Continuum of one-sign solutions of one-dimensional Minkowski-curvature problem with nonlinear boundary conditions

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Abstract: In this work, we investigate the continuum of one-sign solutions of the nonlinear one-dimensional Minkowski-curvature equation

$$-\left(u'/\sqrt{1-\kappa u'^2}\right)' = \lambda f(t,u), \ \ t \in (0,1)$$

with nonlinear boundary conditions $u(0) = \lambda g_1(u(0)), u(1) = \lambda g_2(u(1))$ by using unilateral global bifurcation techniques, where $\kappa > 0$ is a constant, $\lambda > 0$ is a parameter $g_1, g_2 : [0,\infty) \to (0,\infty)$ are continuous functions and $f : [0,1] \times [-\frac{1}{\sqrt{\kappa}}, \frac{1}{\sqrt{\kappa}}] \to \mathbb{R}$ is a continuous function. We prove the existence and multiplicity of one-sign solutions according to different asymptotic behaviors of nonlinearity near zero.

Key words: Minkowski-curvature equation; nonlinear boundary conditions; one-sign solutions; bifurcation theory.

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1. Introduction

In this work we are concerned with the existence and multiplicity of one-sign solutions of the nonlinear boundary value problem

$$\begin{aligned} &-\left(\frac{u'}{\sqrt{1-\kappa u'^2}}\right)' = \lambda f(t,u), \ \ t \in (0,1), \\ &u(0) = \lambda g_1(u(0)), \quad u(1) = \lambda g_2(u(1)), \end{aligned} \tag{1.1}$$

where $\kappa > 0$ is a constant, $\lambda > 0$ is a parameter, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $g_1, g_2 : [0,\infty) \to (0,\infty)$ are continuous functions. This is the one-dimensional versional of the nonlinear problem associated with the Minkowski curvature equation on an annulus

$$\begin{aligned} &-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-\kappa|\nabla u|^2}}\right) = \lambda f(|x|,u) \quad \text{in } 0 < R_1 < |x| < R_2, \\ &u = \lambda g_1(u) \text{ on } |x| = R_1, \quad u = \lambda g_2(u) \text{ on } |x| = R_2. \end{aligned} \tag{1.2}$$

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It's well-known that the Minkowski curvature equation play an important role in differential geometry and in the theory of relativity, see the classical paper of Bartnik and Simon[1], Cheng and Yao[8] and Grenier[18] and the references contained therein.

Note that some general solvability results and global structure of solutions for (1.1) and (1.2) were proved under the assumption that f is continuous with the case $g_1(u) = g_2(u) = 0$, see [1, 2, 3, 4, 9, 10, 12, 13, 23]. 2009, Bereanug and Mawhin [5] obtained the existence results of solutions for (1.1) with $\sqrt{\kappa} = 1$ and nonhomogeneous linear boundary conditions $u(0) = A, u(1) = B$ under the condition $|B - A| < 1$ ([5, Theorem 1]). Yet, as all spacelike solutions are uniformly bounded, however, there is no little information for the existence of positive solution of (1.1) under the nonlinear boundary conditions. Therefore, it may be interesting to investigate in such cases the existence of positive solutions.

When $\kappa = 0$, the problem (1.1) can degenerate to the following nonlinear problem

$$\begin{aligned} & -u'' = \lambda f(t,u), \ \ t \in (0,1), \\ & u(0) = \lambda g_1(u(0)), \quad u(1) = \lambda g_2(u(1)), \end{aligned} \tag{1.3}$$

which has been studied by Dunninger and Haiyan Wang[16] (the method of lower and upper solutions and degree theory). Second-order ordinary differential equation with different nonlinear boundary value conditions has been studied by many authors using different methods, see Drame and Costa [15](phase plane analysis), Shivaji et al. [17, 19] (time maps technique, the method of sub-supersolutions), Ma and Wang [24](degree theory and bifurcation techniques) and their references.

Motivated above papers, we establish the continuum of positive solutions of the problem (1.1) by the bifurcation theorem under the following assumptions

(C1) $f \in C\big([0,1] \times [-\frac{1}{\sqrt{\kappa}}, \frac{1}{\sqrt{\kappa}}], \mathbb{R}\big)$, and $g_1, g_2 : [0,\infty) \to (0,\infty)$ are continuous.

(C2) there exist $g_1^0, g_2^0 \in (\frac{4}{\pi^2}, 1)$, such that $\lim\limits_{s\to 0} \frac{g_1(s)}{s} = g_1^0$ and $\lim\limits_{s\to 0} \frac{g_2(s)}{s} = g_2^0$.

(C3) $f(t,s)s > 0$ for any $(t,s) \in [0,1] \times [-\frac{1}{\sqrt{\kappa}}, \frac{1}{\sqrt{\kappa}}]$ with $s \neq 0$, and there exists $f_0 \in [0,\infty]$ such that

$$f_0 = \lim_{s\to 0} \frac{f(t,s)}{s} \ \text{ uniformly for } t \in [0,1].$$

(C4) $f(t,s)s > 0$ for any $(t,s) \in [0,1] \times [-\frac{1}{\sqrt{\kappa}}, \frac{1}{\sqrt{\kappa}}]$ with $s \neq 0$, and there exist $\bar{f}_0, \bar{f}^0 \in (0,\infty)$ with $\bar{f}_0 \neq \bar{f}^0$, such that

$$\bar{f}_0 = \liminf_{u\to 0} \frac{f(t,u)}{u}, \qquad \bar{f}^0 = \limsup_{u\to 0} \frac{f(t,u)}{u}.$$

Let $P^+ = \{u \in C^1[0,1] \,|\, u(t) \geqslant 0, t \in [0,1]\}$ and $P^- = \{u \in C^1[0,1] \,|\, u(t) \leqslant 0, t \in [0,1]\}$. We also add the point ∞ to the work space $\mathbb{R} \times C^1[0,1]$, and denote the closure set of nontrivial solution pair sets of (1.1) as follows

$$C := \{(\lambda, u) \in \mathbb{R} \times C^1[0,1] \,|\, u \text{ is a nontrivial solution of (1.1) for some } \lambda\}.$$

Then we will prove the following results:

Theorem 1.1 Assume that (C1)-(C3) hold.

(i) If $f_0 \in (0, \infty)$, then the solution set C contains two subcontinuum C^+ and C^- bifurcating from the trivial line at the $(\frac{\pi^2}{f_0}, 0)$, which satisfies $C^\nu \subseteq (\mathbb{R} \times P^\nu) \cup (\frac{\pi^2}{f_0}, 0)$ and $\lambda|g_1(u(0)) - g_2(u(1))| < \frac{1}{\sqrt{\kappa}}$ for any $(\lambda, u) \in C^\nu \backslash (\frac{\pi^2}{f_0}, 0)$, $\nu = \{+, -\}$.

(ii) If $f_0 = \infty$, then the solution set C contains two subcontinuum C^+ and C^- bifurcating from the trivial line at $(0, 0)$, which satisfies $C^\nu \subseteq (\mathbb{R} \times P^\nu) \cup (0, 0)$ and $\lambda|g_1(u(0)) - g_2(u(1))| < \frac{1}{\sqrt{\kappa}}$ for any $(\lambda, u) \in C^\nu \backslash (0, 0)$, $\nu = \{+, -\}$.

(iii) If $f_0 = 0$, then the solution set C contains two subcontinuum C^+ and C^- bifurcating from the trivial line at $(\infty, 0)$, which satisfies $C^\nu \subseteq (\mathbb{R} \times P^\nu) \cup (\infty, 0)$ and $\lambda|g_1(u(0)) - g_2(u(1))| < \frac{1}{\sqrt{\kappa}}$ for any $(\lambda, u) \in C^\nu$, $\nu = \{+, -\}$.

Theorem 1.1 give the continuum of one-sign solutions of (1.1) according to the nonlinearity f is asymptotic linear growth, superlinear growth and sublinear growth near at $u = 0$, respectively. A naturally interesting question is that how to bifurcate the one-sign solution subcontinuum C^ν when $\liminf_{u \to 0} \frac{f(t,u)}{u} \neq \limsup_{u \to 0} \frac{f(t,u)}{u}$, i.e. f is jumping near $u = 0$. A typical example is that $f(t, u) = |u|$ satisfying $-1 = \liminf_{u \to 0^-} \frac{f(t,u)}{u} \neq \limsup_{u \to 0^+} \frac{f(t,u)}{u} = 1$. The following Theorem gives the continuum structure of one-sign solutions of (1.1) under the nonlinearity f is jumping near $u = 0$.

Theorem 1.2 Assume that (C1)-(C2) and (C4) hold. Then the solution set C contains two subcontinuums $\mathscr{C}^\nu$ bifurcating from the trivial line at the interval $[\frac{\pi^2}{f^0}, \frac{\pi^2}{f_0}] \times \{0\}$, which satisfies $\mathscr{C}^\nu \subseteq (\mathbb{R} \times P^\nu) \cup \{[\frac{\pi^2}{f^0}, \frac{\pi^2}{f_0}] \times \{0\}\}$ and $\lambda|g_1(u(0)) - g_2(u(1))| < \frac{1}{\sqrt{\kappa}}$ for any $(\lambda, u) \in C^\nu \backslash \{[\frac{\pi^2}{f^0}, \frac{\pi^2}{f_0}] \times 0\}$, $\nu = \{+, -\}$.

