Biorthogonal Wavelets on the Spectrum

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Abstract: In this article, we introduce the notion of biorthgonoal nonuniform multiresolution analysis on the spectrum $\Lambda = \{0, r/N\} + 2\mathbb{Z}$, where $N \geq 1$ is an integer and r is an odd integer with $1 \leq r \leq 2N-1$ such that r and N are relatively prime. We first establish the necessary and sufficient conditions for the translates of a single function to form the Riesz bases for their closed linear span. We provide the complete characterization for the biorthogonality of the translates of scaling functions of two nonuniform multiresolution analysis and the associated biorthogonal wavelet families. Furthermore, under the mild assumptions on the scaling functions and the corresponding wavelets associated with nonuniform multiresolution analysis, we show that the wavelets can generate Reisz bases.

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1. Introduction

The theory of wavelet transforms have emanated as a broadly used tool in various disciplines of science and engineering including image processing, spectrometry, turbulence, computer graphics, optics and electromagnetism, telecommunications, DNA sequence analysis, quantum physics, solution of differential equations. In context of signal processing, it has been assumed that orthogonality is the key property for synthesis and analysing signals. In order to study a higher-level signal processing, biorthogonality plays a vital role in which two sets are incorporated: one serves for the analysis and the other one for synthesis. Towars the culminating years of 1990's, biorthogonal wavelets are considered as cornerstone technique in image compression due to their natural feature of concentrating energy in a few transform coefficients and advantageous over orthogonal wavelets, by relaxing orthonormal to biorthogonal, additional degrees of freedom are added to design problems. Biorthogonal wavelets in $L^2(\mathbb{R})$ were investigated by Bownik and Garrigos [3], Cohen et al.[5], Chui and Wang [7] and many others.

Multiresolution analysis is the heart of wavelet analysis as it gives a general framework for analysing wavelet systems. These concepts are generalized to various settings [1, 4, 6, 11, 12]. All these concepts are developed on regular lattices, that is the translation set is always a group. All the signals in real life applications are not obtained from the uniform shifts. For the analysis and decomposition of these signals by means of stable mathematical technique, Gabardo and Nashed [8, 9] introduced a notion of nonuniform MRA where the translation set acting on the scaling function associated with the MRA to generate the subspace $\mathcal{V}_0$ is no longer a group, but is the union of $\mathbb{Z}$ and a translate of $\mathbb{Z}$. Shah and Abdullah [13] established NUMRA on non-Archimedean local fields The development of biorthogonal wavelets associated with MRA and NUMRA were carried by various researchers [2, 3, 10, 14].

In this article, we introduce the notion of biorthogonal wavelets on the spectrum and obtain the characterization for the translates of a single function to form the Riesz bases for their closed linear span. We also provide a complete characterization for the biorthogonality of the translates of scaling functions of two NUMRA's and the associated biorthogonal wavelet families. Moreover, under mild assumptions on the scaling functions and the corresponding wavelets, we show that the nonuniform wavelets can generate Reisz bases for $L^2(\mathbb{R})$.

The article is structured as follows. In Section 2, we recall the basic definitions of MRA and NUMRA . In Section 3, we establish necessary and sufficient conditions for the translates of a function to form a Riesz basis for its closed linear span. In the concluding Section, we show that the wavelets associated with dual MRA's are biorthogonal and generate Riesz bases for $L^2(\mathbb{R})$.

2. Preliminaries

Definition 2.1. A sequence of closed subspaces $\{\mathcal{V}_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$ is said to form an MRA of $L^2(\mathbb{R})$ if it satisfies the following conditions:

(a) $\mathcal{V}_j \subset \mathcal{V}_{j+1}\ \forall\, j \in \mathbb{Z}$;

(b) $\bigcup_{j\in\mathbb{Z}} \mathcal{V}_j$ is dense in $L^2(\mathbb{R})$;

(c) $\bigcap_{j\in\mathbb{Z}} \mathcal{V}_j = \{0\}$;

(d) $g(x) \in \mathcal{V}_j \iff g(2x) \in \mathcal{V}_{j+1}\ \forall\, j \in \mathbb{Z}$;

(e) There exists a function $\phi \in \mathcal{V}_0$, known as *scaling function*, such that $\{\phi(x-m) : m \in \mathbb{Z}\}$ forms an orthonormal basis for $\mathcal{V}_0$.

In order to construct wavelets associated with MRA, we define a space $\mathcal{W}_j$ known as *wavelet space* as the orthogonal complement of $\mathcal{V}_j$ in $\mathcal{V}_{j+1}$, i.e., $\mathcal{V}_{j+1} = \mathcal{V}_j \oplus \mathcal{W}_j$, $j \in \mathbb{Z}$, where $\mathcal{W}_j \perp \mathcal{V}_j$, $j \in \mathbb{Z}$. It is easy to check that

$$g(x) \in \mathcal{W}_j \iff g(2x) \in \mathcal{W}_{j+1}, \quad j \in \mathbb{Z}. \tag{2.1}$$

These are mutually orthogonal and admit the following orthogonal decompositions:

$$L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} \mathcal{W}_j = \mathcal{V}_0 \oplus \left(\bigoplus_{j\geq 0} \mathcal{W}_j\right). \tag{2.2}$$

Let $N \geq 1$ be a given integer and r be an odd integer which are relatively prime such that

$1 \le r \le 2N-1$, we consider the translation set Λ as

$$\Lambda = \left\{0, \frac{r}{N}\right\} + 2\mathbb{Z} = \left\{\frac{r\ell}{N} + 2m : m \in \mathbb{Z}, \ell = 0, 1\right\}. \tag{2.3}$$

It can be easily seen that the translation set Λ is not necessarily a group nor a uniform discrete set. The set Λn is the union of $\mathbb{Z}$ and a translate of $\mathbb{Z}$. Furthermore, the translation set Λ is the spectrum for the spectral set $\Gamma_N = \left[0, \frac{1}{2}\right) \cup \left[\frac{N}{2}, \frac{N+1}{2}\right)$ and the pair (Λ, Γ_N) is called a *spectral pair* [8].

Definition 2.2. Let $N \ge 1$ be a given integer and r be an odd integer which are relatively prime such that $1 \le r \le 2N-1$, an associated nonuniform MRA is a sequence of closed subspaces $\{\mathcal{V}_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$ satisfying the following properties:

(a) $\mathcal{V}_j \subset \mathcal{V}_{j+1} \ \forall \ j \in \mathbb{Z}$;

(b) $\bigcup_{j\in\mathbb{Z}} \mathcal{V}_j$ is dense in $L^2(\mathbb{R})$;

(c) $\bigcap_{j\in\mathbb{Z}} \mathcal{V}_j = \{0\}$;

(d) $g(x) \in \mathcal{V}_j \iff g(2Nx) \in \mathcal{V}_{j+1} \ \forall \ j \in \mathbb{Z}$;

(e) There exists a function $\phi \in \mathcal{V}_0$ such that $\{\phi(x-\sigma) : \sigma \in \Lambda\}$, is a complete orthonormal basis for $\mathcal{V}_0$.

