. \tag{4.6}$$

By differentiating on both sides of (4.5), we get

$$\Psi'(t) \quad = 2 \int_\Omega u u_t dx. \tag{4.7}$$

From (4.7) and direct calculation, we obtain

$$\Psi''(t) = 2\|u_t\|^2 + 2 \int_\Omega u u_{tt} dx. \tag{4.8}$$

From (1.1)and (4.8), we have

$$\Psi''(t) = 2(\|u_t(t)\|^2 + \int_\Omega |u|^p \ln |u| dx - \|\nabla u\|_p^p) = 2[\|u_t(t)\|^2 - \mathcal{K}(u)]. \tag{4.9}$$

By $u \in \mathcal{U}$ and (4.9), we receive $\Psi''(t) > 0$. Combining (4.5), (4.7) and (4.9), we get

$$\begin{aligned}
&\Psi(t)\Psi''(t) - \frac{p+2}{4}\Psi'(t)^2 \\
&= 2\Psi(t)\Big[\|u_t(t)\|^2 + \int_\Omega |u|^p \ln |u| dx - \|\nabla u\|_p^p\Big] \\
&-(p+2)\Psi(t)\|u_t(t)\|^2 + (p+2)\Upsilon(t),
\end{aligned} \tag{4.10}$$

where

$$\Upsilon(t) = \|u(t)\|^2 \cdot \|u_t(t)\|^2 - \Big(\int_\Omega u u_t dx\Big)^2. \tag{4.11}$$

By Cauchy-Schwarz inequality, we get

$$\Big(\int_\Omega u u_t dx\Big)^2 \leq \|u(t)\|^2 \|u_t(t)\|^2. \tag{4.12}$$

This inequality (4.12) guarantees $\Upsilon(t) \geq 0$. By (4.10), we have

$$\Psi(t)\Psi''(t) - \frac{p+2}{4}\Psi'(t)^2 \geq \Psi(t)\Pi(t), \tag{4.13}$$

where

$$\Pi(t) = -p\|u_t\|^2 + 2\int_\Omega |u|^p \ln|u|dx - 2\|\nabla u\|_p^p. \tag{4.14}$$

From (2.11) and (4.14), we obtain

$$\Pi(t) = -2p\mathcal{E}(t) + \frac{2}{p}\|u\|_p^p. \tag{4.15}$$

By (4.4), (4.15) and $\mathcal{E}(t) = \mathcal{E}(0) < d$, we get

$$\Pi(t) \geq -2p\mathcal{E}(0) + 2pd = 2p[d - \mathcal{E}(0)] > 0. \tag{4.16}$$

Therefore, there exists $\beta > 0$ such that

$$\Pi(t) \geq \beta > 0. \tag{4.17}$$

Combining (4.6), (4.13) and (4.17), we conclude that

$$\Psi(t)\Psi''(t) - \frac{p+2}{4}\Psi'(t)^2 \geq \alpha\beta > 0,\ \forall t \geq 0. \tag{4.18}$$

Let $\rho = \frac{p-2}{4} > 0$, then, by the differential inequality (4.18) and Lemma 4.1, there is a time T_* such that

$$0 < T_* < \frac{4\Psi(0)}{(p-2)\Psi'(0)}, \tag{4.19}$$

and

$$\lim_{t\to T_*^-} \Psi(t) = +\infty. \tag{4.20}$$

From (4.5) and (4.20), we have $\lim\limits_{t\to T_*^-} \|u(t)\|^2 = +\infty$.

Thus, The proof of Theorem 4.1 has been completed.

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