Deterministic sudden changes and stochastic fluctuation effects on stability and persistence dynamics of two-predator one-prey Model

Jawdat Alebraheem[1], Nasser S. Elazab[3]*, Mogtaba Mohammed[1,2], Ahmad Elmoasry[1], Anis Riyahi[1]

Department of Mathematics, College of Science Al Zufli, Majmaah University, Majmaah 11952, Saudi Arabia[1]

African Institute for Mathematical Sciences (AIMS), Cape Town, South Africa[2]

Department of Mathematics, Faculty of Science, Cairo University, Egypt[3]

j.alebraheem@mu.edu.sa , nazab@sci.cu.edu.eg, mogtaba.m@mu.edu.sa, a.elmoasry@mu.edu.sa **and** anis.r@mu.edu.sa **.**

Abstract: In this paper, we present new results on deterministic sudden changes and stochastic fluctuations effects on the dynamics of a two-predator one-prey model. We purpose to study the dynamics of the model with some impacting factors as problem statement. The methodology depends on investigating the seasonality and stochastic terms that which makes the predator- prey interactions more realistic. A theoretical analysis is introduced for studying the effects of sudden deterministic changes, using three different cases of sudden changes. We show that the system in a good situation presents persistence dynamics only as a stable dynamical behavior. However, the system in a bad situation leads to three main outcomes, as follows: first, constancy at the initial conditions of the prey and predators; second, extinction of the whole system; and third, extinction of both predators, resulting in the growth of the prey population until it reaches a peak carrying capacity. We perform numerical simulations to study effects of stochastic fluctuations, which show that, noise strength leads to an increase in the oscillations in the dynamical behavior and became more complex, finally, leads to extinction when the strength of the noise is high. The random noises transfer the dynamical behavior from the equilibrium case to the oscillation case, which describes some unstable environments.

Keywords: predator-prey model; dynamical behavior; deterministic sudden changes; stochastic; stability.

1. Introduction

1.1. Preface

Theoretical ecology has motivated many mathematicians to discuss different ideas and models from a purely mathematical standpoint [see 1-5]. Mathematical modeling is a useful tool to determine how a process works and to predict what may follow [3]. Many problems taken into consideration in mathematical ecology, seem simple, but are considered complicated problems due to the difficulty of determining the underlying ecological principles [6]. Nonlinear differential equations are used to mathematically describe predator-prey interactions. However, it is typically difficult to find a suitable mathematical analysis, especially when using non-linear terms.

The Lotka- Volterra model, which is a system of non-linear coupled first order ordinary differential equations, has been deemed the basic model for describing predator–prey interactions [7,8]. Two-predator one-prey models have the form of three species interactions and, thus, these systems are described by a system of three equations. Their dynamical behavior has been studied by some researchers [9-11].

Seasonality is an important factor, which plays a vital role in describing the changes and fluctuations in ecological systems with predator-prey interactions [12-18]. Additionally, there are many ecological factors, such as hunting and climate, which have varied effects (positive and/or negative) on the dynamical behaviors of the species. In the literature, a number of studies have investigated the effect of seasonality on the dynamical behavior of predator-prey systems, but most of these studies have focused on the search of chaotic cases in predator-prey systems [12-14, 16]. Several researchers [13, 15, 16] have used impulsive differential equations to describe steep changes, where they studied the systems over a long period. However, we will use novel tool over describing steep changes for a long time or as a new situation is introduced into the system, in this paper.

Deterministic models have been widely used to describe predator-prey interactions and their dynamics. Deterministic models are useful, due to their ability to follow them through mathematical analysis, and are an important mechanism for describing stable environments. However, random fluctuations appear in unstable environments, so deterministic models are difficult to describe these environments. In addition, the random noises are an important tool to conclude some unexpected dynamical behaviors of predator prey interactions. Stochastic models play an important role for describing more realistic dynamical modeling of ecosystems. May [19] introduced an important contribution when he investigated Stochastic differential equations for describing the limits of niche overlap in a randomly fluctuating environment. Recently, stochastic predator-prey models and their dynamics have been studied by some researchers [20-24].

The study of the dynamical behavior of predator-prey interactions has been considered to be an important subject in applied mathematics and mathematical ecology, due to its universal existence and importance [25]. Stability is one of the main important dynamics of predator-prey systems, which is typically the first property considered when studying dynamical behavior.

The persistence and extinction dynamics have also been discussed by many researchers [26–31], due to their importance. The analytical definitions of persistence and extinction are: For a population $p(t)$, if $p(0) > 0$ and $\lim_{t\to\infty} p(t) > 0$, then $p(t)$ persists, while If $p(0) > 0$ and $\lim_{t\to\infty} p(t) = 0$, then $p(t)$ becomes extinct. The geometric meaning of persistence is defined that each trajectory of a system of differential equations is bounded away from the coordinate axes, but the geometric meaning of extinction is that the trajectory of the system of differential equations touches the coordinate axes.

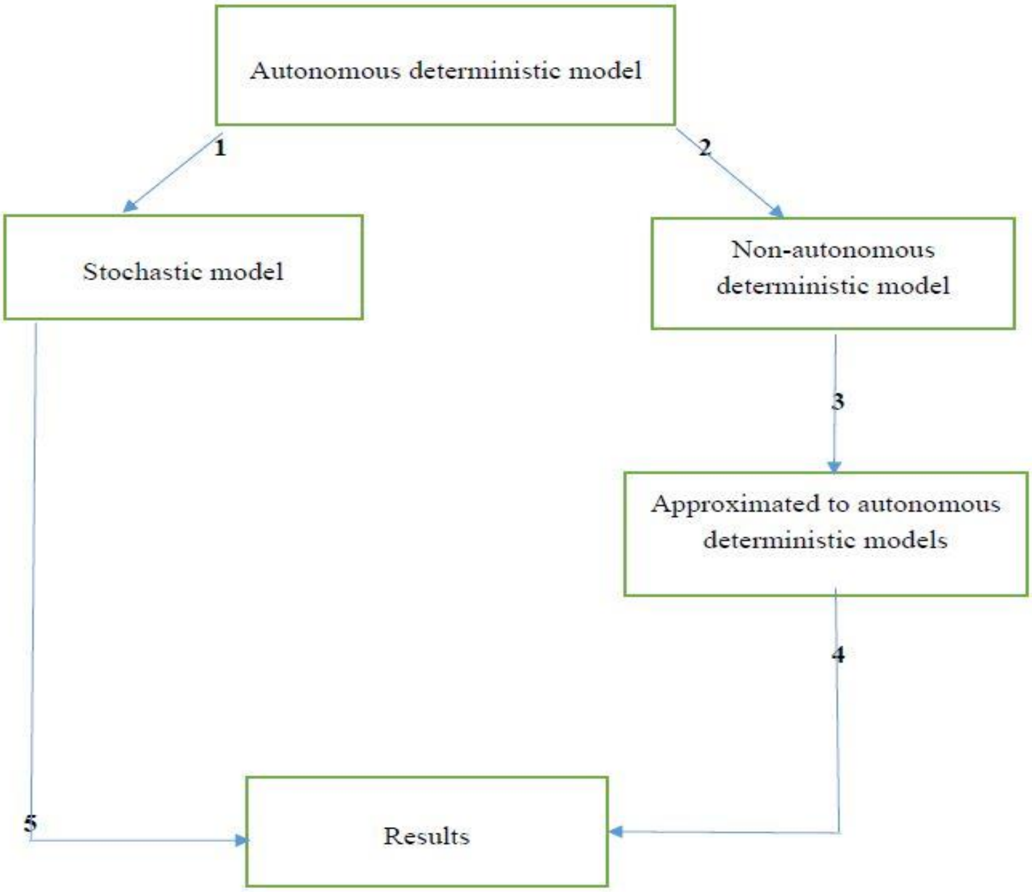
The novelty of our work is on consideration of the deterministic and stochastic models taken in such way, we are to get several results through our analysis. It should be noted that, we transfer the non-autonomous model to autonomous model(s) by using a novel tool that approximates the model to particular cases.