It should be noted that the definition of dyadic dilation multiresolution analysis in one dimension can be deduced from the above definitio when $N = 1$. For $N > 1$, the dilation factor of $2N$ corroborates that $2N\Lambda \subset \mathbb{Z} \subset \Lambda$.

For every $j \in \mathbb{Z}$, define $\mathcal{W}_j$ as the orthogonal complement of $\mathcal{V}_j$ in $\mathcal{V}_{j+1}$. Thus we can write

$$\mathcal{V}_{j+1} = \mathcal{V}_j \oplus \mathcal{W}_j \quad \text{and} \quad \mathcal{W}_m \perp \mathcal{W}_{m'} \quad \text{if } m \ne m'. \tag{2.4}$$

Therefore, it implies that for $j > M$,

$$\mathcal{V}_j = \mathcal{V}_M \oplus \bigoplus_{m=0}^{j-M-1} \mathcal{W}_{j-m}\,. \tag{2.5}$$

By invoking Definition 2.2. (b), this follows that

$$L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} \mathcal{W}_j, \tag{2.6}$$

a decomposition of $L^2(\mathbb{R})$ into mutually orthogonal subspaces.

There exists $2N-1$ functions whose translated and dilated family form an orthonormal basis for $L^2(\mathbb{R})$.

Definition 2.3. A set $\{\psi_\ell : 1 \le \ell \le 2N-1\} \subset L^2(\mathbb{R})$ is said to be a *set of basic wavelets* associated with the nonuniform multiresolution analysis $\{\mathcal{V}_j : j \in \mathbb{Z}\}$ if the family of functions $\{\psi_\ell(x-\sigma) : 1 \le \ell \le 2N-1, \sigma \in \Lambda\}$ forms an orthonormal basis for $\mathcal{W}_0$.

3. Riesz Bases of Translates

Lemma 3.1. *Let $\phi, \widetilde{\phi} \in L^2(\mathbb{R})$ be given. Then the collection $\left\{\phi(x-\sigma) : \sigma \in \Lambda\right\}$ is biorthogonal to $\left\{\widetilde{\phi}(x-\sigma) : \sigma \in \Lambda\right\}$ if and only if*

$$\sum_{\sigma\in\Lambda} \widehat{\phi}(\zeta+\sigma)\overline{\widehat{\widetilde{\phi}}(\zeta+\sigma)} = 1 \quad a.e\ \zeta \in \mathbb{R}.$$

Proof. For $\gamma \in \Lambda$, it follows that $\left\langle \phi(x-\sigma), \widetilde{\phi}(x-\gamma)\right\rangle = \delta_{\sigma,\gamma} \Leftrightarrow \left\langle \phi, \widetilde{\phi}(x-\gamma)\right\rangle = \delta_{0,\gamma}$. Moreover, we have

$$\begin{aligned}
\left\langle \phi, \widetilde{\phi}(x-\gamma)\right\rangle &= \left\langle \widehat{\phi}, \widehat{\widetilde{\phi}}(x-\gamma)\right\rangle \\
&= \int_{\mathbb{R}} \widehat{\phi}(\zeta)\overline{\widehat{\widetilde{\phi}}(\zeta)} e^{-2\pi i\gamma\zeta} d\zeta \\
&= \int_0^{1/2} \left\{ \sum_{m\in\mathbb{Z}} \widehat{\phi}\left(\zeta+\frac{m}{2}\right) \overline{\widehat{\widetilde{\phi}}\left(\zeta+\frac{m}{2}\right)} e^{\pi i\gamma m} \right\} e^{-2\pi i\gamma\zeta} d\zeta.
\end{aligned}$$

Using the fact that $\left\{e^{-2\pi i\gamma\zeta} : \gamma \in \Lambda\right\}$ is an orthonormal basis of $L^2\left[0, \frac{1}{2}\right)$, we obtain the desired result. □

Now we proceed to establish a sufficient condition for the translates of a function to be linearly independent.

Lemma 3.2. *Let $\phi \in L^2(\mathbb{R})$. Suppose there exists two constants $C, D > 0$ such that*

$$C \le \sum_{\sigma\in\Lambda} \left|\widehat{\phi}(\zeta+\sigma)\right|^2 \le D \quad \textit{for a.e } \zeta \in \mathbb{R}. \tag{3.1}$$

Then, the set $\left\{\phi(x-\sigma) : \sigma \in \Lambda\right\}$ is linearly independent.

Proof. For the proof of the lemma, it is sufficient to find another function say $\widetilde{\phi}$ whose translates are biorthogonal to ϕ. To do this, we define the function $\widetilde{\phi}$ by

$$\widehat{\widetilde{\phi}}(\zeta) = \frac{\widehat{\phi}(\zeta)}{\sum\limits_{\sigma\in\Lambda} \left|\widehat{\phi}(\zeta+\sigma)\right|^2}.$$

Equation (3.1) implies that $\widetilde{\phi}$ is well defined and

$$\begin{aligned}
\sum_{\gamma\in\Lambda} \widehat{\phi}(\zeta+\gamma)\overline{\widehat{\widetilde{\phi}}(\zeta+\gamma)} &= \sum_{\gamma\in\Lambda} \widehat{\phi}(\zeta+\gamma) \frac{\overline{\widehat{\phi}(\zeta+\gamma)}}{\sum\limits_{\sigma\in\Lambda} \left|\widehat{\phi}(\zeta+\sigma+\gamma)\right|^2} \\
&= \frac{\sum\limits_{\gamma\in\Lambda} \left|\widehat{\phi}(\zeta+\gamma)\right|^2}{\sum\limits_{\nu\in\sigma} \left|\widehat{\phi}(\zeta+\nu)\right|^2} \\
&= 1.
\end{aligned}$$

Applying Lemma 3.1, it follows that the set $\{\phi(x-\sigma) : \sigma \in \Lambda\}$ is linearly independent. Thus the proof is completed . $\square$

Lemma 3.3. *Assume that the scaling function ϕ satisfies inequality (3.1). Let $g = \sum_{\sigma\in\Lambda} h_\sigma \phi(x-\sigma)$, where $g \in span\{\phi(x-\sigma) : \sigma \in \Lambda\}$ and $\{h_\sigma\}$ is a finite sequence. Define the Fourier transform of h by $\widehat{h}(\zeta) = \sum_{\sigma\in\Lambda} h_\sigma e^{-2\pi i \sigma\zeta}$. Then*

$$C\int_0^{1/2} \left|\widehat{h}(\zeta)\right|^2 d\zeta \le \|g\|_2^2 \le D\int_0^{1/2} \left|\widehat{h}(\zeta)\right|^2 d\zeta.$$

