Asymptotic behavior of the solutions of a Partial Differential Equation with Piecewise Constant Argument

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Abstract

In this paper we study the partial differential equation with piecewise constant argument of the form :

$$\begin{aligned} x_t(t,s) = \quad & A(t)x(t,s) + B(t,s)x([t],s) + C(t,s)x(t,[s]) + \\ & D(t,s)x([t],[s]) + f(x(t,[s])), \quad t,s \in I\!R^+ = (0,\infty) \end{aligned}$$

where $A(t)$ is a $k \times k$ invertible and continuous matrix function on $I\!R^+$, $B(t,s)$, $C(t,s)$, $D(t,s)$ are $k \times k$ continuous and bounded matrix functions on $I\!R^+ \times I\!R^+$, $[t]$, $[s]$ are the integral parts of t, s respectively and $f : I\!R^k \to I\!R^k$ is a continuous function. More precisely under some conditions on the matrices $A(t)$, $B(t,s)$, $C(t,s)$, $D(t,s)$ and the function f we investigate the asymptotic behaviour of the solutions of the above equation.

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1 Introduction

Since differential equations with piecewise constant argument describe hybrid dynamical systems, that is combine properties of both differential and difference equations, interest in studying them has increased (see [1]-[8],[13], [18]-[25], [36]-[38] and the references cited therein). As it is widely known, differential equations have a remarkable ability to describe phenomena or physical systems in the real world. In many of these cases, in formulating the mathematical model, the variables or functions used have to be discontinuous such as piecewise constant, or other types like piecewise linear, stepwise, impulsive etc, in order to translate the actual characteristics of the real world problem. In fact, in many sciences and technology phenomena described by equations with piecewise constant arguments can be found (see [4], [11], [12]). For instance, in [4], Busenberg and Cooke introduced and studied the following mathematical model with a piecewise constant argument

$$\frac{dx(t)}{dt} = F(t, x_t), \quad [t] < t \leq [t] + 1, \quad x_{[t]} = \phi_{[t]}$$
$$\phi_{[t]} = G([t], x_t), \quad [t] \geq 2, \quad \phi_1 = H,$$

for analyzing vertically transmitted diseases. Also, many of physics and engineering systems can be described by a second-order differential equations with piecewise constant arguments, such as

$$m\ddot{x} + kx_1 = \acute{r}sin\Big(2\beta\Big[\frac{t}{T_0}\Big]\Big)$$

which describes an elastic systems impelled by a Geneva wheel used mainly in watches and instruments (for more information see [11]).

In this paper we study the partial differential equation with piecewise constant argument of the form

$$\begin{aligned} x_t(t,s) = \quad & A(t)x(t,s) + B(t,s)x([t],s) + C(t,s)x(t,[s]) + \\ & D(t,s)x([t],[s]) + f(x(t,[s])), \quad t, s \in I\!R^+ = (0,\infty) \end{aligned} \tag{1.1}$$

where $A(t)$ is a $k \times k$ invertible and continuous matrix function on $I\!R^+$, $B(t,s)$, $C(t,s)$, $D(t,s)$ are $k \times k$ continuous matrix functions on $I\!R^+ \times I\!R^+$, $[t]$, $[s]$ are the integral parts of t, s respectively and $f : I\!R^k \to I\!R^k$ is a continuous function which satisfies

$$|f(x) - f(y)| \leq \gamma|x - y|, \quad x, y \in I\!R^k \tag{1.2}$$

where γ is a constant and

$$f(\bar{0}) = \bar{0}, \bar{0}^T = (0, 0, \ldots, 0) \in I\!R^k. \tag{1.3}$$

A function $x(t, s)$ is a solution of (1.1) if the following conditions are satisfied

(i) the function x is continuous on $I\!R^+ \times I\!R^+ \setminus I\!N$ and left continuous at the points (t,m), $t \in I\!R^+, m \in I\!N$.

(ii) the partial derivative with respect to t of x, denoted by x_t, exists except possibly at the points (n,s), $n = 0,1,\ldots,$ $s \in I\!R^+$, where one-sided partial derivatives exist and

(iii) x satisfies (1.1) on every set $[n,n+1)\times[m,m+1)$ where $n=[t], m=[s]$.

In what follows we denote by $|\cdot|$ any convenient norm either for a vector or for a matrix.

We say that a solution $x(t,s)$ of (1.1) tends exponentially to zero as $t \to \infty$ uniformly with respect to s if there exist constants $\Lambda \geq 1$, $\mu > 0$ such that

$$|x(t,s)| \leq \Lambda e^{-\mu t}, \quad t \in I\!R^+.$$

We consider the linear differential equation

$$x' = A(t)x, \ t \in I\!R^+. \tag{1.4}$$

We say that (1.4) is uniformly aysmptotically stable (see [9]) if there exist constant $a > 0$ and $K > 1$ such that for $t,u \in I\!R^+$

$$|X(t)X^{-1}(u)| \leq Ke^{-a(t-u)}, \ t \geq u \tag{1.5}$$

where $X(t)$ is the fundamental matrix solution of (1.4) such that $X(0) = I$, I is the $k \times k$ identity matrix.

We say that the equation (1.4) possesses an exponential dichotomy (see [10]) if there exist a projection P $(P^2 = P)$ and constants $a, K,$ $a > 0$, $K \geq 1$ such that for $t,u \in I\!R^+$

$$\begin{array}{l} |X(t)PX^{-1}(u)| \leq Ke^{-a(t-u)}, \ t \geq u \\ |X(t)(I-P)X^{-1}(u)| \leq Ke^{-a(u-t)}, \ u \geq t. \end{array} \tag{1.6}$$

In the paper [18] the authors consider the differential equation with piecewise constant argument of the form

$$y'(t) = A(t)y(t) + B(t)y([t]) + f(t,y(t)), t \in I\!R^+ \tag{1.7}$$

where $A(t)$, $B(t)$ are $k \times k$ matrices and $f : I\!R^+ \times I\!R^k \to I\!R^k$ is a function which satisfies a Lipschitz condition. More precisely, using some conditions on the matrices $A(t)$, $B(t)$ and the function f he studied the existence and the asymptotic behavior of the solutions of (1.7). For some differential, difference and related equations close to equation (1.7) and their applications see, for example, [26]-[35] and the references cited therein.