Remark 1.1 Note that the condition $\lambda|g_1(u(0)) - g_2(u(1))| < \frac{1}{\sqrt{\kappa}}$ is a sufficient condition for the existence of solutions of (1.1) under the assumption f is continuous, see Corollary 1 of [5]. Therefore, compared with that the one-sign solution subcontinuum C^ν is unbounded in λ-direction under the homogeneous linear boundary conditions $u(0) = 0, u(1) = 0$ (see [9, 12, 13, 23]), the subcontinuum C^ν of (1.1) is more complex, not only depend on the nonlinearity f, but also depend on the boundary function g_1 and g_2. we should divide more details about g_1, g_2 to obtain the direction of the connected branch C^ν.

In addition, $g_1^0, g_2^0 \in (\frac{4}{\pi^2}, 1)$ is a restricted condition for the eigenvalues of corresponding eigenvalue problem of (1.1) don't depend on the boundary(the more details, see reference [7]), which provides a sufficient condition for the existence of one-sign solution components C^ν of (1.1). The best condition looks an interesting but difficult problem.

Remark 1.2 Compared with the results of [2, 3, 5, 9], Theorem 1.1 and Theorem 1.2 obtain the continuum structure of the set of one-sign solutions of (1.1), it gives a clear description of the evolution process of the components of positive solutions as the parameter λ increases. The study of global behavior of the positive solution curves is very useful for computing the numerical solution of (1.1) as it can be used to guide the numerical work. For example, it can be used to estimate the u-interval in advance in applying the finite difference method, and it can be used to restrict the range of initial values we need to consider in applying the shooting method.

As a first step, we present that problem (1.1) can always be reduced to an equivalent one, where the singularity on the left of the equation has been removed and the function on the right is bounded, actually vanishes outside the rectangle $[0,1]\times[-\frac{1}{\sqrt{k}},\frac{1}{\sqrt{k}}]$, and agree with f in a neighborhood of $s=0$. Such a reduction, which is achieved by quite elementary estimates, is depending on bifurcation theory. In fact, we replace the equation in (1.1) by

$$-u''=\lambda\tilde{f}(t,u)h(u'), \tag{1.4}$$

where

$$h(s)=\begin{cases}(1-\kappa s^2)^{\frac{3}{2}}, & \text{if } |s|\leqslant\frac{1}{\sqrt{\kappa}},\\ 0, & \text{if } |s|>\frac{1}{\sqrt{\kappa}}\end{cases} \tag{1.5}$$

has compact support and $\tilde{f}$ is bounded. Note that define the function $\tilde{f}:[0,1]\times\mathbb{R}\to\mathbb{R}$ by setting , for any $t\in[0,1]$,

$$\tilde{f}(t,s)=\begin{cases}f(t,s), & \text{if } -\frac{1}{\sqrt{\kappa}}\leqslant s\leqslant\frac{1}{\sqrt{\kappa}},\\ \textit{linear}, & \text{if } s\in(\frac{1}{\sqrt{\kappa}},\frac{2}{\sqrt{\kappa}})\cup(-\frac{2}{\sqrt{\kappa}},\frac{1}{\sqrt{\kappa}}),\\ 0, & \text{if } |s|\geqslant\frac{2}{\sqrt{\kappa}}.\end{cases} \tag{1.6}$$

Observe that, with the context of one-sign solutions, problem (1.1) is equivalent to the same problem with f by $\tilde{f}$. Indeed, if u is a one-sign solution, then $\|u'\|_\infty<\frac{1}{\sqrt{\kappa}}$ and hence $\|u\|_\infty<\frac{1}{\sqrt{\kappa}}$. In the sequel of the proof we shall replace f with $\tilde{f}$; however, for the sake of simplicity in the notation, the modified function $\tilde{f}$ will still be denoted by f. By the similar argument of [9, Theorem 2.3] and [5, Corollary 1], it follows that $u\in C^2[0,1]$ is a one-sign solution of (1.1) if and only if it is a one-sign solution of the following nonlinear problem

$$\begin{aligned}&-u''(t)=\lambda f(t,u(t))h(u'(t)),\ \ t\in(0,1),\\ &u(0)=\lambda g_1(u(0)),\quad u(1)=\lambda g_2(u(1)).\end{aligned} \tag{1.7}$$

and $\lambda|g_1(u(0))-g_2(u(1))|<\frac{1}{\sqrt{\kappa}}$.

Next, we will show that the one-sign solution branch of (1.1) bifurcate to the trivial line at $(\frac{\pi^2}{f_0},0)$ and develop a bifurcation approach to treat the case $f_0=\infty$ and $f_0=0$, respectively. It's very crucial to this approach is that the construction of a sequence of functions $f^{[n]}$ which is asymptotic linear growth at 0 and satisfies

$$\lim_{n\to\infty}\sup_{s\in[0,\infty)}|f^{[n]}(t,s)-f(t,s)|=0,\ \ (f^{[n]})_0\to\infty\ \ (\text{resp. } (f^{[n]})^0\to 0)\ \ \text{uniformly for } \ t\in[0,1].$$

By means of the corresponding auxiliary equations, we obtain a sequence of unbounded components $\{C_+^{[n]}\}$ via a unilateral global bifurcation theorem, see [20, Section 6.4, 6.5,] or [12, 23] and this enables us to find an unbounded component C^+ (resp. $\mathfrak{C}^+$) satisfying

$$C^+\subset\limsup_{n\to\infty}C_+^{[n]}\qquad(\text{resp.}\mathfrak{C}^+\subset\limsup_{n\to\infty}C_+^{[n]})$$

and joining $(0,0)$ (resp. $(\infty,0)$) with $(\infty,\frac{1}{\sqrt{\kappa}})$. Whereafter, we introduce the one-sign solution component of (1.1) bifurcate to the interval $[\frac{\pi^2}{f^0},\frac{\pi^2}{f_0}]\times\{0\}$.

From Theorem 1.1-Theorem 1.2 and above discussion, we give the following existence results of one-sign solution of (1.1).

Corollary 1 Assume that (C1)-(C3) hold and g_1, g_2 satisfy one of the following assumptions: (a) $|g_1 - g_2| > 0$; (b) $g_1 = g_2$ is strictly monotone function on $[0, \frac{1}{\sqrt{\kappa}}]$; (c) $g_1(u(0)) = g_2(u(1))$, in this case, denote $\frac{1}{\sqrt{\kappa}|g_1-g_2|} = +\infty$.

(i) If $f_0 \in (0, \infty)$ and $|g_1 - g_2| < \frac{f_0}{\sqrt{\kappa}\pi^2}$ for any $u \in [0, \frac{1}{\sqrt{\kappa}})$, then for any $\lambda \in (\frac{\pi^2}{f_0}, \frac{1}{\sqrt{\kappa}|g_1-g_2|})$, the problem (1.1) has at least two one-sign solution $u^+ > 0$ and $u^- < 0$ in $(0, 1)$ such that $(\lambda, u^\nu) \in C^\nu$, here $\nu = \{+, -\}$.

(ii) If $f_0 = \infty$, then for any $\lambda \in (0, \frac{1}{\sqrt{\kappa}|g_1-g_2|})$, the problem (1.1) has at least two one-sign solution $u^+ > 0$ and $u^- < 0$ in $(0, 1)$ such that $(\lambda, u^\nu) \in C^\nu$.

(iii) If $f_0 = 0$ and there exist $\lambda_* > 0$ such that problem (1.7) has at least four one-sign solution $u_1^+ > 0$, $u_2^+ > 0$, $u_1^- < 0$ and $u_2^- < 0$ in $(0, 1)$ such that $(\lambda, u_i^\nu) \in C^\nu$ for any $\lambda \in (\lambda_*, \infty)$, here $i = 1, 2$. Moreover, if $\frac{1}{\sqrt{\kappa}|g_1-g_2|} < \lambda_*$, then problem (1.1) has no one-sign solution; if $\frac{1}{\sqrt{\kappa}|g_1-g_2|} = \lambda_*$, then problem (1.1) has at least two one-sign solution $u^+ > 0$ and $u^- < 0$ in $(0, 1)$ such that $(\lambda, u^\nu) \in C^\nu$ with $\lambda = \lambda_*$; if $\frac{1}{\sqrt{\kappa}|g_1-g_2|} > \lambda_*$, problem (1.1) has at least four one-sign solution $u_1^+ > 0$, $u_2^+ > 0$, $u_1^- < 0$ and $u_2^- < 0$ in $(0, 1)$ such that $(\lambda, u_i^\nu) \in C^\nu$ for any $\lambda \in (\lambda_*, \frac{1}{\sqrt{\kappa}|g_1-g_2|})$.

Corollary 2 Assume that (C1)-(C2) and (C4) hold. If g_1, g_2 satisfy one of the following assumptions:

(i) $0 < |g_1 - g_2| < \frac{\bar{f}_0}{\sqrt{\kappa}\pi^2}$;

(ii) $g_1 = g_2$ is strictly monotone function on $[0, \frac{1}{\sqrt{\kappa}}$ and $|g_1(u(0)) - g_1(u(1))| < \frac{\bar{f}_0}{\sqrt{\kappa}\pi^2}$;

(iii) $g_1(u(0)) = g_2(u(1))$, in this case, denote $\frac{1}{\sqrt{\kappa}|g_1-g_2|} = +\infty$,

then for any $\lambda \in (\frac{\pi^2}{\bar{f}_0}, \frac{1}{\sqrt{\kappa}|g_1-g_2|})$, the problem (1.1) has at least two one-sign solution $u^+ > 0$ and $u^- < 0$ in $(0, 1)$ such that $(\lambda, u^\nu) \in \mathscr{C}^\nu$, here $\nu = \{+, -\}$.