Proof. By using Placherel's theorem, we obtain

$$\begin{aligned}
\int_{\mathbb{R}} \left|g(x)\right|^2 dx &= \int_{\mathbb{R}} \left|\sum_{\sigma\in\Lambda} h_\sigma \phi(x-\sigma)\right|^2 dx \\
&= \int_{\mathbb{R}} \left|\sum_{\sigma\in\Lambda} h_\sigma \widehat{\phi}(\zeta) e^{-2\pi i\sigma\zeta}\right|^2 d\zeta \\
&= \int_{\mathbb{R}} \left|\widehat{\phi}(\zeta)\right|^2 \left|\sum_{\sigma\in\Lambda} h_\sigma e^{-2\pi i\sigma\zeta}\right|^2 d\zeta \\
&= \int_{\mathbb{R}} \left|\widehat{\phi}(\zeta)\right|^2 \left|\widehat{h}(\zeta)\right|^2 d\zeta \\
&= \int_0^{1/2} \sum_{m\in\mathbb{Z}} \left|\widehat{\phi}\left(\zeta + \frac{m}{2}\right)\right|^2 \left|\widehat{h}(\zeta)\right|^2 d\zeta.
\end{aligned}$$

Using identity (3.1), the result follows. $\square$

Theorem 3.4. *Let $\{\phi(x-\sigma) : \sigma \in \Lambda\}$ be a Riesz basis for its closed linear span. Suppose that there exists a function $\{\widetilde{\phi}(x-\sigma) : \sigma \in \Lambda\}$ which is biorthogonal to $\{\phi(x-\sigma) : \sigma \in \Lambda\}$. Then, for every $f \in \overline{span}\{\phi(x-\sigma) : \sigma \in \Lambda\}$, we have*

$$f = \sum_{\sigma\in\Lambda} \left\langle f, \widetilde{\phi}(x-\sigma)\right\rangle \phi(x-\sigma); \tag{3.2}$$

and there exists constants $C, D > 0$ such that

$$C\|f\|_2^2 \le \sum_{\sigma\in\Lambda} \left|\left\langle f, \widehat{\widetilde{\phi}}(\zeta-\sigma)\right\rangle\right|^2 \le D\|f\|_2^2. \tag{3.3}$$

Proof. We first prove (3.2) and (3.3) for any $f \in \text{span}\{\phi(x-\sigma) : \sigma \in \Lambda\}$ and then generalize it to $\overline{\text{span}}\{\phi(x-\sigma) : \sigma \in \Lambda\}$. Let $f \in \text{span}\{\phi(x-\sigma) : \sigma \in \Lambda\}$, then there exists a finite sequence $\{h_\sigma : \sigma \in \Lambda\}$ such that $f = \sum_{\sigma\in\Lambda} h_\sigma \phi(x-\sigma)$. Also, the biorthogonality condition implies that

$$\begin{aligned}
\langle f, \widetilde{\phi}(x-\gamma)\rangle &= \left\langle \sum_{\sigma\in\Lambda} h_\sigma \phi(x-\sigma), \widetilde{\phi}(x-\gamma)\right\rangle \\
&= \sum_{\sigma\in\Lambda} h_\sigma \langle \phi(x-\sigma), \widetilde{\phi}(x-\gamma)\rangle \\
&= h_\sigma,
\end{aligned}$$

which proves (3.2). In order to prove (3.3), we make the use of Lemma 3.3 to get

$$D^{-1}\|f\|_2^2 \le \int_0^{1/2} \left|\widehat{h}(\zeta)\right|^2 d\zeta \le C^{-1}\|f\|_2^2.$$

Using the Placherel formula for Fourier series and the fact that $h_\sigma = \left\langle f, \widetilde{\phi}(x-\sigma)\right\rangle$, we obtain

$$\int_0^{1/2} \left|\widehat{h}(\zeta)\right|^2 d\zeta = \sum_{\sigma\in\Lambda} \left|h_\sigma\right|^2 = \sum_{\sigma\in\Lambda} \left|\left\langle f, \widetilde{\phi}(x-\sigma)\right\rangle\right|^2.$$

This proves (3.3). We now generalize the results to $\overline{\operatorname{span}}\{\phi(x-\sigma) : \sigma\in\Lambda\}$. For $f \in \overline{\operatorname{span}}\left\{\widetilde{\phi}(x-\sigma) : \sigma\in\Lambda\right\}$, there exists a sequence $\{f_m : m\in\mathbb{Z}\}$ in $\operatorname{span}\left\{\widetilde{\phi}(x-\sigma) : \sigma\in\Lambda\right\}$ such that $\|f_m - f\|_2 \to 0$ as $m\to\infty$. Thus, for each $\sigma\in\Lambda$, we have

$$\left\langle f_m, \widetilde{\phi}(x-\sigma)\right\rangle \to \left\langle f, \widetilde{\phi}(x-\sigma)\right\rangle \quad \text{as } m\to\infty.$$

Hence, the result holds for each f_m. Thus, we have

$$\begin{aligned} \sum_{\sigma\in\Lambda} \left|\left\langle f, \widetilde{\phi}(x-\sigma)\right\rangle\right|^2 &= \sum_{\sigma\in\Lambda} \lim_{m\to\infty} \left|\left\langle f_m, \widetilde{\phi}(x-\sigma)\right\rangle\right|^2 \\ &= \lim_{m\to\infty} \sum_{\sigma\in\Lambda} \left|\left\langle f_m, \widetilde{\phi}(x-\sigma)\right\rangle\right|^2 \\ &\le D \lim_{m\to\infty} \left\|f_m\right\|_2^2 \\ &= D\left\|f\right\|_2^2. \end{aligned} \tag{3.4}$$

Moreover, we have

$$\left\{\sum_{\sigma\in\Lambda} \left|\left\langle f_m, \widetilde{\phi}(x-\sigma)\right\rangle\right|^2\right\}^{1/2} \le \left\{\sum_{\sigma\in\Lambda} \left|\left\langle f_m - f, \widetilde{\phi}(x-\sigma)\right\rangle\right|^2\right\}^{1/2} + \left\{\sum_{\sigma\in\Lambda} \left|\left\langle f, \widetilde{\phi}(x-\sigma)\right\rangle\right|^2\right\}^{1/2}.$$

As the upper bound in (3.3) holds for $f_m - f$ and lower bound for each f_m, we infer that

$$C^{1/2}\left\|f\right\|_2 \le D^{1/2}\left\|f_m - f\right\|_2 + \left(\sum_{\sigma\in\Lambda} \left|\left\langle f_m, \widetilde{\phi}(x-\sigma)\right\rangle\right|^2\right)^{1/2},$$

from which we conclude that

$$C\left\|f\right\|_2^2 \le \sum_{\sigma\in\Lambda} \left|\left\langle f, \widetilde{\phi}(x-\sigma)\right\rangle\right|^2. \tag{3.5}$$

Combining (3.4) and (3.5), we obtain (3.3). Similarly, we can prove (3.2) for $f \in \overline{\operatorname{span}}\{\phi(x-\sigma) : \sigma \in$ and the proof is completed. $\square$

4. Properties of Biorthogonal Wavelets on the Spectrum

Let $\{\mathcal{V}_j : j\in\mathbb{Z}\}$ and $\{\widetilde{\mathcal{V}}_j : j\in\mathbb{Z}\}$ be biorthogonal NUMRA's with scaling functions ϕ and $\widetilde{\phi}$. Then there exists integral periodic functions m_0 and $\widetilde{m}_0$ with the property $\widehat{\phi}(\zeta) =$