In this paper, firstly, using some assumptions on the matrices $A(t)$, $B(t,s)$, $C(t,s)$, $D(t,s)$ and the function f we study the existence of the solutions of equation (1.1). Moreover, under the asymptotical stability (1.5) of (1.4) or the exponential dichotomy (1.6) of (1.4) and some further assumptions, we study the asymptotic behavior of the solutions of (1.1).

2 MAIN RESULTS

In the following proposition we study the existence of the solutions of (1.1).

Proposition 2.1 *Consider equation (1.1) where where $A(t)$ is a $k \times k$ invertible and continuous matrix function on $I\!R^+$, $B(t,s)$, $C(t,s)$, $D(t,s)$ are $k \times k$ continuous matrix functions on $I\!R^+ \times I\!R^+$, $[t]$, $[s]$ are the integral parts of t, s respectively and $f : I\!R^k \to I\!R^k$ is a continuous function such that (1.2) holds. Suppose also that*

$$|A(t)| \leq N, \quad |B(t,s)| \leq M, \quad |C(t,s)| \leq M, \quad |D(t,s)| \leq M, \quad t, s \in I\!R^+ \tag{2.1}$$

where M, N are positive constants such that

$$M < \frac{N}{2(e^N - 1)}. \tag{2.2}$$

Then, if $p : I\!R^+ \to I\!R^k$ is a continuous and bounded function, there exists a unique solution $x(t,s)$ of (1.1) such that

$$x(0,s) = p(s), \;\; s \in I\!R^+. \tag{2.3}$$

Proof. We fix an $s \in I\!R^+$. Let $m = [s]$. Then from (1.1) and for $n \leq t < n+1$ we take

$$\begin{aligned} x_t(t,m) = \;\; & A(t)x(t,m) + B(t,m)x(n,m) + C(t,m)x(t,m) + \\ & D(t,m)x(n,m) + f(x(t,m)). \end{aligned} \tag{2.4}$$

Then from (1.2) and (2.4) we get for $n \leq t < n+1$

$$\begin{aligned} x(t,m) = \;\; & \left(X(t)X^{-1}(n) + \int_n^t X(t)X^{-1}(u)(B(u,m) + D(u,m))du \right) x(n,m) + \\ & \int_n^t X(t)X^{-1}(u)\Big(C(u,m)x(u,m) + f(x(u,m))\Big)du \end{aligned} \tag{2.5}$$

where $X(t)$ is a fundamental matrix solution of (1.4). We take $t \to n+1$ in (2.5). Then

$$x(n+1,m) = \Big(X(n+1)X^{-1}(n) + \int_n^{n+1} X(n+1)X^{-1}(u)(B(u,m)+D(u,m))du\Big)x(n,m) + \int_n^{n+1} X(n+1)X^{-1}(u)\Big(C(u,m)x(u,m)+f(x(u,m))\Big)du. \tag{2.6}$$

We consider the linear difference equation

$$w(n+1,m) = S(n,m)w(n,m),$$

$$S(n,m) = X(n+1)X^{-1}(n) + \int_n^{n+1} X(n+1)X^{-1}(u)(B(u,m)+D(u,m))du. \tag{2.7}$$

We prove that $S(n,m)$ is invertible for $n \in I\!N$. We get

$$S(n,m) = X(n+1)X^{-1}(n)\Big(I_k + \int_n^{n+1} X(n)X^{-1}(u)(B(u,m)+D(u,m))du\Big) \tag{2.8}$$

where I_k is the $k \times k$ identity matrix. Moreover, from [9] and (2.1) we take for $u \geq n$

$$|X(n)X^{-1}(u)| \leq e^{N|u-n|} = e^{N(u-n)}. \tag{2.9}$$

Then from (2.1), (2.2) and (2.9) we get

$$\Big|\int_n^{n+1} X(n)X^{-1}(u)(B(u,m)+D(u,m))du\Big| \leq 2M\frac{e^N-1}{N} < 1. \tag{2.10}$$

Therefore relations (2.8) and (2.10) imply that $S(n,m)$ is invertible. Let $W(n,m)$ be the fudamental matrix solution of (2.7) such that $W(0,m) = I_k$. Then from (2.6) we take

$$x(n,m) = W(n,m)x(0,m) + \sum_{r=0}^{n-1} W(n,m)W^{-1}(r+1,m)H(r,m),$$

$$H(r,m) = \int_r^{r+1} X(r+1)X^{-1}(u)\Big(C(u,m)x(u,m)+f(x(u,m))\Big)du. \tag{2.11}$$

From (1.1) we take for $n \leq t < n+1$

$$\begin{aligned} x(t,s) = \quad & \left(X(t)X^{-1}(n) + \int_n^t X(t)X^{-1}(u)B(u,s)du \right) x(n,s) + \\ & \int_n^t X(t)X^{-1}(u)\Big(C(u,s)x(u,m) + D(u,s)x(n,m) + f(x(u,m))\Big)du. \end{aligned} \tag{2.12}$$

We get $t \to n+1$ in (2.12) . Then we take the inhomogenous difference equation

$$\begin{aligned} x(n+1,s) = \quad & \Big(X(n+1)X^{-1}(n) + \\ & \int_n^{n+1} X(n+1)X^{-1}(u)B(u,s)du \Big) x(n,s) + \\ & \int_n^{n+1} X(n+1)X^{-1}(u)\Big(C(u,s)x(u,m) + D(u,s)x(n,m) + f(x(u,m))\Big)du. \end{aligned} \tag{2.13}$$

We consider the linear difference equation

$$\begin{aligned} & z(n+1,s) = R(n,s)z(n,s), \\ & R(n,s) = X(n+1)X^{-1}(n) + \int_n^{n+1} X(n+1)X^{-1}(u)B(u,s)du. \end{aligned} \tag{2.14}$$

Using the same argument to prove that $S(n,m)$ is invertible we can prove that $R(n,s)$ is also invertible. Then if $Z(n,s)$ is a fudamental matrix solution of (2.14) such that $Z(0,s) = I_k$, from (2.13) we have,

$$\begin{aligned} & x(n,s) = Z(n,s)x(0,s) + \sum_{r=0}^{n-1} Z(n,s)Z^{-1}(r+1,s)G(r,s), \\ & G(r,s) = \int_r^{r+1} X(r+1)X^{-1}(u)\Big(C(u,s)x(u,m) + D(u,s)x(r,m) + f(x(u,m))\Big)du. \end{aligned} \tag{2.15}$$

Using (2.5), (2.11), (2.12) and (2.15) $x(t,s)$ is well defined.