Finally, we give some examples to illustrate the main results according to different nonlinear boundary function g_1 and g_2.

The rest of the paper is organized as follows. In Section 2, we introduce some properties of the superior limit of a certain infinity collection of connected sets and preliminary results. Section 3 prove Theorem 1.1 and Corollary 1 and give some examples to illustrate the main results according to different nonlinear boundary function g_1 and g_2. In Section 4, we prove Theorem 1.2, Corollary 2 and introduce some examples as application.

2. Preliminaries and superior limit and component

Let E be a real Banach space with norm $\|\cdot\|$, and let $\mathscr{S}$ be the closure of nontrivial solutions set of the following equation

$$u = \lambda Lu + H_0(\lambda, u),\ u \in E, \tag{2.1}$$

where λ varies in $\mathbb{R}$, $L : E \to E$ is a linear completely continuous operator, $H_0 : \mathbb{R} \times E \to E$ be completely continuous and $H_0(\lambda, u) = \circ(\|u\|)$ at $u = 0$ uniformly on bounded λ intervals.

Let Λ denote the set of real eigenvalues of equation

$$u = \lambda Lu,\ u \in E.$$

Suppose that $\mu \in \Lambda$ has geometric multiplicity 1, let E_0 be a closed subspace of E such that $E = \mathrm{span}\{\varphi_\mu\} \oplus E_0$, where φ_μ is an eigenfunction corresponding to μ with $\|\varphi_\mu\| = 1$.

We give the bifurcation phenomenon of solutions for (2.1).

Lemma 2.1 ([14, 20, 25]) If $\mu \in \Lambda$ isolated and satisfying

$$\deg(I - (\mu - \varepsilon)L, B_r(0), 0) \neq \deg(I - (\mu + \varepsilon)L, B_r(0), 0) \text{ for any } \varepsilon > 0 \text{ small enough,}$$

where $B_r(0) = \{u \in E \,|\, \|u\| < r\}$, then $(\mu, 0)$ is a bifurcation point for equation (2.1). Moreover, $\mathscr{S}$ possesses a maximal continuum $\mathscr{C}_\mu$ such that $(\mu, 0) \in \mathscr{C}_\mu$ and $\mathscr{C}_\mu$ can be decomposed into two sub-continua $\mathscr{C}_\mu^+$ and $\mathscr{C}_\mu^-$, such that each of them either

(i) meets infinity in $\mathbb{R} \times E$; or

(ii) meets $(\tilde{\mu}, 0)$, where $\mu \neq \tilde{\mu} \in \Lambda$; or

(iii) contains a point $(\lambda, v) \in \mathbb{R} \times (E_0 \backslash \{0\})$.

Let M be a metric space and $\{C_n \,|\, n = 1, 2, \cdots\}$ a family of subsets of M. Then the superior limit $\mathscr{D}$ of $\{C_n\}$ is defined by

$$\mathscr{D} := \limsup_{n\to\infty} C_n = \{x \in M \,|\, \exists \{n_k\} \subset \mathbb{N},\ x_{n_k} \in C_{n_k},\ \text{such that } x_{n_k} \to x\}. \tag{2.2}$$

A *component* of a set M means a maximal connected subset of M, see [11] for the detail.

Lemma 2.2 ([11, Theorem 1.2],[22]) Let E be a Banach space and let $\{C_n\}$ be a family of closed connected subsets of E. Assume that

(i) there exist $z_n \in C_n$, $n = 1, 2, \cdots$ and $z_* \in E$ such that $z_n \to z_*$;

(ii) $\lim_{n\to\infty} r_n = \lim_{n\to\infty} \sup\{\|u\| \,|\, u \in C_n\} = \infty$;

(iii) for every $R > 0$, $(\bigcup_{n=1}^{\infty} C_n) \cap B_R$ is a relatively compact of E, $B_R = \{u \in E \,|\, \|u\| < R\}$.

Then there exists an unbounded component $\mathscr{C}$ in $\mathscr{D}$ and $z_* \in \mathscr{C}$.

Lemma 2.3 ([21, Theorem 3],[11], [12]) Let E be a Banach space and $a \in \mathbb{R}$ be a constant. Let $\{C_n\}$ be a family of connected subsets of $\mathbb{R} \times E$. Assume that

(A1) $C_n \cap ((-\infty, a] \times E) = \varnothing$;

(A2) there exist $0 < \sigma < r < \infty$ and $b \in (a, \infty)$ such that

$$C_n \cap \{(\mu, u) \,|\, \mu \geqslant b - \sigma,\ r - \sigma \leqslant \|u\| \leqslant r + \sigma\} = \varnothing; \tag{2.3}$$

(A3) $\mu_k > a$ for all $k \in \mathbb{N}$, $\mu_k \to +\infty$ and C_n meets $(\mu_n, 0)$ and infinity in $([a, \infty) \times E) \backslash \Omega_{b,r}$, here $\Omega_{b,r} := ([0, \infty) \times E) \backslash \{(\mu, u) \in [b, \infty) \times E \,|\, \|u\| < r\}$;

(A4) for every $R > 0$, $(\bigcup_{n=1}^{\infty} C_n) \cap B_R$ is a relatively compact of E.

Then there exists an unbounded component $\mathscr{C}$ in $\mathscr{D}$ such that

(a) both $\mathscr{C} \cap \Omega_{b,r}$ and $\mathscr{C} \cap (([a, \infty) \times E) \backslash \Omega_{b,r})$ are unbounded;

(b) $\mathscr{C} \cap \{(\mu, u) \,|\, \mu \geqslant b, \|u\| = r)\} = \varnothing$.

Let $X = C[0, 1]$ be the Banach space with $\|u\|_\infty = \max_{t\in[0,1]} |u(t)|$.

First, it's easy to verify that problem (1.7) is equivalent to the integral equation

$$u(t) = \lambda(R(\tau(u(t))) + \lambda \int_0^1 G(t, s) f(s, u(s)) h(u'(s)) ds, \tag{2.4}$$

where $\tau : X \to \mathbb{R}^2$ with $\tau(u(t)) = (u(0), u(1))$ is the trace operator, the operator $R : \mathbb{R}^2 \to X$ is defined by

$$R(x, y) = g_2(y)t + g_1(x)(1 - t), \tag{2.5}$$

and $G(t, s)$ is the Green's function of linear problem $-u''(t) = 0,\ t \in (0, 1),\ u(0) = 0,\ u(1) = 0$ with

$$G(t, s) = \begin{cases} (1 - t)s, & 0 \leqslant s \leqslant t \leqslant 1, \\ (1 - s)t, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases} \tag{2.6}$$

It's easy to verify that $G(t, s)$ satisfies the following properties:

(i) $G(t, s) \geqslant 0,\ \forall\, t, s \in [0, 1]$ and $G(t, s) > 0,\ \forall\, t, s \in [\frac{1}{4}, \frac{3}{4}]$;

(ii) $G(t, s) \geqslant \frac{1}{4}G(z, s),\ \forall\, t \in [\frac{1}{4}, \frac{3}{4}],\ z, s \in [0, 1]$;

and the function $R(x, y)$ satisfies the following properties:

(1) $g_2(y)t + g_1(x)(1 - t) \geqslant 0\ \ \forall\, t \in [0, 1], x, y \geqslant 0$;

(2) $g_2(y)t + g_1(x)(1 - t) \leqslant g_2(y) + g_1(x),\ \ \forall\, t \in [0, 1], x, y \geqslant 0$;

(3) $g_2(y)t + g_1(x)(1 - t) \geqslant \frac{1}{4}[g_2(y)z + g_1(x)(1 - z)],\ \forall\, t \in [\frac{1}{4}, \frac{3}{4}],\ z \in [0, 1],\ x, y \geqslant 0.$

Clearly, (2.4) is equivalent to the fixed point equation

$$\lambda A u = u,$$

where $A : X \to X$ is defined by

$$Au(t) = R(\tau(u(t))) + \int_0^1 G(t, s) f(s, u(s)) h(u'(s)) ds. \tag{2.7}$$

Define the cone $K = \{u \in X \,|\, u \geqslant 0\}$ and $K_1 = \{u \in X \,|\, u \geqslant 0, \min\limits_{t\in[1/4,3/4]} u(t) \geqslant \frac{1}{4}\|u\|_\infty\}$, then K is a nonnegative cone of X with int$K \neq 0$ and K_1 is a positive cone of X. It's not very difficult to verify that $A(K) \subset K_1$ and $A : K_1 \to K_1$ is completely continuous, see [16, Lemma 3.2].