$m_0\left(\zeta/2N\right)\widehat{\phi}\left(\zeta/2N\right)$ and $\widehat{\widetilde{\phi}}(\zeta) = \widetilde{m}_0\left(\zeta/2N\right)\widehat{\widetilde{\phi}}\left(\zeta/2N\right)$. Suppose there exists integral periodic functions m_ℓ and $\widetilde{m}_\ell, 1 \le \ell \le 2N-1$ such that

$$\mathcal{M}(\zeta)\overline{\widetilde{\mathcal{M}}(\zeta)} = I, \tag{4.1}$$

where

$$\mathcal{M}(\zeta) = \begin{pmatrix} m_0\left(\frac{\zeta}{2N}\right) & m_0\left(\frac{\zeta}{2N}+\frac{1}{4N}\right) & \cdots & m_0\left(\frac{\zeta}{2N}+\frac{2N-1}{4N}\right) \\ m_1\left(\frac{\zeta}{2N}\right) & m_2\left(\frac{\zeta}{2N}+\frac{1}{4N}\right) & \cdots & m_2\left(\frac{\zeta}{2N}+\frac{2N-1}{4N}\right) \\ \vdots & \vdots & \ddots & \vdots \\ m_{2N-1}\left(\frac{\zeta}{2N}\right) & m_{2N-1}\left(\frac{\zeta}{2N}+\frac{1}{4N}\right) & \cdots & m_{2N-1}\left(\frac{\zeta}{2N}+\frac{2N-1}{4N}\right) \end{pmatrix}$$

and

$$\widetilde{\mathcal{M}}(\zeta) = \begin{pmatrix} \widetilde{m}_0\left(\frac{\zeta}{2N}\right) & \widetilde{m}_0\left(\frac{\zeta}{2N}+\frac{1}{4N}\right) & \cdots & \widetilde{m}_0\left(\frac{\zeta}{2N}+\frac{2N-1}{4N}\right) \\ \widetilde{m}_1\left(\frac{\zeta}{2N}\right) & \widetilde{m}_2\left(\frac{\zeta}{2N}+\frac{1}{4N}\right) & \cdots & \widetilde{m}_2\left(\frac{\zeta}{2N}+\frac{2N-1}{4N}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{m}_{2N-1}\left(\frac{\zeta}{2N}\right) & \widetilde{m}_{2N-1}\left(\frac{\zeta}{2N}+\frac{1}{4N}\right) & \cdots & \widetilde{m}_{2N-1}\left(\frac{\zeta}{2N}+\frac{2N-1}{4N}\right) \end{pmatrix}.$$

For $1 \le \ell \le 2N-1$, define the associated biorthgonal wavelets as ψ_ℓ and $\widetilde{\psi}_\ell$ by

$$\widehat{\psi}_\ell(\zeta) = m_\ell\left(\zeta/2N\right)\widehat{\phi}\left(\zeta/2N\right) \quad \text{and} \quad \widehat{\widetilde{\psi}}_\ell(\zeta) = \widetilde{m}_\ell\left(\zeta/2N\right)\widehat{\widetilde{\phi}}\left(\zeta/2N\right).$$

Definition 4.1. A pair of NUMRA's $\{\mathcal{V}_j : j \in \mathbb{Z}\}$ and $\{\widetilde{\mathcal{V}}_j : j \in \mathbb{Z}\}$ with scaling functions ϕ and $\widetilde{\phi}$, respectively are said to be biorthogonal to each other if $\{\phi(\cdot - \sigma) : \sigma \in \Lambda\}$ and $\{\widetilde{\phi}(\cdot - \sigma) : \sigma \in \Lambda\}$ are biorthogonal.

Definition 4.2. Let ϕ and $\widetilde{\phi}$ be scaling functions for biorthogonal NUMRA's. For each $j \in \mathbb{Z}$, define the operators P_j and $\widetilde{P}_j$ on $L^2(\mathbb{R})$ by

$$P_j f = \sum_{\sigma\in\Lambda} \langle f, \widetilde{\phi}_{j,\sigma}\rangle \phi_{j,\sigma} \quad \text{and} \quad \widetilde{P}_j f = \sum_{\sigma\in\Lambda} \langle f, \phi_{j,\sigma}\rangle \widetilde{\phi}_{j,\sigma},$$

respectively. It is easy to verify that these operators are uniformly bounded on $L^2(\mathbb{R})$ and both the series are convergent in $L^2(\mathbb{R})$.

Remark 4.3. The operators P_j and $\widetilde{P}_j$ satisfy the following properties.

(a) $P_j f = f \iff f \in V_j$ and $\widetilde{P}_j f = f \iff f \in \widetilde{V}_j$.

(b) $\lim\limits_{j\to\infty} \left\|P_j f - f\right\|_2 = 0$ and $\lim\limits_{j\to-\infty} \left\|P_j f\right\|_2 = 0$ for every $f \in L^2(\mathbb{R})$.

Theorem 4.4. *Let ϕ and $\widetilde{\phi}$ be the scaling functions for biorthogonal NUMRA's and ψ_ℓ and $\widetilde{\psi}_\ell, 1 \le \ell \le 2N-1$ be the associated wavelets satisfying (4.1). Then, we have the following*

(a) $\{\psi_{\ell,0,\sigma} : \sigma \in \Lambda\}$ *is biorthogonal to* $\{\widetilde{\psi}_{\ell,0,\gamma} : \gamma \in \Lambda\}$,

(b) $\langle \psi_{\ell,0,\sigma}, \widetilde{\phi}_{0,\gamma} \rangle = \langle \widetilde{\psi}_{\ell,0,\sigma}, \phi_{0,\gamma} \rangle$, for all $\sigma, \gamma \in \Lambda$.

Proof. we have

$$\begin{aligned}
&\sum_{t\in\mathbb{Z}} \widehat{\psi}_\ell\left(\zeta + \frac{t}{2}\right) \overline{\widehat{\widetilde{\psi}}_\ell\left(\zeta + \frac{t}{2}\right)} \\
&= \sum_{t\in\mathbb{Z}} \left\{ m_\ell\left(\frac{\zeta}{2N} + \frac{t}{4N}\right) \widehat{\phi}\left(\frac{\zeta}{2N} + \frac{t}{4N}\right) \overline{\widetilde{m}_\ell\left(\frac{\zeta}{2N} + \frac{t}{4N}\right)}\, \overline{\widehat{\widetilde{\phi}}\left(\frac{\zeta}{2N} + \frac{t}{4N}\right)} \right\} \\
&= \sum_{s=0}^{2N-1} \sum_{t\in\mathbb{Z}} \left\{ m_\ell\left(\frac{\zeta}{2N} + \frac{t}{2} + \frac{s}{4N}\right) \widehat{\phi}\left(\frac{\zeta}{2N} + \frac{t}{2} + \frac{s}{4N}\right) \overline{\widetilde{m}_\ell\left(\frac{\zeta}{2N} + \frac{t}{2} + \frac{s}{4N}\right)}\, \overline{\widehat{\widetilde{\phi}}\left(\frac{\zeta}{2N} + \frac{t}{2} + \frac{s}{4N}\right)} \right\} \\
&= \sum_{s=0}^{2N-1} \left\{ m_\ell\left(\frac{\zeta}{2N} + \frac{s}{4N}\right) \overline{\widetilde{m}_\ell\left(\frac{\zeta}{2N} + \frac{s}{4N}\right)} \right\} \\
&= 1.
\end{aligned}$$

