The intrinsic phenomena of concentration and cavitation on the Riemann solutions for the perturbed macroscopic production model

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Abstract

The exact solutions of the Riemann problems for the two different perturbed macroscopic production models are considered and constructed respectively for all the possible cases. It is found that the asymptotic limits of solutions to the Riemann problem for the first kind of perturbed macroscopic production model do not coverage to those of the pressureless gas dynamics model, because the delta shock wave in the limiting situation has different propagation speed and strength from those for the pressureless gas dynamics model. In order to remedy it, the second kind of perturbed macroscopic production model is proposed, whose asymptotic limits of Riemann solutions are identical with those of the pressureless gas dynamics model.

Keywords: macroscopic production model; pressureless gas dynamics model; Riemann problem; delta shock wave; vacuum state.
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1. Introduction

The fluid-like continuous models [1, 2] haven been extensively used to investigate the high-volume product flows. At the beginning stage, the model consisting of scalar conservation law was proposed [3] by choosing some suitable assumptions on the flux function. It is remarkable that the flux function is known as a clearing function for product flow, which enables us to draw the average of sampled data. Recently, a large amount of works have mainly focused on the so-called second-order model consisting of two conservation laws in order to cope with the data diffusion. In particular, the date-fitted second-order macroscopic production model has been recently introduced in [4], which allows us to predict the production behavior and capture the possible spread based on the observed production data. In the first task of the present paper, we are concerned with the following perturbed macroscopic production model in conservative form

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u + \varepsilon\rho^2 u)_t + (\rho u^2 + \varepsilon\rho^2 u^2)_x = 0, \end{cases} \tag{1.1}$$

where ρ and u represent the density of product and the velocity of product line respectively, which show the work-in-progress at the time t and the production stage x ($x \in [0, 1]$). In addition, here x is the degree of completion or the stage of production instead of a physical position and the parameter ε is taken to be a sufficiently small positive number. For the purpose of explicit study, we shall focus on the Riemann problem for the model (1.1) by considering the Riemann-type initial data

$$(\rho, u)(x, 0) = \begin{cases} (\rho_-, u_-), & x < 0, \\ (\rho_+, u_+), & x > 0. \end{cases} \tag{1.2}$$

It is evident that the Riemann initial data (1.2) should satisfy $\rho_\pm \geq 0$ and $u_\pm \geq 0$ as a result of the expected behavior of a product line.

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If the limit $\varepsilon \to 0$ is taken, then the model (1.1) is formally changed into the following pressureless gas dynamics model

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = 0, \end{cases} \tag{1.3}$$

which has been extensively investigated such as in [5, 6, 7, 8]. It is of great interest to notice that the system (1.3) has also been used in [1, 2] to investigate the product flow of queuing networks and supply chains, in which it was valid for slowly varying influxes and was derived from the closure assumption that parts overtaking each other was a rare event. It is evident that the model (1.3) is a non-strictly hyperbolic and completely linearly degenerate system, whose Riemann solution consists of either a delta shock wave when $u_+ < u_-$ or the combination of two contact discontinuities with the vacuum state between them when $u_+ > u_-$. For example, one may refer to [9, 10, 11, 12, 13, 14] about the related concepts of delta shock wave. In fact, the formation of delta shock wave as well as vacuum state in the Riemann solutions of (1.3) and (1.2) has been extensively investigated by virtue of the vanishing pressure method [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26] and the flux function limit method [27, 28, 29, 30, 31]. However, it is of great interest to discover that although the limit $\varepsilon \to 0$ of Riemann solution of (1.1)-(1.2) is still a delta shock wave for the case $u_+ < u_-$, but the propagation speed and strength of delta shock wave in the limiting $\varepsilon \to 0$ situation are obviously different from those of of delta shock wave solution of the Riemann problem (1.3) and (1.2). That is to say, the Riemann solutions of (1.1) do not coverage to those of (1.3) under the same Riemann initial condition (1.2) when the limit $\varepsilon \to 0$ is taken.

In order to remedy it, we introduce the following perturbed macroscopic production model

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u + \varepsilon \rho^2 u)_t + (\rho u^2 + \frac{3}{2}\varepsilon \rho^2 u^2)_x = 0, \end{cases} \tag{1.4}$$

for the purpose of explicit calculations as far as possible. It is worthwhile to notice that the transient clearing functions incorporating the different dynamic effects have also been taken into account, for example [2]. In other words, different transient clearing functions can be used to describe the different transient behaviors of product flows in the study of production process in the factory. It is interesting to find that the formation of delta shock wave can be captured in the limit $\varepsilon \to 0$ of Riemann solution of (1.4) and (1.2) for the case $u_+ < u_-$ as well as the formation of vacuum state can also be observed in the limit $\varepsilon \to 0$ of Riemann solution of (1.4) and (1.2) for the case $u_+ > u_-$, respectively. Furthermore, from the theory of hyperbolic conservation laws, it can be verified rigorously that the limit $\varepsilon \to 0$ of Riemann solution of (1.4) and (1.2) is indeed identical with the corresponding one of (1.3) and (1.2) when the same Riemann initial condition (1.2) is taken.

Kinetic and fluid model hierarchies for queuing networks and supply chains was first introduced in [2] analogous to the compressible Euler gas dynamics system. For the long production line with many steps, it was useful to deal with the production steps as a continuous variable such that a genuine fluid dynamical description was obtained, in which the factory was regarded as a pipe and the parts flowing through the factory was treated as a fluid. Recently, Armbruster and Wienke [1] have also considered the kinetic models and intrinsic timescales by using the simulation comparison for a second order queueing model. It is worthwhile to notice that Sun [32] has constructed the singular solutions to the Riemann problem for the macroscopic production model (1.1) without the perturbation. Also see [33, 34] about the symmetry analysis of this model (1.1). Hence, it is natural and also necessary to investigate the limits of Riemann solutions for the perturbed macroscopic production model for the actual comparisons of dynamical behaviors among different macroscopic production models.

The outline of the article is as follows. In section 2, we first draw our attention on the construction of Riemann solutions of (1.1) and (1.2) for all the possible cases. Subsequently, we shall focus on the limits $\varepsilon \to 0$ of the solutions to Riemann problem (1.1) and (1.2) under the conditions $u_+ < u_-$ and $u_+ > u_-$ respectively. It is not difficult to investigate the formation of delta shock wave and two-contact-discontinuity solutions, which are different from the corresponding Riemann solutions of (1.1) and (1.2) under the same initial conditions. In section 3, we are interested in the construction of Riemann solutions of (1.4) and (1.2) cases by cases. Furthermore, as the limit $\varepsilon \to 0$ is taken in the Riemann solution of (1.4) and (1.2), the formation of delta shock wave is analyzed when $u_+ < u_-$ as well as the formation of two-contact-discontinuity is derived when $u_+ > u_-$, which are in accordance with the corresponding

Riemann solutions of (1.1) and (1.2) under the same initial conditions. In the end, the conclusions are also drawn in section 4.

2. The limiting $\varepsilon \to 0$ behaviors of Riemann solutions for the perturbed system (1.1)

In this section, we first draw our attention on the constructions of solutions to the Riemann problem (1.1)-(1.2) for all the possible cases. Then, we are interested in the limiting $\varepsilon \to 0$ behaviors of solutions to the Riemann problem (1.1)-(1.2) in detail.

2.1. The Riemann problem for the perturbed system (1.1)

In this subsection, our work is devoted to the properties of elementary waves for the perturbed system (1.1) and then the constructions of solutions to the Riemann problem (1.1)-(1.2). The perturbed system (1.1) is translated into the following form

$$\begin{pmatrix} 1 & 0 \\ u(1+2\varepsilon\rho) & \rho(1+\varepsilon\rho) \end{pmatrix}\begin{pmatrix} \rho \\ u \end{pmatrix}_t + \begin{pmatrix} u & \rho \\ u^2(1+2\varepsilon\rho) & 2\rho u(1+\varepsilon\rho) \end{pmatrix}\begin{pmatrix} \rho \\ u \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.1}$$

so that we have the characteristic equation as follows:

$$\begin{vmatrix} \lambda - u & -\rho \\ u(\lambda-u)(1+2\varepsilon\rho) & \rho(\lambda-2u)(1+\varepsilon\rho) \end{vmatrix} = 0.$$

By means of a simple computation, two real and distinct eigenvalues are taken as

$$\lambda_1(\rho,u) = \frac{u}{1+\varepsilon\rho}, \qquad \lambda_2(\rho,u) = u. \tag{2.2}$$

It can be easily seen that $\lambda_1 < \lambda_2$ holds when $\varepsilon > 0$, $u > 0$ and $\rho > 0$, which means that the perturbed system (1.1) is strictly hyperbolic in the interior of the quarter (ρ, u) phase plane. The right-eigenvectors for (1.1) corresponding to λ_1 and λ_2 are

$$\overrightarrow{r_1} = (1+\varepsilon\rho, -\varepsilon u)^T, \qquad \overrightarrow{r_2} = (1,0)^T. \tag{2.3}$$

We introduce the notation $\nabla = (\frac{\partial}{\partial\rho}, \frac{\partial}{\partial u})$, a direct computation shows that

$$\begin{cases} \nabla\lambda_1 \cdot \overrightarrow{r_1} = \Big(\dfrac{-\varepsilon u}{(1+\varepsilon\rho)^2}, \dfrac{1}{1+\varepsilon\rho}\Big)\cdot(1+\varepsilon\rho, -\varepsilon u)^T = -\dfrac{2\varepsilon u}{1+\varepsilon\rho} \neq 0, \\ \nabla\lambda_2 \cdot \overrightarrow{r_2} = (0,1)\cdot(1,0)^T = 0. \end{cases}$$

Hence, the above computational results demonstrate that the λ_1 characteristic family is genuinely nonlinear and the λ_2 characteristic family is linear degenerate when $\varepsilon > 0$ and $\rho > 0$. To be more exact, the elementary wave with regard to the λ_1 characteristic family is either shock wave or rarefaction wave and the elementary wave related to the λ_2 characteristic family is contact discontinuity.