We prove that $x(t,s)$ is continuous in $\mathbb{R}^+ \times \mathbb{R}^+ - \mathbb{N}$ and left continuous in $\mathbb{R}^+ \times \mathbb{R}^+$. Let $t, t_0 \in [n_0, n_0+1),\ s, s_0 \in [m_0, m_0+1),\ n_0, m_0 \in \mathbb{N}$.

Then from (2.12) we take

$$
\begin{aligned}
&x(t,s)-x(t_0,s_0)=\left(X(t)X^{-1}(n_0)+\int_{n_0}^{t}X(t)X^{-1}(u)B(u,s)du\right)x(n_0,s)-\\
&\left(X(t_0)X^{-1}(n_0)+\int_{n_0}^{t_0}X(t_0)X^{-1}(u)B(u,s_0)du\right)x(n_0,s_0)+\\
&\int_{n_0}^{t}X(t)X^{-1}(u)\Big(C(u,s)x(u,m_0)+D(u,s)x(n_0,m_0)+f(x(u,m_0))\Big)du-\\
&\int_{n_0}^{t_0}X(t_0)X^{-1}(u)\Big(C(u,s_0)x(u,m_0)+D(u,s_0)x(n_0,m_0)+f(x(u,m_0))\Big)du=\\
&(X(t)-X(t_0))X^{-1}(n_0)x(n_0,s)+X(t_0)X^{-1}(n_0)(x(n_0,s)-x(n_0,s_0))+\\
&\Big(\int_{n_0}^{t}(X(t)-X(t_0))X^{-1}(u)B(u,s)du\Big)x(n_0,s)+\\
&\Big(\int_{t_0}^{t}X(t_0)X^{-1}(u)B(u,s)du\Big)x(n_0,s)+\\
&\Big(\int_{n_0}^{t_0}X(t_0)X^{-1}(u)(B(u,s)-B(u,s_0))du\Big)x(n_0,s)+\\
&\Big(\int_{n_0}^{t_0}X(t_0)X^{-1}(u)B(u,s_0)du\Big)(x(n_0,s)-x(n_0,s_0))+\\
&\int_{n_0}^{t}(X(t)-X(t_0))X^{-1}(u))C(u,s)x(u,m_0)du+\\
&\int_{t_0}^{t}X(t_0)X^{-1}(u)C(u,s)x(u,m_0)du+\\
&\int_{n_0}^{t_0}X(t_0)X^{-1}(u)(C(u,s)-C(u,s_0))x(u,m_0)du+\\
&\Big(\int_{n_0}^{t}(X(t)-X(t_0))X^{-1}(u))D(u,s)du\Big)x(n_0,m_0)+\\
&\Big(\int_{t_0}^{t}X(t_0)X^{-1}(u)D(u,s)du\Big)x(n_0,m_0)+\\
&\Big(\int_{n_0}^{t_0}X(t_0)X^{-1}(u)(D(u,s)-D(u,s_0))du\Big)x(n_0,m_0)
\end{aligned}
\tag{2.16}
$$

Moreover, from (2.15) we get

$$
\begin{aligned}
&x(n_0,s)-x(n_0,s_0)=Z(n_0,s)x(0,s)-Z(n_0,s_0)x(0,s)+\\
&Z(n_0,s_0)x(0,s)-Z(n_0,s_0)x(0,s_0)+\\
&\sum_{r=0}^{n_0-1}(Z(n_0,s)Z^{-1}(r+1,s)-Z(n_0,s_0)Z^{-1}(r+1,s_0))G(r,s)+\\
&\sum_{r=0}^{n_0-1}(Z(n_0,s_0)Z^{-1}(r+1,s_0)(G(r,s)-G(r,s_0))= \qquad (2.17)\\
&(Z(n_0,s)-Z(n_0,s_0))x(0,s)+Z(n_0,s_0)(x(0,s)-x(0,s_0))+\\
&\sum_{r=0}^{n_0-1}\Big(Z(n_0,s)Z^{-1}(r+1,s)-Z(n_0,s_0)Z^{-1}(r+1,s_0)\Big)G(r,s)+\\
&\sum_{r=0}^{n_0-1}Z(n_0,s_0)Z^{-1}(r+1,s_0)(G(r,s)-G(r,s_0))
\end{aligned}
$$

where

$$
\begin{aligned}
G(r,s)-G(r,s_0)= \quad & \int_r^{r+1}X(r+1)X^{-1}(u)\Big(C(u,s)-C(u,s_0)\Big)x(u,m)+\\
&(D(u,s)-D(u,s_0))x(r,m)\Big)du.
\end{aligned} \qquad (2.18)
$$

In addition from (1.2), (1.3), (2.5), (2.9) and (2.10) we take

$$
\begin{aligned}
|x(t,m_0)|\le \quad & (e^N+2M\frac{e^N-1)}{N})|x(n_0,m_0)|+\\
&(M+\gamma)\int_{n_0}^t e^{N(t-u)}|x(u,m_0)|du\le\\
&(e^N+2M\frac{e^N-1)}{N})|x(n_0,m_0)|+\\
&(M+\gamma)e^N\int_{n_0}^t|x(u,m_0)|du.
\end{aligned}
$$

Then by Gronwall's Lemma (see [9]) to the interval $[n_0,n_0+1)$ we get

$$
|x(t,m_0)|\le(e^N+2M\frac{e^N-1}{N})|e^{e^N(M+\gamma)}|x(n_0,m_0)|=L|x(n_0,m_0)|. \quad (2.19)
$$