Hence, by Lemma 3.1, $\{\psi_{\ell,0,\sigma} : \sigma \in \Lambda\}$ is biorthogonal to $\{\widetilde{\psi}_{\ell,0,\sigma} : \sigma \in \Lambda\}$. This proves part (a). To prove part (b), we have for, $\sigma, \gamma \in \Lambda$

$$\begin{aligned}
\langle \psi_{\ell,0,\sigma}, \widetilde{\phi}_{0,\gamma} \rangle &= \langle \psi_\ell(x-\sigma), \widetilde{\phi}(x-\gamma) \rangle \\
&= \left\langle \widehat{\psi}_\ell\, e^{-2\pi i\sigma}, \widehat{\widetilde{\phi}}\, e^{-2\pi i\gamma} \right\rangle \\
&= \int_{\mathbb{R}} m_\ell\left(\frac{\zeta}{2N}\right) \widehat{\phi}\left(\frac{\zeta}{2N}\right) e^{-2\pi i\sigma\zeta}\, \overline{\widetilde{m}_0\left(\frac{\zeta}{2N}\right)}\, \overline{\widehat{\widetilde{\phi}}\left(\frac{\zeta}{2N}\right)} e^{2\pi i\gamma\zeta} d\zeta \\
&= \int_0^{1/2} \sum_{p\in\mathbb{Z}} \left\{ m_\ell\left(\frac{\zeta}{2N} + \frac{p}{4N}\right) \widehat{\phi}\left(\frac{\zeta}{2N} + \frac{p}{4N}\right) \right. \\
&\qquad \left. \times \overline{\widetilde{m}_0\left(\frac{\zeta}{2N} + \frac{p}{4N}\right)}\, \overline{\widehat{\widetilde{\phi}}\left(\frac{\zeta}{2N} + \frac{p}{4N}\right)} \right\} e^{2\pi i(\gamma-\sigma)} d\zeta \\
&= \int_0^{1/2} \sum_{s=0}^{2N-1} \sum_{p\in\mathbb{Z}} \left\{ m_\ell\left(\frac{\zeta}{2N} + \frac{p}{2} + \frac{s}{4N}\right) \widehat{\phi}\left(\frac{\zeta}{2N} + \frac{p}{2} + \frac{s}{4N}\right) \right. \\
&\qquad \left. \times \overline{\widetilde{m}_0\left(\frac{\zeta}{2N} + \frac{p}{2} + \frac{s}{4N}\right)}\, \overline{\widehat{\widetilde{\phi}}\left(\frac{\zeta}{2N} + \frac{p}{2} + \frac{s}{4N}\right)} \right\} e^{2\pi i(\gamma-\sigma)} d\zeta \\
&= \int_0^{1/2} \sum_{s=0}^{2N-1} \left\{ m_\ell\left(\frac{\zeta}{2N} + \frac{s}{4N}\right) \overline{\widetilde{m}_0\left(\frac{\zeta}{2N} + \frac{s}{4N}\right)} \right\} e^{2\pi i(\gamma-\sigma)} d\zeta \\
&= 0.
\end{aligned}$$

The dual one can also be shown equal to zero in a similar manner. This proves part (b) and hence the proof is completed. $\square$

Theorem 4.5. *Let $\phi, \widetilde{\phi}, \psi_\ell$ and $\widetilde{\psi}_\ell, 1 \le \ell \le 2N-1$ be as in Theorem 4.1. Let $\psi_0 = \phi$ and $\widetilde{\psi}_0 = \widetilde{\phi}$. Then, for every $f \in L^2(\mathbb{R})$, we have*

$$P_1 f = P_0 f + \sum_{\ell=1}^{2N-1} \sum_{\sigma\in\Lambda} \langle f, \widetilde{\psi}_{\ell,0,\sigma} \rangle \psi_{\ell,0,\sigma} \tag{4.2}$$

and

$$\widetilde{P}_1 f = \widetilde{P}_0 f + \sum_{\ell=1}^{2N-1} \sum_{\sigma\in\Lambda} \langle f, \psi_{\ell,0,\sigma} \rangle \widetilde{\psi}_{\ell,0,\sigma}. \tag{4.3}$$

where the series in (4.2) and (4.3) converges in $L^2(\mathbb{R})$.

Proof. For the sake of convenience, we will only prove (4.2), as (4.3) is an easy consequence. In particular, we will prove it in the weak sense only. For this, let $f, g \in L^2(\mathbb{R})$. Then, we have

$$
\begin{aligned}
&\sum_{\ell=0}^{2N-1} \sum_{\sigma\in\Lambda} \left\langle f, \widetilde{\psi}_{\ell,0,\sigma} \right\rangle \overline{\left\langle g, \psi_{\ell,0,\sigma} \right\rangle} \\
&\quad = \sum_{\ell=0}^{2N-1} \sum_{\sigma\in\Lambda} \left\{ \int_{\mathbb{R}} \widehat{f}(\zeta) \overline{\widehat{\widetilde{\psi}}_\ell(\zeta)} e^{2\pi i\sigma\zeta} d\zeta \right\} \left\{ \int_{\mathbb{R}} \overline{\widehat{g}(\zeta)} \widehat{\psi}_\ell(\zeta) e^{-2\pi i\sigma\zeta} d\zeta \right\} \\
&\quad = \sum_{\ell=0}^{2N-1} \sum_{\sigma\in\Lambda} \left\{ \int_0^{1/2} \sum_{p\in\mathbb{Z}} \widehat{f}\left(\zeta + \frac{p}{2}\right) \overline{\widehat{\widetilde{\psi}}_\ell\left(\zeta + \frac{p}{2}\right)} e^{2\pi i\sigma\zeta} d\zeta \right\} \\
&\qquad\qquad \times \left\{ \int_0^{1/2} \sum_{q\in\mathbb{Z}} \overline{\widehat{g}\left(\zeta + \frac{q}{2}\right)} \widehat{\psi}_\ell\left(\zeta + \frac{q}{2}\right) e^{-2\pi i\sigma\zeta} d\zeta \right\} \\
&\quad = \sum_{\ell=0}^{2N-1} \int_0^{1/2} \left\{ \sum_{p\in\mathbb{Z}} \widehat{f}\left(\zeta + \frac{p}{2}\right) \overline{\widehat{\widetilde{\psi}}_\ell\left(\zeta + \frac{p}{2}\right)} \right\} \left\{ \sum_{q\in\mathbb{Z}} \overline{\widehat{g}\left(\zeta + \frac{q}{2}\right)} \widehat{\psi}_\ell\left(\zeta + \frac{q}{2}\right) \right\} d\zeta \\
&\quad = \int_0^{1/2} \sum_{\ell=0}^{2N-1} \left\{ \sum_{p\in\mathbb{Z}} \widehat{f}\left(\zeta + \frac{p}{2}\right) \overline{\widetilde{m}_\ell\left(\frac{\zeta}{2N} + \frac{p}{4N}\right)} \overline{\widehat{\widetilde{\phi}}\left(\frac{\zeta}{2N} + \frac{p}{4N}\right)} \right. \\
&\qquad\qquad \left. \times \sum_{q\in\mathbb{Z}} \overline{\widehat{g}\left(\zeta + \frac{q}{2}\right)} m_\ell\left(\frac{\zeta}{2N} + \frac{q}{4N}\right) \widehat{\phi}\left(\frac{\zeta}{2N} + \frac{q}{4N}\right) \right\} d\zeta
\end{aligned}
$$

