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RESEARCH PAPER

Blow - up of solutions of wave equation with a nonlinear boundary condition and interior focusing source of variable order of growth †

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Summary

In this paper, being investigated an initial - boundary value problem for a one - dimensional wave equation with a nonlinear source of variable order and nonlinear dissipation at the boundary. The existence of a local solution of the problem under consideration is proved. Then the question of the absence of global solutions is investigated. Depending on the relationship between the order of growth of the nonlinear source and the nonlinear boundary dissipation, different results are obtained on the blow - up of weak solutions in a finite time interval.

KEYWORDS:
nonlinear wave equation, variable exponents, blow - up, boundary damping, interior source

1 | INTRODUCTION

In this paper, we consider initial - boundary value problem for the following nonlinear hyperbolic equation

$$u_{tt} - u_{xx} = a_1 |u|^{p(x)-2} u, \quad 0 \le x \le l, \quad t > 0 \tag{1}$$

with the initial and boundary conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad 0 \le x \le l, \tag{2}$$

$$u(0,t) = 0, \quad t > 0, \tag{3}$$

$$u_x(l,t) + \left[\left| u_t(l,t) \right|^{r-2} + a_2 \right] u_t(l,t) = a_3 |u(l,t)|^{q-2} u(l,t), \quad t > 0, \tag{4}$$

where $(0,l)$ is a bounded open interval in $R = (-\infty, +\infty)$, a_1, a_2, a_3, r, q are real - valued constants to be refined later, $p(x)$, $u_0(x)$, $u_1(x)$ are real - valued functions.

Mathematical models of some physical processes such as the flow of electrorheological fluids or fluids with temperature - dependent viscosity, filtration in porous media, nonlinear viscoelasticity, etc., are reduced to hyperbolic equations with variable growth rates of nonlinearity. More detailed information on these problems can be found in the papers [1,2,3]. However, to best of our knowledge there are only a few works in which hyperbolic problems with nonlinearities of the variable exponent type are investigated (see, for example, [4,5,6,7]). In the papers [4,5], the solvability of initial - boundary value problems with the Dirichlet boundary condition is investigate for nonlinear hyperbolic equations with variable nonlinearity exponents. The absence of global solutions the initial - boundary value problem for the wave equation with variable exponents is investigated in the papers [6,7].

Initial - boundary value problems for nonlinear wave equations is the subject of numerous studies. In these papers, the existence, blow - up and asymptotics of smooth and weak solutions were investigated. See, for example, [8,9,10,11,12,13,14,15,16,17] and the

†This is an example for title footnote.

references indicated in these papers. A wave equation given in the following form

$$u_{tt} - \Delta u + \alpha \left|u_t\right|^{m-2} u_t = \beta \left|u\right|^{p-2} u, \quad x \in \Omega, \quad t > 0, \tag{5}$$

where Ω is a bounded domain in R^n, but α and β are positive constants, together with the initial and boundary conditions of the Dirichlet type has been carefully studied. The case of linear damping, i.e., when $m = 2$, was first considered by Levin and he showed that solutions the problem (5) with negative initial energy blow - up in a finite time[10,11,13,14,15,16]. The main method used in[10] and[11] is the "concavity method", the main idea of which is to construct a positive - definite functional $\theta(t)$ depending on the solution and to show that $\theta^{-\alpha}(t)$ is a concave function depending on t, for some $\alpha > 0$.

Later, this method was developed for the study of nonlinear wave equations, namely, when $m > 2$ or $p > 2$. For example, Georgiev and Todorova[12] extended Levin's result to the nonlinear damping case, when $m > 2$. The interaction between the damping term $\alpha \left|u_t\right|^{m-2} u_t$ and focusing source $\beta \left|u\right|^{p-2} u$ makes the problem more interesting. Similar studies were carried out for various equations (see, for example, [13,14,15,16,17]).

Various problems of mechanics and physics are reduced to the study of initial - boundary value problems for wave equations with a nonlinear boundary condition (see, for example, [18]). Numerous studies have been carried out in this direction (see, for example, [19,20,21,22,23,24,25,26,27,28,29,30] and the references indicated in these works). In the papers[28,29], the phenomena of blow – up solutions of the mixed problem for the one - dimensional wave equations with a non - stationary boundary condition were investigated.

In this paper, we at first investigate the local solvability of the problem (1) - (4). To study the existence and uniqueness of the local solution, well - known theorems from the monograph[31] are used. For this purpose, as in[21,22,23,24,25,26,27], at first problem (1) - (4) is reduced to the Cauchy problem for an operator - differential equation in some Hilbert space.

The purpose of this work is to investigate, blow – up of the solutions in a finite time depending on the interaction between the boundary damping $-\left|u_t(l,t)\right|^{r-2} u_t(l,t)$ and the internal focusing source $\left|u(x,t)\right|^{p(x)-2} u(x,t)$ and the boundary source $\left|u(l,t)\right|^{q-2} u(l,t)$. We will consider this issue in the following cases:

$$a) \quad a_1 > 0, \quad a_2 = 0, \quad a_3 \geq 0, \qquad 2r < p_1 + 2, \quad p_1 = \min_{0 \leq x \leq l} p(x), \quad E(0) < 0;$$

$b) \quad a_1 > 0, \quad a_3 = 0, \quad p'(x) \leq 0, \quad 2r \geq p_1 + 2$ and $E(0) < -\frac{1}{e}\int_0^l \frac{\left|xp'(x)\right|}{p^2(x)} dx.$

This paper, in addition to the introduction, consists of 6 sections. In section 2 we formulate the problem statement and present the main results on the existence of local solutions. In section 3 we recall the definitions of the variable exponent and give some auxiliary lemmas. In section 4 we present the proofs of theorems on the existence and uniqueness of local solutions. In Section 5 present the proof of theorem a blow - up for some solutions with negative initial energy subject to condition *a)*. Section 6 we presents the proof of theorem a blow - up for some solutions subject to condition *b)*. Section 7 contains the proof of the auxiliary lemmas.

2 | STATEMENT OF THE PROBLEM AND THE MAIN RESULTS

Suppose that the $p(\cdot)$ is a measurable function in $[0, l]$ and

$$2 \leq p_1 \leq p(x) \leq p_2 < +\infty, \quad 0 \leq x \leq l, \tag{6}$$

where $p_1 = \operatorname{ess} \inf_{x \in [0,l]} p(x), \quad p_2 = \operatorname{ess} \sup_{x \in [0,l]} p(x)$.

We also assume that $p(x)$ satisfies the log - Hölder continuity condition, i.e. for any $x, y \in [0, l]$,

$$|p(x) - p(y)| \leq \frac{C}{\left|\log|x-y|\right|}, \tag{7}$$

where $|x - y| < \delta, \; C > 0, \; 0 < \delta < 1$.