Since relations (2.1), (2.14) imply that

$$|R(n_0,s)| \leq e^N + M\frac{e^N-1}{N}$$

and

$$Z(n_0,s) = R(n_0-1,s)R(n_0-2,s)\cdots R(0,s)$$

we get

$$|Z(n_0,s)| \leq \Big(e^N + M\frac{e^N-1}{N}\Big)^{n_0}. \tag{2.20}$$

From (1.2), (2.1), (2.15), (2.19) we take

$$\begin{aligned} |G(r,s)| \leq \quad & \int_r^{r+1} e^{N(r+1-u)}\Big((M+\gamma)|x(u,m_0)| + M|x(r,m_0)|\Big)du \leq \\ & \int_r^{r+1} e^{N(r+1-u)}\Bigg((M+\gamma)L + M\Bigg)|x(r,m_0)|du \leq \\ & e^N\Bigg((M+\gamma)L+M\Bigg)|x(r,m_0)|. \end{aligned} \tag{2.21}$$

Therefore relations (2.15), (2.20), (2.21) imply that

$$\begin{aligned} x(n_0,s)| \leq \quad & \Big(e^N + M\frac{e^N-1}{N}\Big)^{n_0}|x(0,s)| + \\ & e^N\Bigg((M+\gamma)L+M\Bigg)\sum_{r=0}^{n_0-1}\Big(e^N + M\frac{e^N-1}{N}\Big)^{n_0-r-1}|x(r,m_0)|. \end{aligned} \tag{2.22}$$

Then since $A(t)$ is a continuous matrix function on $I\!R^+$, $B(t,s)$, $C(t,s)$, $D(t,s)$ are continuous matrix functions on $I\!R^+ \times I\!R^+$, from relations (2.16), (2.17), (2.19) and (2.22) we take that if $t, t_1 \in (n_0, n_0+1),\ s, s_1 \in (m_0, m_0+1)$, $n_0, m_0 \in I\!N$, then

$$\begin{aligned} &\lim_{t\to t_1, s\to s_1} x(t,s) = x(t_1,s_1) \\ &\lim_{t\to t_1, s\to m_0^+} x(t,s) = x(t_1,m_0) \\ &\lim_{t\to n_0^+, s\to s_1} x(t,s) = x(n_0,s_1) \\ &\lim_{t\to n_0^+, s\to m_0^+} x(t,s) = x(n_0,m_0) \end{aligned} \tag{2.23}$$

Using relations (2.12), (2.15) and using the argument as above we can prove that if $t \in [n_0 - 1, n_0)$, $s, s_0 \in [m_0, m_0 + 1)$, $s_1 \in (m_0, m_0 + 1)$, $n_0, m_0 \in I\!N$ then relations (2.16), (2.17), (2.19), (2.22) hold for $n_0 = n_0 - 1$ and $t_0 = n_0$ and so

$$\begin{aligned} \lim_{t \to n_0^-, s \to s_1} x(t,s) &= x(n_0, s_1) \\ \lim_{t \to n_0^-, s \to m_0^+} x(t,s) &= x(n_0, m_0) \end{aligned} \tag{2.24}$$

From (2.23) and (2.24) the function $x(t,s)$ of (2.15) is continous on $I\!R^+ \times I\!R^+ - I\!N$ and left continuous at the points (t, m), $t \in I\!R^+, m \in I\!N$. Thus, the proof of the proposition is completed.

In the next proposition we study the asymptotic behavior of the solutions of (1.1) in the case where (1.5) holds.

Proposition 2.2 *Consider equation (1.1) where $A(t)$ is a $k \times k$ invertible and continuous matrix function on $I\!R^+$, $B(t,s)$, $C(t,s)$, $D(t,s)$ are $k \times k$ continuous matrix functions on $I\!R^+ \times I\!R^+$, $[t]$, $[s]$ are the integral parts of t, s respectively and $f : I\!R^k \to I\!R^k$ is a continuous function such that (1.2), (1.3) hold. Suppose also that relations (1.5), (2.1), (2.2) are fulfilled. Moreover, suppose that*

$$max\{2M, M + \gamma\} < \frac{a^3}{K^2 e^a (a + K(a + 2e^a - 2))}. \tag{2.25}$$

Then, if $x(t,s)$ is a solution of (1.1) where $x(0,s)$ is a bounded and continuous function, we have that $x(t,s)$ tends exponentially to zero as $t \to \infty$ uniformly with respect to s.

Proof. Let $x(t,s)$ be s a solution of (1.1) such that (2.3) is satisfied. We fix an $s \in I\!R^+$. If $m = [s]$, $n = [t]$ from (1.1) we get (2.5). We claim that

$$|x(t,m)| \le L|x(n,m)|, \quad L = K + \frac{2MK}{a}(e^a - 1) \tag{2.26}$$

Using (1.2), (1.3), (1.5), (2.1) , (2.5) we get for $n \leq t < n+1$

$$\begin{aligned} |x(t,m)| \leq \;& Ke^{-a(t-n)}|x(n,m)| + 2MK|x(n,m)|\int_n^t e^{-a(t-u)}du + \\ & K(M+\gamma)\int_n^t e^{-a(t-u)}|x(u,m)|du = \\ & \Big(Ke^{-a(t-n)} - \tfrac{2KM}{a}e^{-a(t-n)} + \tfrac{2KM}{a}\Big)|x(n,m)| + \\ & K(M+\gamma)\int_n^t e^{-a(t-u)}|x(u,m)|du \end{aligned}$$

and so,

$$\begin{aligned} & e^{a(t-n)}|x(t,m)| \\ \leq \;& \Big(K + \tfrac{2KM}{a}(e^{a(t-n)} - 1)\Big)|x(n,m)| + K(M+\gamma)\int_n^t e^{a(u-n)}|x(u,m)|du \\ \leq \;& L|x(n,m)| + K(M+\gamma)\int_n^t e^{a(u-n)}|x(u,m)|du. \end{aligned} \tag{2.27}$$

Then, by setting in (2.27) $\phi_1(t,m) = e^{a(t-n)}|x(t,m)|$ we get

$$\phi_1(t,m) \leq L|x(n,m)| + K(M+\gamma)\int_n^t \phi_1(u,m)du. \tag{2.28}$$

Using Gronwall's Lemma in (2.28) we take

$$\phi_1(t,m) \leq L|x(n,m)|e^{K(M+\gamma)(t-n)}$$

and so

$$|x(t,m)| \leq L|x(n,m)|e^{(-a+K(M+\gamma))(t-n)}. \tag{2.29}$$

Since from (2.25) it is obvious that $-a + K(M+\gamma) < 0$ we have that our claim (2.26) is true.