$$= \int_0^{1/2} \sum_{\ell=0}^{2N-1} \left\{ \sum_{r=0}^{2N-1} \sum_{p' \in \mathbb{Z}} \widehat{f}\left(\zeta + \frac{p'}{2}N + \frac{r}{2}\right) \overline{\widetilde{m}_\ell\left(\frac{\zeta}{2N} + \frac{r}{4N} + \frac{p'}{2}\right)} \overline{\widehat{\widetilde{\phi}}\left(\frac{\zeta}{2N} + \frac{r}{4N} + \frac{p'}{2}\right)} \right.$$

$$\left. \times \sum_{s=0}^{2N-1} \sum_{q' \in \mathbb{N}_0} \overline{\widehat{g}\left(\zeta + \frac{q'}{2}N + \frac{s}{2}\right)} m_\ell\left(\frac{\zeta}{2N} + \frac{s}{4N} + \frac{q'}{2}\right) \widehat{\phi}\left(\frac{\zeta}{2N} + \frac{s}{4N} + \frac{q'}{2}\right) \right\} d\zeta$$

$$= \int_0^{1/2} \sum_{r=0}^{2N-1} \sum_{p' \in \mathbb{N}_0} \sum_{s=0}^{2N-1} \sum_{q' \in \mathbb{N}_0} \left\{ \sum_{\ell=0}^{2N-1} \overline{\widetilde{m}_\ell\left(\frac{\zeta}{2N} + \frac{r}{4N}\right)} m_\ell\left(\frac{\zeta}{2N} + \frac{s}{4N}\right) \right\}$$

$$\times \widehat{f}\left(\zeta + \frac{p'}{2}N + \frac{r}{2}\right) \overline{\widehat{\widetilde{\phi}}\left(\frac{\zeta}{2N} + \frac{r}{4N} + \frac{p'}{2}\right)} \overline{\widehat{g}\left(\zeta + \frac{q'}{2}N + \frac{s}{2}\right)} \widehat{\phi}\left(\frac{\zeta}{2N} + \frac{s}{4N} + \frac{q'}{2}\right) d\zeta$$

$$= \int_0^{1/2} \sum_{p' \in \mathbb{N}_0} \sum_{q' \in \mathbb{N}_0} \sum_{s=0}^{2N-1} \widehat{f}\left(\zeta + \frac{p'}{2}N + \frac{s}{2}\right) \overline{\widehat{\widetilde{\phi}}\left(\frac{\zeta}{2N} + \frac{s}{4N} + \frac{p'}{2}\right)}$$

$$\times \overline{\widehat{g}\left(\zeta + \frac{q'}{2}N + \frac{s}{2}\right)} \widehat{\phi}\left(\frac{\zeta}{2N} + \frac{s}{4N} + \frac{p'}{2}\right) d\zeta$$

$$= \sum_{s=0}^{2N-1} \int_0^{s+1/2} \sum_{p' \in \mathbb{N}_0} \sum_{q' \in \mathbb{N}_0} \widehat{f}\left(\zeta + \frac{p'}{2}N\right) \overline{\widehat{\widetilde{\phi}}\left(\frac{\zeta}{2N} + \frac{p'}{2}\right)} \overline{\widehat{g}\left(\zeta + \frac{q'}{2}N\right)} \widehat{\phi}\left(\frac{\zeta}{2N} + \frac{p'}{2}\right) d\zeta. \quad (4.5)$$

Furthermore, we have

$$\sum_{\sigma \in \Lambda} \left\langle f, \widetilde{\phi}_{1,\sigma} \right\rangle \overline{\langle g, \phi_{1,\sigma} \rangle}$$

$$= \sum_{\sigma \in \Lambda} \left\{ \int_{\mathbb{R}} \widehat{f}(\zeta) \overline{\widehat{\widetilde{\phi}}\left(\frac{\zeta}{2N}\right)} e^{2\pi i \zeta/2N} d\zeta \right\} \left\{ \int_{\mathbb{R}} \overline{\widehat{g}(\zeta)} \widehat{\phi}\left(\frac{\zeta}{2N}\right) e^{-2\pi i \zeta/2N} d\zeta \right\}$$

$$= \int_0^{1/2} \sum_{p \in \mathbb{Z}} \widehat{f}\left(\zeta + \frac{p}{2}N\right) \overline{\widehat{\widetilde{\phi}}\left(\frac{\zeta}{2N} + \frac{p}{2}\right)} d\zeta \int_0^{1/2} \sum_{q \in \mathbb{Z}} \overline{\widehat{g}\left(\zeta + \frac{q}{2}N\right)} \widehat{\phi}\left(\frac{\zeta}{2N} + \frac{q}{2}\right) d\zeta$$

$$= \int_0^{1/2} \sum_{p \in \mathbb{Z}} \widehat{f}\left(\zeta + \frac{p}{2}N\right) \overline{\widehat{\widetilde{\phi}}\left(\frac{\zeta}{2N} + \frac{p}{2}\right)} d\zeta \int_0^{1/2} \sum_{q \in \mathbb{Z}} \overline{\widehat{g}\left(\zeta + \frac{q}{2}N\right)} \widehat{\phi}\left(\frac{\zeta}{2N} + \frac{q}{2}\right) d\zeta$$

$$= \int_0^{1/2} \sum_{p \in \mathbb{Z}} \sum_{q \in \mathbb{Z}} \widehat{f}\left(\zeta + \frac{p}{2}N\right) \overline{\widehat{\widetilde{\phi}}\left(\frac{\zeta}{2N} + \frac{p}{2}\right)} \overline{\widehat{g}\left(\zeta + \frac{q}{2}N\right)} \widehat{\phi}\left(\frac{\zeta}{2N} + \frac{q}{2}\right) d\zeta. \quad (4.5)$$