In this paper, we aim to investigate a cosinusoidal function in a Holling type I two-predator one-prey model, in order to study how sudden changes of the dynamics will effect on the dynamical behavior of the model. Investigating the cosinusoidal function and stochastic terms make our assumptions more realistic by concluding new cases of the model. We transfer the non-autonomous model to autonomous model(s) by using a novel tool which approximates the model to particular cases.

The paper is arranged as follows: we introduce in section 1, the preface and methodology of the paper. In Section 2, we present the mathematical model of the two-predator one-prey system and the seasonality function. In Section 3, we introduce forced deterministic models by sudden changes, divided to two situations: bad and good. In Section 4, we present a mathematical analysis of the deterministic sudden changes. In Section 5, we study the equilibrium points and conduct a stability analysis of these situations. In Section 6, we introduce stochastic model of the two-predator one-prey system. We introduce the numerical simulations in Section 7. In Section 8, we summarize our conclusions.

1.2. The Methodology

I summarize the mechanism that followed in this paper through the following figure



The methodology of arrays:

Array 1: Adding the stochastic term.

Array 2: Adding the seasonality function.

Array 3: Using the approximation method.

Array 4: Theoretical analysis.

Array 5: Numerical simulation.

2. Mathematical Model and Seasonality function

2.1. Mathematical Model

We use a non-dimensional system of Holling type I two-predator one-prey model [31] as follows:

$$\frac{dx}{dt} = x\left(1 - \frac{x}{k}\right) - \alpha xy - \beta xz,$$
$$\frac{dy}{dt} = -uy + e_1\alpha xy - e_1\alpha y^2 - c_1 yz, \tag{1}$$
$$\frac{dz}{dt} = -wz + e_2\beta xz - e_2\beta z^2 - c_2 yz,$$

Subject to initial conditions

$$x(0) = x_0 > 0\ , \mathrm{y}(0) = y_0 > 0\ , z(0) = z_0 > 0.$$

The biological meaning of the variables and parameters is as follows:

x: prey density.

y: first predator density.

z: second predator density.

k: carrying capacity of the system.

α and β: searching and capturing efficiency of predators y and z.

u and w: loss rates of predators y and z.

e_1 and e_2: birth rate of predator for each prey consumed.

c_1 and c_2: interspecific competition between the predators.

The parameters and initial conditions of the model (1) are supposed to be positive values.

Theorem 1. *All the solutions of system (1) which initiate in R_+^3 for $t \geq 0$ are bounded.*

Proof. According to the first equation of the system (1) we prove that it is bounded as follows:

$$\frac{dx}{dt} \leq x\left(1 - \frac{x}{k}\right) \tag{2}$$

The solution of the equation (2) is $\mathrm{x}(t) = \frac{\mathrm{k}e^{t+kc}}{-1\mp e^{t+kc}}$, c is integration constant.

then $0 \leq \lim_{t\to\infty} \sup x(t) \leq \mathrm{k}\ \forall t > 0$.

Then, we prove that $x(t) + y(t) + z(t) \leq Q.\ \forall t \geq 0$.

Let $R(t) = x(t) + y(t) + z(t)$.

The derivative of R with respect to t is

$$\frac{dR}{dt} = \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt} \tag{3}$$

$$\frac{dR}{dt} = \left(\left(1-\frac{x}{k}\right) - \alpha y - \beta z\right)x + (-u + e_1\alpha x - e_1\alpha y - c_1 z)y + (-w + e_2\beta x - e_2\beta z - c_2 y)z. \tag{4}$$

Since all the parameters are positive and the solutions initiating continue in nonnegative quadrant in R_+^3 and; we can suppose the following

$$\frac{dR}{dt} \le \left(\left(1-\frac{x}{k}\right)\right)x + (-u + e_1\alpha x - e_1\alpha y)y + (-w + e_2\beta x - e_2\beta z)z. \tag{5}$$

We have that

$$\max_{R_+}\{x\left(1-\frac{x}{k}\right)\} = \frac{rk}{4}. \tag{6}$$

By substituting in (5) become as follows

$$\frac{dR}{dt} \le \frac{k}{4} + (-u + e_1\alpha x - e_1\alpha y)y + (-w + e_2\beta x - e_2\beta z)z, \tag{7}$$

$$\frac{dR}{dt} \le \frac{k}{4} + (-u + e_1\alpha x - e_1\alpha y)y + (-w + e_2\beta x - e_2\beta z)z \ + R(t) - R(t), \tag{8}$$

The equation (8) can be written as follows

$$\frac{dR}{dt} + R(t) \le \frac{K}{4} + x + (-u + e_1\alpha x - e_1\alpha y + 1)y + (-w + e_2\beta x - e_2\beta z + 1)z. \tag{9}$$

Since $x(t) \le k$, then

$$\frac{dR}{dt} + R(t) \le \frac{5k}{4} + (-u + e_1\alpha - e_1\alpha y + 1)y + (-w + e_2\beta x - e_2\beta z + 1)z. \tag{10}$$

But

$$\max_{R_+}\{(-u + e_1\alpha k - e_1\alpha y + 1)y\} = \frac{-1 + e_1\alpha k^2 - 2e_1\alpha uk + u^2}{4e_1\alpha} \tag{11}$$

and

$$\max_{R_+}\{(-w + e_2\beta k - e_2\beta z + 1)z\} = \frac{-1 + e_2\beta k^2 - 2e_2\beta wk + w^2}{4e_2\beta} \tag{12}$$

So (10) becomes:

$$\frac{dR}{dt} + R(t) \le Q \tag{13}$$

where

$$Q = \frac{1}{4}\left(5k + \frac{-1 + e_1\alpha k^2 - 2e_1\alpha uk + u^2}{e_1\alpha} + \frac{-1 + e_2\beta k^2 - 2e_2\beta wk + w^2}{e_2\beta}\right) \tag{14}$$

Thus,

$R(t) \le Q + \rho e^{-t}$, ρ is a constant of integration.

$$\limsup_{t\to\infty} R(t) \le \lim_{t\to\infty} Q + \rho e^{-t}$$

Then $R(t) \le Q$. ■

2.2. Seasonality function

Cosinusoidal and sinusoidal functions [12, 14, 15] are used for describing the effects of seasonality on the dynamical behavior of the model (1). The cosinusoidal function is:

$$\mathrm{C}(\mathrm{t}) = 1 + \epsilon\cos(\mu\mathrm{t}), \tag{15}$$

where the parameter ϵ indicates the seasonality degree (or strength seasonal degree) and the parameter μ represents the angular frequency of the fluctuations caused by impacts.

3. Forced Deterministic Models by Sudden Changes

Events that happen unexpectedly (i.e., as the result of some environmental factors) on predator-prey interactions are called sudden changes. We apply the approximation method to describe suchchanges, in order to simplify the mathematical analysis of the model and make it biologically sensible. The approximation method has been applied for analyzing SIR models by some researchers [32, 33]. However, Alebraheem [20] has applied this technique to transfer a non-autonomous model containing seasonality terms to autonomous model(s) by approximating the model to particular cases, in order to study the dynamical behavior of predator-prey systems.