Now, we prove that

$$|x(n,m)| \leq Ke^{-bn}|x(0,m)|,\;\; b = b_1 - \frac{K^2L(M+\gamma)}{a}e^{b_1} > 0,\;\; b_1 = a - \frac{2MK^2e^a}{a} > 0. \tag{2.30}$$

Firstly, we prove that

$$b_1 > 0, \quad b > 0. \tag{2.31}$$

Relation (2.25) implies that $b_1 > 0$ and since from (2.25) $M < 1$ we get

$$\begin{aligned} b = \quad & a - \frac{2MK^2e^a}{a} - \frac{K^2L(M+\gamma)}{a}e^{b_1} > \\ & a - \frac{2MK^2e^a}{a} - \frac{K^2L(M+\gamma)}{a}e^{a} > \\ & a - \frac{K^2e^a}{a}(1+L)\max\{2M, M+\gamma\} > \\ & a - \frac{K^2e^a}{a}\Big(1 + K(1 + 2\frac{e^a-1}{a})\Big)\max\{2M, M+\gamma\} > 0. \end{aligned}$$

Let $Y(n)$ be the fudamental matrix solution of the equation

$$y(n+1) = L(n)y(n), \quad L(n) = X(n+1)X^{-1}(n), \quad n = 0, 1, \dots \tag{2.32}$$

such that $Y(0) = I_k$. Then for $n, r \in I\!N$ we get

$$W(n,m)W^{-1}(r,m) = Y(n)Y^{-1}(r) + \sum_{v=r}^{n-1} Y(n)Y^{-1}(v+1)T(v,m)W(v,m)W^{-1}(r,m) \tag{2.33}$$

where $W(n,m)$ is the fundamental matrix solution of (2.7) and

$$T(v,m) = \int_v^{v+1} X(v+1)X^{-1}(u)\Big(B(u,m) + D(u,m)\Big)du.$$

We have

$$\begin{array}{l} Y(n)Y^{-1}(r) = \\ X(n)X^{-1}(n-1)X(n-1)X^{-1}(n-2)\cdots X(1)X^{-1}(0)\times \\ X(0)X^{-1}(1)\cdots X(r-1)X^{-1}(r) = X(n)X^{-1}(r). \end{array} \tag{2.34}$$

Then relations (1.5) and (2.34) imply that

$$|Y(n)Y^{-1}(r)| \le Ke^{-a(n-r)}, \; n \ge r. \tag{2.35}$$

Furthermore from (1.5), (2.1) we take

$$|T(v,m)| \le 2MK\int_v^{v+1} e^{-a(v+1-u)}du \le \frac{2KM}{a}. \tag{2.36}$$

Relations (2.33), (2.35) and (2.36) imply that

$$|W(n,m)W^{-1}(r,m)| \le Ke^{-a(n-r)} + \frac{2MK^2}{a}\sum_{v=r}^{n-1} e^{-a(n-v-1)}|W(v,m)W^{-1}(r,m)|$$

and so

$$e^{a(n-r)}|W(n,m)W^{-1}(r,m)| \le K + \frac{2MK^2e^a}{a}\sum_{v=r}^{n-1} e^{a(v-r)}|W(v,m)W^{-1}(r,m)|.$$

So, if $\phi_2(r,m) = e^{a(n-r)}|W(n,m)W^{-1}(r,m)|$ we get

$$\phi_2(r,m) \le K + \frac{2MK^2e^a}{a}\sum_{v=r}^{n-1}\phi_2(v,m).$$

By discrete Gronwall's lemma (see [15]) we have

$$\phi_2(r,m) \le Ke^{\frac{2MK^2e^a}{a}(n-r)}.$$

Therefore,

$$|W(n,m)W^{-1}(r,m)| \le Ke^{-b_1(n-r)} \tag{2.37}$$

where b_1 is defined in (2.30). From (2.6) we take

$$x(n,m) = W(n,m)W^{-1}(0,m)x(0,m) + \sum_{v=0}^{n-1} W(n,m)W^{-1}(v+1,m)R(v,m),$$
$$R(v,m) = \int_v^{v+1} X(v+1)X^{-1}(u)\Big(C(u,m)x(u,m) + f(x(u,m))\Big)du. \tag{2.38}$$

Relations (1.2), (1.3), (1.5), (2.1), (2.26) and (2.38) imply that

$$|R(v,m)| \le \frac{KL(M+\gamma)}{a}|x(v,m)|. \tag{2.39}$$

Hence, from (2.26), (2.37), (2.38) and (2.39) we get

$$|x(n,m)| \le Ke^{-b_1 n}|x(0,m)| + \frac{K^2L(M+\gamma)}{a}\sum_{v=0}^{n-1} e^{-b_1(n-v-1)}|x(v,m)|.$$

Thus,

$$e^{b_1 n}|x(n,m)| \le K|x(0,m)| + \frac{K^2L(M+\gamma)}{a}e^{b_1}\sum_{v=0}^{n-1} e^{b_1 v}|x(v,m)|.$$

By taking $\phi_3(n,m) = e^{b_1 n}|x(n,m)|$ we take

$$\phi_3(n,m) \le K|x(0,m)| + \frac{K^2L(M+\gamma)}{a}e^{b_1}\sum_{v=0}^{n-1}|\phi_3(v,m)|.$$

By discrete Gronwall's lemma we get

$$\phi_3(n,m) \le K|x(0,m)|e^{\dfrac{K^2L(M+\gamma)}{a}e^{b_1}n}$$

which implies that (2.30) is satisfied.