Combing (4.5) and (4.5), we get the desired result. □

Theorem 4.6. *Let $\phi, \widetilde{\phi}, \psi_\ell$ and $\widetilde{\psi}_\ell, 1 \leq \ell \leq 2N-1$ be as in Theorem 4.1. Then, for every $f \in L^2(\mathbb{R})$, we have*

$$f = \sum_{\ell=1}^{2N-1} \sum_{j \in \mathbb{Z}} \sum_{\sigma \in \Lambda} \left\langle f, \widetilde{\psi}_{\ell,j,\sigma} \right\rangle \psi_{\ell,j,\sigma} = \sum_{\ell=1}^{2N-1} \sum_{j \in \mathbb{Z}} \sum_{\sigma \in \Lambda} \langle f, \psi_{\ell,j,\sigma} \rangle \widetilde{\psi}_{\ell,j,\sigma}, \quad (4.6)$$

where the series converges in $L^2(\mathbb{R})$*.*

Proof. Using Remark 4.3 and Theorem 4.5, proof of Theorem 4.6 follows. $\square$

Theorem 4.7. *Let* ϕ *and* $\widetilde{\phi}$ *be the scaling functions for biorthogonal NUMRA's and* ψ_ℓ *and* $\widetilde{\psi}_\ell, 1 \le \ell \le 2N-1$ *be the associated wavelets satisfying the matrix condition (4.1). Then, the collection* $\{\psi_{\ell,j,\sigma} : 1 \le \ell \le 2N-1, j \in \mathbb{Z}, \sigma \in \Lambda\}$ *and* $\{\widetilde{\psi}_{\ell,j,\sigma} : 1 \le \ell \le 2N-1, j \in \mathbb{Z}, \sigma \in \Lambda\}$ *are biorthogonal. Moreover, if*

$$|\widehat{\phi}(\zeta)| \le K(1+|\zeta|)^{-\frac{1}{2}-\epsilon},\ |\widehat{\widetilde{\phi}}(\zeta)| \le K(1+|\zeta|)^{-\frac{1}{2}-\epsilon},\ |\widehat{\psi}_\ell(\zeta)| \le K|\zeta| \text{ and } |\widehat{\widetilde{\psi}}(\zeta)| \le K|\zeta|, \tag{4.7}$$

for some constant $K > 0$, $\epsilon > 0$ *and for a.e.* $\zeta \in \mathbb{R}$*, then* $\{\psi_{\ell,j,\sigma} : 1 \le \ell \le 2N-1, j \in \mathbb{Z}, \sigma \in \Lambda\}$ *and* $\{\widetilde{\psi}_{\ell,j,\sigma} : 1 \le \ell \le 2N-1, j \in \mathbb{Z}, \sigma \in \Lambda\}$ *form Riesz bases for* $L^2(\mathbb{R})$*.*

Proof. First we show that $\{\psi_{\ell,j,\sigma} : 1 \le \ell \le 2N-1, j \in \mathbb{Z}, \sigma \in \Lambda\}$ and $\{\widetilde{\psi}_{\ell,j,\sigma} : 1 \le \ell \le 2N-1, j \in \mathbb{Z}, \sigma \in \Lambda\}$ are biorthogonal to each other. For this, we will show that for each ℓ, $1 \le \ell \le 2N-1$ and $j \in \mathbb{Z}$,

$$\langle \psi_{\ell,j,\sigma}, \widetilde{\psi}_{\ell,j,\gamma} \rangle = \delta_{\sigma,\gamma}. \tag{4.8}$$

In fact, we have already proved (4.8) for $j = 0$. For $j \neq 0$, we have

$$\langle \psi_{\ell,j,\sigma}, \widetilde{\psi}_{\ell,j,\gamma} \rangle = \langle D_{-j}\psi_{\ell,0,\sigma}, D_{-j}\widetilde{\psi}_{\ell,0,\gamma} \rangle = \langle \psi_{\ell,0,\sigma}, \widetilde{\psi}_{\ell,0,\gamma} \rangle = \delta_{\sigma,\gamma}.$$

Also, for fixed $\sigma, \gamma \in \Lambda$ and $j, j' \in \mathbb{Z}$ with $j < j'$, we claim that

$$\langle \psi_{\ell,j,\sigma}, \widetilde{\psi}_{\ell',j',\gamma} \rangle = 0. \tag{4.9}$$

As $\psi_{\ell,0,\sigma} \in \mathcal{V}_1$, hence $\psi_{\ell,j,\sigma} = D_{-j}\psi_{\ell,0,\sigma} \in \mathcal{V}_{j+1} \subseteq \mathcal{V}_{j'}$. Therefore, it is enough to show that $\widetilde{\psi}_{\ell',j',\gamma}$ is orthogonal to every element of $\mathcal{V}_{j'}$. Let $g \in \mathcal{V}_{j'}$. Since $\{\phi_{j',\sigma} : \sigma \in \Lambda\}$ is a Riesz basis for $\mathcal{V}_{j'}$, hence there exists an l^2-sequence $\{d_\sigma : \sigma \in \Lambda\}$ such that $g = \sum_{\sigma\in\Lambda} d_\sigma \phi_{j',\sigma}$ in $L^2(\mathbb{R})$. Using part (b) of Lemma 4.1, we have

$$\langle \widetilde{\psi}_{\ell',j',\gamma}, \phi_{j',\sigma} \rangle = \langle D_{-j'}\widetilde{\psi}_{\ell',0,\gamma}, D_{-j'}\phi_{0,\sigma} \rangle = 0.$$

Therefore,

$$\langle \widetilde{\psi}_{\ell',j',\gamma}, g \rangle = \Big\langle \widetilde{\psi}_{\ell',j',\gamma}, \sum_{\sigma\in\Lambda} d_\sigma \phi_{j',\sigma} \Big\rangle = \sum_{\sigma\in\Lambda} d_\sigma \langle \widetilde{\psi}_{\ell',j',\gamma}, \phi_{j',\sigma} \rangle = 0.$$

We now show that these two collections form Riesz bases for $L^2(\mathbb{R})$. The linear independence is clear from the fact that these collections are biorthogonal to each other. So, we have to check the frame inequalities only i.e., there exists constants $C, \widetilde{C}, D, \widetilde{D} > 0$ such that

$$C\|g\|_2^2 \le \sum_{\ell=1}^{2N-1} \sum_{j\in\mathbb{Z}} \sum_{\sigma\in\Lambda} |\langle g, \psi_{\ell,j,\sigma} \rangle|^2 \le D\|g\|_2^2, \quad \forall\, f \in L^2(\mathbb{R}), \tag{4.10}$$

and

$$\widetilde{C}\|g\|_2^2 \le \sum_{\ell=1}^{2N-1} \sum_{j\in\mathbb{Z}} \sum_{\sigma\in\Lambda} \Big|\langle g, \widetilde{\psi}_{\ell,j,\sigma} \rangle\Big|^2 \le \widetilde{D}\|g\|_2^2, \quad \forall\, f \in L^2(\mathbb{R}). \tag{4.11}$$