We apply the approximation method by taking the smallest and biggest values of the seasonality degree $\in$, where $0 \leq \in \leq 1$. Hence, we approximate the cosinusoidal function (Eq. 15) by the two following situations:

$$\mathrm{C}(\mathrm{t}) \cong \mathrm{P}(\mathrm{t}) = \begin{cases} 0 & \text{Bad situation} \\ 2 & \text{Good situation} \end{cases} \tag{16}$$

We interpret the 'bad' and 'good' situations as indicating surrounding circumstances are bad or good, respectively.

We investigate the cosinusoidal function (Eq. 15) in the system (2) through three different cases, as follows.

If sudden changes are forced for the whole system, we have:

$$\begin{aligned} \frac{dx}{dt} &= (x\left(1-\frac{x}{k}\right) - \alpha xy - \beta xz)P(t), \\ \frac{dy}{dt} &= (-uy + e_1\alpha xy - e_1\alpha y^2 - c_1 yz)P(t), \\ \frac{dz}{dt} &= (-wz + e_2\beta xz - e_2\beta S(t)z^2 - c_2 yz)P(t), \end{aligned} \tag{17}$$

If sudden changes are forced for the prey species through the growth rate of the prey, we have:

$$\begin{aligned} \frac{dx}{dt} &= P(t)x\left(1-\frac{x}{k}\right) - \alpha xy - \beta xz, \\ \frac{dy}{dt} &= -uy + e_1\alpha xy - e_1\alpha y^2 - c_1 yz, \end{aligned} \tag{18}$$

$$\frac{dz}{dt} = -wz + e_2\beta xz - e_2\beta z^2 - c_2 yz,$$

If sudden changes are forced for both predator' species through the birth rate of predator for each prey consumed, we have:

$$\frac{dx}{dt} = x\left(1 - \frac{x}{k}\right) - \alpha xy - \beta xz,$$
$$\frac{dy}{dt} = -uy + e_1 P(t)\alpha xy - e_1\alpha y^2 - c_1 yz, \tag{19}$$
$$\frac{dz}{dt} = -wz + e_2 P(t)\beta xz - e_2\beta S(t) z^2 - c_2 yz,$$

4. Mathematical Analysis of Deterministic Sudden Changes

In this section, we analyze the sudden changes effects on the system (2) mathematically, so we substitute the values of P(t) (Eq. 16) through three cases (i.e., systems (17), (18) and (19)) as follows:

The first case: If sudden changes have an effect on the whole system.

The bad situation: When we use $P(t) = 0$, the system (17) becomes as follows

$$\frac{dx}{dt} = 0, \tag{20.a}$$
$$\frac{dy}{dt} = 0, \tag{20.b}$$
$$\frac{dz}{dt} = 0. \tag{20.c}$$

The solutions of the equations (20.a), (20.b) and (20.c) are as follows:

$$\lim_{t\to\infty} x(t) = x_0,\ \lim_{t\to\infty} y(t) = y_0 \text{ and } \lim_{t\to\infty} z(t) = z_0.$$

We conclude from the system (20), the system will set at the initial conditions.

The good situation: when we use $P(t)$=2, the system (17) becomes as follows

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{k}\right) - 2\alpha xy - 2\beta xz,$$
$$\frac{dy}{dt} = -2uy + 2e_1\alpha xy - 2e_1\alpha y^2 - 2c_1 yz, \tag{21}$$
$$\frac{dz}{dt} = -2wz + 2e_2\beta xz - 2e_2\beta z^2 - 2c_2 yz.$$

The second case: If sudden changes have an effect on the prey species through the growth rate of the prey.

The bad situation: When we use $P(t) = 0$, the system (18) becomes as follows

$$\frac{dx}{dt} = (0 - \alpha xy - \beta xz), \tag{22.a}$$
$$\frac{dy}{dt} = (-uy + e_1\alpha xy - e_1\alpha y^2 - c_1 yz), \tag{22.b}$$
$$\frac{dz}{dt} = (-wz + e_2\beta xz - e_2\beta z^2 - c_2 yz), \tag{22.c}$$

The solution of the equation (22.a) become as follows:

Because $y > 0$ and $z > 0$, so we can reduce the equation (22.a) to become

$$\frac{dx}{dt} = -\alpha x$$

$x(t) = e^{-\alpha t}$, then $\lim_{t\to\infty} x(t) = 0$.

Since y and z follow x, then

$\lim_{t\to\infty} y(t) = 0$ and $\lim_{t\to\infty} z(t) = 0$.

The good situation: When we use $P(t) = 2$, the system (18) becomes as follows

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{k}\right) - \alpha xy - \beta xz,$$

$$\frac{dy}{dt} = -uy + e_1\alpha xy - e_1\alpha xy^2 - c_1 yz, \tag{23}$$

$$\frac{dz}{dt} = -wz + e_2\beta xz - e_2\beta S(t)z^2 - c_2 y$$

The third case: If sudden changes have an effect on both predator' species through the birth rate of predator for each prey consumed.

The bad situation: when we use $P(t) = 0$, the system (19) becomes as follows

$$\frac{dx}{dt} = x\left(1 - \frac{x}{k}\right) - \alpha xy - \beta xz, \tag{24.a}$$

$$\frac{dy}{dt} = -uy + 0 - 0 - c_1 yz, \tag{24.b}$$

$$\frac{dz}{dt} = -wz + 0 - 0 - c_2 yz, \tag{24.c}$$

For the equations (24.b) and (24.c), we remove the terms $-c_1 yz$ and $-c_2 yz$ because they are negative terms and to simplify the mathematical analysis, so we have

$$\frac{dy}{dt} = -uy - c_1 yz \cong -uy \tag{25}$$

$$\frac{\mathrm{dz}}{\mathrm{dt}} = -\mathrm{wz} - \mathrm{c}_2\mathrm{yz} \cong -\mathrm{wz} \tag{26}$$

The solution of equation (25) is

$$y(t) = y_0 e^{-ut}$$

Then the solution leads to

$$\lim_{t\to\infty} y(t) = 0$$

The solution of the equation (26) is

$$z(t) = z_0 e^{-ut}$$

The solution of this equation leads to

$$\lim_{t\to\infty} z(t) = 0$$

Since $\lim_{t\to\infty} y(t) = 0$ and $\lim_{t\to\infty} z(t) = 0$, then the equation (24.a) becomes as follows

$$\frac{dx}{dt} = x\left(1 - \frac{x}{k}\right) \tag{27}$$

The solution of the equation (24.a) is

$\mathrm{x}(t) = \frac{\mathrm{k}e^{t+kc}}{-1\mp e^{t+kc}}$,

where c is integration constant, then $\lim_{t\to\infty} x(t) = k$.

The good situation: when we use $P(t)$=2, the system (19) becomes as follows

$$\begin{aligned} \frac{dx}{dt} &= x\left(1-\frac{x}{k}\right) - \alpha xy - \beta xz, \\ \frac{dy}{dt} &= -uy + 2e_1\alpha xy - 2e_1\alpha y^2 - c_1 yz, \\ \frac{dz}{dt} &= -wz + 2e_2\beta xz - 2e_2\beta z^2 - c_2 yz, \end{aligned} \tag{28}$$

5. Equilibrium Points and Stability Analysis

One of the main dynamical behaviors is stability. We find the positive equilibrium points to study the stability. To check the local stability, we compute the variational matrices corresponding to each equilibrium point and using the Routh-Hurwitz criterion for studying the stability. To check the global stability, by construction Dulac function and Lyapunov function and using them to prove the global stability. We summarize the results of the equilibrium points of good situations when the sudden changes are forced through three cases in the following table:

Table 1. Description of the equilibrium points, positive equilibrium points conditions, and stability condition(s) to each case for good situation.

