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Reasons for stability in the construction of derivative-free multistep iterative methods

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Abstract

In this paper, a deep dynamical analysis is made using tools from multidimensional real discrete dynamics of some derivative-free iterative methods with memory. They all have good qualitative properties, but one of them (due to Traub) shows the same behavior as Newton's method on quadratic polynomials. Then, the same techniques are employed to analyze the performance of several multipoint schemes with memory, whose first step is Traub's method, but their construction was made using different procedures. Therefore, their stability is analyzed, showing which is the best in terms of the wideness of basins of convergence or the existence of free critical points that would yield convergence towards different elements from the desired zeros of the nonlinear function. Therefore, the best stability properties are linked with the best estimations made in the iterative expressions rather than their simplicity. These results have been checked with a numerical and graphical comparison with many other known methods with and without memory, with different orders of convergence, with excellent performance.

KEYWORDS:

Nonlinear equations; Multipoint root-solver with memory; Derivative-free; Traub's method; Stability analysis.

1 | INTRODUCTION

A wide variety of physical processes observed in real life are nonlinear, as are many systems underlying engineering problems. If in order to simplify the problem, they are linearised, much of the complexity disappears, but the solution obtained is a worse approximation to the real solution. Iterative processes are beneficial in this context, approximating the solution of the nonlinear equations, $f(x) = 0$, that model this type of problem.

Newton's method is the best-known fixed-point iterative method, but it represents only a subclass of numerical procedures: memoryless iterative processes. This kind of scheme uses only the current iteration to compute the next one, building the sequence that eventually converges to the solution. However, there are iterative schemes that use more than one known iterate to calculate the next: these are known as iterative procedures with memory, and the best known is the secant method, whose iterative expression is

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})},\ n = 1, 2, \dots,$$

where x_0 and x_1 are the initial estimations. The simplicity of its expression makes it very useful, but the quadratic order of convergence of Newton's scheme is lost, reaching superlinear convergence. To overload this inconvenient, Traub in [2] designed, among others, the derivative-free scheme (DF, for short) with memory,

$$x_{n+1} = x_n - \frac{f(x_n)}{f[x_{n-1}, x_{n-2}] - f[x_n, x_{n-2}] - f[x_n, x_{n-1}]}, \tag{1}$$

denoted by TM, where $f[x, y] = \dfrac{f(x) - f(y)}{x - y}$, that increases the order of convergence from 1.618 (of secant scheme) up to 1.839. It is lower than other DF methods as Steffensen's scheme (without memory), but it has good numerical properties. This scheme has been used as the first step of several higher-order multipoint methods, with good results in terms of robustness and applicability (see the works by Neta [17, 18]).

In recent years, different iterative schemes with memory have been designed (a good overview can be found in [28]), mostly derivative-free. These have been constructed with increasing order of convergence and, therefore, with increasing computational complexity. In terms of stability, some researchers compared the amplitude of the set of initial points converging to the same attractor using complex discrete dynamics techniques. In [3], the authors observed that iterative schemes with seventh-order memory convergence showed better stability properties than many eighth-order optimal procedures without memory. Different authors subsequently used this graphical comparison; observe, for instance, the work of Wang et al. in [5], and Cordero et al. [4] in 2016 or the investigations of Bakhtiari et al. [6] in 2016. Howk et al. [7] in the following years.

The authors developed in [20] a technique that, using multidimensional real discrete dynamics tools, can study the qualitative performance of iterative with memory schemes, not only in graphical terms but essentially in analytical terms. By using this technique, the stability of the fixed and critical points of secant, Steffensen' and Kurchatov's methods (among others) were studied in [20]. It was also used to analyze other procedures, such as those described in [21], that defined by Choubey et al. in [10], or those by Chicharro et al. in [11, 12, 13]. In this kind of analysis, the performance of the numerical procedure on the most straightforward nonlinear functions (that is, quadratic polynomials) is studied. As many researchers in the area have corroborated it, this kind of study allows us to select those elements of a class of iterative schemes with better qualitative performance. Also, those schemes are shown to be the best also on non-polynomial functions (see, for example, [3, 10, 13]), among others.

The design of high-order multipoint iterative methods is based on the scheme used as the first step: it defines the starting order of convergence, the use of derivatives or not, the employment of only one previous iterate, or the use of memory. We aim to analyze in-depth the qualitative performance of some DF methods with memory to select the one with the best stability properties. Then, that one is used as the first step of different iterative methods designed through diverse techniques. Those multipoint schemes' qualitative behavior is studied to deduce how the qualitative properties are inherited. In this way, we would have objective tools to select which technique is more suitable in the construction of iterative multipoint methods with memory.

In this context, we made in Section 2 a deep dynamical analysis of several DF iterative schemes with memory, defined by using three previous iterates. We find the most stable one and, therefore, compare in Section 3 the performance of several multistep methods based on the previous methods. All this analysis is made by using multidimensional discrete dynamics. By using these results, we select the most stable scheme, and in Section 4, we check the performance of the methods on non-polynomial functions numerically, showing their basins of attraction. Therefore, the applicability of the schemes and the dynamical results are checked.

2 | QUALITATIVE STUDY OF ONE-STEP ITERATIVE WITH MEMORY SCHEMES

An iterative procedure that uses three previous iterates to calculate the next one is

$$x_{n+1} = \Psi(x_{n-2}, x_{n-1}, x_n), \quad n \geq 2,$$

being the starting guesses x_0, x_1 and x_2. The authors described in [20, 21] a procedure that allows us to describe any iterative with memory scheme as a multidimensional real discrete dynamical system so that its stability performance can be studied.

To get the fixed points of an iterative scheme defined by Ψ, we define a multidimensional fixed point function $H : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, related to Ψ as

$$H\left(x_{n-2}, x_{n-1}, x_n\right) = (x_{n-1}, x_n, \Psi(x_{n-2}, x_{n-1}, x_n)),$$

for $n = 1, 2, \ldots$, where x_0, x_1, and x_2 are the initial guesses. Then, any fixed point of H must satisfy $x_{n+1} = x_n$, $x_{n-2} = x_n$ and $x_{n-1} = x_n$.

From function $H : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, the associate discrete dynamical system in $\mathbb{R}^3$ is defined by $H\left(x_{n-2}, x_{n-1}, x_n\right) = (x_{n-1}, x_n, x_{n+1})$, where Ψ is the operator of the iterative method with memory. Let us define the sequence of vectors $\bar{x}_n =$

(x_{n-1}, x_n, x_{n+1}) by taking three consecutive iterates. The fixed points $\bar{x}$ of H satisfy $\bar{x} = \Psi(\bar{x})$ and all three components are identical. This notation implies $x_{n-2} = x_{n-1} = x_n$. Now, let us introduce some definitions (see [1]).