We start by considering the rarefaction curve. As usual, we look for the self-similar solutions of the form

$$(\rho,u)(x,t) = (\rho,u)(\xi), \qquad \xi = \frac{x}{t}. \tag{2.4}$$

Conventionally, the Riemann problem (1.1)-(1.2) can be formally rewritten as the following boundary value problem of ODEs

$$\begin{cases} -\xi\rho_\xi + (\rho u)_\xi = 0, \\ -\xi(\rho u + \varepsilon\rho^2 u)_\xi + (\rho u^2 + \varepsilon\rho^2 u^2)_\xi = 0. \end{cases} \tag{2.5}$$

For smooth solutions, the equations (2.5) are equivalent to

$$\begin{pmatrix} u-\xi & \rho \\ u(u-\xi)(1+2\varepsilon\rho) & \rho(2u-\xi)(1+\varepsilon\rho) \end{pmatrix}\begin{pmatrix} \rho \\ u \end{pmatrix}_\xi = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.6}$$

It is evident that the trivial solution (ρ, u) is a constant state, which exists if and only if $(d\rho, du) = (0, 0)$. If $(d\rho, du) \neq (0, 0)$, one gets immediately the singular solution that is rarefaction curve in light of a simple calculation. That is to say, given a left state (ρ_-, u_-), it suffices to observe that the rarefaction curve can be expressed by

$$R(\rho_-, u_-): \quad \xi = \lambda_1(\rho, u) = \frac{u}{1+\varepsilon\rho}, \qquad (1+\varepsilon\rho)u = (1+\varepsilon\rho_-)u_-, \qquad \rho < \rho_-, \quad u > u_-. \tag{2.7}$$

Moreover, a straightforward computation yields that $\dfrac{d\rho}{du} = -\dfrac{(1+\varepsilon\rho_-)u_-}{\varepsilon u^2} < 0$, which means u decreases as ρ increases for the curve $R(\rho_-, u_-)$. It is sufficient to compute that the rarefaction curve $R(\rho_-, u_-)$ intersects the positive u-axis at the point $(0, u_-(1+\varepsilon\rho_-))$ in the quarter (ρ, u) phase plane.

In what follows, we want to discuss the shock curve. The following Rankine-Hugoniot conditions

$$\begin{cases} \sigma[\rho] = [\rho u], \\ \sigma[\rho u + \varepsilon\rho^2 u] = [\rho u^2 + \varepsilon\rho^2 u^2], \end{cases} \tag{2.8}$$

can be established, in which $[\rho] = \rho - \rho_-$ and $\sigma = \frac{dx}{dt}$ represent the jump across the discontinuity and the speed of the discontinuity, respectively. If $\sigma = 0$, then we can infer immediately that $(\rho, u) = (\rho_-, u_-)$. Otherwise, if $\sigma \neq 0$, then eliminating σ in (2.8) and simplifying the result lead to

$$\rho\rho_-(u - u_-)\{(u - u_-) + \varepsilon(\rho u - \rho_- u_-)\} = 0. \tag{2.9}$$

In a word, given a left state (ρ_-, u_-), the shock wave and the contact discontinuity can be written respectively as

$$S(\rho_-, u_-): \quad \sigma = \frac{\rho u - \rho_- u_-}{\rho - \rho_-}, \quad (1+\varepsilon\rho)u = (1+\varepsilon\rho_-)u_-, \quad \rho > \rho_-, \quad u < u_-, \tag{2.10}$$

$$J(\rho_-, u_-): \quad \tau = u = u_-. \tag{2.11}$$

It is obvious that the shock curve $S(\rho_-, u_-)$ has the positive ρ-axis as its asymptote. Furthermore, the monotonicity of the shock curve $S(\rho_-, u_-)$ is the same as that of the rarefaction curve $R(\rho_-, u_-)$.

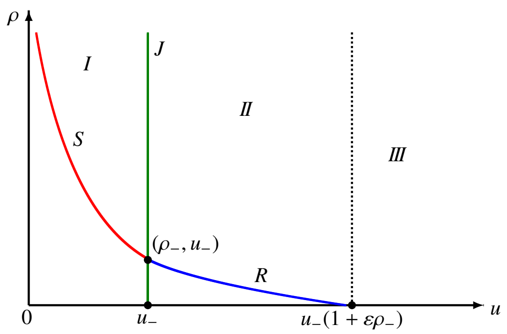


Fig.1 The elementary wave curves starting from the given left state (ρ_-, u_-) are demonstrated in the quarter (ρ, u) phase plane for the Riemann problem (1.1)-(1.2).

As shown in Fig.1, according to the formulae (2.7), (2.10) and (2.11), the shock curve $S(\rho_-, u_-)$, the rarefaction curve $R(\rho_-, u_-)$ and the line of contact discontinuity $J(\rho_-, u_-)$ divide the quarter phase plane into three parts, namely I, II and III. More specifically, for a fixed left state (ρ_-, u_-), the solutions of Riemann problem (1.1)-(1.2) are expressed as $S + J$, $R + J$ and $R + Vac + J$ if and only if the varying right state (ρ_+, u_+) is situated in I, II and III respectively. It is noteworthy that $S + J$ denotes a shock wave S followed by a contact discontinuity J, etc.

2.2. The limiting behavior of Riemann solutions for (1.1) as $\varepsilon \to 0$

As shown in Fig.1, the solution of the Riemann problem (1.1)-(1.2) consists of a 1-shock wave and a 2-contact discontinuity when $u_- > u_+$ and ε is sufficiently small. In this section, we are going to obtain the limit $\varepsilon \to 0$ of above solution for the situation $u_- > u_+$. Let us use (ρ_*, u_*) to denote the intermediate state between the 1-shock wave S_1 and the 2-contact discontinuity J_2 (see Fig.2(a)). It is obvious to show that (ρ_-, u_-) and (ρ_*, u_*) are jointed by the 1-shock wave S_1 with the speed σ_1, and then (ρ_*, u_*) and (ρ_+, u_+) are jointed by the 2-contact discontinuity J_2 with the speed τ_2. More exactly, we have

$$S_1: \quad \sigma_1 = \frac{\rho_* u_* - \rho_- u_-}{\rho_* - \rho_-}, \quad (1+\varepsilon\rho_*)u_* = (1+\varepsilon\rho_-)u_-, \quad \rho_* > \rho_-, \quad u_* < u_-, \tag{2.12}$$

$$J_2: \quad \tau_2 = u_+ = u_*. \tag{2.13}$$

By virtue of (2.12) and (2.13), we immediately get

$$(\rho_*, u_*) = \Big(\frac{u_-(1+\varepsilon\rho_-) - u_+}{\varepsilon u_+}, u_+\Big). \tag{2.14}$$

In what follows, we briefly give some conclusions which are related to the limiting $\varepsilon \to 0$ behavior of Riemann solution for the case $u_- > u_+$. It is clear from (2.14) that

$$\lim_{\varepsilon\to 0} \rho_* = \lim_{\varepsilon\to 0} \frac{u_-(1+\varepsilon\rho_-) - u_+}{\varepsilon u_+} = \lim_{\varepsilon\to 0} \frac{u_- - u_+}{\varepsilon u_+} = +\infty. \tag{2.15}$$

Furthermore, it can be deduced from (2.12) that

$$\lim_{\varepsilon\to 0} \sigma_1 = \lim_{\varepsilon\to 0} \frac{\rho_* u_* - \rho_- u_-}{\rho_* - \rho_-} = \lim_{\varepsilon\to 0} \Big(u_* + \frac{\rho_-(u_* - u_-)}{\rho_* - \rho_-}\Big) = u_+ = \tau_2, \tag{2.16}$$

which implies that the 1-shock wave S_1 and the 2-contact discontinuity J_2 coincide with each other on the line $x = u_+ t$ in the limiting $\varepsilon \to 0$ situation (see Fig.2(b)). Let us use the following lemma to describe the formation of singularity on the line $x = u_+ t$ as $\varepsilon \to 0$.

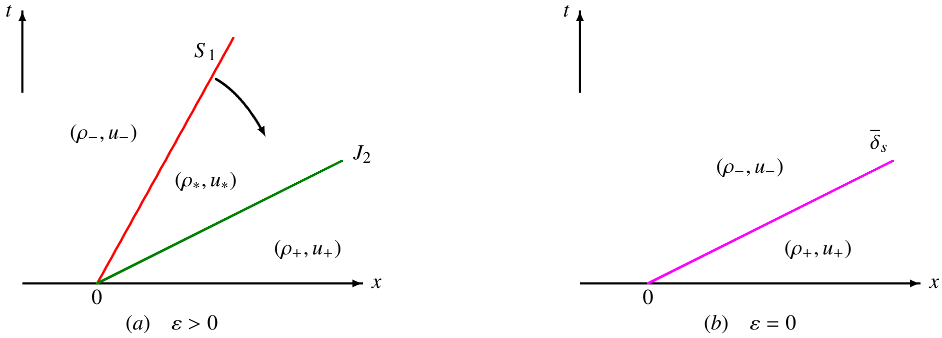


Fig.2 The perturbed solution and the limiting situation are demonstrated for the Riemann problem (1.1)-(1.2) for the case $u_- > u_+$.