The equilibrium points	**positive equilibrium points conditions**	**The Dynamical behavior**
The first case		
$E_0 = (0,0,0)$	No conditions	Saddle point
$E_1 = (k,0,0)$	No conditions	Globally stable
$E_2 = (x, \mathrm{y}, 0) = (\frac{k(u+e_1)}{e_1\alpha k + e_1}, \frac{e_1\alpha k - u}{e_1\alpha^2 k + e_1\alpha}, 0)$	$e_1\alpha k > u$	Globally stable
$E_3 = (x,0,z) = (\frac{k(w+e_2)}{e_2\beta k + e_2}, 0, \frac{e_2\beta k - w}{e_2\beta^2 k + e_2\beta})$	$e_2\beta k > w$	Globally stable
$E_4 = (x,y,z) = (\frac{kc_1c_2 - kwc_1\alpha + \mathrm{k}c_2\beta u - \mathrm{k}e_1\alpha\beta w - \mathrm{k}ue_2\alpha\beta - \mathrm{k}e_1e_2\alpha\beta}{c_1c_2 + \mathrm{k}c_2e_1\alpha\beta + \mathrm{k}c_1e_2\alpha\beta - e_1e_2\alpha\beta - \mathrm{k}e_1e_2\alpha^2\beta - \mathrm{k}e_1e_2\alpha\beta^2},$	$e_1\alpha k > u$	Globally stable

$\frac{c_1w + ke_1\alpha\beta w - ke_2c_1\beta - e_2\beta u - ke_2\beta^2u + ke_1e_2\alpha\beta}{-c_1c_2 - kc_2e_1\alpha\beta - kc_1e_2\alpha\beta + e_1e_2\alpha\beta + ke_1e_2\alpha^2\beta + ke_1e_2\alpha\beta^2},$ $\frac{c_2u - ke_1\alpha c_2 - e_1\alpha w - ke_1\alpha^2w + ke_2\alpha\beta u + ke_1e_2\alpha\beta}{-c_1c_2 - kc_2e_1\alpha\beta - kc_1e_2\alpha\beta + e_1e_2\alpha\beta + ke_1e_2\alpha^2\beta + ke_1e_2\alpha\beta^2})$	$e_2\beta k > w$	
The second case		
$E_0 = (0,0,0)$	No conditions	Saddle point
$E_1 = (k,0,0)$	No conditions	Globally stable
$E_2 = (x,y,0) = (\frac{k(u+2e_1)}{e_1\alpha k + 2e_1}, \frac{2e_1\alpha k - 2u}{e_1\alpha^2k + 2e_1\alpha}, 0)$	$e_1\alpha k > u$	Globally stable
$E_3 = (x,0,z) = (\frac{k(w+2e_2)}{wk + 2e_2}, 0, \frac{2e_2\beta k - 2w}{e_2\beta^2k + 2e_2\beta})$	$e_2\beta k > w$	Globally stable
$E_4 = (x,y,z) = (\frac{2c_1c_2k - wc_1\alpha k + c_2\beta uk - ke_1\alpha\beta w - kue_2\alpha\beta - 2ke_1e_2\alpha\beta}{2c_1c_2 + c_2e_1\alpha\beta k + c_1e_2\alpha\beta k - 2e_1e_2\alpha\beta - ke_1e_2\alpha^2\beta - ke_1e_2\alpha\beta^2},$ $\frac{2c_1w + ke_1\alpha\beta w - 2ke_2c_1\beta - 2e_2\beta u - ke_2\beta^2u + 2ke_1e_2\alpha\beta}{-2c_1c_2 - kc_2e_1\alpha\beta - kc_1e_2\alpha\beta + 2e_1e_2\alpha\beta + ke_1e_2\alpha^2\beta + ke_1e_2\alpha\beta^2},$ $\frac{2c_2u - 2ke_1\alpha c_2 - 2e_1\alpha w - ke_1\alpha^2w + ke_2\alpha\beta u + 2ke_1e_2\alpha\beta}{-c_1c_2 - kc_2e_1\alpha\beta - kc_1e_2\alpha\beta + e_1e_2\alpha\beta + ke_1e_2\alpha^2\beta + ke_1e_2\alpha\beta^2})$	$e_1\alpha k > u$ $e_2\beta k > w$	Globally stable
The third case		
$E_0 = (0,0,0)$	No conditions	Saddle point
$E_1 = (k,0,0)$	No conditions	Globally stable
$E_2 = (x,y,0) = (\frac{k(u+2e_1)}{2e_1\alpha k + 2e_1}, \frac{2e_1\alpha k - u}{2e_1\alpha^2k + 2e_1\alpha}, 0)$	$2e_1\alpha k > u$	Globally stable
$E_3 = (x,0,z) = (\frac{k(w+2e_2)}{2e_2\beta k + 2e_2}, 0, \frac{2e_2\beta k - w}{2e_2\beta^2k + 2e_2\beta})$	$2e_2\beta k > w$	Globally stable
$E_4 = (x,y,z) = (\frac{c_1c_2k - wc_1\alpha k + c_2\beta uk - 2ke_1\alpha\beta w - 2kue_2\alpha\beta - 4ke_1e_2\alpha\beta}{c_1c_2 + 2c_2e_1\alpha\beta k + 2c_1e_2\alpha\beta k - 4e_1e_2\alpha\beta - 4ke_1e_2\alpha^2\beta - 4ke_1e_2\alpha\beta^2},$ $\frac{c_1w + 2ke_1\alpha\beta w + 2ke_2c_1\beta + 2e_2\beta u + 2ke_2\beta^2u - 2ke_1e_2\alpha\beta}{-c_1c_2 - 2kc_2e_1\alpha\beta - 2kc_1e_2\alpha\beta + 4e_1e_2\alpha\beta + 4ke_1e_2\alpha^2\beta + 4ke_1e_2\alpha\beta^2}$ $\frac{c_2u - 2ke_1\alpha c_2 - 2e_1\alpha w - 2ke_1\alpha^2w + 2ke_2\alpha\beta u + 4ke_1e_2\alpha\beta}{-c_1c_2 - 2kc_2e_1\alpha\beta - 2kc_1e_2\alpha\beta + 4e_1e_2\alpha\beta + 4ke_1e_2\alpha^2\beta + 4ke_1e_2\alpha\beta^2}$	$e_1\alpha k > u$ $e_2\beta k > w$	Globally stable

We present only the proof of the first case and, in the same manner, the proofs of the second and third cases will be followed, so the proofs of second and third cases will be omitted.

Theorem 2. *(i) The trivial equilibrium point $E_0 = (0,0,0)$ is a saddle point.*

(ii) The peak equilibrium point $E_1 = (k,0,0)$is locally asymptotically stable in x-direction, but it is locally asymptotically stable in $y - z$ plane if it holds the conditions (29) and (30).

Proof. (i) We compute the variational matrix of $E_0$is given as follows

$$M_1 = \begin{pmatrix} \mathbf{2} & 0 & 0 \\ 0 & -2u & 0 \\ 0 & 0 & -2w \end{pmatrix}$$

Through the variational matrix M_1, we see that the eigenvalues of y-direction and z-direction is negative, but the eigenvalue of x-direction is positive, this explains that the manifold is unstable along x-direction, but stable along y-direction and along z-direction. Then, the trivial equilibrium point E_0 is saddle point.