Let us consider the vectorial rational function $H : \mathbb{R}^3 \to \mathbb{R}^3$, usually obtained by applying an iterative method on a scalar polynomial $q(x)$. Then, if a fixed point $\bar{x}$ of operator H is different from (r, r, r), being r a zero of $q(x)$, it is called strange fixed point. Moreover, the orbit of a point $x^* \in \mathbb{R}^3$ is defined as the set of successive images from x^* by the vector function, that is, $\text{orbit}(x^*) = \{x^*, H(x^*), \dots, H^n(x^*), \dots\}$. Indeed, if a point $\bar{x}^* \in \mathbb{R}^3$ satisfy $H^k(\bar{x}^*) = \bar{x}^*$ and $H^p(\bar{x}^*) \neq \bar{x}^*$, $p = 1, 2, \dots, k-1$ is called k-periodic point. Let us remark that a k-periodic point x^* is a fixed point if $k = 1$.

The qualitative performance of a point of $\mathbb{R}^3$ is classified depending on its asymptotic performance. So, in order to declare the stability of multidimensional fixed points, the following result from Robinson [22] is used.

Theorem 1. Let H be a function of class C^2, defined from $\mathbb{R}^m$ to $\mathbb{R}^m$. Let us also assume that x^* is a k-periodic point. If we denote by $\lambda_1, \lambda_2, \dots, \lambda_m$ the eigenvalues of $H'(x^*)$, then

a) x^* is attracting if $|\lambda_j| < 1$, for all $j = 1, 2, \dots, m$.

b) If $\exists j_0 \in \{1, 2, \dots, m\}$ such that $|\lambda_{j_0}| > 1$, then x^* is unstable (repelling or saddle).

c) x^* is repelling if $|\lambda_j| > 1$, for all $j = 1, 2, \dots, m$.

Moreover, a fixed point $\bar{x} \in \mathbb{R}^3$ is said to be hyperbolic if $|\lambda_j| \neq 1$ for all $j = 1, 2, \dots, m$. Specifically, if $\exists i, j \in \{1, 2, \dots, m\}$ satisfying $|\lambda_i| < 1$ and $|\lambda_j| > 1$, then the fixed point is a saddle point.

Nevertheless, sometimes the Jacobian is not well-defined at the fixed points. In these cases, we impose to the rational operator H the condition that all components are identical so that it is reduced to a real-valued function. Therefore, the stability of the fixed point can be inferred from the absolute value of its first derivative at the fixed point, as it is done in scalar complex dynamics.

By considering $\bar{x}$ an attracting fixed point of function H, we define its basin of attraction $\mathcal{B}(\bar{x})$ as the set

$$\mathcal{B}(\bar{x}) = \left\{ \bar{x} \in \mathbb{R}^3 \,:\, H^m(\bar{x}) \to \bar{x}, \text{for } m \to \infty \right\}.$$

A key element in the stability analysis of an iterative method is the set of critical points of its associated rational function H: if $H'(\bar{x})$ satisfies $\det(H'(\bar{x})) = 0$, $\bar{x}$ is said to be a critical point. This definition usually does not provide a finite set of points but one or several curves in the domain of the rational function or even that all points are critical. Therefore, we calculate them by finding those points satisfying that H' has zero eigenvalues; this is a more restrictive definition but often necessary. Moreover, if the critical points are also fixed points, they are called superattracting points; if not, they are called free critical points (let us remark that components of critical points can be different). Indeed, Julia and Fatou [1] proved that there is at least one critical point associated with each basin of attraction. Therefore, all the attracting elements can be found by studying the orbit of the free critical points.

2.1 | Preliminary analysis: how to select the first step

In this section, we analyze the performance of quadratic polynomials of three different schemes with memory due to Traub [2], (1), denoted by TM, that of Jarratt and Nudds [14],

$$x_{n+1} = x_n - f(x_n)\frac{(x_{n-1} - x_n)(x_{n-2} - x_n)(f(x_{n-2}) - f(x_{n-1}))}{(x_{n-1} - x_n)(f(x_{n-2}) - f(x_n))f(x_{n-1}) + (x_{n-2} - x_n)(f(x_n) - f(x_{n-1}))f(x_{n-2})}, \tag{2}$$

denoted by JNM, and the procedure presented by Popovski et al. in [15],

$$x_{n+1} = x_n - f(x_n)\frac{(x_n - x_{n-2})(f(x_{n-2}) - f(x_{n-1}))(x_n - x_{n-1})}{(f(x_{n-2}) - f(x_n))(x_{n-2} - x_{n-1})(f(x_{n-1}) - f(x_n))}, \tag{3}$$

denoted by PM.

All these schemes have similar iterative expressions and the same order of convergence ($p = 1.839$). Our first aim is to decide, under qualitative considerations, which is the most stable one to add two more steps, increasing its convergence order and showing the best performance in terms of the wideness of the sets of initial estimations converging to the roots.

To extend the results to any polynomial of second degree, this study is constructed on $q(x) = x^2 - c$ so that the value of c yields to a situation with real, complex, or multiple roots depending on $c > 0$, $c < 0$ or $c = 0$, respectively. This analysis can be summarized in the following results.

Theorem 2. The multidimensional rational operator associated with Traub's scheme TM, when it is mapped on polynomial $q(x) = x^2 - c$, $c \neq 0$ is

$$T(w, z, x) = \left(z, x, \frac{c + x^2}{2x} \right),$$

and it is

$$T(w, z, x) = \left(z, x, \frac{x}{2} \right),$$

for $c = 0$. Moreover, TM satisfies:

a) The only fixed points are the roots of $q(x)$.

b) The only critical points are the roots of $q(x)$.

So, there is no other possible performance of TM scheme than convergence to the roots.

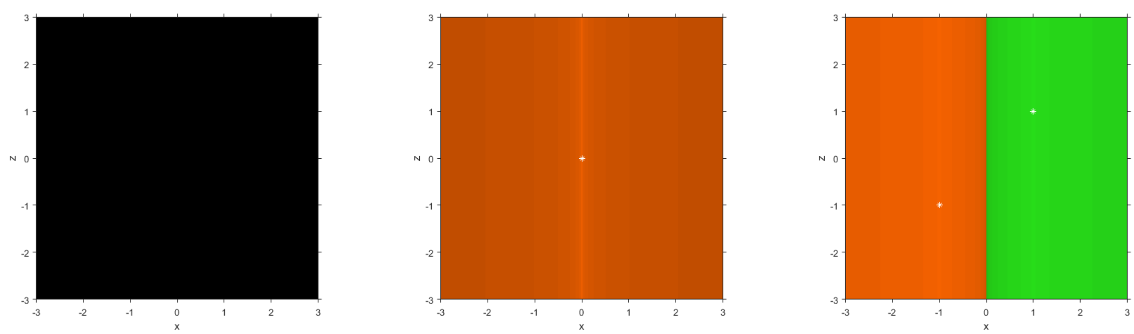
Proof. Let us remark that the third component of operator $T(w, z, x)$ is equal to the rational function obtained when classical Newton's method is applied on polynomial $q(x)$. This is the reason why, when we force the three consecutive iterates to be equal $(x = z = w)$ in order to get the fixed points, then the only fixed points are the roots $x = \pm\sqrt{c}$.