Let us first check the existence of the upper bounds in (4.10) and (4.1). For this, we have

$$
\begin{aligned}
\sum_{\sigma\in\Lambda}\left|\langle g,\widetilde{\psi}_{\ell,j,\sigma}\rangle\right|^2 &= \sum_{\sigma\in\Lambda}\left|\int_{\mathbb{R}}\widehat{g}(\zeta)(2N)^{-j/2}\,\overline{\widehat{\psi}_\ell\big((2N)^{-j}\zeta\big)}e^{2\pi i\sigma(2N)^{-j}\zeta}d\zeta\right|^2\\
&= (2N)^{-j}\sum_{\sigma\in\Lambda}\left|\int_0^{1/2}\sum_{p\in\mathbb{Z}}\widehat{g}\left(\zeta+(2N)^j\frac{p}{2}\right)\overline{\widehat{\psi}_\ell\left((2N)^{-j}\zeta+\frac{p}{2}\right)}e^{2\pi i\sigma(2N)^{-j}\zeta}d\zeta\right|^2\\
&= \int_0^{1/2}\left|\sum_{p\in\mathbb{Z}}\widehat{g}\left(\zeta+(2N)^j\frac{p}{2}\right)\overline{\widehat{\psi}_\ell\left((2N)^{-j}\zeta+\frac{p}{2}\right)}\right|^2d\zeta\\
&= \int_0^{1/2}\left\{\sum_{p\in\mathbb{Z}}\left|\widehat{g}\left(\zeta+(2N)^j\frac{p}{2}\right)\right|^2\left|\widehat{\psi}_\ell\left((2N)^{-j}\zeta+\frac{p}{2}\right)\right|^{2\delta}\right\}\\
&\qquad\qquad\times\left\{\sum_{q\in\mathbb{Z}}\left|\widehat{\psi}_\ell\left((2N)^{-j}\zeta+\frac{q}{2}\right)\right|^{2(1-\delta)}\right\}d\zeta\\
&= \int_{\mathbb{R}}|\widehat{g}(\zeta)|^2\left|\widehat{\psi}_\ell\left((2N)^{-j}\zeta\right)\right|^{2\delta}\sum_{q\in\mathbb{Z}}\left|\widehat{\psi}_\ell\left((2N)^{-j}\zeta+\frac{q}{2}\right)\right|^{2(1-\delta)}d\zeta.
\end{aligned}
$$

By our assumption (4.7), we have $|\widehat{\psi}_\ell(\zeta)|\leq K\left(1+|(2N)^{-1}\zeta|\right)^{-1/2-\epsilon}$ and therefore, it follows that $\sum_{q\in\mathbb{Z}}\left|\widehat{\psi}_\ell\left((2N)^{-j}\zeta+q/2\right)\right|^{2(1-\delta)}$ is uniformly bounded if $\delta<2\epsilon(1+2\epsilon)^{-1}$. Thus, there exists a constant $K>0$ such that

$$
\begin{aligned}
\sum_{\sigma\in\Lambda}\left|\langle g,\widetilde{\psi}_{\ell,j,\sigma}\rangle\right|^2 &\leq K\int_{\mathbb{R}}|\widehat{g}(\zeta)|^2\sum_{j\in\mathbb{Z}}\left|\widehat{\psi}_\ell\left((2N)^{-j}\zeta\right)\right|^{2\delta}d\zeta\\
&\leq K\sup\left\{\sum_{j\in\mathbb{Z}}\left|\widehat{\psi}_\ell\left((2N)^{-j}\zeta\right)\right|^{2\delta}:1\leq\zeta\leq 2N\right\}\|g\|_2^2.
\end{aligned}
$$

Also for $\zeta\in[1,2N]$, we have

$$
\begin{aligned}
\sum_{j=-\infty}^{0}\left|\widehat{\psi}_\ell\left((2N)^{-j}\zeta\right)\right|^{2\delta} &\leq \sum_{j=-\infty}^{0}\frac{K^{2\delta}}{\left(1+|(2N)^{j-1}\zeta|\right)^{\delta(1+2\epsilon)}}\\
&\leq \sum_{j=-\infty}^{0}\frac{K^{2\delta}}{(2N)^{(j-1)\delta(1+2\epsilon)}}\\
&\leq K^{2\delta}\frac{q^{\delta(1+2\epsilon)}}{1-(2N)^{-\delta(1+2\epsilon)}}.
\end{aligned}
$$

Furthermore, we have

$$
\sum_{j=1}^{\infty}\left|\widehat{\psi}_\ell\left((2N)^{-j}\zeta\right)\right|^{2\delta}\leq\sum_{j=1}^{\infty}\left(K(2N)^{-j}|\zeta|\right)^{2\delta}\leq K^{2\delta}\sum_{j=1}^{\infty}(2N)^{(-j+1)2\delta}=K^{2\delta}\frac{1}{1-(2N)^{-2\delta}},
$$

and hence, it follows that $\sup\left\{\sum_{j\in\mathbb{Z}}\left|\widehat{\psi}_\ell\left((2N)^{-j}\zeta\right)\right|^{2\delta}:1\leq\zeta\leq 2N\right\}$ is finite. Therefore, there exist $D>0$ such that of (4.10) holds. Similarly, we can show for dual one also. The existence of lower bounds for both the cases can be shown in similar fashion. Using Theorem 4.6, it follows that if $g \in L^2(\mathbb{R})$, then (4.6) holds. Thus, we have

$$
\begin{aligned}
\|g\|_2^2 &= \langle g, g\rangle \\
&= \left\langle \sum_{\ell=1}^{2N-1} \sum_{j\in\mathbb{Z}} \sum_{\sigma\in\Lambda} \langle g, \widetilde{\psi}_{\ell,j,\sigma}\rangle \psi_{\ell,j,\sigma}, g \right\rangle \\
&= \sum_{\ell=1}^{2N-1} \sum_{j\in\mathbb{Z}} \sum_{\sigma\in\Lambda} \langle g, \widetilde{\psi}_{\ell,j,\sigma}\rangle \langle \psi_{\ell,j,\sigma}, g\rangle \\
&\leq \left(\sum_{\ell=1}^{2N-1} \sum_{j\in\mathbb{Z}} \sum_{\sigma\in\Lambda} \left|\langle g, \widetilde{\psi}_{\ell,j,\sigma}\rangle\right|^2 \right)^{1/2} \left(\sum_{\ell=1}^{2N-1} \sum_{j\in\mathbb{Z}} \sum_{\sigma\in\Lambda} \left|\langle g, \psi_{\ell,j,\sigma}\rangle\right|^2 \right)^{1/2} \\
&\leq (\widetilde{D})^{1/2} \|g\|_2 \left(\sum_{\ell=1}^{2N-1} \sum_{j\in\mathbb{Z}} \sum_{\sigma\in\Lambda} \left|\langle g, \psi_{\ell,j,\sigma}\rangle\right|^2 \right)^{1/2}.
\end{aligned}
$$

Hence,

$$
\frac{1}{\widetilde{D}} \|g\|_2^2 \leq \sum_{\ell=1}^{2N-1} \sum_{j\in\mathbb{Z}} \sum_{\sigma\in\Lambda} \left|\langle g, \psi_{\ell,j,\sigma}\rangle\right|^2.
$$

The dual case can be proved in similar lines. This completes the proof. □

5. Conclusion

In this paper, we develop the comprehensive theory of biorthogonal wavelets on the spectrum. We provide the complete characterization for the translates of a single function to form Reisz basis and the associated biorthogonal families with respect to NUMRA. Under some mild assuptions on wavelets associated with NUMRA and the scaling function, we show the wavelets can generate Reisz bases.The results established in this paper are theoretical in nature and will definitely provide new directions to the development of Wavelet analysis and widen its field of applications.

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