Let $\rho > 0$ is a constant and let us consider the linear eigenvalue problem

$$\begin{aligned} &-u''(t) = \lambda\rho u(t),\ \ t \in (0, 1), \\ &u(0) = \lambda g_1^0 u(0), \quad u(1) = \lambda g_2^0 u(1). \end{aligned} \tag{2.8}$$

It's easy to verify that (2.8) has only trivial solution $u \equiv 0$ if $\lambda \leqslant 0$, and (2.8) has the nontrivial solutions $\varphi_k(t) = \sin(k\pi t)$ which changes its sign $k - 1$ on $(0, 1)$ with $\lambda = \lambda_k = \frac{(k\pi)^2}{\rho}, k = 1, 2, \cdots$ if $\lambda > 0$. In addition, it's not difficult verify that if $\lambda = \frac{1}{g_1^0}$, then the boundary value condition degenerates to $u(0) \in \mathbb{R}, u(1) = 0$, (2.8) has nontrivial solution if and only if $\frac{1}{g_1^0} = (k\pi)^2$ or $\frac{1}{g_1^0} = (\frac{(2k-1)\pi}{2})^2$, $k = 1, 2, \cdots$; if $\lambda = \frac{1}{g_2^0}$, then the boundary value condition degenerates to $u(0) = 0, u(1) \in \mathbb{R}$, (2.8) has nontrivial solution if and only if $\frac{1}{g_2^0} = (k\pi)^2$.

Thus, under the condition (C2), $g_1^0, g_2^0 > \frac{4}{\pi^2}$ implies that $\frac{1}{g_1^0}, \frac{1}{g_2^0} < \frac{\pi^2}{4}$, which means that (2.8) has no other eigenvalues except for $\lambda = \lambda_k$. Especially, the principal eigenvalue of (2.8) is $\frac{\pi^2}{\rho}$ and the corresponding eigenfunction $\varphi_1(t) = \sin \pi t > 0, t \in (0, 1)$. The more general spectrum results of (2.8), we refer to the reference [7].

3. The continuum bifurcate from point in the case of f is linear near zero

Let $E = C^1[0,1]$ denote the Banach space equipped with the norm $\|u\| = \max\{\|u\|_\infty,\ \|u'\|_\infty\}$.

Let $L_\rho : E \to E$ be the linear operator

$$L_\rho u(t) = R_0(\tau(u(t))) + \int_0^1 G(t,s)\rho u(s)ds,\ t \in [0,1],$$

here $R_0(\tau(u(t))) = g_2^0 u(1)t + g_1^0 u(0)(1-t)$, $\rho > 0$ is a constant. Then $L_\rho : E \to E$ is completely continuous and the linear eigenvalue problem (2.8) is equivalent to the operator equation

$$u = \lambda L_\rho(u),$$

so that the eigenvalues of the problem (2.8) are precisely the characteristic values of L_ρ.

Now we give the important proposition of the bifurcation from trivial solution axis.

Proposition 3.1 Suppose that (C1)-(C3) hold. If $f_0 \in (0,\infty)$, then the pair $(\frac{\pi^2}{f_0}, 0)$ is a bifurcation point of problem (1.7). Moreover, there exist two unbounded continua $\mathscr{C}^+$ and $\mathscr{C}^-$ of the set of nontrivial solutions of problem (1.7) in $\mathbb{R} \times E$ bifurcating from $(\frac{\pi^2}{f_0}, 0)$, such that $\mathscr{C}^\nu \subseteq [\{(\frac{\pi^2}{f_0}, 0)\} \cup (\mathbb{R} \times P^\nu)]$.

Proof Let $\zeta \in C([0,1]\times\mathbb{R},\ \mathbb{R})$ and $\xi_1, \xi_2 \in C(\mathbb{R},\mathbb{R})$ such that

$$f(t,u) = f_0 u + \zeta(t,s),\ g_1(u) = g_1^0 u + \xi_1(u),\ g_2(u) = g_2^0 u + \xi_2(u). \tag{3.1}$$

Clearly, $\lim\limits_{|s|\to 0} \frac{\zeta(t,s)}{s} = 0$ uniformly for $t \in [0,1]$. Let $\tilde{\zeta}(t,u) := \max\limits_{0\leqslant|s|\leqslant u} |\zeta(t,s)|$ for any $t \in [0,1]$, then $\tilde{\zeta}$ is nondecreasing with respect to u and $\lim\limits_{|s|\to 0} \frac{\tilde{\zeta}(t,s)}{s} = 0$ uniformly for $t \in [0,1]$. It follows that

$$\left|\frac{\zeta(t,u)}{u}\right| \leqslant \frac{\tilde{\zeta}(t,|u|)}{|u|} \leqslant \frac{\tilde{\zeta}(t,\|u\|)}{\|u\|} \to 0 \text{ as } \|u\| \to 0.$$

Note that (C2) implies that $\lim\limits_{|s|\to 0} \frac{\xi_i(s)}{s} = 0, i = 1,2$. Denote $\tilde{R}(\tau(u)) = R(\tau(u)) - R_0(\tau(u))$, then $\tilde{R}(\tau(u)) = \xi_2(u(1))t + \xi_1(u(0))(1-t) \to 0$ as $u \to 0$. Let us set, for convenience, $k(v) = h(v) - 1$ for $v \in \mathbb{R}$. It's easy to see that

$$\lim_{v\to 0} \frac{k(v)}{v} = 0. \tag{3.2}$$

Define the operator $H : C^1[0,1] \to C^1[0,1]$ as follows

$$H(u(t)) = \tilde{R}(\tau(u(t))) + \int_0^1 G(t,s)[f_0 u k(u') + k(u')\zeta(s,u) + \zeta(s,u)]ds. \tag{3.3}$$

It's easy to verify that $\lim\limits_{\|u\|\to 0} \frac{\|H(u)\|}{\|u\|} = 0$.

Let us consider the problem

$$\begin{aligned} u(t) &= \lambda R(\tau(u(t))) + \lambda \int_0^1 G(t,s) f(s,u(s)) h(u'(s)) ds \\ &= \lambda[R_0(\tau(u(t))) + \int_0^1 G(t,s) f_0 u(s) ds] + \lambda H(u) \\ &= \lambda L_{f_0} u(t) + \lambda H(u) \end{aligned} \tag{3.4}$$

as a bifurcation problem from the trivial solution axis. Then (3.4) is equivalent to a completely continuous operator equation from $\mathbb{R} \times E$ into E as follows:

$$u(t) = \lambda[L_{f_0}u(t) + H(u(t))]. \tag{3.5}$$

Clearly, $\lambda H(u) = \circ(\|u\|)$ near $u = 0$ uniformly on bounded λ intervals. By a similar argument of [12, 13, 23] and applying Lemma 2.1 to problem (1.7), $(\frac{\pi^2}{f_0}, 0)$ is a bifurcation point of (1.7) and there exists a continuum $\mathscr{C}$ of nontrivial solutions of problem (1.7) bifurcating from $(\frac{\pi^2}{f_0}, 0)$. Since $(0,0)$ is the only solution of problem (1.7) for $\lambda = 0$, it yields that $\mathscr{C} \cap (\{0\} \times E) = \emptyset$ and

$$\mathscr{C} \subseteq [\{(\frac{\pi^2}{f_0}, 0)\} \cup (\mathbb{R} \times P^+) \cup (\mathbb{R} \times P^-)]$$

and $\mathscr{C}$ is unbounded in $\mathbb{R} \times E$.

By Lemma 2.1, $\mathscr{C}$ can be split into $\mathscr{C}^+$ and $\mathscr{C}^-$ such that $\mathscr{C}^+ \cap \mathscr{C}^- = \{(\frac{\pi^2}{f_0}, 0)\}$. According the similar argument of [13, Theorem 1.1] and [20, Sections 6.4-6.5], we can show $\mathscr{C}^+ \subseteq [\{(\frac{\pi^2}{f_0}, 0)\} \cup (\mathbb{R} \times P^+)]$ and $\mathscr{C}^- \subseteq [\{(\frac{\pi^2}{f_0}, 0)\} \cup (\mathbb{R} \times P^-)]$. Moreover, it conclude that $\mathscr{C}^\nu$ is a unbounded continua of the set of nontrivial solutions of (1.7) joining $(\frac{\pi^2}{f_0},\ 0)$ to infinity, and $\mathscr{C}^\nu \backslash \{(\frac{\pi^2}{f_0},\ 0)\} \subset P^\nu$, $\nu \in \{+, -\}$. This together the fact $\|u\|_\infty < \frac{1}{\sqrt{\kappa}}$ for any fixed $(\lambda, u) \in \mathscr{C}^\nu$ implies that the projection of $\mathscr{C}^\nu$ on $\mathbb{R}$ is unbounded. Therefore,

$$(\frac{\pi^2}{f_0}, +\infty) \subseteq \mathrm{Proj}_{\mathbb{R}}(\mathscr{C}^\nu), \tag{3.6}$$

here $\mathrm{Proj}_{\mathbb{R}}(\mathscr{C}^\nu)$ is the projection of $\mathscr{C}^\nu$ on $\mathbb{R}$. □

Now, we will develop a bifurcation approach to treat the case $f_0 = \infty$ (resp. $f_0 = 0$).

Proposition 3.2 Suppose that (C1)-(C3) hold. If $f_0 = \infty$, then the pair $(0,0)$ is a bifurcation point of problem (1.7). Moreover, there exist two unbounded continua $\mathscr{C}^+$ and $\mathscr{C}^-$ of the set of nontrivial solutions of problem (1.7) in $\mathbb{R} \times E$ bifurcating from $(0,0)$, such that $\mathscr{C}^\nu \subseteq [\{(0,0)\} \cup (\mathbb{R} \times P^\nu)]$ and $\lim\limits_{\lambda \to +\infty} \|u\| = \frac{1}{\sqrt{\kappa}}$ for any $(\lambda, u) \in \mathscr{C}^\nu \subseteq \{(0,0)\}$.