Lemma 2.1. *Let us denote $\overline{\sigma}_\delta = u_+$, then the following limiting relations can be established as*

$$\lim_{\varepsilon\to 0} (\tau_2 - \sigma_1)\rho_* = \rho_-(u_- - u_+) = \overline{\sigma}_\delta[\rho] - [\rho u], \tag{2.17}$$

$$\lim_{\varepsilon\to 0} (\tau_2 - \sigma_1)(\rho_* u_* + \varepsilon\rho_*^2 u_*) = \rho_- u_-(u_- - u_+) = \overline{\sigma}_\delta[\rho u] - [\rho u^2], \tag{2.18}$$

in which $[\rho] = \rho_+ - \rho_-$ denotes the jump of ρ.

Proof. Owing to the fact $\lim_{\varepsilon\to 0} \rho_* = +\infty$, it can be derived from (2.12) and (2.13) that

$$\begin{aligned} \lim_{\varepsilon\to 0} (\tau_2 - \sigma_1)\rho_* &= \lim_{\varepsilon\to 0} \Big(u_+ - \frac{\rho_* u_* - \rho_- u_-}{\rho_* - \rho_-}\Big)\rho_* = \lim_{\varepsilon\to 0} \Big(u_+ - u_* - \frac{\rho_-(u_* - u_-)}{\rho_* - \rho_-}\Big)\rho_* \\ &= \lim_{\varepsilon\to 0} \frac{\rho_*\rho_-(u_- - u_+)}{\rho_* - \rho_-} = \rho_-(u_- - u_+) = \overline{\sigma}_\delta[\rho] - [\rho u]. \end{aligned} \tag{2.19}$$

Subsequently, the second equation of (2.8) also yields

$$\begin{cases} \sigma_1(\rho_* u_* + \varepsilon\rho_*^2 u_* - \rho_- u_- - \varepsilon\rho_-^2 u_-) = \rho_* u_*^2 + \varepsilon\rho_*^2 u_*^2 - \rho_- u_-^2 - \varepsilon\rho_-^2 u_-^2, \\ \tau_2(\rho_+ u_+ + \varepsilon\rho_+^2 u_+ - \rho_* u_* - \varepsilon\rho_*^2 u_*) = \rho_+ u_+^2 + \varepsilon\rho_+^2 u_+^2 - \rho_* u_*^2 - \varepsilon\rho_*^2 u_*^2, \end{cases}$$

which enables us to get

$$\begin{aligned} \lim_{\varepsilon\to 0}(\tau_2 - \sigma_1)(\rho_* u_* + \varepsilon\rho_*^2 u_*) &= \lim_{\varepsilon\to 0}(-\sigma_1\rho_- u_- + \tau_2\rho_+ u_+ - \sigma_1\varepsilon\rho_-^2 u_- + \tau_2\varepsilon\rho_+^2 u_+ \\ &\quad + \rho_- u_-^2 + \varepsilon\rho_-^2 u_-^2 - \rho_+ u_+^2 - \varepsilon\rho_+^2 u_+^2) \\ &= -\overline{\sigma}_\delta\rho_- u_- + \overline{\sigma}_\delta\rho_+ u_+ + \rho_- u_-^2 - \rho_+ u_+^2 = \overline{\sigma}_\delta[\rho u] - [\rho u^2]. \end{aligned} \tag{2.20}$$

In addition, in term of (2.14), one also has $\lim_{\varepsilon\to 0}\varepsilon\rho_* = \frac{u_- - u_+}{u_+}$. Which, together with (2.20), yields

$$\begin{aligned} \lim_{\varepsilon\to 0}(\tau_2 - \sigma_1)\rho_* u_* &= \lim_{\varepsilon\to 0}\frac{(\tau_2 - \sigma_1)\rho_* u_*(1 + \varepsilon\rho_*)}{1 + \varepsilon\rho_*} = \frac{\lim_{\varepsilon\to 0}(\tau_2 - \sigma_1)(\rho_* u_* + \varepsilon\rho_*^2 u_*)}{\lim_{\varepsilon\to 0}(1 + \varepsilon\rho_*)} \\ &= \frac{\overline{\sigma}_\delta[\rho u] - [\rho u^2]}{1 + \frac{u_- - u_+}{u_+}} = \rho_- u_+(u_- - u_+), \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} \lim_{\varepsilon\to 0}(\tau_2 - \sigma_1)\varepsilon\rho_*^2 u_* &= \lim_{\varepsilon\to 0}(\tau_2 - \sigma_1)\rho_* u_* \cdot \varepsilon\rho_* = \lim_{\varepsilon\to 0}(\tau_2 - \sigma_1)\rho_* u_* \cdot \lim_{\varepsilon\to 0}\varepsilon\rho_* \\ &= \rho_- u_+(u_- - u_+) \cdot \frac{u_- - u_+}{u_+} = \rho_-(u_- - u_+)^2. \end{aligned} \tag{2.22}$$

The proof is finished. □

As a result of (2.17), the strength of delta shock wave obtained from the limit $\varepsilon \to 0$ of Riemann solution of (1.1) and (1.2) can be calculated by

$$\overline{\beta}_\delta(t) = \lim_{\varepsilon\to 0}\int_{\sigma_1 t}^{\tau_2 t}\rho_* dx = \rho_-(u_- - u_+)t. \tag{2.23}$$

In brief, for the case $u_+ < u_-$, the delta shock wave solution obtained from the limit $\varepsilon \to 0$ of Riemann solution of (1.1) and (1.2) can be summarized as

$$(\rho, u)(x,t) = \begin{cases} (\rho_-, u_-), & x < u_+ t, \\ (\overline{\beta}_\delta(t)\delta(x - u_+ t), u_+), & x = u_+ t, \\ (\rho_+, u_+), & x > u_+ t, \end{cases} \tag{2.24}$$

which is obviously different from the delta shock wave solution for the Riemann problem (1.3) and (1.2) in the from [5]

$$(\rho, u)(x,t) = \begin{cases} (\rho_-, u_-), & x < \sigma_\delta t, \\ (\beta(t)\delta(x - \sigma_\delta t), u_\delta(t)), & x = \sigma_\delta t, \\ (\rho_+, u_+), & x > \sigma_\delta t, \end{cases} \tag{2.25}$$

in which

$$\sigma_\delta = u_\delta(t) = \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}}, \qquad \beta(t) = \sqrt{\rho_-\rho_+}(u_- - u_+)t. \tag{2.26}$$

It is evident to see that the propagation speed and strength of delta shock wave are different from the limits $\varepsilon \to 0$ of the perturbed system (1.1) here and the perturbed gas dynamical system in [15]. In fact, it is not surprising to see that different approximations yield the different over-compressive entropy conditions $u_+ = \overline{\sigma}_\delta < u_-$ here and $u_+ < \sigma_\delta < u_-$ in [15], which leads to the different propagation speeds and strengths of delta shock waves respectively. Here, it can be still verified rigorously that the constructed delta shock wave solution (2.24) also satisfies the system (1.3) in the weak sense of distributions.

Theorem 2.2. *Let $u_+ < u_-$ and ε be sufficiently small, then the Riemann problem (1.1)-(1.2) has a solution consisting of a 1-shock wave and a 2-contact discontinuity. When $\varepsilon \to 0$, the solution converges to the delta shock wave solution (2.24) in the sense of distributions. Besides, it is shown that*

$$\lim_{\varepsilon\to 0}\rho = \rho_- + [\rho]H(x-\overline{\sigma}_\delta t) + t(\overline{\sigma}_\delta[\rho]-[\rho u])\delta(x-\overline{\sigma}_\delta t), \tag{2.27}$$

$$\lim_{\varepsilon\to 0}(\rho u+\varepsilon\rho^2 u) = \rho_- u_- + [\rho u]H(x-\overline{\sigma}_\delta t) + t(\overline{\sigma}_\delta[\rho u]-[\rho u^2])\delta(x-\overline{\sigma}_\delta t). \tag{2.28}$$

Proof. For ε adequately small under the case $u_+ < u_-$, the Riemann solution consisting of a 1-shock wave and a 2-contact discontinuity gives immediately

$$(\rho,u)(\xi) = \begin{cases} (\rho_-,u_-), & -\infty<\xi<\sigma_1,\\ (\rho_*,u_*), & \sigma_1<\xi<\tau_2,\\ (\rho_+,u_+), & \tau_2<\xi<+\infty, \end{cases} \tag{2.29}$$

in which (ρ_*,u_*) is given by (2.14). The solution (2.29) obeys the following weak forms

$$-\int_{-\infty}^{+\infty}\rho(\xi)(u(\xi)-\xi)\phi'(\xi)d\xi + \int_{-\infty}^{+\infty}\rho(\xi)\phi(\xi)d\xi = 0, \tag{2.30}$$

and

$$-\int_{-\infty}^{+\infty}(\rho(\xi)u(\xi)+\varepsilon\rho^2(\xi)u(\xi))(u(\xi)-\xi)\phi'(\xi)d\xi + \int_{-\infty}^{+\infty}(\rho(\xi)u(\xi)+\varepsilon\rho^2(\xi)u(\xi))\phi(\xi)d\xi = 0, \tag{2.31}$$

for any $\phi(\xi)\in C_0^\infty(-\infty,+\infty)$.