(ii) The variational matrix of E_1 is given as follows

$$M_2 = \begin{pmatrix} -\mathbf{2} & -2\mathrm{k}\alpha & -2\mathrm{k}\beta \\ 0 & -2u + 2e_1\alpha k & 0 \\ 0 & 0 & -2w + 2e_2\beta k \end{pmatrix}$$

Through the variational matrix M_2, we notice that the equilibrium point E_1 is locally asymptotically stable, if the following conditions are satisfied:

$u > e_1\alpha k$ (29)

$w > e_2\beta k$ (30)

Theorem 3. *The peak equilibrium point* $E_1 = (k, 0, 0)$ *is globally asymptotically stable under the following conditions:*

$\mathrm{u} > 4e_1\alpha^2 k$ (31)

$\mathrm{w} > 4e_2\beta^2 k$ (32)

Proof. Consider the following Lyapunov function about E_1;

$$V_1 = (x - \mathrm{k} - \mathrm{k}\ln(\tfrac{x}{\mathrm{k}})) + \frac{y}{2e_1\alpha} + \frac{z}{2e_2\beta} \quad (33)$$

V_1 is a continuously differentiable real valued function defined on R_+^3. Therefore, we have

$$\frac{dV_1}{dt} = (1 - \frac{\bar{x}}{x})\frac{dx}{dt} + \frac{1}{2e_1\alpha}\frac{dy}{dt} + \frac{1}{2e_2\beta}\frac{dz}{dt} \quad (34)$$

$$\frac{dV_1}{dt} = B_1\frac{(x-\mathrm{k})}{x}x\left[1 - \frac{x}{k} - 2\alpha y - 2\beta z\right] + \frac{y}{2e_1\alpha}[-u + 2e_1\alpha x - 2e_1\alpha y - c_1 z] + \frac{z}{2e_2\beta}[-w + 2e_2\beta x - 2e_2\beta z - c_2 y] \quad (35)$$

$$\frac{dV_1}{dt} = -\frac{1}{k}(x - \mathrm{k})^2 - y^2 - z^2 - 2\alpha xy - 2\beta xz - \left(\frac{2e_2\beta c_1 + 2e_1\alpha c_2}{4e_1e_2\alpha\beta}\right)yz - \left(\frac{\mathrm{u} - 4e_1\alpha^2 k}{2e_1\alpha}\right)y - \left(\frac{\mathrm{w} - 4e_2\beta^2 k}{2e_2\beta}\right)z \quad (36)$$

If the conditions (31) and (32) are satisfied, then we obtain that $\frac{dV_1}{dt} < 0$ **for any point in** R_+^3**.**

Theorem 4. (i) The equilibrium point E_2 is globally asymptotically stable in the interior of the positive quadrant of $x - y$ plane.

(ii) The equilibrium point E_3 is globally asymptotically stable in the interior of the positive quadrant of $x - z$ plane.

We prove part (i) and, in the same manner, part (ii) can be proved.

Proof. Let $G(x,y) = \frac{1}{xy}$. G is a Dulac function. It is continuously differentiable in the positive quadrant of the $x - y$ plane $A = \{\{(x,y)|x > 0, y > 0\}$.

$$N_1(x,y) = 2x(1 - \frac{x}{k}) - 2\alpha xy,$$

$$N_2(x,y) = -2uy + 2e_1\alpha xy - 2e_1\alpha y^2.$$

Thus, $\Delta(GN_1, GN_2) = \frac{\partial(GN_1)}{\partial x} + \frac{\partial(GN_2)}{\partial y} = \frac{-2}{yk} - \frac{2e_1\alpha}{x}$.

We find that $\Delta(\mathrm{GN}_1, \mathrm{GN}_2) < 0$ for all $x > 0$ and $y > 0$ in the positive quadrant of the $x - y$ plane. By using Bendixson–Dulac criterion, there is no periodic solution in the interior of the positive quadrant of the $x - y$ plane. E_2 is globally asymptotically stable in the interior of the positive quadrant of the $x - y$ plane.

Theorem 5. The persistence equilibrium point $\hat{E} = (\hat{x}, \hat{y}, \hat{z})$ of the system (21) is globally asymptotically stable.

Proof. We use Lyapunov function to prove the global stability of positive equilibrium point $\hat{E}$ as follows:

$$V = B_1(x - \hat{x} - \hat{x}\ln(\tfrac{x}{\hat{x}})) + B_2(y - \hat{y} - \hat{y}\ln(\tfrac{y}{\hat{y}})) + B_3(z - \hat{z} - \hat{z}\ln(\tfrac{z}{\hat{z}})) \tag{37}$$

The equation (37) can be expressed as follows:

$$V = B_1h_1(x,\bar{x}) + B_2h_2(y,\bar{y}) + B_3h_3(z,\bar{z}) \tag{38}$$

where $h_1(x,\bar{x}) = x - \bar{x} - \bar{x}\ln(\frac{x}{\bar{x}})$

$$h_2(y,\bar{y}) = y - \bar{y} - \bar{y}\ln(\tfrac{y}{\bar{y}})$$

$$h_3(z,\bar{z}) = z - \bar{z} - \bar{z}\ln(\tfrac{z}{\bar{z}})$$

System (21) can be written as:

$$\frac{dx}{dt} = xJ(x,y,z)$$

$$\frac{dy}{dt} = yL_1(x,y,z) \tag{39}$$

$$\frac{dz}{dt} = zL_2(x,y,z)$$

where

$$J(x,y,z) = 2 - 2\frac{x}{k} - 2\alpha y - 2\beta z$$

$$L_1(x,y,z) = -2u + 2e_1\alpha x - 2e_1\alpha y - 2c_1 z \tag{40}$$

$$L_2(x,y,z) = -2w + 2e_2\beta x - 2e_2\beta z - 2c_2 y$$

Let

$$h'(\lambda,\bar{\lambda}) = \frac{\partial h}{\partial \lambda}(\lambda,\bar{\lambda}) = 1 - \frac{\bar{\lambda}}{\lambda} \tag{41}$$

We compute the derivative of V along the trajectories of system (21).

$$\frac{dV}{dt} = B_1 {h_1}'(x,\bar{x})\frac{dx}{dt} + B_2 {h_2}'(y,\bar{y})\frac{dy}{dt} + B_3 {h_3}'(z,\bar{z})\frac{dz}{dt} \tag{42}$$

which is

$$\frac{dV}{dt} = B_1(1-\frac{\bar{x}}{x})\frac{dx}{dt} + B_2(1-\frac{\bar{y}}{y})\frac{dy}{dt} + B_3(1-\frac{\bar{z}}{z})\frac{dz}{dt} \tag{43}$$

$$\frac{dV}{dt} = B_1\frac{(x-\bar{x})}{x}x[J(x,y,z)] + B_2\frac{(y-\bar{y})}{y}y[L_1(x,y,z)] + B_3\frac{(z-\bar{z})}{z}z[L_2(x,y,z)] \tag{44}$$

The equation (44) can be expressed as follows:

$$= B_1\frac{(x-\bar{x})}{x}x[J(x,y,z) - J(\bar{x},\bar{y},\bar{z})] + B_2\frac{(y-\bar{y})}{y}y[L_1(x,y,z) - L_1(\bar{x},\bar{y},\bar{z})] + B_3\frac{(z-\bar{z})}{z}z[L_2(x,y,z) - L_2(\bar{x},\bar{y},\bar{z})] \tag{45}$$

where $J(\bar{x},\bar{y},\bar{z}) = 0$, $L_1(\bar{x},\bar{y},\bar{z}) = 0$ and $L_2(\bar{x},\bar{y},\bar{z}) = 0$, so we have

$$\frac{dV}{dt} = B_1(x-\hat{x})\left[\left[2-2\frac{x}{k}-2\alpha y-2\beta z\right]-\left[2-2\frac{\hat{x}}{k}-2\alpha\hat{y}\ -2\beta\hat{z}\right]\right] + B_2(y-\hat{y})[[-2u+2e_1\alpha x - 2e_1\alpha y - 2c_1 z] - [-2u+2e_1\alpha\hat{x}\ -2e_1\alpha\hat{y}-2c_1\hat{z}]] + B_3(z-\hat{z})[[-2w+2e_2\beta x - 2e_2\beta z - 2c_2 y] - [-2w+2e_2\beta\hat{x}\ -2e_2\beta\hat{z} - 2c_2\hat{y}]] \tag{46}$$