Let's

$$q \geq 2. \tag{8}$$

Let us introduce the notation: $H^k = W_2^k(0, l)$ and ${}_0H^1 = \left\{ v \, : \, v \in H^1, \, v(0) = 0 \right\}$.

The energy functional corresponding to the problem (1) - (4) is determined by the equality

$$E(t) = E_0(t) - G_0(u(\cdot, t)), \tag{9}$$

where

$$E_0(t) = \frac{1}{2}\left[\int_0^l |u_t(\cdot,t)|^2\,dx + \int_0^l |u_x(\cdot,t)|^2\,dx\right], \tag{10}$$

$$G_0(u(\cdot,t)) = a_1 \int_0^l \frac{|u(x,t)|^{p(x)}}{p(x)}dx + \frac{a_3}{q}\,|u(l,t)|^q\,. \tag{11}$$

A strong solution to problem (1) - (4) is a function $u(x,t)$ defined in the domain $(0,T)\times(0,l)$, such that $u(\cdot) \in L_\infty(0,T;\ H^2\bigcap{}_0H^1), u_t(\cdot) \in L_\infty(0,T;{}_0H^1)$, $u_{tt}(\cdot) \in L_\infty(0,T;L_2(0,l))$ and for almost all $(x,t) \in (0,T)\times(0,l)$ satisfying equation (1), boundary conditions (2), (3) and initial conditions (4).

By a weak solution to problem (1) - (4) we mean such a function $u(\cdot)$ defined in the domain $(0,T)\times(0,l)$ such that

1) $u(\cdot) \in C_w\left([0,T];\ {}_0H^1\right), \quad u_t(\cdot) \in C_w\left([0,T];L_2(0,l)\right);$
2) The trace of $u(\cdot)$ in $(0,T)\times\{l\}$ that exists by the trace theorem[32], has a distributional time derivative on $(0,T)\times\{l\}$ and belongs to $L^r(0,T)$, i.e. $u_t(l,t) \in L^r(0,T)$;
3) For all $\eta(\cdot) \in C_w\left([0,T];{}_0H^1\right)$, where $\eta_t(\cdot) \in C_w\left([0,T];L_2(0,l)\right)$,$\eta_t(l,\cdot) \in L^r(0,T),\ \ \eta(x,T)=0,\ \ 0\le x\le l$ the following equalities hold

$$\int_0^T\int_0^l \left[-u_t(x,t)\eta_t(x,t) + u_x(x,t)\eta_x(x,t)\right]dxdt +$$

$$+\int_0^T \left[|u_t(l,t)|^{r-2}\,u_t(l,t) + a_2u_t(l,t)\right]\eta(l,t)dt + \int_0^l u_1(x)\eta(x,0)dx =$$

$$= \int_0^T\int_0^l a_1\,|u(x,t)|^{p(x)}\,u(x,t)\eta(x,t)dxdt;$$

$$\lim_{t\to 0}\langle u(\cdot,t)-u_0(\cdot),\eta(\cdot,t)\rangle_{{}_0H^1} = 0.$$

Here $C_w([0,T];Y)$ denotes the space of weakly continuous functions with values in a Banach space Y.

We introduce the following function space:

$$C_1(T') = \{u(\cdot)\,:\,u(\cdot)\in C([0,T'];{}_0H^1\ ),$$

$$u_t(\cdot)\in C([0,T'];L_2(0,l)), u_t(l,t)\in L^r(0,T'),$$

$$\|u(\cdot)\|_{C_1(T')} = \|u(\cdot)\|_{C([0,T'];{}_0H^1)} + \|u_t(\cdot)\|_{C([0,T'];L_2(0,l))} + \|u_t(l,t)\|_{L^r(0,T')}\Big\}.$$

We denote the following class of functions:

$$C_2(T') = \Big\{u(\cdot)\,:\,u(\cdot)\in C([0,T'];H^2\bigcap{}_0H^1\ ),$$

$$u_t(\cdot)\in C([0,T'];{}_0H^1),\quad u_{tt}(\cdot)\in C([0,T'];L_2(0,l))\}.$$

The following theorems on the local solvability of the problem (1) - (4) are valid.

Theorem 1. Let the conditions (6), (7) be satisfied, and assume that

$$a_3 = 0 \ \text{ or }\ a_3\neq 0,\ a_2>0. \tag{12}$$

Then for any initial data

$$u_0 \in \left({}_0H^1\bigcap H^2\right)\times{}_0H^1,\quad u_1\in{}_0H^1, \tag{13}$$

where

$$u_{0x}(l) + |u_1(l)|^{r-2}\,u_1(l) + a_2u_1(l) - a_3\,|u_0(l)|^{q-2}\,u_0(l) = 0 \tag{14}$$

there exists such $T'\in(0,T]$ that problem (1) - (4) has a unique strong solution $u(\cdot)$ defined in the domain $(0,T')\times(0,l)$ and $u(\cdot)\in C_2(T')$.

Theorem 2. Suppose that the conditions (6), (7) are satisfied, and assume that

$$a_1, a_3 \in R,\ a_2 \geq 0. \tag{15}$$

Then for any initial data

$$u_0 \in {}_0H^1, \quad u_1 \in L_2(0,l), \tag{16}$$

there exists such $T' \in (0,T]$ that problem (1) - (4) has a weak solution $u(\cdot)$ in $(0,T') \times (0,l)$ and $u(\cdot) \in C_1(T')$.

If condition (12) is additionally satisfied, then this solution is unique. Moreover, weak local solutions are the limits of strong solutions in the space $C_1(T')$, and for weak solutions the following equality hold:

$$E(t) + \int_0^t \left\{ \left|u_\tau(l,\tau)\right|^r + a_2 \left|u_\tau(l,\tau)\right|^2 \right\} d\tau = E(0), t \in \left[0, T'\right]. \tag{17}$$

It can be proved that in the case when $a_1 \leq 0,\ a_3 \leq 0$, the solutions defined by Theorems 1,2 can be extended to the entire region $[0,T] \times (0,l)$.

In this paper, the problem (6) - (9) is studied in the case of a focusing source, i.e. when $a_1 > 0, \quad a_3 > 0$, and it is shown that in this case, for certain initial data, the solutions defined by Theorems 1,2 blow-up in a finite time.

Depending on the relationship between $r(x)$ and $p(x)$, different results are obtained on the absence of global solutions to problem (1) - (4).

In the case when $2r < p_1 + 2$, the following result is obtained.

Theorem 3. Let the conditions of Theorem 2 be satisfied. Assume that

$$a_1 > 0, \quad a_2 = 0, \quad a_3 \geq 0, \tag{18}$$

$$2r < p_1 + 2, \tag{19}$$

$$E(0) < 0. \tag{20}$$

Then the solution to problem (1) - (4) blows - up in finite time.