Decomposing the first term in (2.31), it can be seen that

$$\int_{-\infty}^{+\infty}(\rho(\xi)u(\xi)+\varepsilon\rho^2(\xi)u(\xi))(u(\xi)-\xi)\phi'(\xi)d\xi = \Big(\int_{-\infty}^{\sigma_1}+\int_{\sigma_1}^{\tau_2}+\int_{\tau_2}^{+\infty}\Big)(\rho(\xi)u(\xi)+\varepsilon\rho^2(\xi)u(\xi))(u(\xi)-\xi)\phi'(\xi)d\xi. \tag{2.32}$$

Combining the first and last integrals of (2.32), it is sufficient to have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{-\infty}^{\sigma_1}(\rho(\xi)u(\xi)+\varepsilon\rho^2(\xi)u(\xi))(u(\xi)-\xi)\phi'(\xi)d\xi + \lim_{\varepsilon\to 0}\int_{\tau_2}^{+\infty}(\rho(\xi)u(\xi)+\varepsilon\rho^2(\xi)u(\xi))(u(\xi)-\xi)\phi'(\xi)d\xi\\
&= \lim_{\varepsilon\to 0}\int_{-\infty}^{\sigma_1}(\rho_-u_-+\varepsilon\rho_-^2u_-)(u_--\xi)\phi'(\xi)d\xi + \lim_{\varepsilon\to 0}\int_{\tau_2}^{+\infty}(\rho_+u_++\varepsilon\rho_+^2u_+)(u_+-\xi)\phi'(\xi)d\xi\\
&= \lim_{\varepsilon\to 0}\Big((\rho_-u_-^2+\varepsilon\rho_-^2u_-^2)\phi(\sigma_1) - (\rho_-u_-+\varepsilon\rho_-^2u_-)\sigma_1\phi(\sigma_1) - (\rho_+u_+^2+\varepsilon\rho_+^2u_+^2)\phi(\tau_2) + (\rho_+u_++\varepsilon\rho_+^2u_+)\tau_2\phi(\tau_2)\\
&\quad +(\rho_-u_-+\varepsilon\rho_-^2u_-)\int_{-\infty}^{\sigma_1}\phi(\xi)d\xi + (\rho_+u_++\varepsilon\rho_+^2u_+)\int_{\tau_2}^{+\infty}\phi(\xi)d\xi\Big)\\
&= (\overline{\sigma}_\delta[\rho u]-[\rho u^2]\,)\phi(\overline{\sigma}_\delta) + \int_{-\infty}^{+\infty}(\rho_-u_-+[\rho u]H(\xi-\overline{\sigma}_\delta))\phi(\xi)d\xi.
\end{aligned} \tag{2.33}$$

Similarly, due to Lemma 2.1, the second part of (2.32) yields

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\sigma_1}^{\tau_2}(\rho(\xi)u(\xi)+\varepsilon\rho^2(\xi)u(\xi))(u(\xi)-\xi)\phi'(\xi)d\xi\\
&= \lim_{\varepsilon\to 0}\int_{\sigma_1}^{\tau_2}(\rho_*u_*+\varepsilon\rho_*^2u_*)(u_*-\xi)\phi'(\xi)d\xi\\
&= \lim_{\varepsilon\to 0}(\rho_*u_*+\varepsilon\rho_*^2u_*)(\tau_2-\sigma_1)\Big(u_*\Big(\frac{\phi(\tau_2)-\phi(\sigma_1)}{\tau_2-\sigma_1}\Big)-\Big(\frac{\tau_2\phi(\tau_2)-\sigma_1\phi(\sigma_1)}{\tau_2-\sigma_1}\Big)+\frac{\int_{\sigma_1}^{\tau_2}\phi(\xi)d\xi}{\tau_2-\sigma_1}\Big)\\
&= (\overline{\sigma}_\delta[\rho u]-[\rho u^2]\,)(\overline{\sigma}_\delta\phi'(\overline{\sigma}_\delta)-\overline{\sigma}_\delta\phi'(\overline{\sigma}_\delta)-\phi(\overline{\sigma}_\delta)+\phi(\overline{\sigma}_\delta)) = 0,
\end{aligned} \tag{2.34}$$

where we take into account the facts that $\phi(\xi) \in C_0^\infty(-\infty, +\infty)$ and $\lim\limits_{\varepsilon\to 0} \sigma_1 = \lim\limits_{\varepsilon\to 0} \tau_2 = \lim\limits_{\varepsilon\to 0} u_* = u_+$.

Substituting (2.33), (2.34) into (2.31) indicates that

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{+\infty} ((\rho(\xi)u(\xi) + \varepsilon\rho^2(\xi)u(\xi)) - (\rho_-u_- + [\rho u]H(\xi - \overline{\sigma}_\delta)))\phi(\xi)d\xi = (\overline{\sigma}_\delta[\rho u] - [\rho u^2])\phi(\overline{\sigma}_\delta). \tag{2.35}$$

More precisely, according to formulae (2.21) and (2.22), we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_{-\infty}^{+\infty} \rho(\xi)u(\xi)\phi(\xi)d\xi = \lim_{\varepsilon\to 0} \Big(\int_{-\infty}^{\sigma_1} + \int_{\sigma_1}^{\tau_2} + \int_{\tau_2}^{+\infty} \Big)\rho(\xi)u(\xi)\phi(\xi)d\xi \\
&= \lim_{\varepsilon\to 0} \int_{-\infty}^{\sigma_1} \rho_-u_-\phi(\xi)d\xi + \lim_{\varepsilon\to 0} \int_{\tau_2}^{+\infty} \rho_+u_+\phi(\xi)d\xi + \lim_{\varepsilon\to 0} \int_{\sigma_1}^{\tau_2} \rho_*u_*\phi(\xi)d\xi \\
&= \lim_{\varepsilon\to 0} \Big(\rho_-u_- \int_{-\infty}^{\sigma_1} \phi(\xi)d\xi + \rho_+u_+ \int_{\tau_2}^{+\infty} \phi(\xi)d\xi\Big) + \lim_{\varepsilon\to 0} \rho_*u_*(\tau_2 - \sigma_1)\frac{\int_{\sigma_1}^{\tau_2} \phi(\xi)d\xi}{\tau_2 - \sigma_1} \\
&= \int_{-\infty}^{+\infty} (\rho_-u_- + [\rho u]H(\xi - \overline{\sigma}_\delta))\phi(\xi)d\xi + \rho_-u_+(u_- - u_+)\phi(\overline{\sigma}_\delta),
\end{aligned} \tag{2.36}$$

and

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_{-\infty}^{+\infty} \varepsilon\rho^2(\xi)u(\xi)\phi(\xi)d\xi = \lim_{\varepsilon\to 0} \Big(\int_{-\infty}^{\sigma_1} + \int_{\sigma_1}^{\tau_2} + \int_{\tau_2}^{+\infty} \Big)\varepsilon\rho^2(\xi)u(\xi)\phi(\xi)d\xi \\
&= \lim_{\varepsilon\to 0} \int_{-\infty}^{\sigma_1} \varepsilon\rho_-^2u_-\phi(\xi)d\xi + \lim_{\varepsilon\to 0} \int_{\tau_2}^{+\infty} \varepsilon\rho_+^2u_+\phi(\xi)d\xi + \lim_{\varepsilon\to 0} \int_{\sigma_1}^{\tau_2} \varepsilon\rho_*^2u_*\phi(\xi)d\xi \\
&= \lim_{\varepsilon\to 0} \varepsilon\rho_*^2u_*(\tau_2 - \sigma_1)\frac{\int_{\sigma_1}^{\tau_2} \phi(\xi)d\xi}{\tau_2 - \sigma_1} = \rho_-(u_- - u_+)^2\phi(\overline{\sigma}_\delta).
\end{aligned} \tag{2.37}$$

Furthermore, using the same method on (2.30) leads to

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{+\infty} (\rho(\xi) - (\rho_- + [\rho]H(\xi - \overline{\sigma}_\delta)))\phi(\xi)d\xi = (\overline{\sigma}_\delta[\rho] - [\rho u])\ \phi(\overline{\sigma}_\delta). \tag{2.38}$$

Our goal is to get the limits of ρ and $\rho u + \varepsilon\rho^2 u$ as $\varepsilon \to 0$. Let $\psi(x, t) \in C_0^\infty(R \times R_+)$, then (2.35) gives rise to

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_0^{+\infty} \int_{-\infty}^{+\infty} (\rho u + \varepsilon\rho^2 u)\Big(\frac{x}{t}\Big)\psi(x, t)dxdt \\
&= \lim_{\varepsilon\to 0} \int_0^{+\infty} t \Big(\int_{-\infty}^{+\infty} (\rho(\xi)u(\xi) + \varepsilon\rho^2(\xi)u(\xi))\psi(\xi t, t)d\xi\Big)dt \\
&= \int_0^{+\infty} t \Big(\lim_{\varepsilon\to 0} \int_{-\infty}^{+\infty} (\rho(\xi)u(\xi) + \varepsilon\rho^2(\xi)u(\xi))\psi(\xi t, t)d\xi\Big)dt \\
&= \int_0^{+\infty} t \Big(\int_{-\infty}^{+\infty} (\rho_-u_- + [\rho u]H(\xi - \overline{\sigma}_\delta))\psi(\xi t, t)d\xi + (\overline{\sigma}_\delta[\rho u] - [\rho u^2])\ \psi(\overline{\sigma}_\delta t, t)\Big)dt \\
&= \int_0^{+\infty} t \Big(t^{-1} \int_{-\infty}^{+\infty} (\rho_-u_- + [\rho u]H(\xi - \overline{\sigma}_\delta))\psi(x, t)dx + (\overline{\sigma}_\delta[\rho u] - [\rho u^2])\ \psi(\overline{\sigma}_\delta t, t)\Big)dt \\
&= \int_0^{+\infty} \int_{-\infty}^{+\infty} (\rho_-u_- + [\rho u]H(\xi - \overline{\sigma}_\delta))\psi(x, t)dxdt + \int_0^{+\infty} t(\overline{\sigma}_\delta[\rho u] - [\rho u^2])\ \psi(\overline{\sigma}_\delta t, t)dt,
\end{aligned}$$

which allows us to establish the limiting relation (2.28). In terms of (2.36) and (2.37), one further has

$$\lim_{\varepsilon\to 0}\int_0^{+\infty}\int_{-\infty}^{+\infty}\rho u\Big(\frac{x}{t}\Big)\psi(x,t)dxdt$$
$$=\int_0^{+\infty}\int_{-\infty}^{+\infty}(\rho_-u_-+[\rho u]H(\xi-\overline{\sigma}_\delta))\psi(x,t)dxdt+\int_0^{+\infty}t(\rho_-u_+(u_--u_+))\ \psi(\overline{\sigma}_\delta t,t)dt,$$

and

$$\lim_{\varepsilon\to 0}\int_0^{+\infty}\int_{-\infty}^{+\infty}\varepsilon\rho^2u\Big(\frac{x}{t}\Big)\psi(x,t)dxdt=\int_0^{+\infty}t(\rho_-(u_--u_+)^2)\ \psi(\overline{\sigma}_\delta t,t)dt.$$