Arguing as in (2.37) and using (2.25) we get

$$|Z(n,m)Z^{-1}(r,m)| \le Ke^{-b_2(n-r)}, \quad b_2 = a - \frac{MK^2e^a}{a} > 0 \tag{2.40}$$

where $Z(n,m)$ is defined in Proposition 2.1. In addition from (1.2), (1.3), (1.5), (2.1), (2.15), (2.26) we obtain

$$\begin{aligned} |G(r,s)| \le \quad & K(ML+\gamma L+M)|x(r,m)|\int_r^{r+1} e^{-a(r+1-u)}du \le \\ & \frac{K(ML+\gamma L+M)}{a}|x(r,m)|. \end{aligned} \tag{2.41}$$

where $G(r,s)$ is defined in (2.15). Thus, from (2.15), (2.30), (2.40), (2.41) we get

$$\begin{aligned} |x(n,s)| \quad & \le Ke^{-b_2n}|x(0,s)| + \tfrac{K^2(ML+M+\gamma L)}{a}\sum_{r=0}^{n-1} e^{-b_2(n-r-1)}|x(r,m)| \\ & \le Ke^{-b_2n}|x(0,s)| + \tfrac{K^3(ML+M+\gamma L)}{a}|x(0,m)|\sum_{r=0}^{n-1} e^{-b_2(n-r-1)}e^{-br} \\ & = Ke^{-b_2n}|x(0,s)| + \tfrac{K^3(ML+M+\gamma L)e^{b_2}}{a}\frac{e^{-bn}-e^{-b_2n}}{e^{b_2-b}-1}|x(0,m)|. \end{aligned} \tag{2.42}$$

In addition from (1.2), (1.3), (1.5), (2.1), (2.12), (2.25) and (2.26) we get

$$\begin{aligned} |x(t,s)| \le \quad & \Big(Ke^{-a(t-n)} + \frac{KM}{a}(1-e^{-a(t-n)}\Big)|x(n,s)| + \\ & K(ML+M+\gamma L)\frac{1}{a}(1-e^{-a(t-n)})|x(n,m)| \le \\ & K\Big(1+\frac{M}{a}\Big)|x(n,s)| + \frac{K(ML+M+\gamma L)}{a}|x(n,m)|. \end{aligned} \tag{2.43}$$

Hence, using (2.25), (2.30), (2.31), (2.42), (2.43) the proof of the proposition follows immediately.

Finally, using the exponential dichotomy of (1.4) we study the asymptotic behaviour of the solutions of (1.1).

Proposition 2.3 *Consider the equation (1.1) where where $A(t)$ is a $k \times k$ invertible and continuous matrix function on $I\!R^+$, $B(t,s)$, $C(t,s)$, $D(t,s)$ are $k \times k$ continuous matrix functions on $I\!R^+ \times I\!R^+$, $[t]$, $[s]$ are the integral parts of t, s respectively and $f : I\!R^k \to I\!R^k$ is a continuous function such that (1.2), (1.3) are fulfilled. Suppose also that relations (1.6), (2.1), (2.2) hold. Furthermore, suppose that*

$$
\begin{aligned}
&(3M+\gamma)\Big(\frac{Ke^{2N}(1+e^a)}{N(e^a-1)} + \frac{e^N}{N}\Big) < 1,\\
&M < \tfrac{N}{2Ke^N}\min\{\tfrac{e^a-1}{1+e^a},\ \tfrac{e^\epsilon-1}{(1+e^{-\epsilon})e^\alpha}\},\quad 0<\epsilon<a,\\
&\lambda = \frac{Ke^N(2M+c(M+\gamma))}{N} < \frac{1-e^{-a}}{1+e^{-a}},\quad c = (e^N + \frac{2Me^N}{N})e^{\frac{e^N(M+\gamma)}{N}},\\
&\mu = a - \frac{\lambda e^a}{1-d} > 0,\quad d = \lambda\frac{1+e^{-a}}{1-e^{-a}}.
\end{aligned}
\tag{2.44}
$$

Then there exist solutions $x(t,s)$ of (1.1) which tend exponentially to zero as $t \to \infty$ uniformly with respect to s. Moreover, every bounded solution $x(t,s)$ of (1.1), where $x(0,s)$ is a bounded and continuous function, tends exponentially to zero as $t \to \infty$ uniformly with respect to s.

Proof. Let E be the set of all bounded and continuous functions from $I\!R^+$ into $I\!R^k$. Let $\xi \in I\!R^k$. For a fixed $s \in I\!R$ and a function $x(t,m) \in E$,

$n \le t < n+1$, $m = [s]$, we define the operator T_m on E as follows:

$$
\begin{aligned}
T_m x(t,m) = \quad & X(t)X^{-1}(n)b(n,m) + \int_n^t X(t)X^{-1}(u)\Big(B(u,m)x(n,m) + \\
& C(u,m)x(u,m) + D(u,m)x(n,m) + f(x(u,m))\Big) du, \\
& b(n,m) = X(n)P\xi + \sum_{v=0}^{n-1} X(n)PX^{-1}(v+1)H_1(v,m) - \\
& \sum_{v=n}^{\infty} X(n)(I-P)X^{-1}(v+1)H_1(v,m), \\
& H_1(v,m) = \int_v^{v+1} X(v+1)X^{-1}(u)\Big(B(u,m)x(v,m) + \\
& C(u,m)x(u,m) + D(u,m)x(v,m) + f(x(u,m))\Big) du.
\end{aligned}
\tag{2.45}
$$

Using (1.2), (1.3), (1.6), (2.1), (2.9) and if $|x|_m = \sup\{|x(t,m)|, t \in I\!R^+\}$ we take for $n \le t < n+1$

$$
\begin{aligned}
& |T_m x(t,m)| \le e^{N(t-n)}|b(n,m)| + (3M+\gamma)|x|_m \int_n^t e^{N(t-u)} du \le \\
& e^N |b(n,m)| + \tfrac{e^N}{N}(3M+\gamma)|x|_m, \\
& |b(n,m)| \le Ke^{-an}|\xi| + K\sum_{v=0}^{n-1} e^{-a(n-v-1)}|H_1(v,m)| + K\sum_{v=n}^{\infty} e^{-a(v+1-n)}|H_1(v,m)|, \\
& |H_1(v,m)| \le (3M+\gamma)|x|_m \int_v^{v+1} e^{N(v+1-u)} du \le \frac{e^N}{N}(3M+\gamma)|x|_m.
\end{aligned}
\tag{2.46}
$$

So, relations (2.46) imply that

$$
|T_m x(t,m)| \le e^N\Big(Ke^{-an}|\xi| + K\frac{e^a+1}{e^a-1}\frac{e^N}{N}(3M+\gamma)|x|_m \Big) + \frac{e^N}{N}(3M+\gamma)|x|_m.
$$

Hence, we have that $T_m x(t,m)$ is a bounded function.