Proof Define $f^{[n]} : [0,1] \times \mathbb{R} \to \mathbb{R}$ as follows

$$f^{[n]}(t,s) = \begin{cases} ns, & \text{if } s \in [-\frac{1}{n}, \frac{1}{n}], \\ (f(t, \frac{2}{n}) - 1) \cdot ns + 2 - f(t, \frac{2}{n}), & \text{if } s \in (\frac{1}{n}, \frac{2}{n}), \\ -(f(t, -\frac{2}{n}) + 1) \cdot ns - 2 - f(t, -\frac{2}{n}), & \text{if } s \in (-\frac{2}{n}, -\frac{1}{n}), \\ f(t,s), & \text{if } s \in (-\infty, \frac{2}{n}] \cup [\frac{2}{n}, \infty). \end{cases} \tag{3.7}$$

Then $f^{[n]}$ satisfies (C1),(C3) and

$$(f^{[n]})_0 = n. \tag{3.8}$$

Now, let us consider the auxiliary family of the problems

$$\begin{aligned} -u''(t) &= \lambda f^{[n]}(t, u(t)) h(u'(t)), \quad t \in (0,1), \\ u(0) &= \lambda g_1(u(0)), u(1) = \lambda g_2(u(1)). \end{aligned} \tag{3.9}$$

From (3.8), it follows that for $t \in [0,1]$ and every $u \in \mathbb{R}$,

$$f^{[n]}(t,s) = ns + \zeta^{[n]}(t,s),$$

where $\zeta^{[n]} \in C([0,1] \times \mathbb{R}, \mathbb{R})$ satisfies

$$\lim_{s\to 0} \frac{\zeta^{[n]}(t,s)}{s} = 0 \quad \text{uniformly for } t \in [0,1]. \tag{3.10}$$

Define the operator $\mathcal{H}^{[n]} : \mathbb{R} \times C^1[0,1] \to C^1[0,1]$ by

$$\mathcal{H}^{[n]}(u(t)) = [\tilde{R}(\tau(u(t))) + \int_0^1 G(t,s)[nuk(u') + k(u')\zeta^{[n]}(s,u) + \zeta^{[n]}(s,u)]ds.$$

Clearly, $\mathcal{H}^{[n]}$ is completely continuous and by the similar argument of (3.2) and (3.3), it follows that

$$\lim_{\|u\|\to 0} \frac{\|\mathcal{H}^{[n]}(u)\|}{\|u\|} = 0.$$

Observe that, for any λ, the couple $(\lambda, u) \in \mathbb{R} \times C^1[0,1]$ is a solution of the equation

$$u(t) = \lambda L_n u(t) + \lambda H^{[n]}(u(t)) \tag{3.11}$$

if and only if u is a solution of (3.9).

Let $\Sigma^{[n]} \subset \mathbb{R} \times C^1[0,1]$ be the closure of the set of all nontrivial solutions (λ, u) of (3.11) with $\lambda > 0$. As a straightforward consequences of Proposition 3.1, for each fixed n, there exists an unbounded component $C^\nu_{[n]} \subset \Sigma^{[n]}$ of one-sign solutions of (3.11) joining $(\frac{\pi^2}{n}, 0) \in C^\nu_{[n]}$ to infinity in $[0,\infty) \times P^\nu$. Moreover, $(\frac{\pi^2}{n}, 0) \in C^\nu_{[n]}$ is the only positive bifurcation point of (3.11) lying on a trivial solution line $u \equiv 0$ and the component $C^\nu_{[n]}$ joins the infinity in the direction of λ, here $\nu \in \{+,-\}$.

Note that from the compactness of the embedding $C^1[0,1] \hookrightarrow C[0,1]$, it concluded that $C^\nu_{[n]} \subset [0,\infty) \times C[0,1]$. By a similar argument of [12, Theorem 1.2] or [23], it is not difficult to verify that $C^\nu_{[n]}$ satisfies all conditions in Lemma 2.2 and consequently $\limsup_{n\to\infty} C^\nu_{[n]}$ contains a component C^ν which is unbounded. Moreover, from (3.7) and (3.8), we have that the component C^ν joins $(0,0)$ with $(\infty, \frac{1}{\sqrt{\kappa}})$ in $[0,\infty) \times P^\nu$. $\square$

Proposition 3.3 Suppose that (C1)-(C3) hold. If $f_0 = 0$, then the pair $(\infty, 0)$ is a bifurcation point of problem (1.7). Moreover, there exist two unbounded continua $\mathscr{C}^+$ and $\mathscr{C}^-$ of the set of nontrivial solutions of problem (1.7) in $\mathbb{R} \times E$ bifurcating from $(\infty, 0)$, such that $\mathscr{C}^\nu \subseteq [\{(\infty,0)\} \cup (\mathbb{R} \times P^\nu)]$ and $\lim_{\lambda\to+\infty} \|u\| = \frac{1}{\sqrt{\kappa}}$ or $\lim_{\lambda\to+\infty} \|u\| = 0$ for any $(\lambda, u) \in \mathscr{C}^\nu$.

Proof Define $f_{[n]} : [0,1] \times \mathbb{R} \to \mathbb{R}$ as follows

$$f_{[n]}(t,s) = \begin{cases} \frac{1}{n}s, & \text{if } s \in [-\frac{1}{n}, \frac{1}{n}], \\ (f(t,\frac{2}{n}) - \frac{1}{n^2}) \cdot ns + 2\frac{1}{n^2} - f(t,\frac{2}{n}), & \text{if } s \in (\frac{1}{n}, \frac{2}{n}), \\ -(f(t,-\frac{2}{n}) + \frac{1}{n^2}) \cdot ns - 2\frac{1}{n^2} - f(t,-\frac{2}{n}), & \text{if } s \in (-\frac{2}{n}, -\frac{1}{n}), \\ f(t,s), & \text{if } s \in (-\infty, \frac{2}{n}] \cup [\frac{2}{n}, \infty). \end{cases} \tag{3.12}$$

Then $f_{[n]}$ satisfies (C1),(C3) and

$$(f_{[n]})_0 = \frac{1}{n}. \tag{3.13}$$

Now, let us consider the auxiliary family of the problems

$$\begin{aligned} -u''(t) &= \lambda f_{[n]}(t,u(t))h(u'(t)), \quad t \in (0,1), \\ u(0) &= \lambda g_1(u(0)), u(1) = \lambda g_2(u(1)). \end{aligned} \tag{3.14}$$

From (3.12), it follows that for $t \in [0,1]$ and every $u \in \mathbb{R}$,

$$f_{[n]}(t,s) = \frac{1}{n}s + \zeta_{[n]}(t,s),$$

where $\zeta_{[n]} \in C([0,1] \times \mathbb{R}, \mathbb{R})$ satisfies $\lim\limits_{s\to 0} \frac{\zeta_{[n]}(t,s)}{s} = 0$ uniformly for $t \in [0,1]$.

As a straightforward consequences of Proposition 3.1, for each fixed n, there exists an unbounded component $C^{\nu}_{[n]}$ of one-sign solutions of (3.11) joining $(n\pi^2, 0) \in C^{\nu}_{[n]}$ to infinity in $[0,\infty) \times P^{\nu}$. Moreover, $(n\pi^2, 0) \in C^{\nu}_{[n]}$ is the only positive bifurcation point of (3.11) lying on a trivial solution line $u \equiv 0$ and the component $C^{\nu}_{[n]}$ joins the infinity in the direction of λ, here $\nu \in \{+,-\}$. By a similar argument of [12, Theorem 1.3] or [23], it is not difficult to verify that $C^{\nu}_{[n]}$ satisfies all conditions in Lemma 2.3 and consequently $\limsup\limits_{n\to\infty} C^{\nu}_{[n]}$ contains a component C^{ν} bifurcating from $(\infty,0)$, such that $\mathscr{C}^{\nu} \subseteq [\{(\infty,0)\} \cup (\mathbb{R} \times P^{\nu})]$ and $\lim\limits_{\lambda\to+\infty} \|u\| = \frac{1}{\sqrt{\kappa}}$ or $\lim\limits_{\lambda\to+\infty} \|u\| = 0$ for any $(\lambda,u) \in \mathscr{C}^{\nu}$. □

Proof of Theorem 1.1 (i) From Proposition 3.1, the pair $(\frac{\pi^2}{f_0}, 0)$ is a bifurcation point of problem (1.7). Moreover, there exist two unbounded continua $\mathscr{C}^+$ and $\mathscr{C}^-$ of the set of nontrivial solutions of problem (1.7) in $\mathbb{R} \times E$ bifurcating from $(\frac{\pi^2}{f_0}, 0)$, such that $\mathscr{C}^{\nu} \subseteq [\{(\frac{\pi^2}{f_0}, 0)\} \cup (\mathbb{R} \times P^{\nu})]$ and

$$\mathrm{Proj}_{\mathbb{R}}\mathscr{C}^{\nu} \supset [\frac{\pi^2}{f_0}, \infty).$$

For any one-sign solution $(\lambda,u) \in \mathbb{R} \times P^{\nu}$ of (1.7), only if $\lambda|g_1(u(0)) - g_1(u(1))| < \frac{1}{\sqrt{\kappa}}$, then u is the one-sign solution of (1.1). In fact, set $0 < \varepsilon < \min\{\frac{2f_0}{5\sqrt{\kappa}\pi^2}, \frac{2}{5\sqrt{\kappa}}\}$, there exists $\delta = \frac{\varepsilon}{2}$, such that $\lambda|g_1(u) - g_2(u)| \leqslant \lambda[|g_1^0 - g_2^0|\|u\| + 2\varepsilon] < \frac{1}{\sqrt{\kappa}}$ for any $(\lambda,u) \in B_\delta \cap \mathscr{C}^{\nu}$, where $B_\delta = \{u \in E \mid \|u\| < \delta\}$.