In addition, by virtue of (2.38), one also has

$$\lim_{\varepsilon\to 0}\int_0^{+\infty}\int_{-\infty}^{+\infty}\rho\Big(\frac{x}{t}\Big)\psi(x,t)dxdt=\int_0^{+\infty}\int_{-\infty}^{+\infty}(\rho_-+[\rho]H(\xi-\overline{\sigma}_\delta))\psi(x,t)dxdt+\int_0^{+\infty}t(\overline{\sigma}_\delta[\rho]-[\rho u])\ \psi(\overline{\sigma}_\delta t,t)dt,$$

which leads to the limiting relation (2.27). The proof is finished. □

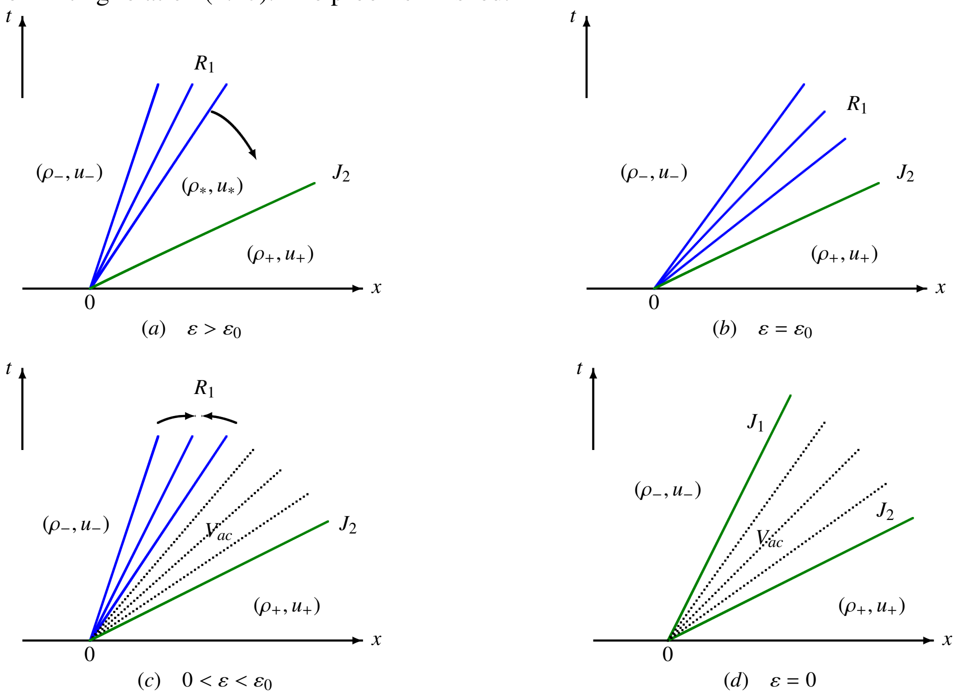


Fig.3 The perturbed solution and the limit solution are demonstrated for the Riemann problem (1.1)-(1.2) for the case $u_+>u_-$.

In what follows, we shall focus on the limiting $\varepsilon\to 0$ behavior of Riemann solution of (1.1) and (1.2) when $u_+>u_-$. Let us introduce the critical value $\varepsilon_0=\dfrac{u_+-u_-}{\rho_-u_-}$ based on $u_-(1+\varepsilon\rho_-)=u_+$. If $\varepsilon>\varepsilon_0$, then it is easy to obtain that the Riemann solution of (1.1) and (1.2) includes a 1-rarefaction wave R_1 and a 2-contact discontinuity J_2 with non-vacuum intermediate state (ρ_*,u_*) in the form (see Fig.3(a))

$$(\rho,u)(\xi)=\begin{cases}(\rho_-,u_-), & -\infty<\xi<\dfrac{u_-}{1+\varepsilon\rho_-},\\ R_1, & \dfrac{u_-}{1+\varepsilon\rho_-}\le\xi\le\dfrac{u_*}{1+\varepsilon\rho_*},\\ (\rho_*,u_*), & \dfrac{u_*}{1+\varepsilon\rho_*}\le\xi\le u_+,\\ (\rho_+,u_+), & u_+<\xi<+\infty,\end{cases}\tag{2.39}$$

in which

$$(\rho_*, u_*) = (\frac{u_-(1+\varepsilon\rho_-) - u_+}{\varepsilon u_+}, u_+).$$

It is worthy mentioned that a composite wave RJ is formed at the critical value $\varepsilon = \varepsilon_0$, in which the wave front of the 1-rarefaction wave R_1 coincides with the 2-contact discontinuity J_2 in the (x,t) plane (see Fig.3(b)). Furthermore, if $0 < \varepsilon < \varepsilon_0$, then the 1-rarefaction curve $R_1(\rho_-, u_-)$ intersects the positive $u-$axis at the point $(0, \widetilde{u}_*)$ in the quarter (ρ, u) phase plane, in which $\widetilde{u}_* = u_-(1+\varepsilon\rho_-)$. As a consequence, it suffices to find that there exists vacuum state in the Riemann solution of (1.1) and (1.2) as follows (see Fig.3(c)):

$$(\rho, u)(\xi) = \begin{cases} (\rho_-, u_-), & -\infty < \xi < \dfrac{u_-}{1+\varepsilon\rho_-}, \\ R_1, & \dfrac{u_-}{1+\varepsilon\rho_-} \leq \xi \leq u_-(1+\varepsilon\rho_-), \\ \text{Vacuum}, & u_-(1+\varepsilon\rho_-) \leq \xi \leq u_+, \\ (\rho_+, u_+), & u_+ < \xi < +\infty. \end{cases} \tag{2.40}$$

In the end, it is noticeable from (2.7) that

$$\lim_{\varepsilon\to 0} \lambda_1(\rho_-, u_-) = \lim_{\varepsilon\to 0} \frac{u_-}{1+\varepsilon\rho_-} = u_-, \qquad \lim_{\varepsilon\to 0} \lambda_1(0, \widetilde{u}_*) = \lim_{\varepsilon\to 0} \widetilde{u}_* = \lim_{\varepsilon\to 0}(1+\varepsilon\rho_-)u_- = u_-, \tag{2.41}$$

which enables us to infer that the 1-rarefaction wave R_1 degenerates to be the 1-contact discontinuity $J_1 : u = u_-$ when ε tends to 0 and the region between two contact discontinuities J_1 and J_2 is filled up with the vacuum state (see Fig.3(d)).

3. The limiting $\varepsilon \to 0$ behaviors of Riemann solutions for the system (1.4)

In this section, we are mainly concerned with the limiting $\varepsilon \to 0$ behaviors of solutions to the Riemann problem (1.4) and (1.2) in detail. More precisely, we take a step further to discuss the formation of delta shock wave as well as the formation of vacuum state when the limits $\varepsilon \to 0$ are taken in the solutions to the Riemann problem (1.4) and (1.2) for all the possible cases.

3.1. The Riemann problem for the system (1.4)

The system (1.4) is converted into the following form

$$\begin{pmatrix} 1 & 0 \\ u(1+2\varepsilon\rho) & \rho(1+\varepsilon\rho) \end{pmatrix} \begin{pmatrix} \rho \\ u \end{pmatrix}_t + \begin{pmatrix} u & \rho \\ u^2(1+3\varepsilon\rho) & \rho u(2+3\varepsilon\rho) \end{pmatrix} \begin{pmatrix} \rho \\ u \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.1}$$

whose characteristic equation is written as

$$\begin{vmatrix} \lambda - u & -\rho \\ \lambda u(1+2\varepsilon\rho) - u^2(1+3\varepsilon\rho) & \lambda\rho(1+\varepsilon\rho) - \rho u(2+3\varepsilon\rho) \end{vmatrix} = 0.$$

One can obtain explicitly two eigenvalues

$$\lambda_1(\rho, u) = u\Big(1 - \sqrt{\frac{\varepsilon\rho}{1+\varepsilon\rho}}\Big), \qquad \lambda_2(\rho, u) = u\Big(1 + \sqrt{\frac{\varepsilon\rho}{1+\varepsilon\rho}}\Big). \tag{3.2}$$

The system (1.4) is strictly hyperbolic as a consequence of $\lambda_1 < \lambda_2$ when $\rho > 0$ and $\varepsilon > 0$ are satisfied. The corresponding right-eigenvectors are given by

$$\overrightarrow{r_1} = \Big(-\rho, u\sqrt{\frac{\varepsilon\rho}{1+\varepsilon\rho}}\Big)^T, \qquad \overrightarrow{r_2} = \Big(\rho, u\sqrt{\frac{\varepsilon\rho}{1+\varepsilon\rho}}\Big)^T. \tag{3.3}$$