Rearrange the terms of equation (46):

$$\frac{dV}{dt} = \frac{B_1}{k}(x-\hat{x})[-2(x\ -\hat{x}) - 2\alpha(y-\hat{y}) - 2\beta(z-\hat{z})] + B_2(y-\hat{y})[2e_1\alpha(x-\hat{x}) - 2e_1\alpha(y-\bar{y}) - 2c_1(z-\hat{z})] + B_3(z-\hat{z})[2e_2\beta(x-\hat{x}) - 2e_2\beta(z-\hat{z}) - 2c_2(y-\hat{y}) \tag{47}$$

$$\frac{dV}{dt} = \frac{B_1}{k}[-2(x-\hat{x})^2 - 2\alpha(x-\hat{x})(y-\hat{y}) - 2\beta(x-\hat{x})(z-\hat{z})] + B_2[2e_1\alpha(y-\hat{y})(x-\hat{x}) - 2e_1\alpha(y-\hat{y})^2 - 2c_1(y-\hat{y})(z-\hat{z})] + B_3[2e_2\beta(z-\hat{z})(x-\hat{x}) - 2e_2\beta(z-\hat{z})^2 - 2c_2(z-\hat{z})(y-\hat{y})] \tag{48}$$

By selecting $B_1 = 1$, $B_2 = \frac{1}{e_1}$, and $B_3 = \frac{1}{e_2}$, so

$$\frac{dV}{dt} = -\frac{2}{k}(x-\hat{x})^2 - 2\alpha(y-\hat{y})^2 - 2\frac{c_1}{e_1}(y-\hat{y})(z-\hat{z}) - 2\beta(z-\hat{z})^2 - 2\frac{c_2}{e_2}(z-\hat{z})(y-\hat{y})] \tag{49}$$

We find that $\frac{dV}{dt}$ is negative under no conditions (i.e. no restrictions on parameters).

From the theorem (5), we notice that the persistence dynamical behaviors of the system (21) is globally stable.

6. Stochastic Model

In this section, we give numerical simulation to the stochastic version of our model. This consideration is due to the prevalence of randomness in almost all wild animal life, which makes the use of stochastic differential equations more realistic and efficient to describe some predictions of dynamical behaviors, see Figures 2-4.

The standard Ito stochastic differential equation is written as follows [24]:

$$dx(t) = F\big(t, x(t)\big)dt + G\big(t, x(t)\big)dW(t), \quad x(\mathrm{t}_0) = x_0$$

where the first term represents the drift coefficient and the second term represents the random noise in environment, which is sometimes called Gaussian white noise.

We use a stochastic term in the deterministic model (2) as in the reference [24], so we have the following model:

$$\begin{aligned}
\frac{dx}{dt} &= x\left(1 - \frac{x}{k}\right) - \alpha xy - \beta xz + \sigma_1 x dW_1,\\
\frac{dy}{dt} &= -uy + e_1\alpha xy - e_1\alpha y^2 - c_1 yz + \sigma_2 y dW_2,\\
\frac{dz}{dt} &= -wz + e_2\beta xz - e_2\beta z^2 - c_2 yz + \sigma_3 z dW_3,
\end{aligned} \tag{50}$$

where σ_i, $i = 1,2,3$ represent the strength of noise, and dW_i, $i = 1,2,3$ is a standard Wiener or Brownian motion processes.

We have theoretically proven that the dynamical behavior of deterministic sudden changes is globally stable. In this section, we present the effects stochastic fluctuations on the dynamical behavior. The MATHEMATICA program was used to perform the numerical simulations. The values of parameters were selected to fulfil the positive values of a non-trivial equilibrium point, called co-existence point (i.e., to satisfy the conditions 29 and 30) in deterministic models. In addition, the values of $\sigma_i = \sigma$ were set as in [24], to represent three levels of noise strength; that is, low, medium and high noise strengths. The parameters and initial conditions values were taken as follows:

$k = 2.0$, $\alpha = 1.0$, $\beta = 1.4$, $e_1 = 0.6$, $e_2 = 0.65$, $c_1 = 0.07$, $c_2 = 0.04$, $u = 0.45$, $w = 0.6$, $x(0) = 0.6$, $y(0) = 0.3$, $z(0) = 0.25$

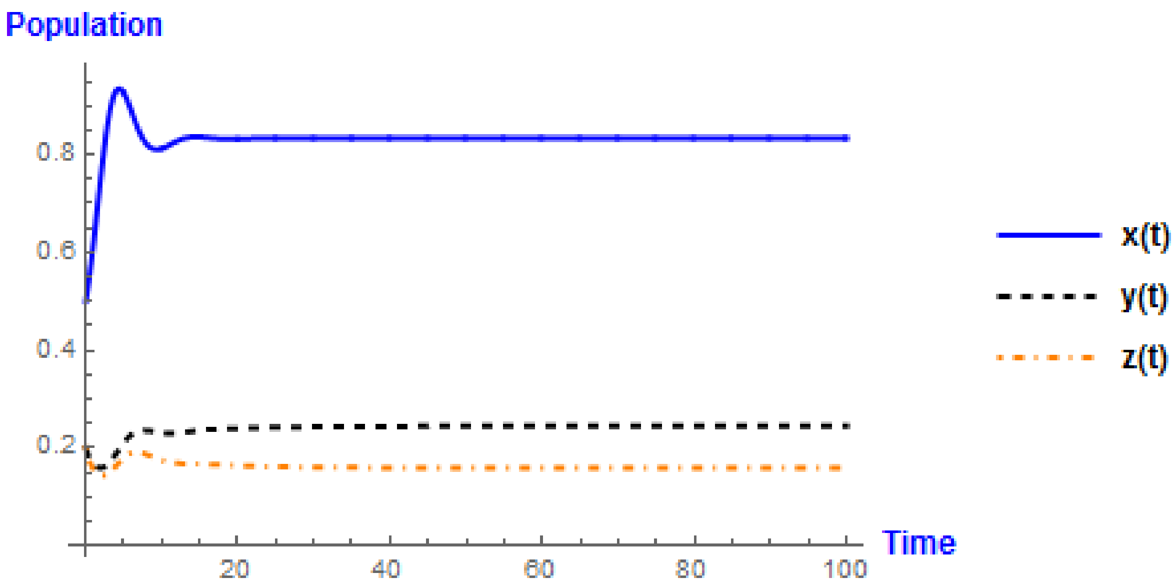


Figure 1. Time series of system (50) without noise ($\sigma = \mathbf{0.0}$).