Now, we prove that $T_m x(t,m)$ is continuous at $t \in I\!R^+$. From (2.45) we can prove that

$$
b(n+1,m) = L(n)b(n,m) + H_1(n,m), \quad L(n) = X(n+1)X^{-1}(n). \tag{2.47}
$$

Therefore, from (2.45) and (2.47) we have

$$\begin{aligned}
&\lim_{t\to n+1} T_m x(t,m) = X(n+1)X^{-1}(n)b(n,m)+\\
&\int_n^{n+1} X(n+1)X^{-1}(u)\Bigg(B(u,m)x(n,m)+\\
&C(u,m)x(u,m)+D(u,m)x(n,m)+f(x(u,m))\Bigg)du=\\
&L(n)b(n,m)+H_1(n,m)=b(n+1,m)=T_m x(n+1,m).
\end{aligned} \tag{2.48}$$

Therefore, from (2.45), (2.48) we have that $T_m x(t,m)$ is continuous for any $t \in I\!R^+$.

To continue, we prove that T_m is a contraction on E. Let $x_1(t,m), x_2(t,m) \in E$, then

$$\begin{aligned}
&T_m x_1(t,m) - T_m x_2(t,m) = X(t)X^{-1}(n)(b_1(n,m)-b_2(n,m))+\\
&\int_n^t X(t)X^{-1}(u)\Bigg(B(u,m)(x_1(n,m)-x_2(n,m))+C(u,m)(x_1(u,m)-x_2(u,m))+\\
&D(u,m)(x_1(n,m)-x_2(n,m))+f(x_1(u,m))-f(x_2(u,m))\Bigg)du,\\
&b_i(n,m) = X(n)P\xi + \sum_{v=0}^{n-1} X(n)PX^{-1}(v+1)H_1^{(i)}(v,m)-\\
&\sum_{v=n}^{\infty} X(n)(I-P)X^{-1}(v+1)H_1^{(i)}(v,m),\ i=1,2\\
&H_1^{(i)}(v,m) = \int_v^{v+1} X(v+1)X^{-1}(u)\Bigg(B(u,m)x_i(v,m)+\\
&C(u,m)x_i(u,m)+D(u,m)x_i(v,m)+f(x_i(u,m))\Bigg)du,\ i=1,2.
\end{aligned} \tag{2.49}$$

Hence, from (1.2), (1.3), (2.1), (2.9) and (2.49) we get

$$|T_m x_1(t,m)-T_m x_2(t,m)| \le e^N|b_1(n,m)-b_2(n,m)|+\frac{e^N}{N}(3M+\gamma)|x_1-x_2|_m. \tag{2.50}$$

Moreover,

$$
\begin{aligned}
|b_1(n,m) - b_2(n,m)| \leq \quad & K\sum_{v=0}^{n-1} e^{-a(n-v-1)} |H_1^{(1)}(v,m) - H_1^{(2)}(v,m)| + \\
& K\sum_{v=n}^{\infty} e^{-a(v+1-n)} |H_1^{(1)}(v,m) - H_1^{(2)}(v,m)|,
\end{aligned}
\tag{2.51}
$$

and

$$
\begin{aligned}
|H_1^{(1)}(v,m) - H_1^{(2)}(v,m)| \leq \quad & (3M+\gamma)|x_1 - x_2|_m \int_v^{v+1} e^{N(v+1-u)} du \leq \\
& \frac{e^N}{N}(3M+\gamma)|x_1 - x_2|_m.
\end{aligned}
\tag{2.52}
$$

So, from (2.51) and (2.52) we have,

$$
|b_1(n,m) - b_2(n,m)| \leq K \frac{e^a+1}{e^a-1} \frac{e^N}{N}(3M+\gamma)|x_1 - x_2|_m. \tag{2.53}
$$

Relations (2.44), (2.50), (2.53) imply that T_m is a contraction on E. Hence, there exists a unique $x(t,m)$ such that

$$
T_m x(t,m) = x(t,m). \tag{2.54}
$$

Therefore, relations (2.45), (2.54) imply that

$$
x(n,m) = b(n,m), \ n = 0, 1, \dots \ . \tag{2.55}
$$

Furthermore, from (2.45) and (2.55) for $n=0$ we get

$$
x(0,m) = b(0,m) = P\xi - \sum_{v=0}^{\infty} (I-P)X^{-1}(v+1)H_1(v,m)
$$

and so,

$$
Px(0,m) = P\xi. \tag{2.56}
$$

Therefore, from (2.45), (2.54), (2.55) and (2.56) we take

$$
\begin{aligned}
x(t,m) = \quad & X(t)X^{-1}(n)x(n,m) + \int_n^t X(t)X^{-1}(u)\Bigg(B(u,m)x(n,m) + \\
& C(u,m)x(u,m) + D(u,m)x(n,m) + f(x(u,m))\Bigg) du, \\
& x(n,m) = X(n)Px(0,m) + \sum_{v=0}^{n-1} X(n)PX^{-1}(v+1)H_1(v,m) - \\
& \sum_{v=n}^{\infty} X(n)(I-P)X^{-1}(v+1)H_1(v,m).
\end{aligned} \tag{2.57}
$$

We can easily prove that $x(n,m)$ satisfies (2.6).

Since $X(n)$ is the fundamental matrix solution of the difference equation (2.32) from (1.6) we have that (2.32) has an exponential dichotomy (see [14], [16] and the references cited therein)

$$
\begin{array}{l}
|X(n)PX^{-1}(v)| \leq Ke^{-a(n-v)}, \;\; n \geq v \\
|X(n)(I-P)X^{-1}(v)| \leq Ke^{-a(v-n)}, \;\; v \geq n
\end{array} \tag{2.58}
$$

where $n, v \in I\!N$. Using the roughness of exponential dichotomies for difference equations (see [16]) and using (2.44) we have that there exists a projection Q_s, rankQ_s =rank P such that the difference equation (2.14) has an exponential dichotomy

$$
\begin{array}{l}
|Z(n,s)Q_sZ^{-1}(v,s)| \leq K_1e^{-b(n-v)}, \;\; n \geq v \\
|Z(n,s)(I-Q_s)Z^{-1}(v,s)| \leq K_1e^{-b(v-n)}, \;\; v \geq n
\end{array} \tag{2.59}
$$

where $K_1 = 2K\frac{e^{\epsilon}+1}{1-e^{-\epsilon}}, b = a - \epsilon, 0 < \epsilon < a$.