Introducing the notation $\nabla = (\frac{\partial}{\partial\rho}, \frac{\partial}{\partial u})$, one has

$$\begin{cases} \nabla\lambda_1 \cdot \overrightarrow{r_1} & = \Big(\dfrac{-\varepsilon u}{2(1+\varepsilon\rho)\sqrt{\varepsilon\rho(1+\varepsilon\rho)}}, 1 - \sqrt{\dfrac{\varepsilon\rho}{1+\varepsilon\rho}}\Big) \cdot \Big(-\rho, u\sqrt{\dfrac{\varepsilon\rho}{1+\varepsilon\rho}}\Big)^T \\ & = u\sqrt{\dfrac{\varepsilon\rho}{1+\varepsilon\rho}}\Big(\dfrac{1}{2(1+\varepsilon\rho)} + 1 - \sqrt{\dfrac{\varepsilon\rho}{1+\varepsilon\rho}}\Big) \neq 0, \\ \nabla\lambda_2 \cdot \overrightarrow{r_2} & = \Big(\dfrac{\varepsilon u}{2(1+\varepsilon\rho)\sqrt{\varepsilon\rho(1+\varepsilon\rho)}}, 1 + \sqrt{\dfrac{\varepsilon\rho}{1+\varepsilon\rho}}\Big) \cdot \Big(\rho, u\sqrt{\dfrac{\varepsilon\rho}{1+\varepsilon\rho}}\Big)^T \\ & = u\sqrt{\dfrac{\varepsilon\rho}{1+\varepsilon\rho}}\Big(\dfrac{1}{2(1+\varepsilon\rho)} + 1 + \sqrt{\dfrac{\varepsilon\rho}{1+\varepsilon\rho}}\Big) \neq 0, \end{cases}$$

provided that $\rho > 0$ and $u > 0$ are required and ε is taken to be a sufficiently small positive number. Indeed, the above-mentioned results inform us that the λ_1 and λ_2 characteristic families are genuinely nonlinear under the above suitable assumptions. More precisely, owing to the above results, it is sufficient to show that the elementary waves are either shock waves or rarefaction waves.

Let us consider the rarefaction curves in the following. Hence, we shall focus on settling the self-similar solutions of the form

$$(\rho, u)(x, t) = (\rho, u)(\xi), \qquad \xi = \frac{x}{t}. \tag{3.4}$$

The Riemann problem (1.4) and (1.2) is equivalent to

$$\begin{cases} -\xi\rho_\xi + (\rho u)_\xi = 0, \\ -\xi(\rho u + \varepsilon\rho^2 u)_\xi + (\rho u^2 + \frac{3}{2}\varepsilon\rho^2 u^2)_\xi = 0. \end{cases} \tag{3.5}$$

From the viewpoint of smooth solutions, (3.5) can be interpreted as

$$\begin{pmatrix} u - \xi & \rho \\ u^2(1+3\varepsilon\rho) - \xi u(1+2\varepsilon\rho) & \rho u(2+3\varepsilon\rho) - \xi\rho(1+\varepsilon\rho) \end{pmatrix} \begin{pmatrix} \rho \\ u \end{pmatrix}_\xi = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.6}$$

In particular, we have the trivial solution that (ρ, u) is a constant state if and only if $(d\rho, du) = (0, 0)$. Otherwise, if $(d\rho, du) \neq (0, 0)$, then one has to prove that the singular solution is rarefaction curve. To be more precise, given a left state (ρ_-, u_-), it is easy to conclude that the rarefaction curves are expressed by

$$R_1(\rho_-, u_-) : \begin{cases} \xi = \lambda_1(\rho, u) = u\Big(1 - \sqrt{\dfrac{\varepsilon\rho}{1+\varepsilon\rho}}\Big), \\ \ln u - \ln u_- = -\displaystyle\int_{\rho_-}^{\rho} \frac{\sqrt{\varepsilon}}{\sqrt{s(1+\varepsilon s)}} ds, \\ \rho < \rho_-, \qquad u > u_-, \end{cases} \tag{3.7}$$

and

$$R_2(\rho_-, u_-) : \begin{cases} \xi = \lambda_2(\rho, u) = u\Big(1 + \sqrt{\dfrac{\varepsilon\rho}{1+\varepsilon\rho}}\Big), \\ \ln u - \ln u_- = \displaystyle\int_{\rho_-}^{\rho} \frac{\sqrt{\varepsilon}}{\sqrt{s(1+\varepsilon s)}} ds, \\ \rho > \rho_-, \qquad u > u_-. \end{cases} \tag{3.8}$$

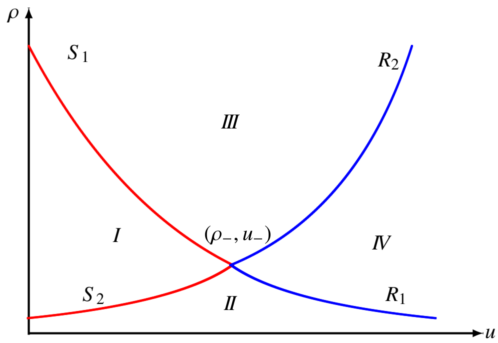


Fig.4 The elementary wave curves are demonstrated in the quarter (ρ, u) phase plane for the Riemann problem (1.4) and (1.2) with the given left state (ρ_-, u_-).

Let us turn to the shock curves, which should satisfy the following Rankine-Hugoniot conditions

$$\begin{cases} \sigma[\rho] = [\rho u], \\ \sigma[\rho u + \varepsilon\rho^2 u] = [\rho u^2 + \dfrac{3}{2}\varepsilon\rho^2 u^2], \end{cases} \tag{3.9}$$

in which $[\rho] = \rho - \rho_-$ and $\sigma = \frac{dx}{dt}$ are similar as those in (2.8). In particular, if $\sigma = 0$, then it is evident that $(\rho, u) = (\rho_-, u_-)$. Otherwise, if $\sigma \neq 0$, then the elimination of σ from (3.9) brings about

$$(\rho u - \rho_- u_-)(\rho u + \varepsilon\rho^2 u - \rho_- u_- - \varepsilon\rho_-^2 u_-) = (\rho - \rho_-)(\rho u^2 + \frac{3}{2}\varepsilon\rho^2 u^2 - \rho_- u_-^2 - \frac{3}{2}\varepsilon\rho_-^2 u_-^2),$$

which enables us to show immediately that

$$\rho\rho_-(\varepsilon\rho + \varepsilon\rho_- + 2)(u - u_-)^2 - (\varepsilon\rho u^2 + \varepsilon\rho_- u_-^2)(\rho - \rho_-)^2 = 0. \tag{3.10}$$

That is to say, given a left state (ρ_-, u_-), the shock curves can be expressed by

$$S_1(\rho_-, u_-): \ \sigma = \frac{\rho u - \rho_- u_-}{\rho - \rho_-}, \qquad u = u_- - \sqrt{\frac{(\varepsilon\rho u^2 + \varepsilon\rho_- u_-^2)(\rho - \rho_-)^2}{\rho\rho_-(\varepsilon\rho + \varepsilon\rho_- + 2)}}, \qquad \rho > \rho_-, \quad u < u_-, \tag{3.11}$$

and

$$S_2(\rho_-, u_-): \ \sigma = \frac{\rho u - \rho_- u_-}{\rho - \rho_-}, \qquad u = u_- - \sqrt{\frac{(\varepsilon\rho u^2 + \varepsilon\rho_- u_-^2)(\rho - \rho_-)^2}{\rho\rho_-(\varepsilon\rho + \varepsilon\rho_- + 2)}}, \qquad \rho < \rho_-, \quad u < u_-. \tag{3.12}$$

It can be shown, from (3.7), (3.8), (3.11) and (3.12), that the shock curves $S_1(\rho_-, u_-)$, $S_2(\rho_-, u_-)$ and the rarefaction curves $R_1(\rho_-, u_-)$, $R_2(\rho_-, u_-)$ divide the quarter (ρ, u) phase plane into four regions, namely I, II, III and IV (see Fig.4). In other words, for a static left state (ρ_-, u_-), the solutions of Riemann problem (1.4) and (1.2) are given by $S_1 + S_2$, $R_1 + S_2$, $S_1 + R_2$ and $R_1 + R_2$ if and only if the right state (ρ_+, u_+) is located in the regions I, II, III and IV respectively, in which the representation of symbol is the same as before.

3.2. Formation of delta shock wave

When $u_- > u_+$, it suffices to get that the solution of Riemman problem (1.4) and (1.2) includes two shock waves (see Fig.5(a)) provided that ε is adequately small. In this subsection, we draw our attention on the limiting $\varepsilon \to 0$ behavior of two-shock-wave solution of Riemman problem (1.4) and (1.2) when the condition $u_- > u_+$ is fulfilled. Let us define (ρ_*, u_*) be the intermediate state between two shock waves, then it can be easily shown that (ρ_-, u_-) and (ρ_*, u_*) are connected by a 1-shock wave S_1 with the speed σ_1 as well as (ρ_*, u_*) and (ρ_+, u_+) are connected by a 2-shock wave S_2 with the speed σ_2 respectively. To be more exact, one arrives at

$$
S_1:\begin{cases}\sigma_1=\dfrac{\rho_*u_*-\rho_-u_-}{\rho_*-\rho_-},\\[2ex] u_*=u_--\sqrt{\dfrac{(\varepsilon\rho_*u_*^2+\varepsilon\rho_-u_-^2)(\rho_*-\rho_-)^2}{\rho_*\rho_-(\varepsilon\rho_*+\varepsilon\rho_-+2)}},\\[2ex] \rho_*>\rho_-,\qquad u_*<u_-,\end{cases}\tag{3.13}
$$

and

$$
S_2:\begin{cases}\sigma_2=\dfrac{\rho_+u_+-\rho_*u_*}{\rho_+-\rho_*},\\[2ex] u_+=u_*-\sqrt{\dfrac{(\varepsilon\rho_+u_+^2+\varepsilon\rho_*u_*^2)(\rho_+-\rho_*)^2}{\rho_+\rho_*(\varepsilon\rho_++\varepsilon\rho_*+2)}},\\[2ex] \rho_+<\rho_*,\qquad u_+<u_*.\end{cases}\tag{3.14}
$$