Further we investigate the absence of global solutions in the case $2r \geq p_1 + 2$.

Theorem 4. Let the conditions of Theorem 1 be satisfied and assume that

$$a_1 > 0, \quad a_2 = a_3 = 0, \tag{21}$$

$$p'(x) \leq 0, \tag{22}$$

$$2r \geq p_1 + 2, \tag{23}$$

$$E(0) < -\frac{1}{e} \int_0^l \frac{|xp'(x)|}{p^2(x)} dx, \tag{24}$$

where $e = \sum_{n=0}^{\infty} \frac{1}{n!}$,

$$l > \frac{2(p_1 - 1)}{p_1} \max \left\{ \frac{1}{a_1}, 1 \right\} \max_{0 \leq x \leq l} \frac{2p(x) - xp'(x)}{p(x)\left[p(x) - 2\right]}. \tag{25}$$

Then the solution to problem (1) - (4) blows - up in finite time.

3 | NECESSARY NOTATION OF A LEBESGUE SPACE WITH VARIABLE EXPONENTS AND SOME TECHNICAL LEMMAS

First, we note that in the future, some constants that do not depend on the solution of the problem will be denoted by $c_i,\ i = 1, 2, \ldots$.

Let us present some information about Lebesgue and Sobolev spaces with variable exponents set out in various monographs (see[33]) and articles (see, for example,[34,35,36]).

For a measurable function $p(\cdot)\,:\,[0,l]\to[1,+\infty)$ the Lebesgue space $L_{p(\cdot)}(0,l)$ with variable index $p(\cdot)$ is defined as follows:

$$L_{p(\cdot)}(0,l)=\{v:\ \ v:[0,l]\to R,\quad \text{measurable function and}$$

$$\rho_{p(\cdot)}(v)=\int\limits_0^l |v(x)|^{p(x)}\,dx<+\infty\}$$

.

It is known that $L_{p(\cdot)}(0,l)$ with the Luxembourg - type norm

$$\|v\|_{p(\cdot)}=\inf\left\{\lambda>0;\quad \int\limits_0^l\left|\frac{v(x)}{\lambda}\right|^{p(x)}dx\le 1\right\}$$

is a Banach space[29,30,31,33].

The Sobolev space with the variable exponents $p(\cdot)$, i.e. $W^1_{p(\cdot)}(0,l)$ is defined as follows:

$$W^1_{p(\cdot)}(0,l)=\left\{v:\quad v,v_x\in L_{p(\cdot)}(0,l)\right\},\|v\|_{W^1_{p(\cdot)}(0,l)}=\|v\|_{L_{p(\cdot)}(0,l)}+\left\|v_x\right\|_{L_{p(\cdot)}(0,l)}.$$

$W^1_{p(\cdot)}(0,l)$ is a Banach space and $W^1_{p(\cdot)}[0,l]\subset C[0,l]$, where $\|v\|_{C[0,l]}=\max\limits_{0\le x\le l}|v(x)|$.

Between $\rho_{p(\cdot)}(v)$ and the norm $\|v\|_{p(\cdot)}$ the relation

$$\min\left\{\|v\|^{p_1}_{L_{p(\cdot)}(0,l)},\|v\|^{p_2}_{L_{p(\cdot)}(0,l)}\right\}\le\rho_{p(\cdot)}(v)\le\max\left\{\|v\|^{p_1}_{L_{p(\cdot)}(0,l)},\|v\|^{p_2}_{L_{p(\cdot)}(0,l)}\right\}$$

is satisfied.

If $p(x)$ satisfies conditions (6), (7), then for any $u\in L_{p(\cdot)}(0,l),\ v\in L_{p'(\cdot)}(0,l)$ the following Hölder inequality holds:

$$\int\limits_0^l|u(x)v(x)|\,dx\le 2\,\|u\|_{L_{p(\cdot)}(0,l)}\cdot\|v\|_{L_{p'(\cdot)}(0,l)},$$

where $p'(x)=\frac{p(x)}{p(x)-1}$[33].

Applying Hölder's inequality and embedding theorems, we prove the following lemmas (see[6,30]).

Lemma 1. Let for $p(x)$ the conditions (6), (7) be satisfied, then for any $v\in L_{p(\cdot)}(0,l)$ the following inequality holds:

$$\|v\|^{p_1}_{p_1}\le\rho_{p(\cdot)}(v)+l^{\frac{p_2-p_1}{p_2}}\left\{\rho_{p(\cdot)}(v)\right\}^{\frac{p_1}{p_2}}. \tag{26}$$

Lemma 2. Let for $p(x)$ the conditions (6), (7) be satisfied, then for any $v\in{}_0H^1\bigcap L_{p(\cdot)}(0,l)$ the following inequality holds:

$$\rho_{p(\cdot)}(v)\le l\cdot\max\left\{\|v\|^{p_1}_{C[0,l]},\|v\|^{p_2}_{C[0,l]}\right\} \tag{27}$$

By virtue of the embedding theorems, from (27) we obtain

$$\rho_{p(\cdot)}(v)\le l\cdot\max\left\{l^{\frac{p_1}{2}}\,\|v\|^{p_1}_{{}_0H^1},l^{\frac{p_2}{2}}\,\|v\|^{p_2}_{{}_0H^1}\right\}. \tag{28}$$

From (17) we have

$$E(t)\le E(0),\quad t>0. \tag{29}$$

Let us introduce the following notations:

$$V(t)=-E(t), \tag{30}$$

$$\rho(u)=\rho_{p(\cdot)}(u)+|u(l,t)|^q. \tag{31}$$

It follows from (29) and (30) that

$$V(t)\ge -E(0)>0,\quad t\in[0,\infty). \tag{32}$$

By virtue of (6), we have

$$c_1\rho(u)\le G_0(u(\cdot,t))\le c_2\rho(u), \tag{33}$$

where $c_1 = \min\left\{\frac{a_1}{p_2}, \frac{a_3}{q}\right\}$, $\quad c_2 = \max\left\{\frac{a_1}{p_1}, \frac{a_3}{q}\right\}$. Thus,

$$\rho(u(t)) \geq \frac{1}{c_2} V(0) > 0. \tag{34}$$

By virtue of Lemma 1

$$\|u\|_{p_1}^{p_1} + |u(l,t)|^q \leq \rho(u)\left\{1 + l^{\frac{p_2-p_1}{p_2}} (\rho(u))^{\frac{p_1-p_2}{p_2}}\right\}. \tag{35}$$