From (2.12) and (2.57) we define $x(t,s)$ where $x(n,s)$ satisfies the relation

$$
\begin{aligned}
x(n,s) = \quad & Z(n,s)Q_sx(0,s) + \sum_{v=0}^{n-1} Z(n,s)Q_sZ^{-1}(v+1,s)G(v,s) - \\
& \sum_{v=n}^{\infty} Z(n,s)(I-Q_s)Z^{-1}(v+1,s)G(v,s), \\
& G(v,s) = \int_v^{v+1} X(v+1)X^{-1}(u)\Bigg(C(u,s)x(u,m) + \\
& D(u,s)x(v,m) + f(x(u,m))\Bigg) du
\end{aligned} \tag{2.60}
$$

We can easily prove that $x(n, s)$ satisfies (2.13). Since $x(t, m), x(t, s)$ satisfy relations (2.5), (2.12) respectively and $x(n, m), x(n, s)$ satisfy the difference equations (2.6), (2.13) correspondingly, then arguing as in Proposition 2.1 we have that $x(t, s)$ is continuous in $I\!R^+ \times I\!R^+ \setminus I\!N$ and left continuous at the points (t, m), $t \in I\!R^+, m \in I\!N$. Therefore, the function $x(t, s)$ defined from relations (2.12), (2.57), (2.60) is a solution of (1.1).

From (1.2), (1.3), (2.1), (2.9), (2.57) we take

$$|x(t,m)| \leq \left(\frac{2Me^N}{N} + e^N\right)|x(n,m)| + (M+\gamma)\int_n^t e^{N(t-u)}|x(u,m)|du.$$

Then by Gronwall's lemma we take

$$|x(t,m)| \leq c|x(n,m)|. \tag{2.61}$$

Furthermore, from (1.6), (2.45), (2.57), (2.61) we take

$$\begin{aligned}
&|x(n,m)| \leq Ke^{-an}|x(0,m)| + K\sum_{v=0}^{\infty} e^{-a|n-v-1|}|H_1(v,m)|,\\
&|H_1(v,m)| \leq \int_v^{v+1} e^{N(v+1-u)}(2M|x(v,m)| + (M+\gamma)|x(u,m)|)du \leq\\
&\frac{(2M + c(M+\gamma))e^N}{N}|x(v,m)|.
\end{aligned} \tag{2.62}$$

Hence, relations (2.62) imply that

$$|x(n,m)| \leq Ke^{-an}|x(0,m)| + \lambda\sum_{v=0}^{\infty} e^{-a|n-v-1|}|x(v,m)|. \tag{2.63}$$

By applying Lemma 6 of [17] to (2.63) and from (2.44)we have

$$|x(n,m)| \leq \frac{K}{1-d}e^{(\frac{\lambda e^a}{1-d}-a)n}|x(0,m)| = \frac{K}{1-d}e^{-\mu n}. \tag{2.64}$$

Moreover, from (2.12), (2.59), (2.60) and (2.61) we take

$$|x(t,s)| \le (e^N + \frac{Me^N}{N})|x(n,s)| + \frac{e^N}{N}(M+(M+\gamma)c)|x(n,m)|,$$

$$|x(n,s)| \le K_1 e^{-bn}|x(0,s)| + K_1\frac{e^N}{N}(M+c(M+\gamma))\left(\sum_{v=0}^{n-1} e^{-b(n-v-1)}|x(v,m)|+\right.$$
$$\left.\sum_{v=n}^{\infty} e^{-b(v+1-n)}|x(v,m)|\right) = K_1 e^{-bn}|x(0,s)|+$$
$$K_1\frac{e^N}{N}(M+c(M+\gamma))\frac{K}{1-d}\Big(\frac{e^{-bn+\mu}-e^{-n\mu+\mu}}{e^{\mu-b}-1} + \frac{e^{-n\mu-b}}{1-e^{-b-\mu}}\Big). \tag{2.65}$$

So, from (2.44), (2.59), (2.64) and (2.65) the solution $x(t,s)$ of (1.1) defined by (2.57), (2.12) and (2.60) tends exponentially to zero as $n \to \infty$.

Now, let $x(t,s)$ be a bounded solution of (1.1). Then $x(t,s)$ satisfies relations (2.5), (2.12), where $x(n,m)$, $x(n,s)$ satisfy equations (2.6), (2.13) respectively. We take

$$\begin{aligned}\bar{x}(n,m) = \quad & x(n,m) - X(n)Px(0,m) - \sum_{v=0}^{n-1} X(n)PX^{-1}(v+1)H_1(v,m)+ \\ & \sum_{v=n}^{\infty} X(n)(I-P)X^{-1}(v+1)H_1(v,m).\end{aligned}$$

Then, we can easily prove that $\bar{x}(n,m)$ is a bounded solution of (2.32). For $n=0$ we get $P\bar{x}(0,m)=\bar{0}$. Then $\bar{x}(n,m)=\bar{0},\ n \in I\!N$ since (2.32) has an exponential dichotomy (2.58). So,

$$\begin{aligned}x(n,m) = \quad & X(n)Px(0,m) + \sum_{v=0}^{n-1} X(n)PX^{-1}(v+1)H_1(v,m)- \\ & \sum_{v=n}^{\infty} X(n)(I-P)X^{-1}(v+1)H_1(v,m).\end{aligned} \tag{2.66}$$

Similarly, since equation (2.14) has an exponential dichotomy (2.59) we can prove that

$$\begin{aligned}x(n,s) = \quad & Z(n,s)Qx(0,s) + \sum_{v=0}^{n-1} Z(n,s)Q_s Z^{-1}(v+1,s)G(v,s)- \\ & \sum_{v=n}^{\infty} Z(n,s)(I-Q_s)Z^{-1}(v+1,s)G(v,s).\end{aligned} \tag{2.67}$$

Hence, from (2.5), (2.12), (2.58), (2.59), (2.66), (2.67) and arguing as above we can prove that $x(t,s)$ tends exponentially to zero as $n \to \infty$ uniformly with respect to s. This completes the proof of the proposition.

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