It follows from the second equations of (3.13) and (3.14) that

$$
u_--u_+=\sqrt{\frac{(\varepsilon\rho_*u_*^2+\varepsilon\rho_-u_-^2)(\rho_*-\rho_-)^2}{\rho_*\rho_-(\varepsilon\rho_*+\varepsilon\rho_-+2)}}+\sqrt{\frac{(\varepsilon\rho_+u_+^2+\varepsilon\rho_*u_*^2)(\rho_+-\rho_*)^2}{\rho_+\rho_*(\varepsilon\rho_++\varepsilon\rho_*+2)}}.\tag{3.15}
$$

Lemma 3.1. *The limiting relations are established as follows:*

$$
\lim_{\varepsilon\to0}\rho_*=+\infty,\qquad \lim_{\varepsilon\to0}\varepsilon\rho_*=0,\qquad \lim_{\varepsilon\to0}\varepsilon\rho_*^2=\frac{2\rho_-\rho_+(u_--u_+)^2}{(u_-\sqrt{\rho_-}+u_+\sqrt{\rho_+})^2},\tag{3.16}
$$

$$
\lim_{\varepsilon\to0}\sigma_1=\lim_{\varepsilon\to0}\sigma_2=\lim_{\varepsilon\to0}u_*=\frac{u_-\sqrt{\rho_-}+u_+\sqrt{\rho_+}}{\sqrt{\rho_-}+\sqrt{\rho_+}}=\sigma_\delta.\tag{3.17}
$$

Proof. Let us first assume that $\lim\limits_{\varepsilon\to0}\rho_*$ is bounded, namely $\lim\limits_{\varepsilon\to0}\rho_*=a\in(max(\rho_-,\rho_+),+\infty)$. Then, taking the limit of (3.15) as $\varepsilon\to0$ yields $u_--u_+=0$, which must violate the fact $u_->u_+$. Therefore, it is easily inferred that $\lim\limits_{\varepsilon\to0}\rho_*=+\infty$. Owing to the fact $\lim\limits_{\varepsilon\to0}\rho_*=+\infty$, one has (see Fig.5(b))

$$
\lim_{\varepsilon\to0}\sigma_1=\lim_{\varepsilon\to0}\frac{\rho_*u_*-\rho_-u_-}{\rho_*-\rho_-}=\lim_{\varepsilon\to0}u_*,\qquad \lim_{\varepsilon\to0}\sigma_2=\lim_{\varepsilon\to0}\frac{\rho_+u_+-\rho_*u_*}{\rho_+-\rho_*}=\lim_{\varepsilon\to0}u_*.
$$

Taking into account $\lim\limits_{\varepsilon\to0}\rho_*=+\infty$, it is easy to arrive at

$$
\lim_{\varepsilon\to0}\sqrt{\frac{(\varepsilon\rho_*u_*^2+\varepsilon\rho_-u_-^2)(\rho_*-\rho_-)^2}{\rho_*\rho_-(\varepsilon\rho_*+\varepsilon\rho_-+2)}}=\frac{1}{\sqrt{\rho_-}}\cdot\lim_{\varepsilon\to0}\sqrt{\frac{\varepsilon\rho_*^2u_*^2}{\varepsilon\rho_*+2}},
$$

$$
\lim_{\varepsilon\to0}\sqrt{\frac{(\varepsilon\rho_+u_+^2+\varepsilon\rho_*u_*^2)(\rho_+-\rho_*)^2}{\rho_+\rho_*(\varepsilon\rho_++\varepsilon\rho_*+2)}}=\frac{1}{\sqrt{\rho_+}}\cdot\lim_{\varepsilon\to0}\sqrt{\frac{\varepsilon\rho_*^2u_*^2}{\varepsilon\rho_*+2}}.
$$

As a result, we takes a step further in (3.15) to get

$$
u_--u_+=\frac{1}{\sqrt{\rho_-}}\cdot\lim_{\varepsilon\to0}\sqrt{\frac{\varepsilon\rho_*^2u_*^2}{\varepsilon\rho_*+2}}+\frac{1}{\sqrt{\rho_+}}\cdot\lim_{\varepsilon\to0}\sqrt{\frac{\varepsilon\rho_*^2u_*^2}{\varepsilon\rho_*+2}}=(\frac{1}{\sqrt{\rho_-}}+\frac{1}{\sqrt{\rho_+}})\lim_{\varepsilon\to0}\sqrt{\frac{\varepsilon\rho_*^2u_*^2}{\varepsilon\rho_*+2}},
$$

which brings about

$$
\lim_{\varepsilon\to0}\sqrt{\frac{\varepsilon\rho_*^2u_*^2}{\varepsilon\rho_*+2}}=\frac{u_--u_+}{\frac{1}{\sqrt{\rho_-}}+\frac{1}{\sqrt{\rho_+}}}=\frac{(u_--u_+)\sqrt{\rho_-\rho_+}}{\sqrt{\rho_-}+\sqrt{\rho_+}}.\tag{3.18}
$$

In terms of (3.13) and (3.18), we can immediately verify that

$$\lim_{\varepsilon\to 0} u_* = u_- - \lim_{\varepsilon\to 0}\sqrt{\frac{(\varepsilon\rho_* u_*^2+\varepsilon\rho_- u_-^2)(\rho_*-\rho_-)^2}{\rho_*\rho_-(\varepsilon\rho_*+\varepsilon\rho_-+2)}} = u_- - \frac{1}{\sqrt{\rho_-}}\lim_{\varepsilon\to 0}\sqrt{\frac{\varepsilon\rho_*^2u_*^2}{\varepsilon\rho_*+2}}$$
$$= u_- - \frac{(u_- - u_+)\sqrt{\rho_-\rho_+}}{\sqrt{\rho_-}(\sqrt{\rho_-}+\sqrt{\rho_+})} = u_- - \frac{(u_- - u_+)\sqrt{\rho_+}}{\sqrt{\rho_-}+\sqrt{\rho_+}} = \frac{u_-\sqrt{\rho_-}+u_+\sqrt{\rho_+}}{\sqrt{\rho_-}+\sqrt{\rho_+}} = \sigma_\delta. \tag{3.19}$$

It follows from (3.18) and (3.19) that

$$\lim_{\varepsilon\to 0}\sqrt{\frac{\varepsilon\rho_*^2}{\varepsilon\rho_*+2}} = \lim_{\varepsilon\to 0}\sqrt{\frac{\varepsilon\rho_*^2u_*^2}{\varepsilon\rho_*+2}}\cdot\frac{1}{\lim_{\varepsilon\to 0}u_*} = \frac{(u_- - u_+)\sqrt{\rho_-\rho_+}}{\sqrt{\rho_-}+\sqrt{\rho_+}}\cdot\frac{\sqrt{\rho_-}+\sqrt{\rho_+}}{u_-\sqrt{\rho_-}+u_+\sqrt{\rho_+}} = \frac{(u_- - u_+)\sqrt{\rho_-\rho_+}}{u_-\sqrt{\rho_-}+u_+\sqrt{\rho_+}},$$

which means that

$$\lim_{\varepsilon\to 0}\varepsilon\rho_* = 0, \qquad \lim_{\varepsilon\to 0}\varepsilon\rho_*^2 = \frac{2\rho_-\rho_+(u_- - u_+)^2}{(u_-\sqrt{\rho_-}+u_+\sqrt{\rho_+})^2}. \tag{3.20}$$

The proof of lemma is established. □

Lemma 3.2. *The limits $\varepsilon\to 0$ of mass and momentum between the two shock waves are as follows:*

$$\lim_{\varepsilon\to 0}\int_{\sigma_1}^{\sigma_2}\rho_* d\xi = \sigma_\delta[\rho]-[\rho u], \tag{3.21}$$

$$\lim_{\varepsilon\to 0}\int_{\sigma_1}^{\sigma_2}\rho_* u_* d\xi = \sigma_\delta[\rho u]-[\rho u^2]. \tag{3.22}$$

Proof. By virtue of the first equation of the Rankine-Hugoniot conditions (3.9), it follows from the two shock waves S_1 and S_2 mentioned in (3.11) and (3.12) that

$$\begin{cases}\sigma_1(\rho_*-\rho_-) = \rho_*u_* - \rho_-u_-,\\ \sigma_2(\rho_+-\rho_*) = \rho_+u_+ - \rho_*u_*,\end{cases}$$

which leads to

$$\lim_{\varepsilon\to 0}(\sigma_2-\sigma_1)\rho_* = \lim_{\varepsilon\to 0}(\sigma_2\rho_+ - \sigma_1\rho_- - \rho_+u_+ + \rho_-u_-) = \sigma_\delta[\rho]-[\rho u]. \tag{3.23}$$

As before, for the second equation of (3.9), we also obtain

$$\begin{cases}\sigma_1(\rho_*u_* + \varepsilon\rho_*^2u_* - \rho_-u_- - \varepsilon\rho_-^2u_-) = \rho_*u_*^2 + \frac{3}{2}\varepsilon\rho_*^2u_*^2 - \rho_-u_-^2 - \frac{3}{2}\varepsilon\rho_-^2u_-^2,\\ \sigma_2(\rho_+u_+ + \varepsilon\rho_+^2u_+ - \rho_*u_* - \varepsilon\rho_*^2u_*) = \rho_+u_+^2 + \frac{3}{2}\varepsilon\rho_+^2u_+^2 - \rho_*u_*^2 - \frac{3}{2}\varepsilon\rho_*^2u_*^2,\end{cases}$$

which suffices to identify that

$$\lim_{\varepsilon\to 0}(\sigma_2-\sigma_1)\rho_*u_* = \lim_{\varepsilon\to 0}(\sigma_1\varepsilon\rho_*^2u_* - \sigma_2\varepsilon\rho_*^2u_* - \sigma_1\rho_-u_- + \sigma_2\rho_+u_+ - \sigma_1\varepsilon\rho_-^2u_- + \sigma_2\varepsilon\rho_+^2u_+$$
$$+\rho_-u_-^2 + \frac{3}{2}\varepsilon\rho_-^2u_-^2 - \rho_+u_+^2 - \frac{3}{2}\varepsilon\rho_+^2u_+^2)$$
$$= -\sigma_1\rho_-u_- + \sigma_2\rho_+u_+ + \rho_-u_-^2 - \rho_+u_+^2 = \sigma_\delta[\rho u]-[\rho u^2]. \tag{3.24}$$