Taking into account (34) from (35), we obtain the following statement

Lemma 3. Let conditions (6) - (8), (13) be satisfied and $u(\cdot)$ be a solution to problem (1) - (4), then

$$\|u(\cdot,t)\|_{p_1}^{p_1} + |u(l,t)|^q \leq c_3 \rho(u(\cdot,t)). \tag{36}$$

Lemma 4. Let conditions (6) - (8), (13) be satisfied and $\max\left\{\frac{2}{p_1}, \frac{2}{q}\right\} < \varkappa < 1$, then for any $u \in {}_0H^1$ the following inequality holds:

$$[\rho(u)]^{\varkappa} \leq c_4 \left\{\|u_x\|_2^2 + \rho(u)\right\}. \tag{37}$$

In particular, it is also true the inequality:

$$\|u\|_{p_1}^{\varkappa p_1} \leq c_5 \left\{\|u_x\|_2^2 + \|u\|_{p_1}^{p_1}\right\}. \tag{38}$$

Lemma 5. Let conditions (6) - (8), (13) be satisfied, and

$$\max\left\{\frac{2}{p_1}, \frac{2}{q}, \frac{r-1}{p_1-r+1}\right\} < \alpha < 1.$$

Then the solution to problem (1) - (4) satisfies the inequality

$$|u(l,t)|^r \leq c_6 \left[(\rho(u))^{\frac{r-1}{p_1}} + (\rho(u))^{\frac{\alpha+1}{2}} + (\rho(u))^{\frac{(r-1)(\alpha+1)}{\alpha p_1}}\right], \tag{39}$$

so that

$$\frac{1}{p_1} < \frac{r-1}{p_1}, \frac{\alpha+1}{2}, \frac{(r-1)(\alpha+1)}{\alpha p_1} < 1. \tag{40}$$

4 | PROOF OF LOCAL SOLVABILITY THEOREMS

Proof of Theorem 1. We use the standard method of reducing this problem to a Cauchy problem for differential equations in a Hilbert space $\mathcal{H} = {}_0H^1 \times L_2(0,l)$[22,23,24,25,26,27].

For this in the space $L_2(0,l)$ define the linear operator ${}_0\Delta$:

$$D({}_0\Delta) = \left\{f \,:\, f \in H^2(0,l),\ f(0) = 0,\ f'(l) = 0\right\},\ {}_0\Delta f(x) = f''(x),\ x \in (0,l).$$

Define also the operator $N\colon \alpha \to h = N\alpha \,:\, (-\infty, +\infty) \to H^2 = W_2^2(0,l)$,
where $h''(x) = 0,\ 0 < x < l,\ h(0) = 0,\ h'(l) = \alpha \quad (N\alpha = \alpha x)$[22,23,24,25,26,27].

We define the function $\phi_K(\xi)$ in the following way

$$\phi_K(\xi) = \begin{cases} \phi(\xi), & if \quad |\xi| \leq K \\ \phi(K\frac{\xi}{|\xi|}), & if \quad |\xi| > K \end{cases}, \tag{41}$$

where $K > 0,\ \phi(\xi) = a_3 |\xi|^{q-2} \xi$.

It is easy to prove that $\phi_K(\cdot)$ satisfies the Lipschitz condition, i.e., for any ξ_1 and ξ_2, the following inequality holds

$$\left|\phi_K(\xi_2) - \phi_K(\xi_1)\right| \leq C_\phi(K) \left|\xi_2 - \xi_1\right|. \tag{42}$$

In the space $\mathcal{H}$ we define the operator $A_K(\cdot)$:

$$D(A_K) = \left\{(u,v) \in \mathcal{H},\, u + N\left[g(\gamma v) + \phi_K(\gamma u)\right] \in D({}_0\Delta)\right\},$$

$$A_K(w) = \left\{-v, -{}_0\Delta\left(u + N\left[g(\gamma v) - \phi_K(\gamma u)\right]\right)\right\},$$

where $g(s) = |s|^{r-2} s + a_2 s$.

Moreover, we define the nonlinear operator $w \to F_K(w) = (0,\ f_{K_1}(x,u))$, where $w = (u,v) \in \mathcal{H}$, $\quad K_1 = Kl^{\frac{1}{2}}$,

$$f_{K_1}(x,u) = \begin{cases} f(x,u), & \|u\|_{{}_0H^1} \le K_1, \\ f(x, K_1 \frac{u}{\|u\|_{{}_0H^1}}), & \|u\|_{{}_0H^1} > K_1 \end{cases}, \quad f(x,u) = a_1 |u|^{p(x)-2} u. \tag{43}$$

Then the problem (1) - (4) can be written in the form

$$\begin{cases} w' + A_K(w) + F_K(w) = 0, \\ w(0) = w_0, \end{cases} \tag{44}$$

where $w_0 = (u_0(x), u_1(x)), \quad 0 \le x \le l.$

At first we prove that $A_K(\cdot)$ is an accretive operator. Indeed, for every $w_1, w_2 \in D(A_K)$ we have

$$\langle A_K(w_2) - A_K(w_1), w_2 - w_1 \rangle_{\mathcal{H}} = -\int_0^l (v_2 - v_1)_x (u_2 - u_1)_x dx -$$

$$-\int_0^l \left\{ {}_0\Delta \left[u_2 - Ng(\gamma v_2) + N\phi_K(\gamma u_2) \right] - {}_0\Delta [u_1 - Ng(\gamma v_1) + N\phi_K(\gamma u_1)] \right\} \times$$

$$\times (v_2 - v_1) dx = \left[g(v_2(l)) - g(v_1(l)) \right] (v_2(l) - v_1(l)) -$$

$$-[\phi_K(u_2(l)) - \phi_K(u_1(l))](v_2(l) - v_1(l)).$$

Hence, it is clear that if $a_3 = 0$, then

$$\langle A_K(w_2) - A_K(w_1), w_2 - w_1 \rangle_{\mathcal{H}} \ge 0.$$

If $a_3 \ne 0$ and $a_2 > 0$, then using the Hölder and Young's inequality, we obtain

$$\langle A_K(w_2) - A_K(w_1), w_2 - w_1 \rangle_{\mathcal{H}} \ge l(a_2 - \varepsilon) \left| v_2(l) - v_1(l) \right|^2 - \frac{lC_\phi(K)}{\varepsilon} \left| u_2(l) - u_1(l) \right|^2,$$

where $0 < \varepsilon < a_2$.

Therefore, in both cases $A_K(\cdot) + \omega I$ is an accretive operator in $\mathcal{H}$, for any $K > 0$, where $\omega = \frac{lC_\phi(K)}{\varepsilon}$.