Regarding the critical points, the Jacobian matrix T' is

$$T'(w, z, x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2} - \frac{c}{2x^2} \end{pmatrix},$$

with eigenvalues $\left\{ 0, 0, \frac{1}{2} - \frac{c}{2x^2} \right\}$. So, there are no free critical points. □

A handy tool to visualize the analytical results is the dynamical plane of the system, composed of the set of the different basins of attraction. It can be drawn employing the programs presented in [16] after some changes to adapt them to schemes with memory. The dynamical plane of a method is built by calculating the orbit of a mesh of 400×400 starting points (z, x) (y does not appear in the rational function T). Then, we paint each of them in different colors (orange and green in this case) depending on the attractor they converge to (marked as a white star), with a tolerance of 10^{-3}. Also, they appear in black if the orbit has not reached any attracting fixed point in a maximum of 80 iterations. In Figure 1, we show the dynamical planes of this method for selected values of c in order to show its performance.



(a) $c = -1$ **(b)** $c = 0$ **(c)** $c = 1$

Figure 1 Dynamical planes of scheme TM on $q(x)$

Let us remark that, by definition, all the fixed points have equal components; they will always appear in the main diagonal of the dynamical plane. It can be observed that when there is no real root ($c < 0$, Figure 1a), no other attracting element appears; when $c = 0$, the only root is multiple, and the convergence is linear, so there is global convergence to $x = 0$ as can be seen in Figure 1b. In Figure 1c, the convergence to the roots is also observed to be global, being their basins of attraction two symmetrical half-planes.

Now, we analyze the performance of Jarratt-Nudds method with memory [14] on quadratic polynomials.

Theorem 3. The multidimensional rational operator associated with method JNM, when it is applied on polynomial $q(x) = x^2 - c$, $c \neq 0$ is

$$JN(w,z,x) = \left(z, x, \frac{c(x+z+w)+xzw}{c+x(z+w)+zw}\right),$$

and it is

$$JN(w,z,x) = \left(z, x, \frac{xzw}{x(z+w)+zw}\right),$$

for $c = 0$. Moreover, JNM satisfies:

a) There are no attracting strange fixed points. If $c \neq 0$, $x = 0$ is a strange fixed point, that is a saddle point. If $c = 0$, $x = 0$ is an attracting fixed point, as it is a multiple zero of $q(x)$.

b) There exists an infinite set of free critical points (w, z, x), defined by the lines $x = \pm\sqrt{c}$ or $z = \pm\sqrt{c}$, being $c > 0$ and w arbitrary, provided that $c + x(z+w) + zw \neq 0$.

Proof. By applying Jarratt-Nudds' method on $q(x)$ and constructing the auxiliary multidimensional operator, $JN(w,z,x)$ is found. To get the fixed points of JN, we solve $JN(x,x,x) = (x,x,x)$ and find

$$\frac{2x\left(c-x^2\right)}{c+3x^2} = 0,$$

so the fixed points are those whose three components coincide at $x = \pm\sqrt{c}$ and $x = 0$, provided that $c + 3x^2 \neq 0$. To study their qualitative behavior, we calculate

$$JN'(w,z,x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{(c-x^2)(c-z^2)}{(c+x(z+w)+zw)^2} & \frac{(c-x^2)(c-w^2)}{(c+x(z+w)+zw)^2} & \frac{(c-z^2)(c-w^2)}{(c+x(z+w)+zw)^2} \end{pmatrix}$$

and its eigenvalues at the fixed points are $(0,0,0)$ in the case of $y = z = x = \pm\sqrt{c}$ and (approximately) $\{1.83929, -0.419643 + 0.606291i, -0.419643 - 0.606291i\}$ for $w = z = x = 0$. So, the roots of $q(x)$ are superattracting fixed points and $(0,0,0)$ is saddle, since $|\lambda_1| = 1.83929 > 1$ and $|\lambda_2| = |\lambda_2| \approx 0.737353 < 1$.

Regarding the critical points, it is not possible to gent an analytical expression of the eigenvalues of $JN'(w,z,x)$. Then, it can be checked that

$$\det(JN'(w,z,x)) = \frac{\left(c-x^2\right)\left(c-z^2\right)}{(c+x(z+w)+zw)^2},$$

and, therefore, $x = \pm\sqrt{c}$ or $z = \pm\sqrt{c}$ are curves of critical points, provided that $c + x(z+w) + zw \neq 0$ and they are free as the third component w is not fixed. □

In Figure 2, we show the dynamical planes of this method for selected values of c in order to show its performance. For all the dynamical planes, different values of w have been used to observe the dependence of the wideness of the basins of attraction on it.

It can be noticed that, for $c \geq 0$, global convergence to the roots is found, being slower in case of multiplicity (see Figure 2b). Moreover, a symmetry is observed for opposite values of w in the wideness of the basins of attraction of both roots (see Figures 2c and 2d).

Finally, by means of a similar analysis, we found the main result about the stability of Popovski's scheme [15]. The proof is omitted as it is similar to the previous ones.

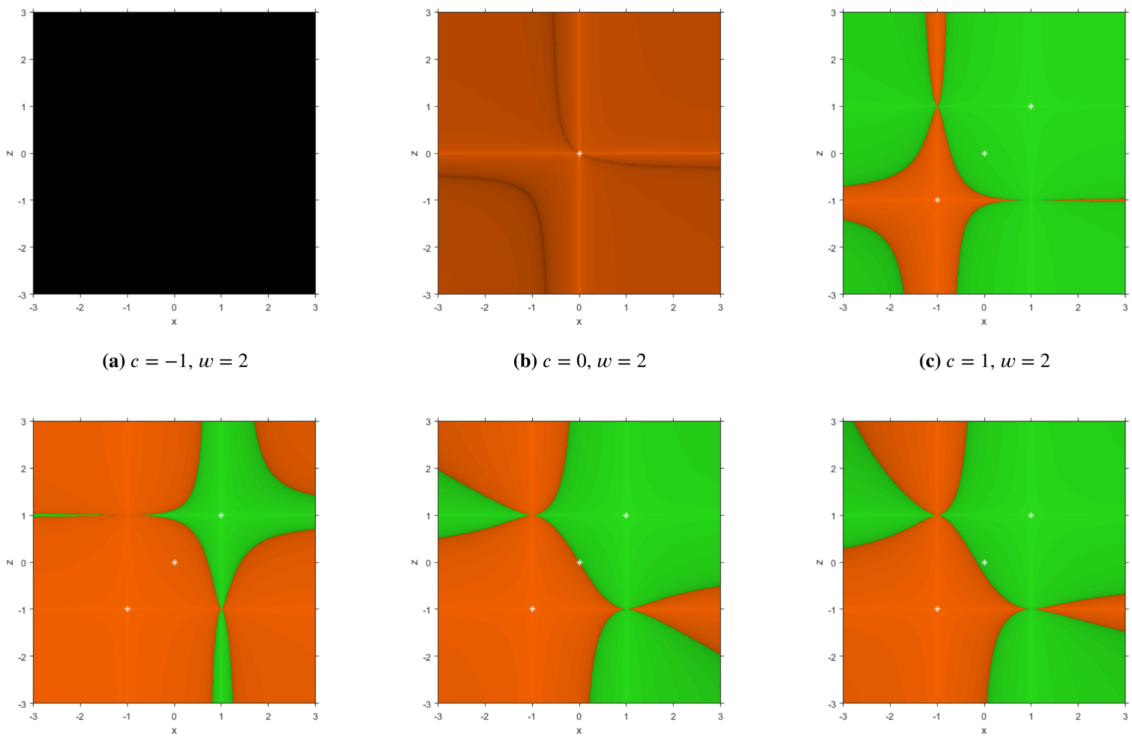
Theorem 4. The multidimensional rational operator associated with method PM, when it is applied on polynomial $q(x) = x^2 - c$, $c \neq 0$ is

$$P(y,z,x) = \left(z, x, \frac{c(z+y)+x^3+xzy}{(x+z)(x+y)}\right),$$

and it is

$$P(y,z,x) = \left(z, x, \frac{x^3+xzy}{(x+z)(x+y)}\right),$$

for $c = 0$. Moreover, PM satisfies:



(a) $c = -1$, $w = 2$ **(b)** $c = 0$, $w = 2$ **(c)** $c = 1$, $w = 2$

(d) $c = 1$, $w = -2$ **(e)** $c = 1$, $w = 0$ **(f)** $c = 1$, $w = 10$

Figure 2 Dynamical planes of scheme JNM method on $q(x)$, for different values of w and c

a) The only fixed points are the roots of $q(x)$.

b) There exists an infinite set of free critical points (w, z, x), defined by the lines $z = x$, provided that $x \neq z$ and $x \neq w$.

In Figure 3, we show the performance of this method with memory for several values of c. For all the dynamical planes, different values of w has been used, in order to observe the dependence of the wideness of the basins of attraction on it.

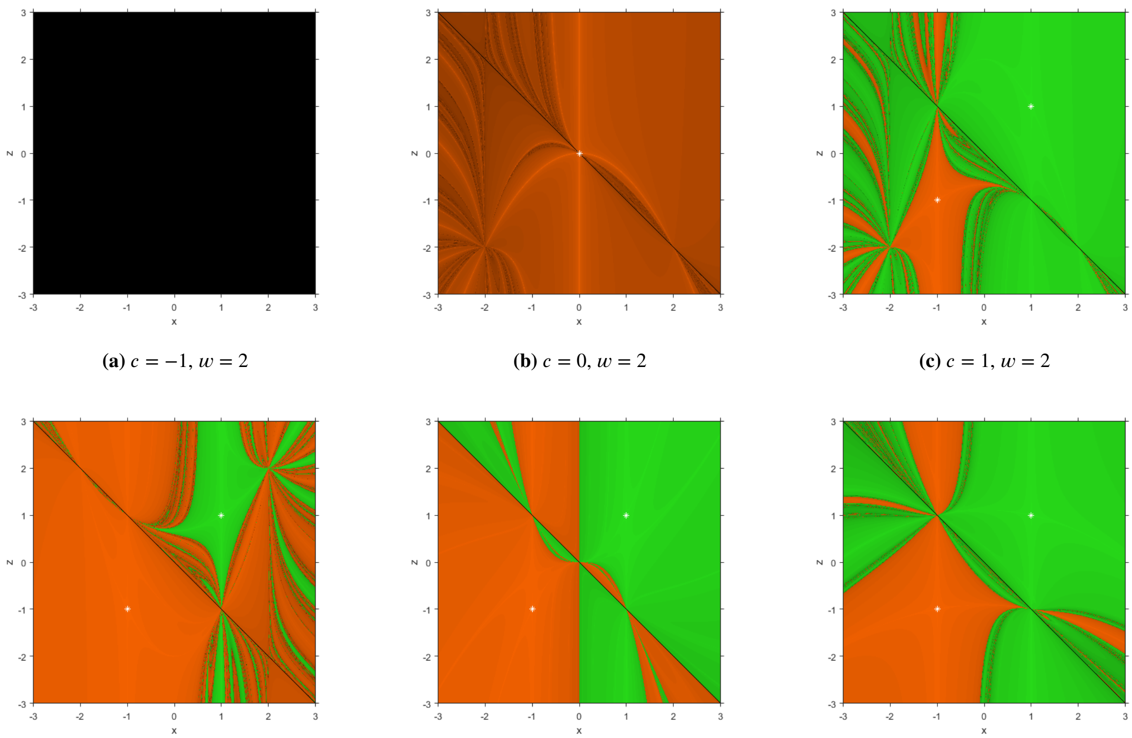
Similar performance to the case of JNM is observed (in terms of symmetry and convergence to the roots). However, their basins of attraction have more connected components, and the Julia set (the boundary among the basins of attraction) is much more complicated. Also, slow convergence to the multiple root in case of $c = 0$ is observed.

So, it can be concluded that the stability of Traub's scheme with memory is much better than the other methods with similar shape and order of convergence under analysis. Therefore, we study in the following section how is the qualitative behavior of two iterative schemes with three steps based on Traub's procedure as the first step.

3 | QUALITATIVE PERFORMANCE OF MULTIPOINT METHODS WITH THE SAME FIRST STEP

As it has been previously stated, we analyze the qualitative properties of two iterative with memory schemes based on Traub's scheme. We denote by method M1 that scheme with iterative expression

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f[x_{n-2}, x_n] + f[x_{n-1}, x_n] - f[x_{n-2}, x_{n-1}]}, \\ z_n &= y_n - \frac{f(y_n)}{f[y_n, x_n] + f[y_n, x_n, x_{n-1}](y_n - x_n) + f[y_n, x_n, x_{n-1}, x_{n-2}](y_n - x_n)(y_n - x_{n-1})}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f[z_n, y_n] + f[z_n, y_n, x_n](z_n - y_n) + f[z_n, y_n, x_n, x_{n-1}](z_n - y_n)(z_n - x_n)}, \end{aligned} \tag{4}$$



(a) $c = -1$, $w = 2$ **(b)** $c = 0$, $w = 2$ **(c)** $c = 1$, $w = 2$

(d) $c = 1$, $w = -2$ **(e)** $c = 1$, $w = 0$ **(f)** $c = 1$, $w = 10$

Figure 3 Dynamical planes of scheme PM on $q(x)$, for different values of w and c

presented in [17], with order of convergence 7.356. Also in [18], the scheme with memory that we denote by M2 was constructed,

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f[x_{n-2}, x_n] - f[x_{n-2}, x_{n-1}] + f[x_{n-1}, x_n]}, \\ z_n &= y_n - \frac{f(y_n)}{\alpha_1 f(x_n) + \alpha_2 f(x_{n-1}) + \alpha_3 f(y_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{\beta_1 f(x_n) + \beta_2 f(y_n) + \beta_3 f(z_n)}. \end{aligned} \tag{5}$$

showing the order of convergence is 6.219.

Although both schemes are based on the same first step, they reach different orders of convergence with also significant divergence between their computational complexity: as M1 uses in the denominator of the second and third steps, high-order estimations of the derivatives $f'(y_n)$ and $f'(z_n)$, respectively, its expression are more complicated, but it reaches higher order of convergence than its partner M2, with a simpler iterative expression but lower order of convergence. Moreover, the highest is the order of convergence, higher is the need to be close to the root to converge; so, it would be possible to get better stability properties for lower-order methods. Our aim is not to classify them through their convergence order but their stability. In what follows, we construct the multidimensional discrete dynamical system associated with both schemes and analyze the existence of strange attracting fixed points or free critical points that might yield undesirable numerical performances.