Hence, it suffices to observe that Lemma 3.2 can be achieved directly from (3.23) and (3.24). □

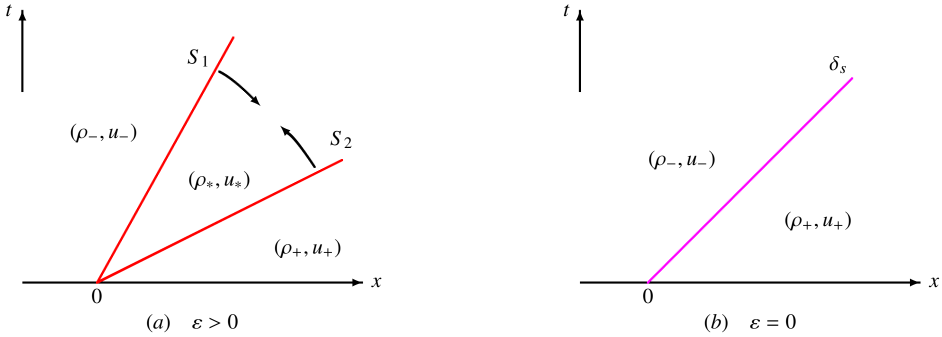


Fig.5 The perturbed solution with $\varepsilon > 0$ in (a) and the limiting solution with $\varepsilon = 0$ in (b) are demonstrated for the Riemann problem (1.4) and (1.2) under the case $u_- > u_+$.

Theorem 3.3. *Let $u_+ < u_-$ and ε be sufficiently small, then the Riemann problem (1.4) and (1.2) exists a solution consisting of two shock waves. If $\varepsilon \to 0$, then the solution converges exactly to the delta shock wave solution of the Riemann problem (1.3) and (1.2) in the sense of distributions. In addition, it follows immediately that*

$$\lim_{\varepsilon\to 0} \rho = \rho_- + [\rho]H(x - \sigma_\delta t) + t(\sigma_\delta[\rho] - [\rho u])\delta(x - \sigma_\delta t), \tag{3.25}$$

$$\lim_{\varepsilon\to 0} \rho u = \rho_- u_- + [\rho u]H(x - \sigma_\delta t) + t(\sigma_\delta[\rho u] - [\rho u^2])\delta(x - \sigma_\delta t). \tag{3.26}$$

The proof of Theorem 3.3 is completely similar to that of Theorem 2.2 and thus the details are omitted here.

3.3. Formation of vacuum state

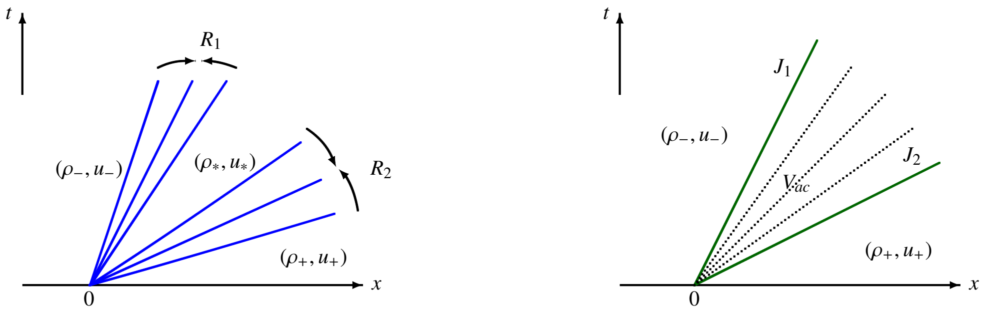
In this subsection, we shall turn our attention to concern the case $u_+ > u_-$. When ε is sufficiently small, it suffices to obtain that the Riemann solution of the system (1.4) includes two rarefaction waves (see Fig.6(a)). The vacuum state is formed in the limit $\varepsilon \to 0$ of solution to the Riemann problem (1.4) and (1.2) under the case $u_+ > u_-$. To be more specifical, if (ρ_*, u_*) is the intermediate state between two rarefaction waves, then it can be easily seen that

$$R_1 : \begin{cases} \xi = \lambda_1(\rho, u) = u\Big(1 - \sqrt{\dfrac{\varepsilon\rho}{1 + \varepsilon\rho}}\Big), \\ \ln u_* - \ln u_- = -\displaystyle\int_{\rho_-}^{\rho_*} \frac{\sqrt{\varepsilon}}{\sqrt{s(1 + \varepsilon s)}} ds, \\ \rho_* < \rho_-, \qquad u_* > u_-, \end{cases} \tag{3.27}$$

and

$$R_2 : \begin{cases} \xi = \lambda_2(\rho, u) = u\Big(1 + \sqrt{\dfrac{\varepsilon\rho}{1 + \varepsilon\rho}}\Big), \\ \ln u_+ - \ln u_* = \displaystyle\int_{\rho_*}^{\rho_+} \frac{\sqrt{\varepsilon}}{\sqrt{s(1 + \varepsilon s)}} ds, \\ \rho_+ > \rho_*, \qquad u_+ > u_*. \end{cases} \tag{3.28}$$

Theorem 3.4. *Let $u_- < u_+$ and ε be sufficiently small, then the Riemann problem (1.4) and (1.2) exists a solution consisting of two rarefaction waves. If $\varepsilon \to 0$, then the solution converges to a two-contact-discontinuity solution with vacuum state between them, which is the same as that for the Riemann problem (1.3) and (1.2) with the same Riemann initial data.*



$(a)\quad \varepsilon > 0$ $\qquad\qquad$ $(b)\quad \varepsilon = 0$

Fig.6 The perturbed solution with $\varepsilon > 0$ in (a) and the limiting solution with $\varepsilon = 0$ in (b) are demonstrated for the Riemann problem (1.4) and (1.2) under the case $u_+ > u_-$.

Proof. With the help of the fact $\rho_* \le \min(\rho_-, \rho_+)$, it follows from the second equations of (3.27) and (3.28) that

$$\begin{aligned}
\ln u_+ - \ln u_- &= \int_{\rho_*}^{\rho_-} \frac{\sqrt{\varepsilon}}{\sqrt{s(1+\varepsilon s)}} ds + \int_{\rho_*}^{\rho_+} \frac{\sqrt{\varepsilon}}{\sqrt{s(1+\varepsilon s)}} ds \\
&\le \sqrt{\varepsilon}\Big(\int_{\rho_*}^{\rho_-} \frac{1}{\sqrt{\rho_*(1+\varepsilon\rho_*)}} ds + \int_{\rho_*}^{\rho_+} \frac{1}{\sqrt{\rho_*(1+\varepsilon\rho_*)}} ds\Big) \\
&= \frac{\sqrt{\varepsilon}}{\sqrt{\rho_*(1+\varepsilon\rho_*)}}(\rho_- - \rho_* + \rho_+ - \rho_*) \le \frac{\sqrt{\varepsilon}}{\sqrt{\rho_*(1+\varepsilon\rho_*)}}(\rho_- + \rho_+).
\end{aligned} \tag{3.29}$$

If $\lim\limits_{\varepsilon\to 0}\rho_* > 0$, then it is easy to check that $u_+ - u_- = 0$ from the above result, which contradicts with the fact $u_- < u_+$. Hence, one has $\lim\limits_{\varepsilon\to 0}\rho_* = 0$. That is to say, the intermediate state turns out to be vacuum state as $\varepsilon \to 0$. As a result, we take a step further to get

$$\lim_{\varepsilon\to 0}\lambda_1(\rho_-, u_-) = \lim_{\varepsilon\to 0}\lambda_1(\rho_*, u_*) = u_-, \qquad \lim_{\varepsilon\to 0}\lambda_2(\rho_*, u_*) = \lim_{\varepsilon\to 0}\lambda_2(\rho_+, u_+) = u_+, \tag{3.30}$$

which infer us to verify that the rarefaction waves R_1 and R_2 degenerate to the contact discontinuities J_1 and J_2 separately, in which the velocities are u_- and u_+ respectively as $\varepsilon \to 0$ (see Fig.6(b)). The proof is finished. $\square$

4. Conclusions

In the present work, we carefully investigated the limiting behaviors of Riemann solutions to the macroscopic production model as the stacking product vanishes. For the case $u_+ < u_-$, it is of great interest to notice that the limits of Riemann solutions to the two different perturbed systems exactly converge to the delta shock wave solution of the pressureless gas dynamics model (1.3) under the different over-compressive entropy conditions $u_+ = \overline{\sigma}_\delta < u_-$ and $u_+ < \sigma_\delta < u_-$ respectively. During the process of taking the limit, the singularities of all the solutions are analyzed in fully explicit forms and then some interesting nonlinear phenomena are discovered. In each of the two limiting processes, the formation of vacuum state can also be observed and investigated in details. Oh the one hand, in the actual system of industry production, different approximations may be used to illustrate different production situations. On the other hand, in the theory of hyperbolic conservation laws, different approximations usually lead to different over-compressive entropy conditions and thus obtain the delta shock wave solutions with the different strengths and speeds in the limiting situations.

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