Secondly, we prove that $A_K(\cdot) + \omega I + \mu I$ is a surjective operator for some $\mu > 0$, i.e. for any $E = (\eta_1, \eta_2) \in \mathcal{H}$, there exists such $w = (u,v) \in D(A_K)$ that

$$A_K(w) + \lambda w = E$$

is equivalent to the following equations

$$\begin{cases} -v + \lambda u = \eta_1 \\ -{}_0\Delta(u - N[g(\gamma v) - \phi_K(\gamma u)]) + \lambda v = \eta_2 \end{cases}.$$

It's clear that $u = \frac{1}{\lambda} v + \frac{1}{\lambda}\eta_1$ and the function $z = u - N[g(\gamma v) - \phi_K(\gamma u)]$ is solutions to the boundary value problem

$$z'' - \lambda^2 z = \eta(x), \tag{45}$$

$$z(0) = 0, \quad z'(l) = 0, \tag{46}$$

where $\eta(x) = -\eta_2 + \lambda\eta_1 + \lambda^2 N[g(\gamma v) - \phi_K(\gamma u)]$.

Solving the problem (45), (46), we have

$$z(x) = A(x,\lambda) \left\{ \lambda^2 [g(v(l)) - \phi_K(\frac{1}{\lambda}\eta_1(l) - \frac{1}{\lambda} v(l))] \right\} + B(x,\lambda), \tag{47}$$

where

$$A(x,\lambda) = -\frac{1}{2\lambda} e^{-\lambda x} \int_0^x e^{\lambda s} s ds + \frac{1}{2\lambda} e^{\lambda x} \int_0^x e^{-\lambda s} s ds +$$

$$+ \frac{e^{-\lambda x}}{2\lambda(e^{\lambda x} + e^{-\lambda x})} \left\{ e^{-\lambda l} \int_0^l e^{\lambda s} s ds + e^{\lambda l} \int_0^l e^{-\lambda s} s ds \right\} -$$

$$-\frac{e^{\lambda x}}{2\lambda(e^{\lambda x}+e^{-\lambda x})}\left\{e^{-\lambda l}\int_0^l e^{\lambda s}sds+e^{\lambda l}\int_0^l e^{-\lambda s}sds\right\},$$

$$B(x,\lambda)=-\frac{1}{\lambda}\int_0^x Sh(x-s)(\eta_2(s)+\lambda\eta_1(s))ds+$$

$$+\frac{1}{\lambda}th\lambda x\int_0^l Ch\lambda(l-s)(\eta_2(s)+\lambda\eta_1(s))ds.$$

From a simple calculation we get

$$A(l,\lambda)=-\frac{1}{\lambda^2}\left[l-\frac{1-e^{-2\lambda l}}{\lambda(1+e^{-2\lambda l})}\right]. \tag{48}$$

By using (47) and (48), we have

$$z(l)=A(l,\lambda)\left\{\lambda^2\left[g(v(l))-\phi_K\left(\frac{1}{\lambda}\eta_1(l)-\frac{1}{\lambda}v(l)\right)\right]\right\}+B(l,\lambda),$$

from which it follows that

$$g(v(l))+\frac{1+e^{-2\lambda l}}{1-e^{-2\lambda l}}v(l)+\phi_K\left(\frac{1}{\lambda}\eta_1(l)-\frac{1}{\lambda}v(l)\right)=B_1(l,\lambda), \tag{49}$$

where $B_1(l,\lambda)=\frac{1+e^{-2\lambda l}}{1-e^{-2\lambda l}}B(l,\lambda)$.

From (42) we obtain that for any ξ_1 and ξ_2 the following inequality is true

$$\left|\phi_{1K}(\xi_2)-\phi_{1K}(\xi_1)\right|\le\frac{C_\varphi(K)}{\lambda}\left|\xi_2-\xi_1\right|, \tag{50}$$

where $\phi_{1K}(\xi)=\phi_K(\frac{1}{\lambda}\eta_1(l)-\frac{1}{\lambda}\xi)$. By virtue of (49) and (50) $\tau(\xi)=g(\xi)+\frac{1+e^{-2\lambda l}}{1-e^{-2\lambda l}}\xi+\phi_{1K}(\xi)$ is continuous function and monotonically decreases for $\lambda\ge\frac{C_\varphi(K)}{a_2}$.

Let ξ_0 be a solution to the equation $\tau(\xi)=B_1(l,\lambda)$. Taking $v(l)=\xi_0$, and using (47), we find the function $z(x)$. So we find the function $u(x)=z(x)+N[g(\xi_0)-\varphi_K(\frac{1}{\lambda}\xi_0+\frac{1}{\lambda}\eta_1(x))]$ and $v(x)=\lambda u(x)-\eta_1(x)$.

Thus for$\lambda\ge\frac{c_\varphi(k)}{a_2}$, $J_m(A_K(\cdot)+\lambda I)=\mathcal{H}$ and therefore $A_K(\cdot)+\omega I$ is the maximal accretive operator for each $K>0$.

Using Lemma 4, one can easily prove that the nonlinear operator $F_K(\cdot)$ satisfies the Lipschitz condition, i.e. for any $w_1,w_2\in\mathcal{H}$ the following inequality holds:

$$\left\|F_K(w_2)-F_K(w_1)\right\|_{\mathcal{H}}\le C_F(K)\left\|w_2-w_1\right\|_{\mathcal{H}},$$

where $C_F(K)>0,\ F_K(\cdot)$ is a Lipshtsev operator. Thus, $A_K(\cdot)+F_K(\cdot)$ is the maximal accretive operator[31,37].

If $a_3=0$ or $a_3\neq 0,\quad a_2>0$, then, according to the[31], the following statements are true:

(i) For any $w_0\in D(A_K)$ and $K>0$ there exists such $T_K=T_K(w_0)>0$ that problem (44) has a unique strong solution $w(\cdot)\in C^1([0,T_K];\mathcal{H})\bigcap C([0,T_K];D(A_K))$ ([31]: Theorem 4.1);

(ii) If $w_0\in\mathcal{H}$, then problem (44) has a weak solution ([31]: Theorem 4.1A, corollary 4.1A), and this solution is the limit of strong solutions in the space $C([0,T_K];\mathcal{H})$.

According to reference[31] (Theorem 4.2) for any $u_0\in{}_0H^1,\ u_1\in L_2(0,l)$ the problem (44) has a unique solution in the space $C([0,T_K];\mathcal{H})$, and this solution is equal to the limit of strong solutions in the same space.

Identity (17) for strong solutions is proved by direct differentiation taking into account (1) - (4). For weak solutions, the same identity is obtained by approximating them by strong solutions.