3.1 | Qualitative study of $M1$

We now analyze the performance of the rational operator related to $M1$ on quadratic polynomials. As in the previous section, this analysis is made on $q(x) = x^2 - c$. The results are condensed in the following result. It can be observed that the third component of the vectorial rational function does not depend on the two previous iterations, w, and z, as it happened in Traub's method.

Theorem 5. The multidimensional rational operator associated with method $M1$, when it is applied on $q(x) = x^2 - c$, $c \neq 0$ is

$$M1(w,z,x) = \left(z, x, \frac{c^4 + 28c^3x^2 + 70c^2x^4 + 28cx^6 + x^8}{8c^3x + 56c^2x^3 + 56cx^5 + 8x^7} \right),$$

and it is

$$M1(w,z,x) = \left(z, x, \frac{x}{8} \right),$$

for $c = 0$. Moreover, $M1$ satisfies:

a) There are no strange attracting fixed points. If $c < 0$, there exist six real strange fixed points that are saddle points. If $c = 0$, $x = 0$ is the unique fixed point, that is only attracting; finally, for $c > 0$, the only fixed points are the roots of $q(x)$.

b) There exists no critical points different from the roots of $q(x)$.

So, method $M1$ has global convergence.

Proof. We calculate the fixed points of operator $M1$ by solving $M1(w,z,x) = (w,z,x)$, that must satisfy $w = x = z$. Specifically,

$$M1(w,z,x) = \left(z, x, \frac{c^4 + 28c^3x^2 + 70c^2x^4 + 28cx^6 + x^8}{8c^3x + 56c^2x^3 + 56cx^5 + 8x^7} \right) = (w,z,x),$$

if and only if $w = z = x$ and

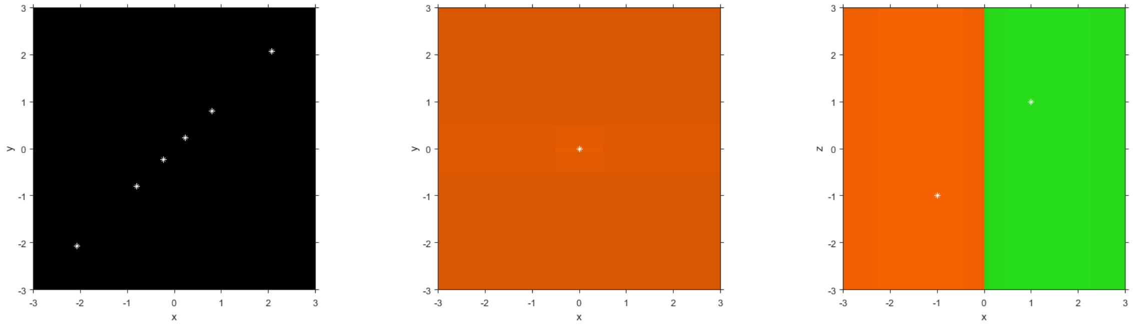
$$-\frac{\left(x^2 - c\right)\left(c^3 + 21c^2x^2 + 35cx^4 + 7x^6\right)}{8x\left(c + x^2\right)\left(c^2 + 6cx^2 + x^4\right)} = 0.$$

So, the fixed points of $M1(y,z,x)$ are the roots of $q(x)$ and also the zeros of the sixth-degree polynomial $c^3+21c^2x^2+35cx^4+7x^6$ (that are real if $c < 0$), meanwhile $c^2 + 6cx^2 + x^4 \neq 0$. Let us remark that in case $c > 0$, there are no strange fixed points, and when $c = 0$, the rational function is reduced, and the only fixed point is $x = 0$, which is attracting but not superattracting. The Jacobian matrix $M1'(w,z,x)$ is defined as

$$M1'(w,z,x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \frac{\left(x^2-c\right)^7}{8x^2\left(c+x^2\right)^2\left(c^2+6cx^2+x^4\right)^2} \end{pmatrix}.$$

It can be checked that the first two eigenvalues of $M1'$ evaluated at each one of these strange fixed points are null. Then, their character would be attracting or saddle depending on the absolute value of the third eigenvalue. In all cases, $|\lambda_3| = 8$, so they are saddle.

By calculating the eigenvalues of $M1'(w,z,x)$, we get $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = -\frac{\left(c-x^2\right)^7}{8x^2\left(c+x^2\right)^2\left(c^2+6cx^2+x^4\right)^2}$. So, we conclude that the only critical points are the roots of $q(x)$, proving the global convergence for quadratic polynomials. □



(a) $c = -1$ **(b)** $c = 0$ **(c)** $c = 1$

Figure 4 Dynamical planes of scheme $M1$ on $q(x)$

Figure 4 shows the behavior stated at Theorem 5. Let us notice that only the convergence to the roots is reached (it is clear from the absence of free critical points), showing the best possible behavior in terms of stability. Fixed points are represented as white stars.

3.2 | Qualitative study of $M2$

A similar study is made for $M2$; the rational function involved depends on the two previous iterations (in this case, $w = x_{n-2}$ has not any role). The proof is omitted as it can be developed in a similar way as in Theorem 5.

Theorem 6. The multidimensional rational operator associated with method $M2$, when it is applied on $q(x) = x^2 - c$, $c \neq 0$ is

$$M2(y,z,x) = \left(z, x, \frac{-c^5(3x+z)^3 + c^4x^2q_1(z,x) + c^3q_2(z,x) - 2c^2x^6q_3(z,x) + cx^8q_4(z,x) + x^{10}q_5(z,x)}{8x\left(c(x+z) + x^2(x-3z)\right)\left(-c^3(3x+z)^2 + c^2x^2r_1(z,x) + cx^4r_2(z,x) + x^6r_3(z,x)\right)} \right),$$

where

$$\begin{aligned}
q_1(z,x) &= -141x^3 - 77x^2z + 17xz^2 + 9z^3,\\
q_2(z,x) &= -222x^7 + 610x^6z + 438x^5z^2 + 70x^4z^3,\\
q_3(z,x) &= 69x^3 - 411x^2z + 471xz^2 + 319z^3,\\
q_4(z,x) &= -23x^3 + 425x^2z - 1373xz^2 + 1163z^3\\
q_5(z,x) &= 39x^3 - 217x^2z + 333xz^2 - 91z^3,\\
r_1(z,x) &= -17x^2 + 18xz + 15z^2,\\
r_2(z,x) &= -11x^2 + 62xz - 35z^2,\\
r_3(z,x) &= 5x^2 - 10xz - 11z^2
\end{aligned}$$

and it is

$$M2(y,z,x) = \left(z, x, \frac{x\left(39x^3 - 217x^2z + 333xz^2 - 91z^3\right)}{8(x-3z)\left(5x^2 - 10xz - 11z^2\right)} \right),$$

for $c = 0$. Indeed, $M2$ satisfies:

a) There are no strange attracting fixed points. If $c < 0$, there are two real strange fixed points $\left(-\frac{\sqrt{-c}}{\sqrt{3}}, -\frac{\sqrt{-c}}{\sqrt{3}}\right)$ and $\left(\frac{\sqrt{-c}}{\sqrt{3}}, \frac{\sqrt{-c}}{\sqrt{3}}\right)$ that are saddle points. If $c = 0$, the unique fixed point is $x = 0$ that is attracting but not superattracting; finally, for $c > 0$, the only fixed points are the roots of $q(x)$.