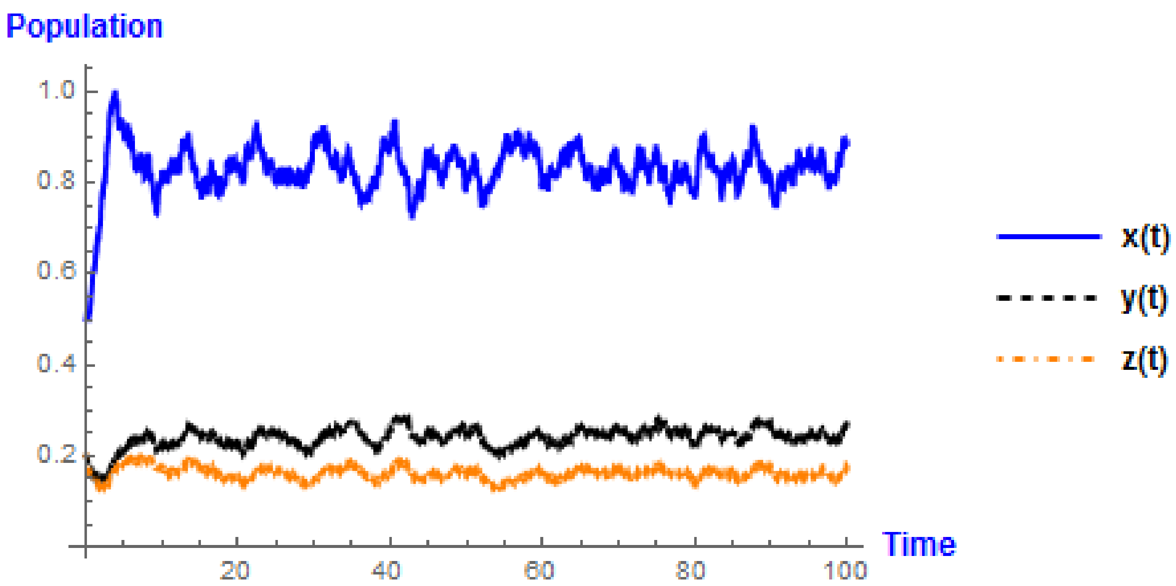


Figure 2. Time series of system (50) with low noise strength ($\sigma = \mathbf{0.05}$).

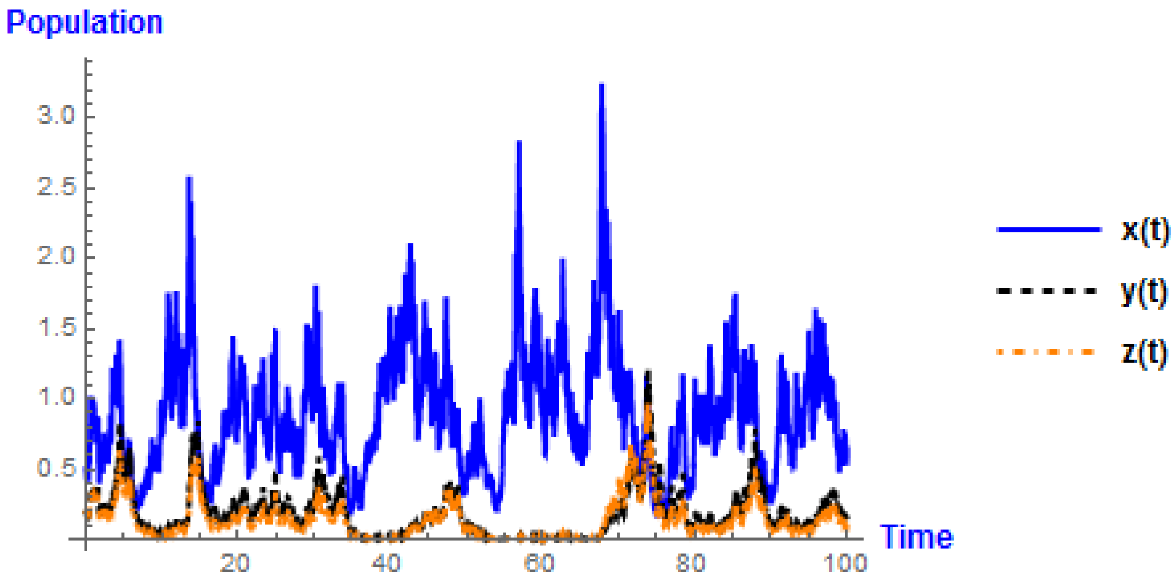


Figure 3. Time series of system (50) with strength medium noise strength ($\sigma = \mathbf{0.5}$).

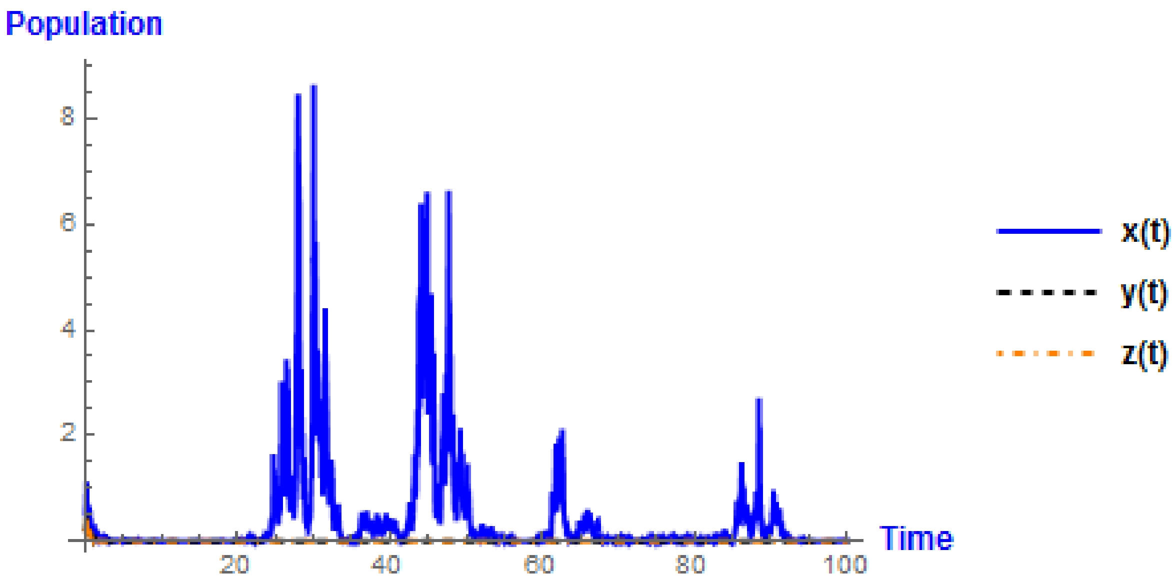


Figure 4. Time series of system (50) with strength high noise strength ($\sigma = \mathbf{1.4}$).

Figure 1 represents the dynamical behavior of the model (50) without noise (i.e., $\sigma = 0$), which gives the deterministic model. In figure 1, the dynamical behavior of the species was stable co-existence, which corresponds with the theoretical analysis of the deterministic model. Figure 2 represents the dynamical behavior of the model (50) when the strength of the noise was low. Figure 2 shows that the dynamical behavior of the species was co-existence with smooth oscillations. However, with an increase in the strength of noise, such as the medium-noise situation shown in Figure 3, the dynamical behavior of the species was co-existence with sharp oscillations, which may lead to extinction. Figure 4 represents the dynamical behavior of the model (50) when the noise strength was high. The dynamical behavior of the species became more complex and they tended to extinction. The random noises transfer the dynamical behavior from the equilibrium case to the oscillation case, which describes some unstable environments.

We conclude that increasing the noise strength led to an increase of oscillation in the dynamical behavior, which can be interpreted biologically as increasing the probability of extinction, representing the worst-case scenario of dynamical behavior. This result corresponds with the numerical simulations. These results correspond well with the results of reference [24] with the difference being the mathematical model used, whereas increasing the noise strength led to an increase in oscillations in the dynamical behavior, finally leading to extinction when the noise strength was high.