Since $A_K(\cdot)+\omega I$ is a maximally accretive operator, from (44) we have

$$\frac{d}{dt}\left\|w(\cdot,t)\right\|^2_{\mathcal{H}}\le\left[\omega+C_K\right]\left\|w(\cdot,t)\right\|^2_{\mathcal{H}},$$

$$w(x,0)=w_0(x).$$

Hence we get if $\left\|w_0(\cdot)\right\|_{\mathcal{H}}<K$, then

$$\left\|w(\cdot,t)\right\|^2_{\mathcal{H}}\le\left\|w_0(\cdot)\right\|^2_{\mathcal{H}}e^{[\omega+C_K]t}<K,0\le t\le t^*, \tag{51}$$

where $t^*=\frac{1}{\omega+C_K}Ln\frac{K^2}{\left\|w_0(\cdot)\right\|^2_H}$. It follows from (51) the validity of the inequality $\left\|u(x,t)\right\|_{H^1}<K,\ \left\|u_t(x,t)\right\|_{L_2(0;l)}<K$, so $\varphi_k(u_t(x,t))=\varphi(u_t(x,t)),\ f_k(u(x,t))=f(x,u(x,t))$.

Hence, the function $u(x,t)$ is the solution of the problem (1) - (4).

Theorem 1 is proved.

Proof of Theorem 2. Let $w_0 \in \mathcal{H}$ then there exist $w_{n0} = (u_{n0}(\cdot), u_{n1}(\cdot)) \in D(A_K),\ n = 1, 2,$ such that

$$w_{n0} \to w_0 \text{ in } \mathcal{H} \text{ at } n \to \infty. \tag{52}$$

By Theorem 1, for each $n \in N$ exists $T_n \in (0, T]$, such that problem

$$u_{ntt} - u_{nxx} - a_1 \left|u_n\right|^{p(x)-2} u_n = 0, \quad 0 < x < l,\ \ 0 \le t \le T_n, \tag{53}$$

$$u_n(0,t) = 0, \quad 0 \le t \le T_n, \tag{54}$$

$$u_{nx}(l,t) + \left|u_{nt}(l,t)\right|^{r-2} u_{nt}(l,t) + \\ + \frac{1}{n} u_{nt}(l,t) - a_3 \left|u_n(l,t)\right|^{q-2} u_n(l,t) = 0, \quad 0 \le t \le T_n, \tag{55}$$

$$u_n(x,0) = u_{n0}(x), \quad u_{nt}(x,0) = u_{n1}(x), \quad 0 < x < l, \tag{56}$$

has a unique strong solution $u_n(x,t)$ in the domain $(0,l) \times (0,T_n)$ where $u_n \in C^2_{T_n}$, and the following identity is true

$$E_n(t) + \int_0^t \left\{ \left|u_{n\tau}(l,\tau)\right|^{r_i+1} + \frac{1}{n} \left|u_{n\tau}(l,\tau)\right|^2 \right\} d\tau = E_n(0), 0 \le t \le T_n. \tag{57}$$

Here $E_n(t) = E_{n0}(t) - G_0(u_n)$, $E_{n0}(t) = \frac{1}{2}\left[\left\|u_{nt}(\cdot,t)\right\|_2^2 + \left\|u_{nx}(\cdot,t)\right\|_2^2\right]$,
$G_n(t) = a_1 \int_0^l \frac{\left|u_n(x,t)\right|^{p(x)}}{p(x)} dx + \frac{a_3}{q}\left|u_n(l,t)\right|^q$,where

$$T_n = \frac{1}{\omega + C_F(K)} Ln \frac{K}{\left\|w_{n0}\right\|_{\mathcal{H}}}. \tag{58}$$

From (57) it follows that

$$E'_{n0}(t) + \left[\left|u_{nt}(l,t)\right|^{r+1} + \frac{1}{n}\left|u_{nt}(l,t)\right|^2\right] = \\ = a_1 \int_0^l \left|u_n(x,t)\right|^{p(x)-2} u_n(x,t) u_{nt}(x,t) dx + a_3 \left|u_n(l,t)\right|^{q-2} u_n(l,t) u_{nt}(l,t) = \\ = J_1 + J_2. \tag{59}$$

Applying the Hölder and Young inequalities and taking into account (27), we have:

$$\left|J_1\right| \le \int_0^l \left|u_n(x,t)\right|^{p(x)-1} \left|u_{nt}(x,t)\right| dx \le \\ \le \max\left\{\left(\int_0^l \left|u_n(x,t)\right|^{2(p_1-1)} dx\right)^{\frac{1}{2}}, \left(\int_0^l \left|u_n(x,t)\right|^{2(p_2-1)} dx\right)^{\frac{1}{2}}\right\}\left(\int_0^l \left|u_{nt}(x,t)\right|^2 dx\right)^{\frac{1}{2}} \le \\ \le \left\{\left(\int_0^l \left|u_n(x,t)\right|^{2(p_1-1)} dx\right)^{\frac{1}{2}} + \left(\int_0^l \left|u_n(x,t)\right|^{2(p_2-1)} dx\right)^{\frac{1}{2}}\right\}\left(\int_0^l \left|u_{nt}(x,t)\right|^2 dx\right)^{\frac{1}{2}} \le \\ \le \left\|u_n(\cdot,t)\right\|_{{}_0H^1}^{p_1-1} \left\|u_{nt}(\cdot,t)\right\|_{L_2(0,l)} + \left\|u_n(\cdot,t)\right\|_{{}_0H^1}^{p_2-1} \left\|u_{nt}(\cdot,t)\right\|_{L_2(0,l)} \le \\ \le c_7 \left\{\left\|u_n(\cdot,t)\right\|_{{}_0H^1}^{p_1} + \left\|u_n(\cdot,t)\right\|_{{}_0H^1}^{p_2} + \left\|u_{nt}(\cdot,t)\right\|_{L_2(0,l)}^{p_1} + \left\|u_{nt}(\cdot,t)\right\|_{L_2(0,l)}^{p_2}\right\} \le \\ \le c_8 \left\{E_{n0}^{p_1}(t) + E_{n0}^{p_2}(t)\right\}.$$

Similarly, applying Young's inequalities, we have:

$$\left|J_2\right| \leq \left|u_n(l,t)\right|^{q-1}\left|u_{nt}(l,t)\right| \leq c_9\left\|u_n(\cdot,t)\right\|_{{}_0H^1}^{q-1}\left|u_{nt}(l,t)\right| \leq$$

$$\leq \frac{rc_9^{\frac{r+1}{r}}}{(r+1)^{\frac{1}{r}}\varepsilon^{\frac{1}{r}}}\left\|u_n(\cdot,t)\right\|_{{}_0H^1}^{(q-1)\frac{r+1}{r}} + \varepsilon\left|u_{nt}(l,t)\right|^{r+1} \leq$$

$$\leq \frac{rc_9^{\frac{r+1}{r}}}{(r+1)^{\frac{1}{r}}\varepsilon^{\frac{1}{r}}}E_{n0}^{(q-1)\frac{r+1}{r}}(t) + \varepsilon\left|u_{nt}(l,t)\right|^{r+1},$$

where $0 < \varepsilon < \frac{1}{2}$.