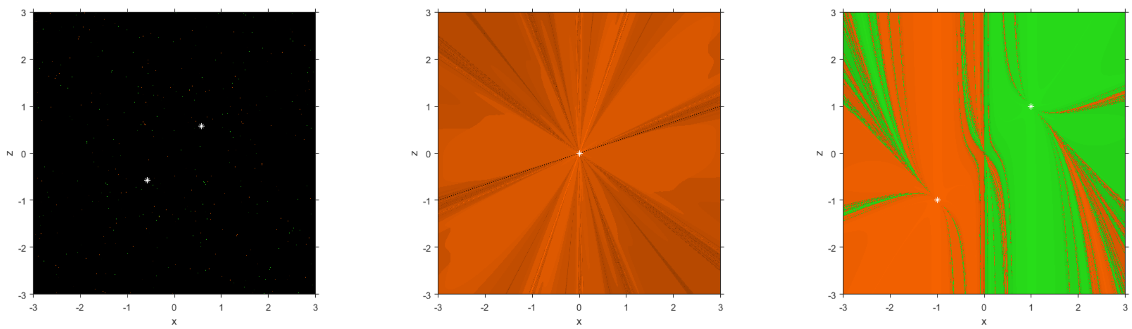
b) If $c > 0$, there are two infinite sets of free critical points, $(w, \frac{9}{11}\sqrt{\frac{5}{17}}\sqrt{c}, -\frac{\sqrt{c}}{\sqrt{85}})$ and $(w, -\frac{9}{11}\sqrt{\frac{5}{17}}\sqrt{c}, \frac{\sqrt{c}}{\sqrt{85}})$, for any real value of w.

The existence of free critical points led us to infer the possibility of convergence to attracting elements (points, orbits,...) different from the roots. As a first step to check if it other performances are possible, some dynamical planes can be seen in Figure 5.

In Figure 5, $M2$ scheme is found to have a very stable performance. In case there exist strange fixed points, they are repelling or neutral. Global convergence is observed, despite free critical points that lie inside the basins of attraction of the roots in Figure 5c, where $c > 0$. The observed performance is similar to that of $M1$, but the attraction basins are divided into infinite connected components. However, free critical points do not assure that there are other values of c with convergence to attracting periodic orbits or even with chaotical performance. So, it should be possible that for any value of $c > 0$, those free critical points were not in the basins of attraction of the roots but inside the basin of any other attractor, maybe a periodic orbit or a strange attractor. In order to detect this performance, we use Feigenbaum's diagram.

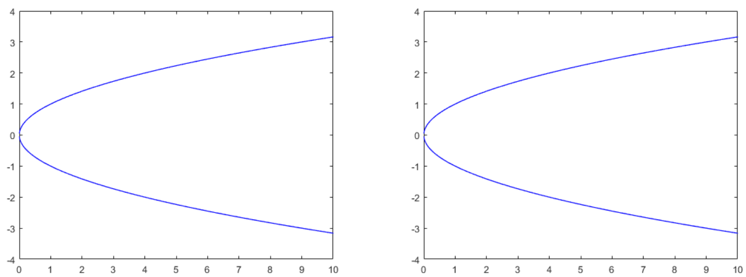
3.2.1 | Feigenbaum's diagrams

We use bifurcation diagrams of $M2$, depending on the value of c, by means of the use of each real critical point $s_1(c) = (w, \frac{9}{11}\sqrt{\frac{5}{17}}\sqrt{c}, -\frac{\sqrt{c}}{\sqrt{85}})$ and $s_2(c) = (w, -\frac{9}{11}\sqrt{\frac{5}{17}}\sqrt{c}, \frac{\sqrt{c}}{\sqrt{85}})$ as a starting point, w arbitrary, (described in Theorem 6) and observing the range $[0, 10]$ of the parameter c, where free critical points are real.



(a) $c = -1$ **(b)** $c = 0$ **(c)** $c = 1$

Figure 5 Dynamical planes of $M2$ method on $q(x)$for different values of c



(a) $z = \frac{9}{11}\sqrt{\frac{5}{17}}\sqrt{c},\ x = -\frac{\sqrt{c}}{\sqrt{85}}$ **(b)** $z = -\frac{9}{11}\sqrt{\frac{5}{17}}\sqrt{c},\ x = \frac{\sqrt{c}}{\sqrt{85}}$

Figure 6 Bifurcation diagrams of $M2$ for real critical points

Both Feigenbaum's diagrams can be observed in Figures 6a and 6b, with the same performance. We use blue color for plotting the last 100 from 500 iterations, for each $c \in [0, 10]$ (if a wider interval is used, the results are the same). We notice that the same curve appears in both. It corresponds with the real roots of $q(x)$ in this interval.

So, both schemes have shown good stability properties on quadratic polynomials. The performance of M1, despite the higher complexity of its iterative expression, has shown to be globally convergent due to the absence of free critical points. However, the final performance of M2 has been similar. The best estimation of the derivatives has a crucial role in the qualitative properties. In what follows, these schemes are numerically checked on some other nonlinear functions to test the applicability of these qualitative results.

4 | NUMERICAL EXPERIMENTS

In this section, we compare 9 methods of various orders, some of which are derivative-free (DF, for short) and other are optimal eighth-order schemes without memory. The methods and their order of convergence are:

1. TM, Traub's DF method (1) of order 1.839 [2] (Method 7a on page 234)
2. JNM, Jarrat-Nudds's DF method (2) of order 1.839 [14]
3. PM, Popovski's DF method (3) of order 1.839 [15]
4. NM, Newton's second order method
5. SM, Steffensen's DF second order method [30]

6. M1, Neta's DF method of order 7.356 [17]
7. M2, Neta's DF method of order 6.219 [18]
8. ZOM, Zhanlav-Otgondroj's DF method of optimal order 8 [31]
9. SAM, Sharma-Arora's method of optimal order 8 [29]

We ran these methods on 3 examples on a 6×6 square with center at $(0, 0)$. The functions are:

1. Wilkinson-type polynomial
$$f_1(x) = x(x^2 - 1/4)(x^2 - 1)(x^2 - 9/4)(x^2 - 4) \tag{6}$$
2. A function vanishing at $\pm 3, \pm 2, \pm 1, 0, 3/2$ on [-3, 3]
$$f_2(x) = \sin(\pi x)\left(e^{x-1.5} - 1\right) \tag{7}$$
3. A function vanishing at $\pm 2.5, \pm 1.5, -1, \pm 1/2$ on [-3,3]
$$f_3(x) = \cos(\pi x)\left(e^{x+1} - 1\right) \tag{8}$$

The square is divided into a mesh of initial points of the complex plane to apply the iterative procedures to them. For those methods requiring additional starting values, we have taken $x_{-1} = x_0 + 0.01$ and $x_{-2} = x_0 + 0.02$. The number of function evaluations to converge within a tolerance of 10^{-7} is collected; Also, the searched root the sequence has converged to. If the iterates have not converged in 40 iterations, we denote it as a divergent point. The color corresponding colors each point to the root. Note that we have used 6 different colors therefore some roots will have the same color, but they are far apart. Moreover, the color is brighter for lower number of iterations needed to converge to the root. A divergent point is colored black. We also have annotated the CPU time needed to run the code on all initial guesses of the mesh using MacBook Pro computer.