Thus, (1.1) has two continua C^+ and C^- of the set of nontrivial solutions of problem (1.1) satisfying $\lambda|g_1(u(0)) - g_2(u(1))| < \frac{1}{\sqrt{\kappa}}$ for any $(\lambda,u) \in C^{\nu}\backslash(\frac{\pi^2}{f_0}, 0)$, $\nu = \{+,-\}$.

(ii) From Proposition 3.2, there exist two unbounded continua $\mathscr{C}^+$ and $\mathscr{C}^-$ of the set of nontrivial solutions of problem (1.7) in $\mathbb{R} \times E$ bifurcating from $(0,0)$, such that

$$\mathrm{Proj}_{\mathbb{R}}\mathscr{C}^{\nu} = (0,\infty).$$

Thus,(1.1) has two continua C^+ and C^- of the set of nontrivial solutions of problem (1.1) satisfying $\lambda|g_1(u(0)) - g_2(u(1))| < \frac{1}{\sqrt{\kappa}}$ for any $(\lambda,u) \in C^{\nu}\backslash(0,0)$, $\nu = \{+,-\}$.

(iii) From Proposition 3.3, the pair $(\infty,0)$ is a bifurcation point of problem (1.7). Moreover, there exist two unbounded continua $\mathscr{C}^+$ and $\mathscr{C}^-$ of the set of nontrivial solutions of problem (1.7) in $\mathbb{R} \times E$ bifurcating from $(\infty,0)$, such that $\mathscr{C}^{\nu} \subseteq [\{(\infty,0)\} \cup (\mathbb{R} \times P^{\nu})]$ and $\lim\limits_{\lambda\to+\infty} \|u_\lambda\| = \frac{1}{\sqrt{\kappa}}$ or $\lim\limits_{\lambda\to+\infty} \|u_\lambda\| = 0$ for $(\lambda,u_\lambda) \in \mathscr{C}^{\nu}$. Moreover, there exists $\lambda_* > 0$, such that

$$\mathrm{Proj}_{\mathbb{R}}\mathscr{C}^{\nu} = [\lambda_*, \infty).$$

Hence, only the condition $\lambda|g_1(u(0)) - g_2(u(1))| < \frac{1}{\sqrt{\kappa}}$ for any $(\lambda, u) \in C^\nu \backslash (\infty, 0)$ hold, (1.1) has two continua C^+ and C^- of the set of nontrivial solutions of problem (1.1). □

Proof of Corollary 1 It's the directly consequences of Theorem 1.1, we omit it. □

Now, we give some examples to illustrate the main results.

Example 3.1 Let us consider the problem

$$-(\frac{u'}{\sqrt{1-u'^2}}) = \lambda f(u), \ \ u(0) = \frac{3\lambda}{5}(e^{u(0)} - 1), \quad u(1) = \frac{2\lambda}{3}u(1). \tag{3.15}$$

where $f(u) = \begin{cases} 20\ln(1+u) & u \geqslant 0; \\ -20\ln(1-u) & u \leqslant 0. \end{cases}$ Obviously, $\kappa = 1$, f satisfies (C1), (C3) and $f_0 = 20$ and $g_1(u) = \frac{3(e^u-1)}{5}$, $g_2(u) = \frac{2}{3}u$ satisfy (C2), $g_1^0 = \frac{3}{5} \in (\frac{4}{\pi^2}, 1)$, $g_2^0 = \frac{2}{3} \in (\frac{4}{\pi^2}, 1)$ and $|g_1^0 - g_2^0| = \frac{1}{15} < 1$. Since $|g_1(u) - g_2(u)| = |\frac{3(e^u-1)}{5} - \frac{2}{3}u| \leqslant |\frac{3(e^u-1)}{5}| \leqslant 1.64$ for any $u \in [0, 1)$, it follows that $|g_1(u) - g_2(u)| < \frac{f_0}{\pi^2} \approx 2.03$. Therefore, from Theorem 1.1 and Corollary 1, for any $\lambda \in (\frac{\pi^2}{20}, 0.61)$, (3.15) has at least two one-sign solution $u^+ > 0$ and $u^- < 0$ in $(0, 1)$ such that $(\lambda, u^\nu) \in C^\nu$, here $\nu = \{+, -\}$.

Example 3.2 Let us consider the problem

$$-(\frac{u'}{\sqrt{1-u'^2}}) = \lambda f(u), \ \ u(0) = \frac{3\lambda}{4}\sin u(0), \quad u(1) = \frac{3\lambda}{4}\sin u(1). \tag{3.16}$$

where $f(u) = \begin{cases} e^u & u \geqslant 0; \\ -e^{-u} & u \leqslant 0. \end{cases}$ Obviously, $\kappa = 1$, f satisfies (C1), (C3) and $f_0 = \infty$ and $g_1(u) = g_2(u) = \frac{3}{4}\sin u$ satisfy (C2), $g_1^0 = g_2^0 = \frac{3}{4} \in (\frac{4}{\pi^2}, 1)$, and $\frac{3}{4}\sin u$ is strictly increasing on $[0, 1]$. Since $|g_1(u(0)) - g_2(u(1))| = \frac{3}{4}|\sin u(0) - \sin u(1)| < \frac{3}{4}$, from from Theorem 1.1 and Corollary 1, for any $\lambda \in (0, \frac{4}{3})$, (3.16) has at least two one-sign solution $u^+ > 0$ and $u^- < 0$ in $(0, 1)$ such that $(\lambda, u^\nu) \in C^\nu$, here $\nu = \{+, -\}$.

Example 3.3 Let us consider the problem

$$-(\frac{u'}{\sqrt{1-u'^2}}) = \lambda f(u), \ \ u(0) = \frac{1}{2}\lambda, \quad u(1) = \frac{1}{2}\lambda. \tag{3.17}$$

where $f(u) = \begin{cases} u^p & u \geqslant 0; \\ -(-u)^p & u \leqslant 0, \end{cases}$ and $p > 1$ is a constant. Obviously, $\kappa = 1$, f satisfies (C1), (C3) and $f_0 = 0$ and $g_1(u) = g_2(u) = \frac{1}{2}$ satisfy (C2), $g_1^0 = g_2^0 = \frac{1}{2} \in (\frac{4}{\pi^2}, 1)$, and $|g_1(u(0)) - g_2(u(1))| = 0$. Hence, from Theorem 1.1 and Corollary 1, there exist $\lambda_* > 0$ such that problem (3.17) has at least four one-sign solution $u_1^+ > 0$, $u_2^+ > 0$, $u_1^- < 0$ and $u_2^- < 0$ in $(0, 1)$, such that $(\lambda, u_i^\nu) \in C^\nu$ for any $\lambda \in (\lambda_*, \infty)$, here $i = 1, 2$.

4. The continuum bifurcate from interval in the case of f is jumping at zero

In this section, we present the one-sign solution component of (1.1) bifurcate to the interval $[\frac{\pi^2}{\bar{f}^0}, \frac{\pi^2}{\bar{f}_0}] \times \{0\}$. From (C4), it's easy to see that $\bar{f}_0 u + \zeta_1(t, u) \leqslant f(t, u) \leqslant \bar{f}^0 u + \zeta_2(t, u)$ as $u \to 0$, where $\lim\limits_{u \to 0} \frac{\zeta_i(t,u)}{u} = 0$ uniformly for $u \to 0$, $i = 1, 2$.

Define the operators $H_i : C^1[0,1] \to C^1[0,1]$ as follows

$$H_1(u(t)) = \tilde{R}(\tau(u(t))) + \int_0^1 G(t,s)l_1(s,u,u')ds, \tag{4.1}$$

$$H_2(u(t)) = \tilde{R}(\tau(u(t))) + \int_0^1 G(t,s)l_2(s,u,u')ds, \tag{4.2}$$

where $l_1(s,u,u') = \bar{f}_0 uk(u') + k(u')\zeta_1(s,u) + \zeta_1(s,u)$, $l_2(s,u,u') = \bar{f}^0 uk(u') + k(u')\zeta_2(s,u) + \zeta_2(s,u)$ satisfying $\lim\limits_{\|u\|\to 0} \frac{l_i(t,u,u')}{u} = 0$ uniformly for $s \in [0,1]$, $i = 1,2$, and define the operator $G : C^1[0,1] \to C^1[0,1]$ as follows

$$G(u(t)) = \tilde{R}(\tau(u(t))) + \int_0^1 G(t,s)[f(s,u(s))h(u'(s)) - \rho u(s)]ds, \tag{4.3}$$

where $f_0 \leqslant \rho \leqslant f^0$ and by the similar argument of (3.2) and (3.3), $\lim\limits_{u\to 0} \frac{G(u)}{\|u\|} = 0$.