Therefore, for $y = y(t) = E_{n0}(t) + 1$ we have the inequality

$$y' \leq c_{10}y^P, \tag{60}$$

where $y(0) = y_{0n} = E_{n0}(0) + 1$, $P = \max\left\{p_1, p_2, \frac{(q-1)(r+1)}{r}\right\}$.

From (60) we obtain that $y(t) \leq 2^{\frac{1}{P-1}}y_{0n}$, $\quad 0 \leq t \leq \tilde{T}_n$, where $\tilde{T}_n = \min\left\{T_n, \frac{1}{2c_{10}y_{0n}^{P-1}(P-1)}\right\}$, i.e.

$$E_{n0}(t) \leq 2^{\frac{1}{P-1}}\left[E_{n0}(0) + 1\right], \quad 0 \leq t \leq \tilde{T}_n. \tag{61}$$

It follows from (52) that $\lim\limits_{n\to\infty} E_{n0}(0) = E_0(0) = \frac{1}{2}\left[\left\|u_1\right\|_{L_2(0,l)}^2 + \left\|u_{0_x}\right\|_{L_2(0,l)}^2\right]$. For this reason, there exists a natural number N such that, for any $n \geq N$, the inequality $E_{n0}(0) \leq 2E_0(0)$, holds. Taking this into account, from (61) we find that

$$E_{n0}(t) \leq 2^{\frac{1}{P-1}}\left[2E_0(0) + 1\right], \quad 0 \leq t \leq T_0, \tag{62}$$

where $T_0 = \min\left\{\frac{1}{\omega+C_F(K)}\ln\frac{K}{\sqrt{E_0(0)}}, \frac{1}{2c_{10}\left[2E_0(0)+1\right]^{P-1}(P+1)}\right\}$.

It follows from (57), (62) that

$$\int_0^t\left|u_{n_\tau}(\tau,t)\right|^r d\tau \leq c_{11}, \quad 0 \leq t \leq T_0, \quad i = 1,2 \tag{63}$$

and

$$\frac{1}{n}\int_0^t\left|u_{n_\tau}(\tau,t)\right|^2 d\tau \leq c_{11}, \quad 0 \leq t \leq T_0, \quad i = 1,2. \tag{64}$$

From (55) - (58), we see that there exists a function $u(t)$ and a subsequence of the sequence $\left\{u_n\right\}$, which, we still denote by $\left\{u_n\right\}$, where as $n \to \infty$ (see[32]).

$$u_n(\cdot) \to u(\cdot), \text{ weakly star in } L_\infty(0,T_0;{}_0H^1), \tag{65}$$

$$u_{nt}(\cdot) \to u_t(\cdot), \text{ weakly star in } L_\infty(0,T_0;L_2(0,l)), \tag{66}$$

$$\left|u_{nt}(l,\cdot)\right|^{r-2}u_{nt}(l,\cdot) \to \zeta(\cdot) \text{ weakly in } L_{\frac{r}{r-1}}(0,T_0), \tag{67}$$

$$\frac{1}{n}u_{nt}(l,\cdot) \to 0 \text{ weakly in } L_2(0,T_0). \tag{68}$$

Due to the monotonicity of the operator $g_0(v) = |v|^{r-2}v$, $\quad L_r(0,T_0) \to L_{\frac{r}{r-1}}(0,T_0)$ we have

$$\left|u_{nt}(\cdot)\right|^{r-1}u_{nt}(\cdot) \to \left|u_t(l,t)\right|^{r-2}u_t(l,t) \text{ weakly in } L_r(0,T_0). \tag{69}$$

If we write down the problem (53) - (56) for $u_n(\cdot)$ and take into account (65) - (69), we get that in the case $a_2 = 0$ the limit function $u(\cdot)$ satisfies (1) - (4). Then, using[26] (Lemma 1), we obtain $u(\cdot) \in C_{T_0}^1$.

Theorem 2 is proved.

5 | SCHEMA PROOF OF THEOREM 3

Theorem 3 is proved in a similar way as the proof of Theorem 2 from[30]. For this reason, here we only give a scheme for the proof of this theorem.

It follows from (30), (33) and Lemma 5 that

$$V^{\sigma(r-1)}(t)\cdot|u(l,t)|^r \le c_{12}\rho(u), \tag{70}$$

where $0<\sigma<\min\left\{\frac{p_1-r}{p_1(r-1)},\frac{1-\alpha_0}{2(r-1)},\frac{\alpha_0 p_1-(r-1)(\alpha_0+1)}{\alpha_0 p_1(r-1)},\frac{p_1-2}{2p_1}\right\}$, $0<\max\left\{\frac{2}{p_1},\frac{2}{q},\frac{r-1}{p_1-r+1}\right\}<$ $<\alpha_0<1$.

Let us introduce the notation:

$$L(t)=V^{1-\sigma}(t)+\varepsilon\int_0^l u(x,t)u_t(x,t)dx. \tag{71}$$

Taking into account (29)-(34), Lemmas 1-4 and applying Hölder's inequality, as well as Young's inequalities with the parameter $\delta=\mathcal{K}^{-\frac{r-1}{r}}V^{\frac{\sigma(r-1)}{r}}(t)$, we obtain that

$$L'(t)\ge\left(1-\sigma-\frac{\varepsilon(r-1)}{r}\mathcal{K}\right)V^{-\sigma}(t)V'(t)+2\varepsilon(1-\eta)(p_1+1)V(t)+$$
$$+c_{13}\left[\int_0^l|u_t|^2dx+\int_0^l|u_x|^2dx\right]+\varepsilon\left(c_{14}-\frac{\mathcal{K}^{1-r}}{r}c_{15}\right)\rho(u).$$

where $\mathcal{K}>0$ is sufficiently large number and $0<\eta<1$ (see[30]).

Taking into account Lemmas 1-5 and using the Hölder and Young inequalities, we obtain that

$$L'(t)\ge c_{16}\left[V(t)+\|u_t\|_2^2+\rho(u)\right]\ge c_{17}\left[V(t)+\|u_t\|_2^2+\|u\|_{p_1}^{p_1}\right]\ge 0. \tag{72}$$

On the other hand, by virtue of (20), for sufficiently small $\varepsilon>0$ we have

$$L(0)>0. \tag{73}$$

From (71), (72), and (73) it follows

$$L(t)\ge L(0)>0. \tag{74}$$

Applying the Hölder and Young inequalities and taking into accound (38) we have

$$L^{\frac{1}{1-\sigma}}(t)\le c_{18}\left[V(t)+\|u_t\|_2^2+\|u\|_{p_1}^{p_1}\right]. \tag{75}$$

Comparing (72) and (75), we get

$$L'(t)\ge c_{19}\left[L(t)\right]^{\frac{1}{1-\sigma}},\quad t>0.$$

This implies that $L(t)\ge\left[[L(0)]^{-\frac{\sigma}{1-\sigma}}-c_{19}t\frac{\sigma}{1-\sigma}\right]^{\frac{\sigma-1}{\sigma}}$. Obviously, $\lim\limits_{t\to T^*}L(t)=+\infty$, where $T^*=\frac{1-\sigma}{\sigma c_{19}[L(0)]^{\frac{\sigma}{1-\sigma}}}$.