In Figure 7 we have depicted the basins of attraction for the 9 methods of the first function. SM and ZOM (having SM as first step) have too many divergent points. Also, the basins of M1, M2, and SAM are brighter than the rest, showing the fastest convergence. In this cases, the basins of attraction of the roots are similar (in terms of wideness) to those of Newton's method.

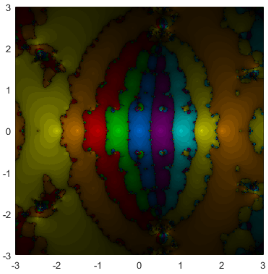
We have also collected in Tables 1-3 the average number of function-evaluation per point for each scheme, the CPU run time in seconds, and the percentage of divergent points. The methods SM and ZOM use the highest number of functional evaluations per point. The CPU runtime for these schemes is the highest since they have the most divergent points. The methods M1, M2, JNM, and SAM have no divergent points. TM and NM have very few divergent points.

Table 1 Average number of function evaluations per point for each example and each of the methods

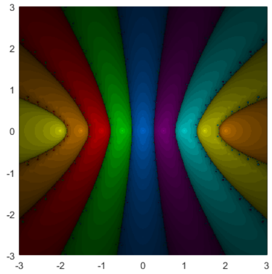
Method	Ex1	Ex2	Ex3	average
TM	16.11	14.31	14.94	15.12
JNM	11.62	9.48	9.76	10.29
PM	14.92	12.81	13.18	13.64
NM	23.25	18.63	19.71	20.53
SM	63.90	39.36	49.17	50.81
M1	13.67	11.84	12.20	12.57
M2	16.72	14.38	14.98	15.36
ZOM	84.23	53.03	72.46	69.91
SAM	14.57	13.41	13.66	13.88

The basins of attraction for the methods in the second example are given in Figure 8. Again SM and ZOM are inferior. The methods M2, SAM, and M1 are the fastest.

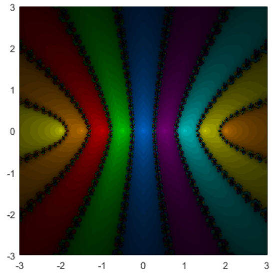
We average the numerical results over the 3 examples, and we can conclude that JNM is the top scheme in all 3 categories, followed by M1. In a previous comparison of TM, JNM and PM using 4 polynomials of degrees 2–5 and one non-polynomial function [18], we found that TM was best.



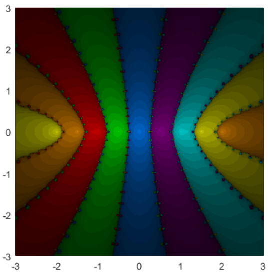
(a) TM



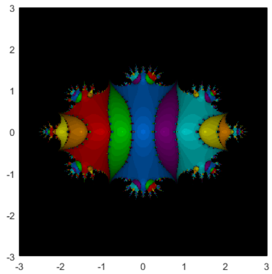
(b) JNM



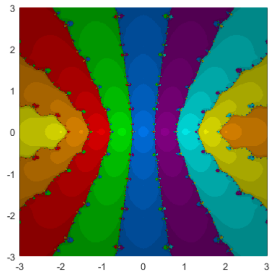
(c) PM



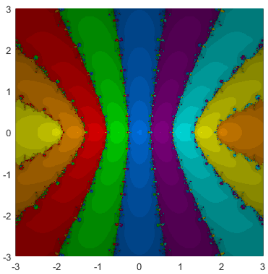
(d) NM



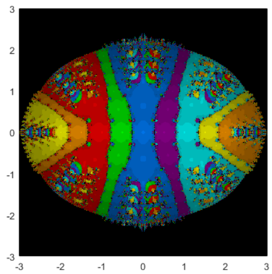
(e) SM



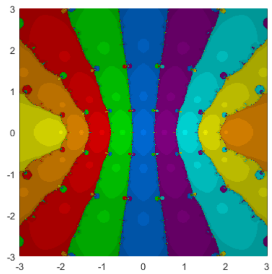
(f) M1



(g) M2



(h) ZOM



(i) SAM

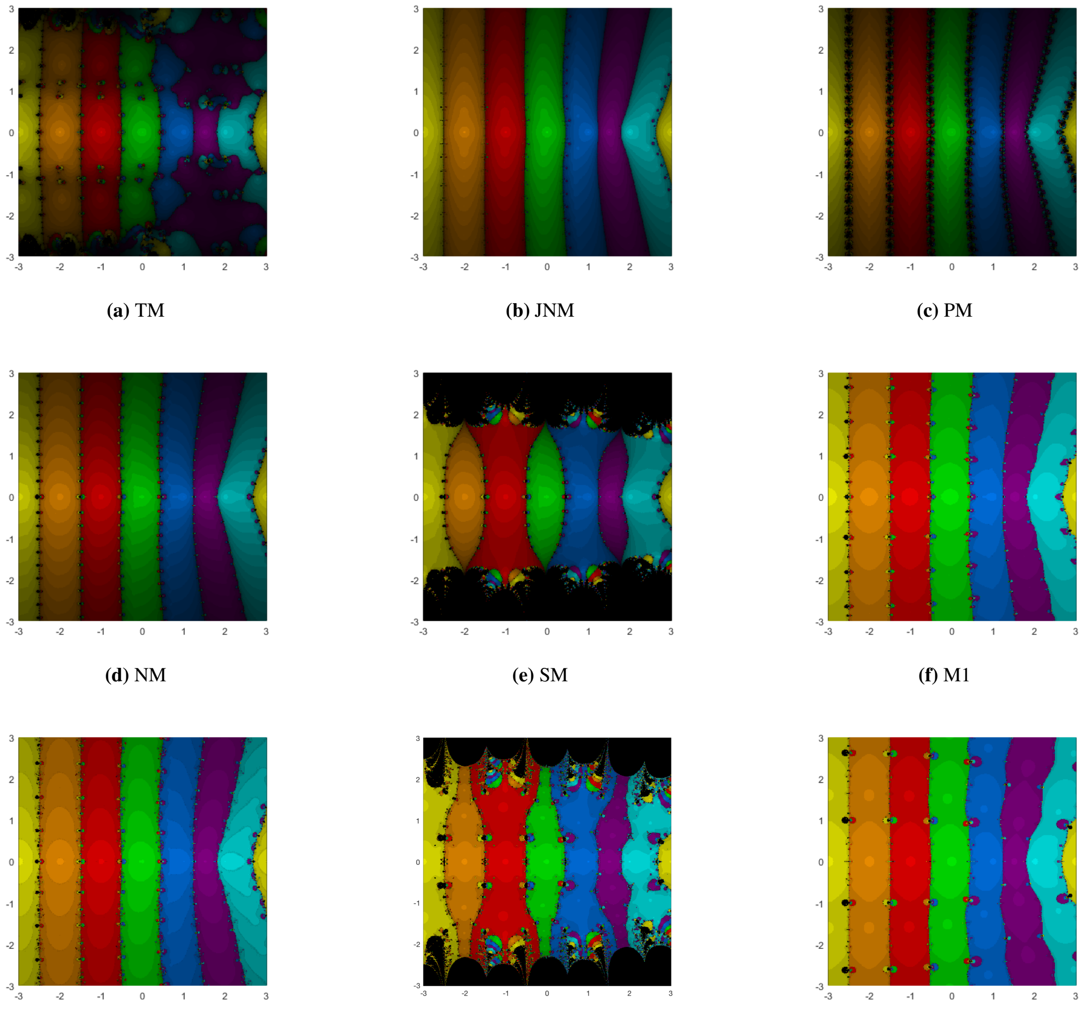
Figure 7 Dynamical planes of analyzed methods for the roots of the function $f_1(x)$