Let us consider the following problem

$$u(t) = \lambda[L_\rho u(t) + G(u(t))] = 0, \quad u \in E$$

as a bifurcation problem from the trivial solution $u \equiv 0$.

First, we shall show a priori bounds for the solutions of the following approximation problem

$$u = \lambda L_\rho u + \lambda G(u|u|^\epsilon) = 0, \quad u \in E \tag{4.4}$$

near the trivial solution.

Let $\mathfrak{S}$ be the closure of the set of nontrivial solution pairs of (4.4) in $\mathbb{R} \times E$. From Proposition 3.1, $(\frac{\pi^2}{\rho}, 0)$ is a bifurcation point of problem (4.4) and $\mathfrak{S}$ possesses two unbounded continuum $\mathscr{C}^+$ and $\mathscr{C}^-$, which join $(\frac{\pi^2}{\rho}, 0)$ to infinity, and $(C^\nu \backslash \{(\frac{\pi^2}{\rho}, 0)\}) \subseteq \mathbb{R} \times P^\nu$, here $\frac{\pi^2}{\rho}$ satisfies $\frac{\pi^2}{\bar{f}^0} \leqslant \frac{\pi^2}{\rho} \leqslant \frac{\pi^2}{\bar{f}_0}$, see [7].

Lemma 4.1 Let (C1)-(C2), (C4) hold and $\epsilon_n (0 \leqslant \epsilon_n \leqslant 1)$ be a sequence converging to 0. For each $k \in \mathbb{N}$, if there exists a sequence $(\lambda_n, u_n) \in \mathbb{R} \times P^\nu (\nu \in \{+,-\})$ such that (λ_n, u_n) is a solution of problem (4.4) with $\epsilon = \epsilon_n$ and (λ_n, u_n) converges to $(\lambda_0, 0)$ in $\mathbb{R} \times E$, then $\lambda_0 \in [\frac{\pi^2}{\bar{f}^0}, \frac{\pi^2}{\bar{f}_0}] =: I$.

Proof It's easy to see that

$$H_1(u_n) \leqslant \frac{G(u_n)}{u_n} \leqslant H_2(u_n), \qquad n \to \infty. \tag{4.5}$$

Hence, there exists a subsequence, relabeling if necessary, such that (4.4) can be rewritten as follows

$$u_n = \lambda_n L_{\tilde{\rho}} u_n + \lambda_n G(u_n|u_n|^{\epsilon_n}) = 0, \quad u_n \in E \tag{4.6}$$

where $\bar{f}_0 \leqslant \tilde{\rho} \leqslant \bar{f}^0$ and

$$\lim_{n\to+\infty} \frac{G(u_n|u_n|^{\epsilon_n})}{\|u_n\|} = 0.$$

Let $v_n = \frac{u_n}{\|u_n\|}$. Then it follows from (4.5) and (4.6) that

$$\lambda_n\big[L_{f_0} v_n + \frac{H_1(u_n)}{\|u_n\|}\big] \leqslant v_n = \lambda_n L_{\tilde{\rho}} v_n + \lambda_n \frac{G(u_n|u_n|^{\epsilon_n})}{\|u_n\|} \leqslant \lambda_n\big[L_{f^0} v_n + \frac{H_1(u_n)}{\|u_n\|}\big].$$

Hence, v_n is a relatively compact set in E by the compactness of A. Suppose that there is a subsequence of $\{v_n\}$, which is still written $\{v_n\}$ if necessary, such that $v_n \to v_* \in C^1[0,1],\ n \to +\infty$ and $v_* \in E$ with $\|v_*\| = 1$. By the similar argument of the proof of Proposition 3.1, it concludes that $v_* = \lambda_0 \tilde{\rho} v_*$, i.e.

$$\begin{aligned} &v_*'' + \lambda_0 \tilde{\rho} v_* = 0, \\ &v_*(0) = \lambda g_1^0 v_*(0),\ v_*(1) = \lambda g_2^0 v_*(1). \end{aligned} \tag{4.7}$$

Since $v_n \in P^\nu$, it's follows that $v_* \in P^\nu$ is the nontrivial solution of (4.7), which means that $\lambda_0 \tilde{\rho} = \pi^2$. This together with the comparison theorem of eigenvalue in [7], it follows that

$$\frac{\pi^2}{\bar{f}^0} \leqslant \lambda_0 = \frac{\pi^2}{\tilde{\rho}} \leqslant \frac{\pi^2}{\bar{f}_0}.$$

That is, $\lambda_0 \in I$.

Proposition 4.1 Let (C1)-(C2), (C4) hold and $I = [\frac{\pi^2}{\bar{f}^0}, \frac{\pi^2}{\bar{f}_0}]$. Then there are two unbounded components $\mathscr{C}^+$ and $\mathscr{C}^-$ of C, which bifurcates from $I \times \{0\}$ and satisfies $\mathscr{C}^\nu \subseteq [\mathbb{R} \times P^\nu] \cup \{I \times \{0\}\}$.

Proof Let $\mathscr{S}^\nu$ be the closure in $\mathbb{R} \times E$ of the set of all solutions (λ, u) of problem (1.7) with $u \in P^\nu$. We only to show the case $\nu = +$, the case $\nu = -$ can be shown similarly, we omit it. Let $\mathscr{C}^+$ be the component of $\mathscr{S}^+$ containing $I \times \{0\}$ and u is positive near $t = 0$.

We show that $\mathscr{C}^+ \subseteq (\mathbb{R} \times P^+) \cup \{I \times \{0\}\}$. For any $(\tilde{\lambda}, \tilde{u}) \in \mathscr{C}^+$, we have that $\tilde{u} \in P^+$. If $\tilde{u} \equiv 0$, there exists a solution sequence $(\lambda_n, u_n) \in \mathbb{R} \times P^+$, such that (λ_n, u_n) converges to $(\tilde{\lambda}, 0)$ in $\mathbb{R} \times E$. It follows from Lemma 4.1 implies that $\tilde{\lambda} \in I$. If $\tilde{u} \geqslant 0$ and $\tilde{u} \not\equiv 0$, then by the similar argument in the proof of Theorem 2.3 of [9], it concludes that $\tilde{u} > 0$ in $(0,1)$. By the similar argument of Theorem 1 in [6] with obvious changes, it follows that $\mathscr{C}^+$ is unbounded and there exists a neighborhood $\mathscr{U}$ of $\mathscr{C}^+$ such that $\partial\mathscr{U} \cap \mathscr{S}^+ = \emptyset$.

Let us consider the approximation problem (4.4) again. For any $\epsilon > 0$, it's not difficult to verify that $G(u|u|^\epsilon) = \circ(|u|)$ near $u = 0$. By the Rabinowitz's global bifurcation theorem [25, Theorem 1.3], there exists two continuum $\mathscr{C}_\epsilon^\nu$ of the set of nontrivial solutions pairs of (4.4) bifurcating from $(\pi^2/\tilde{\rho}, 0)$, such that they are both unbounded or $\mathscr{C}_\epsilon^+ \cap \mathscr{C}_\epsilon^- \neq \{(\pi^2/\tilde{\rho}, 0)\}$. By the similar argument of Proposition 3.1, $\mathscr{C}_\epsilon^+$ and $\mathscr{C}_\epsilon^-$ are both unbounded and

$$(\mathscr{C}_\epsilon^\nu \backslash \{(\pi^2/\tilde{\rho}), 0)\}) \subseteq \mathbb{R} \times P^\nu.$$

Since $\frac{\pi^2}{\tilde{\rho}} \in I$, there exists $(\lambda_\epsilon, u_\epsilon) \in \mathscr{C}_\epsilon^+ \cap \partial\mathscr{U}$ for all $\epsilon > 0$. Thus, there exists a sequence $\epsilon_n \to 0$ such that $(\lambda_{\epsilon_n}, u_{\epsilon_n})$ converges to a solution (λ, u) of problem (1.7). Clearly, $u \in \bar{P}^+$. If $u \equiv 0$, it follows from Lemma 3.1 implies that $\lambda \in I$, which contradict the definition of $\mathscr{U}$. Hence, $u \geqslant 0$ and $u \not\equiv 0$, It follows from Theorem 2.3 of [9] that $u > 0$ in $(0,1)$. Thus, $(\lambda, u) \in \mathscr{S}^+ \cap \mathscr{U}$, which contradicts the fact of $\partial\mathscr{U} \cap \mathscr{S}^+ = \emptyset$. $\square$.

Proof of Theorem 1.2 From Proposition 4.1, there exist two unbounded continua $\mathscr{C}^+$ and $\mathscr{C}^-$ of the set of nontrivial solutions of problem (1.7) in $\mathbb{R} \times E$ bifurcating from $I \times \{0\}$, such that $\mathscr{C}^\nu \subseteq [\mathbb{R} \times P^\nu] \cup \{I \times \{0\}\}$ and

$$\mathrm{Proj}_{\mathbb{R}} \mathscr{C}^\nu \supset [\frac{\pi^2}{\bar{f}_0}, \infty).$$

For any one-sign solution $(\lambda, u) \in \mathbb{R} \times P^\nu$ of (1.7), only if $\lambda|g_1(u(0)) - g_1(u(1))| < \frac{1}{\sqrt{\kappa}}$, then u is the one-sign solution of (1.1). Thus, (1.1) has two unbounded continua C^+ and C^- of the set of nontrivial solutions of problem (1.1) satisfying $\lambda|g_1(u(0)) - g_2(u(1))| < \frac{1}{\sqrt{\kappa}}$ for any $(\lambda, u) \in C^\nu \backslash I \times \{0\}$, $\nu = \{+, -\}$.