7. Conclusion

We investigated the seasonality effects in a Holling type I two-predator one-prey model, which can more realistically describe the species of interaction more realistic. We transferred the non-autonomous models to autonomous models by approximating the model to particular cases representing sudden changes. We classified the situations to bad and good situations, according to the surrounding circumstances. We introduced a mathematical analysis of sudden changes and discussed the equilibrium points and stability. We made the following conclusions:

For the bad situations, we obtained the following outcomes:

- If sudden fluctuations have an effect on the whole system, then the system will remain at the initial conditions.
- If sudden fluctuations have an effect on the prey species, then both predators species and the prey species will go extinct.
- If sudden fluctuations have an effect on both predator' species, then the prey species will reach carrying capacity, while both of the predator' species will go extinct.
- The equilibrium points of each case were obtained and found to be stable.

For the good situations, we obtained the following outcomes:

- The one-prey two-predators system interacted through three different systems (21, 23, and 28) which represented the three cases.
- We obtained five positive equilibrium points, in each case.
- We proved that the general dynamical behavior is globally stable, except for the trivial equilibrium point (which was a saddle point).
- The dynamical behavior in the case of a good situations presented the persistence dynamic is only a stable dynamical behavior.

Through numerical simulations, we presented effects of stochastic fluctuations on interactions, which showed that noise strength led to an increase in the oscillations in dynamical behavior and became more complex, finally leading to extinction when the noise strength was high.

Acknowledgments: This work is funded by the Basic Science Research Unit, Scientific Research Deanship at Majmaah University under the research project No. RGP-2019-4. The author is extremely grateful to Majmaah University, Deanship of Scientific Research and Basic Science Research Unit, Majmaah University.

Conflicts of Interest: The author declares no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, and in the decision to publish the results.

Data Availability: The data used to support the findings of this study are included within the article.

References

1. Maynard-Smith, J. *Models in ecology*; Cambridge University Press, London, 1978.
2. Haberman, R. *Mathematical Models Mechanical Vibrations, Population Dynamics, and Traffic Flow*; Society for Industrial and Applied Mathematics: Philadelphia, PA, USA, 1998.
3. Murray, JD. *Mathematical biology*; Vol. 2, springer, New York, 2002.
4. Rockwood, LL. *Introduction to population ecology*; Blackwell Publishing, Malden, Massachusetts, 2006.
5. Pastor, J. *Mathematical ecology of populations and ecosystems*; Wiley-Blackwell, USA, 2008.
6. Upadhyay R; Naji, R; Kumari, N. Dynamical complexity in some ecological models: effect of toxin production by phytoplankton. *Nonlinear Analysis: Modelling and Control* **2007**. 12(1), 123–138.
7. Lotka. Elements of Physical Biology. Math Meth Appl Sci **1925**. 36: 1263- 1280,
8. Volterra. *"Variazioni e fluttuazioni del numerod'individui in specie animaliconviventi"*. *Mem. Acad. Lincei Roma* **1926**, 2, 31-113.
9. Gakkhar, S.; Singh, B.; Naji, R.K. Dynamical behavior of two predators competing over a single prey. *Biosystems* **2007**, *90*, 808–817.
10. Alebraheem, J; Abu-Hassan, Y. Coexistence in different dynamics of two predators-one prey model. Proceeding of International Seminar on the Application of Science and Mathematics 2011 (ISASM 2011), Code number. MTH013. 1-3 November 2011.
11. Alebraheem, J.; Abu-Hassan, Y. Persistence of Predators in a Two Predators-One Prey Model with Non-Periodic Solution. *J. Appl. Sci*. **2012**, *6*, 943–956.
12. Gakkhar, S; Naji, RK. Seasonally perturbed prey-predator system with predator-dependent functional response. *Chaos, Solitons & Fractals* **2003**. 18(5), 1075–1083.
13. Zhang, S; Tan, D; Chen, LS. Chaos in periodically forced Holling type II predator-prey system with impulsive perturbations. *Chaos, Solitons and Fractals* **2006**. vol. 28, pp. 367-376.

14. Upadhyay, RK; Iyengar, S. Effect of seasonality on the dynamics of 2 and 3 species prey-predator systems. *Nonlinear Analysis: Real World Applications* **2007**. 6(3), 509–530.

15. Baek, H; Do, Y. Seasonal effects on a beddington-deangelis type predator-prey system with impulsive perturbations. *Abstract and Applied Analysis* **2009**. Article ID 695121 (2009), 19 pages. doi:10.1155/2009/695121.

16. Yu, H; Zhou, S; Agarwal, R; Sen, S. Effect of seasonality on the dynamical behavior of an ecological system with impulsive control strategy. *Journal of the Franklin Institute* **2011**. vol. 348, pp. 652-670.

17. Alebraheem, J.; Abu-Hassan, Y. Simulation of complex dynamical behaviour in prey predator model. In Proceedings of the 2012 International Conference on Statistics in Science, Business and Engineering, Langkawi, Malaysia, 10–12 September 2012. doi:10.1109/ICSSBE.2012.6396618

18. Alebraheem, J. Fluctuations in interactions of prey predator systems. *Sci. Int.* **2016**, *28*, 2357–2362.

19. R.M. May, *Stability and Complexity in Model Ecosystems*, Princeton: Princeton University Press, 1974.

20. Samanta, G.P. Stochastic analysis of a prey–predator system. Int. J. Math. Edu. Sci. Technol. 1994, 25, 797–803.

21. Bandyopadhyay M, Chakrabarti CG. Deterministic and stochastic analysis of a nonlinear prey-predator system. *J Biol Syst* **2003**, 11: 161-172.

22. Cai, G.Q.; Lin, Y.K. Stochastic analysis of the Lotka-Volterra model for ecosystems. Phys. Rev. E 2004, 70, 041910.

23. A. Maiti & G. P. Samanta. Deterministic and stochastic analysis of a ratio-dependent prey-predator system, *International Journal of Systems Science* **2006**, 37:12, 817-826, DOI: 10.1080/00207720600879252

24. L.M.Addison, Analysis of a predator-prey model: a deterministic and stochastic approach, *Journal of Biometrics & Biostatistics* **2017**, vol.8(4).

25. Kuang, Y; Beretta, E. Global qualitative analysis of a ratio-dependent predator-prey system. *Journal of Mathematical Biology* **1998**,36(4), 389–406.

26. Freedman, H.; Waltman, P. Persistence in models of three interacting predator-prey populations. *Math. Biosci.* **1984**, *68*, 213–231.

27. Dubey, B; Upadhyay R. Persistence and extinction of one-prey and two-predator system. *Nonlinear Analysis* **2004**. 9(4), 307–329.

28. Alebraheem, J.; Abu-Hassan, Y. The Effects of Capture Efficiency on the Coexistence of a Predator in a Two Predators-One Prey Model. *J. Appl. Sci.* **2011**, *11*, 3717–3724.

29. Alebraheem, J.; Abu-Hassan, Y. Efficient Biomass Conversion and its Effect on the Existence of Predators in a Predator-Prey System. *Res. J. Appl. Sci.* **2013**, *8*, 286–295.

30. Alebraheem, J.; Abu-Hassan, Y. Dynamics of a two predator–one prey system. *Comput. Appl. Math.* **2014**, *33*, 767–780, doi:10.1007/s40314-013-0093-8.

31. Alebraheem, J. Relationship between the Paradox of Enrichment and the Dynamics of Persistence and Extinction in Prey-Predator Systems. *Symmetry* **2018**, 10, 532.

32. Keeling, MJ; Rohani, P; Grenfell, BT. Seasonally forced disease dynamics explored as switching between attractors. *Physica D: Nonlinear Phenomena* **2001**. 148(3), 317–335.

33. Olinky, R; Huppert, A; Stone, L. Seasonal dynamics and thresholds governing recurrent epidemics. *Journal of Mathematical Biology* **2008**, 56(6), 827–839.