Table 2 CPU time (msec) for each example and each of the methods

Method	Ex1	Ex2	Ex3	average
TM	1050.964	842.572	774.008	889.181
JNM	568.898	574.472	551.134	564.835
PM	717.363	656.711	665.299	679.791
NM	926.910	530.669	522.581	660.054
SM	1498.093	920.992	975.164	1131.416
M1	582.13	532.055	582.332	565.506
M2	726.71	354.122	685.238	655.357
ZOM	1404.49	1110.924	1148.6	1221.338
SAM	805.488	512.445	492.982	603.638

Table 3 Number of black points for each example and each of the methods and average across examples

Method	Ex1	Ex2	Ex3	average
TM	278	11515	9022	6938
JNM	0	174	152	109
PM	2446	10625	11515	8195
NM	20	1742	1806	1189
SM	273404	140192	192616	202071
M1	0	1730	1679	1136
M2	0	1600	1900	1167
ZOM	166138	94779	142562	134439
SAM	0	1894	1827	1240



(a) TM **(b)** JNM **(c)** PM

(d) NM **(e)** SM **(f)** M1

(g) M2 **(h)** ZOM **(i)** SAM

Figure 8 Dynamical planes of analyzed methods for the roots of the function $f_2(x)$

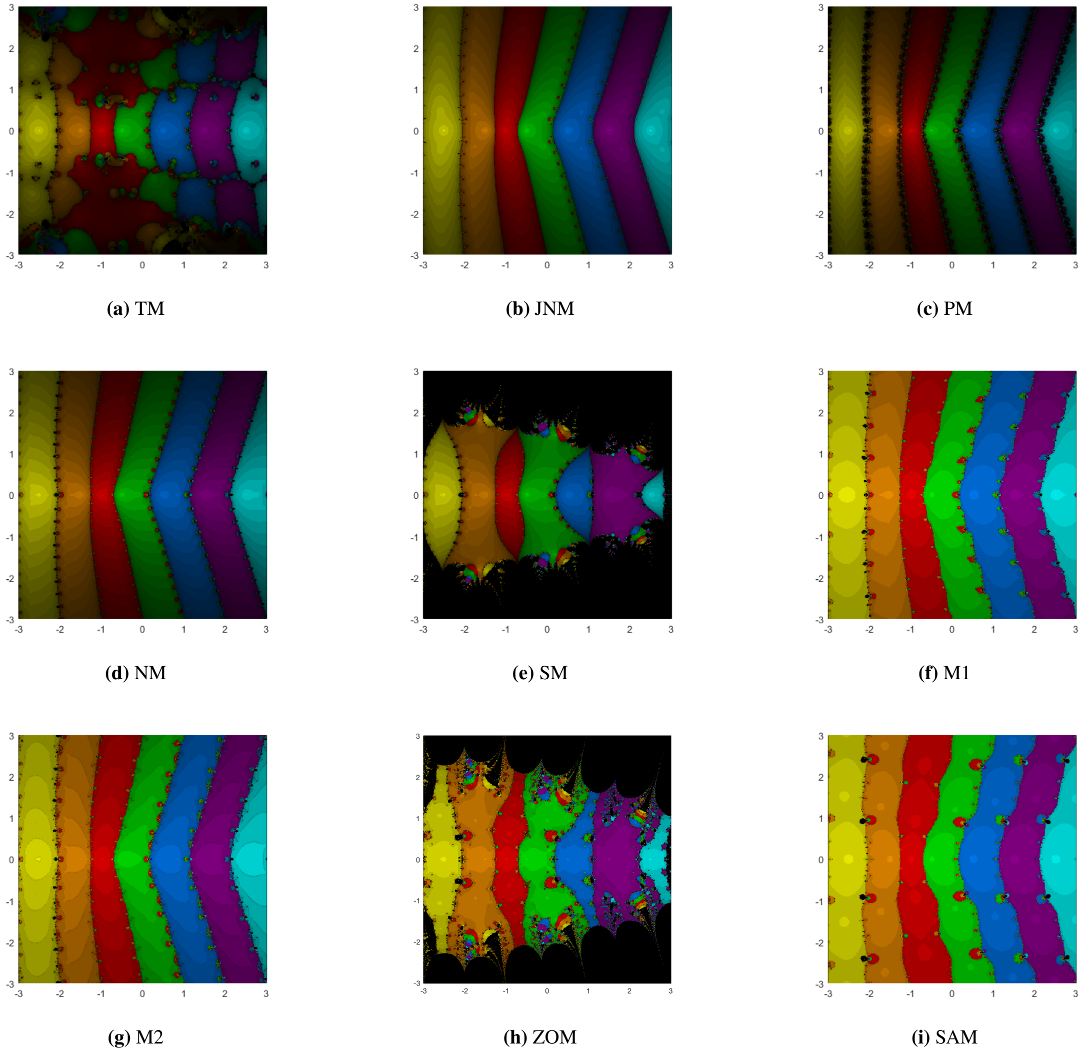


Figure 9 Dynamical planes of analyzed methods for the roots of the function $f_3(x)$

5 | CONCLUSIONS

In this manuscript, we have deepened the reasons for the better stability of derivative-free iterative methods with memory based on DF Traub's scheme regarding other methods of the same kind but based on other schemes as the first step. The absence of critical points different from the roots in the case of Traub's method yields global convergence on quadratic polynomials, precisely the same performance as Newton's scheme. Other procedures also under analysis show stable behavior, but the complexity of the basins of attraction is much higher. Once Traub's method is selected as the most stable, some schemes constructed with this method as the first step are also analyzed with the same dynamical technique, finding only convergence to the roots but global convergence in the case of M1. This scheme was designed by using high-order estimations of the derivatives in the iterative expression versus a simpler construction of the denominators in the design of M2. Therefore, the dynamical analysis has shown that the computational complexity is not a key fact in the stability, even if it is a sufficient element to assure a good performance from good estimations of the derivatives. Numerically, these schemes show this good performance, compared with other schemes with memory of different convergence orders. It shows better than optimal iterative procedures without memory, of higher order of convergence. Even in these cases, the performance of M1 shows lower computational time and better efficiency.

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