Proof of Corollary 2 It's the directly consequences of Theorem 1.2, we omit it. $\square$

Finally, we present some examples to illustrate the results of Theorem 1.2 and Corollary 2.

Example 4.1 Let us consider the problem

$$-\Big(\frac{u'}{\sqrt{1-u'^2}}\Big) = \lambda f(u), \;\; u(0) = \frac{3\lambda}{5}(e^{u(0)} - 1), \quad u(1) = \frac{2\lambda}{3}u(1). \tag{4.8}$$

where $f(u) = 20u + 2|u|$. Obviously, $\kappa = 1$, f satisfies (C1), (C4) and $\overline{f}_0 = 18 \neq \overline{f}^0 = 20$, and $g_1(u) = \frac{3(e^u-1)}{5}$, $g_2(u) = \frac{2}{3}u$ satisfy (C2), $g_1^0 = \frac{3}{5} \in (\frac{4}{\pi^2}, 1)$, $g_2^0 = \frac{2}{3} \in (\frac{4}{\pi^2}, 1)$ and $|g_1^0 - g_2^0| = \frac{1}{15} < 1$. Since $|g_1(u) - g_2(u)| = |\frac{3(e^u-1)}{5} - \frac{2}{3}u| \leqslant |\frac{3(e^u-1)}{5}| \leqslant 1.64$ for any $u \in [0, 1)$, it follows that $|g_1(u)-g_2(u)| < \frac{18}{\pi^2} \approx 1.83$. Therefore, from Theorem 1.2 and Corollary 2, for any $\lambda \in (\frac{\pi^2}{18}, 0.61)$, (4.8) has at least two one-sign solution $u^+ > 0$ and $u^- < 0$ in $(0, 1)$ such that $(\lambda, u^\nu) \in C^\nu$, here $\nu = \{+, -\}$.

Example 4.2 Let us consider the problem

$$-\Big(\frac{u'}{\sqrt{1-u'^2}}\Big) = \lambda f(u), \;\; u(0) = \frac{3\lambda}{4}\sin u(0), \quad u(1) = \frac{3\lambda}{4}\sin u(1). \tag{4.9}$$

where $f(u) = 20u + 2|u|$. Obviously, $\kappa = 1$, f satisfies (C1), (C4) and $\overline{f}_0 = 18 \neq \overline{f}^0 = 20$, and $g_1(u) = g_2(u) = \frac{3}{4}\sin u$ satisfy (C2), $g_1^0 = g_2^0 = \frac{3}{4} \in (\frac{4}{\pi^2}, 1)$, and $\frac{3}{4}\sin u$ is strictly increasing on $[0, 1]$. Since $|g_1(u(0)) - g_2(u(1))| = \frac{3}{4}|\sin u(0) - \sin u(1)| < \frac{3}{4} < \frac{18}{\pi^2}$, from from Theorem 1.2 and Corollary 2, for any $\lambda \in (\frac{\pi^2}{18}, \frac{4}{3})$, (4.9) has at least two one-sign solution $u^+ > 0$ and $u^- < 0$ in $(0, 1)$ such that $(\lambda, u^\nu) \in C^\nu$, here $\nu = \{+, -\}$.

Example 4.3 Let us consider the problem

$$-\Big(\frac{u'}{\sqrt{1-u'^2}}\Big) = \lambda f(u), \;\; u(0) = \frac{1}{2}\lambda, \quad u(1) = \frac{1}{2}\lambda. \tag{4.10}$$

where $f(u) = 20u + 2|u|$. Obviously, $\kappa = 1$, f satisfies (C1), (C4) and $\overline{f}_0 = 18 \neq \overline{f}^0 = 20$, and $g_1(u) = g_2(u) = \frac{1}{2}$ satisfy (C2), $g_1^0 = g_2^0 = \frac{1}{2} \in (\frac{4}{\pi^2}, 1)$, and $|g_1(u(0)) - g_2(u(1))| = 0$. Hence, from Theorem 1.2 and Corollary 2, for any $\lambda \in (\frac{\pi^2}{18}, +\infty)$, (4.10) has at least two one-sign solution $u^+ > 0$ and $u^- < 0$ in $(0, 1)$ such that $(\lambda, u^\nu) \in C^\nu$, here $\nu = \{+, -\}$.

References

[1] R. Bartnik, L. Simon, Spacelike hypersurfaces with prescribed boundary values and mean curvature, Comm. Math. Phys. 87(1) (1982), 131-152.

[2] C. Bereanu, P. Jebelean, P.J. Torres, Multiple positive radial solutions for a Dirichlet problem involving the mean curvature operator in Minkowski space, J. Funct. Anal. 265(4) (2013), 644-659.

[3] C. Bereanu, P. Jebelean, P.J. Torres, Positive radial solutions for Dirichlet problems with mean curvature operators in Minkowski space, J. Funct. Anal. 264 (2013), 270-287.

[4] C. Bereanu, J. Mawhin, Existence and multiplicity results for some nonlinear problems with singular ϕ-Laplacian, J. Differential Equations 243(20) (2007), 536-557.

[5] C. Bereanu, J. Mawhin, Nonhomogeneous boundary value problems for some nonlinear equations with singular ϕ-Laplacian, J. Math. Anal. Appl. 352 (2009), 218-233.

[6] H.Berestycki, On some nonlinear Sturm-Liouville problems, J. Differ. Equ. 26, (1977), 375-90.

[7] P.A. Binding, P.J. Browne, Sturm-Liouville problems with non-separated eigenvalue dependent boundary conditions, Proceedings of the Royal Society of Edinburgh, 130A, (2000), 239-247.

[8] S.Y. Cheng, S.T. Yao, Maximal spacelike hypersurface in the Lorente-Minkowski spaces, Ann. Math. 104 (1976), 407-419.

[9] I. Coelho, C. Corsato, F. Obersnel, P. Omari, Positive solutions of the Dirichlet problem for the one-dimensional Minkowski-Curvature equation, Adv. Nonlinear stud. 12(3) (2012), 621-638.

[10] C.Corsato, F. Obersnel, P. Omari, S. Rivetti, Positive solutions of the Dirichlet problem for the prescribed mean curvature equation in Minkowski space, J. Math. Anal. Appl. 405 (2013), 227-239.

[11] G.W. Dai, Bifurcation and one-sign solutions of the p-Laplacian involving a nonlinearity with zeros, Discrete Contin. Dyn. Syst. 36 (2016), 5323-5345.

[12] G.W. Dai, Bifurcation and nonnegative solutions for problems with mean curvature operator on general domain, Indiana Univ. Math. J. 67(6) (2018), 2103-2121.

[13] G.W. Dai, Global structure of one-sign solutions for problem with mean curvature operator, Nonlinearity 31(11) (2018), 5309-5328.

[14] E.N. Dancer, Bifurcation from simple eigenvalues and eigenvalues of geometric multiplicity one, Bull. Lond. Math. Soc. 34 (2002), 533-538.

[15] A.K. Drame, D.G. Costa, On positive solutions of one-dimensional semipositone equations with nonlinear boundary conditions, Appl. Math. Lett. 25 (2012) 2411-2416.

[16] D.R. Dunninger, Haiyan Wang, Multiplicity of positive solutions for a nonlinear differential equation with nonlinear boundary conditions, Ann. Polon. Math. 69(2) (1998), 155-165.

[17] J. Goddard, Q. Morris, R. Shivaji, Byungjae Son, Bifurcation curves for singular and nonsingular problems with nonlinear boundary conditions, Electron. J. Differential Equations 2018, Paper No. 26, 12 pp.

[18] W. Grenier, Classical Mechanics-Point Particles and Relativity, Springer, Berlin, 2004.

[19] Eun Kyoung Lee, R. Shivaji, Inbo Sim, Byungjae Son, Analysis of positive solutions for a class of semipositone p-Laplacian problems with nonlinear boundary conditions, Commun. Pure Appl. Anal. 18(3) (2019), 1139-1154.

[20] J. López-Gómez, Spectral Theory and Nonlinear Functional Analysis, Chapman & Hall/ CRC Res. Notes Math., vol. 426, Chapman & Hall/CRC, Boca Raton, FL, 2001.

[21] H. Luo, R.Y. Ma, The existence and applications of unbounded connected components, J. Appl. Math. 2014 (2014), 7 pages, http://doc.doi.org/10.1155/2014/780486.

[22] R.Y. Ma, Y.L. An, Global structure of positive solutions for nonlocal boundary value problems involving integral conditions, Nonlinear Anal. 71 (2009), 4364-4376.

[23] R.Y. Ma, H.L. Gao, Y.Q. Lu, Global structure of radial positive solutions for a prescribed mean curvature problem in a ball, J. Funct. Anal. 270(7) (2016), 2430-2455.

[24] R.Y. Ma, S.Y. Wang, Positive solutions for some semi-positone problems with nonlinear boundary conditions via bifurcation theory, Mediterr. J. Math. 17(1) (2020), 12 pp.

[25] P.H. Rabinowitz, Some global results for nonlinear eigenvalue problems, J. Funct. Anal. 7(1971), 487-513.