6 | PROOF OF THEOREM 4

By virtue of (22) $p_1=p(l)$ and $p_2=p(0)$.

Let us introduce the notation:

$$y(t)=-E(t)+\varepsilon k\int_0^l u(x,t)\cdot u_t(x,t)dx+\varepsilon\int_0^l xu_t(x,t)\cdot u_x(x,t)dx,$$

where

$$\max_{0\le x\le l}\frac{2p(x)-xp'(x)}{p(x)(p(x)-2)}<k<\frac{lp_1}{2(p_1-1)}. \tag{76}$$

By virtue of (25), such k exists. It follows from (1) – (4) that

$$y'(t)=\left|u_t(l,t)\right|^r-\varepsilon(1+2k)E(t)+2\varepsilon k\int_0^l\left|u_t(x,t)\right|^2dx+$$

$$+\varepsilon a_1 \int_0^l \left\{ k\frac{p(x)-2}{p(x)} - \frac{2p(x)-xp'(x)}{p^2(x)} \right\} |u(x,t)|^{p(x)}\, dx -$$

$$-\varepsilon a_1 \int_0^l \frac{xp'(x)}{p(x)} |u(x,t)|^{p(x)} \ln |u(x,t)|\, dx + \frac{\varepsilon l}{2} \left|u_t(l,t)\right|^2 + \frac{\varepsilon l}{2} \left|u_t(l,t)\right|^{2(r-1)} +$$

$$+\frac{a_1 l \varepsilon}{p_1} |u(l,t)|^{p_1} - \varepsilon k \left|u_t(l,t)\right|^{r-2} u_t(l,t) u(l,t). \tag{77}$$

Using Young's inequality with exponents $\theta = p_1, \quad \theta' = \frac{p_1}{p_1-1}$,we have

$$\left|u_t(l,t)\right|^{r-1} |u(l,t)| \le \frac{p_1-1}{p_1} \left|u_t(l,t)\right|^{\frac{p_1}{p_1-1}(r-1)} + \frac{1}{p_1} |u(l,t)|^{p_1}. \tag{78}$$

Since $(r-1)\frac{p_1}{p_1-1} \le 2(r-1)$ and $(r-1)\frac{p_1}{p_1-1} - 2 \ge \frac{p_1}{2}\frac{p_1}{p_1-1} - 2 = \frac{p_1^2-4(p_1-1)}{2(p_1-1)} > 0$,
we have

$$\left|u_t(l,t)\right|^{\frac{p_1}{p_1-1}(r-1)} \le \left|u_t(l,t)\right|^2 + \left|u_t(l,t)\right|^{2(r-1)}. \tag{79}$$

From (76), (77) - (79) we obtain

$$y'(t) \ge \left|u_t(l,t)\right|^r - \varepsilon(1+2k)E(t) + 2\varepsilon k \int_0^l \left|u_t(x,t)\right|^2 dx +$$

$$+\varepsilon a_1 \int_0^l \left\{ k\frac{p(x)-2}{p(x)} - \frac{2p(x)-xp'(x)}{p^2(x)} \right\} |u(x,t)|^{p(x)}\, dx -$$

$$-\varepsilon a_1 \int_0^l \frac{xp'(x)}{p(x)} |u(x,t)|^{p(x)} \ln |u(x,t)|\, dx + \frac{\varepsilon l}{2} \left|u_t(l,t)\right|^2 +$$

$$+\varepsilon(\frac{l}{2} - \frac{p_1-1}{p_1}k) \left|u_t(l,t)\right|^2 + \varepsilon(\frac{l}{2} - \frac{p_1-1}{p_1}k) \left|u_t(l,t)\right|^{2(r-1)} +$$

$$+\varepsilon(\frac{a_1 l}{p_1} - \frac{1}{p_1}k) |u(l,t)|^{p_1}. \tag{80}$$

For $t > 0$ we denote $\Sigma_{1t} = \{x : \ |u(x,t)| \ge 1, \quad 0 \le x \le l\}$ and $\Sigma_{2t} = \{x : \ |u(x,t)| < 1, \quad 0 \le x \le l\}$, then

$$I_t = \int_0^l \frac{xp'(x)}{p(x)} |u(x,t)|^{p(x)} \ln |u(x,t)|\, dx = I(\Sigma_{1t}) + I(\Sigma_{2t}),$$

where

$$I(\Sigma_{1t}) = \int_{x\in\Sigma_{1t}} \frac{xp'(x)}{p(x)} |u(x,t)|^{p(x)} \ln |u(x,t)|\, dx,$$

$$I(\Sigma_{2t}) = \int_{x\in\Sigma_{2t}} \frac{xp'(x)}{p(x)} |u(x,t)|^{p(x)} \ln |u(x,t)|\, dx.$$

By virtue of (20)

$$I(\Sigma_{1t}) \le 0. \tag{81}$$

On the other hand, $|\alpha|^{p(x)} \ln |\alpha| \ge -\frac{1}{p(x)} e^{-1}$, if $|\alpha| < 1$. Hence,

$$I(\Sigma_{2t}) \le e^{-1} \int_0^l \frac{|xp'(x)|}{p^2(x)} dx = M_1. \tag{82}$$

From (81) and (82) we have

$$-\varepsilon a_1 I_t \ge -\varepsilon M_1. \tag{83}$$

Moreover, from (25), (76), and (83) we obtain the following inequalities

$$k\frac{p(x)-2}{p(x)}-\frac{2p(x)-xp'(x)}{p^2(x)}\geq 0,$$

$$-\varepsilon(1+2k)E(t)-a_1\varepsilon I_t\geq-\varepsilon\left\{(1+2k)E(0)+M_1\right\}\geq 0.$$

On the other hand, it follows from (76) that

$$\xi=\varepsilon\min\left\{\frac{l}{2}-k\frac{p_1-1}{p_1},\frac{l}{p_1}-k\frac{1}{p_1}\right\}>0.$$

Then, taking into account the last inequalities, from (80) we have

$$y'(t)\geq\xi\left\{\left|u_t(l,t)\right|^2+\left|u_t(l,t)\right|^{2(r-1)}+|u(l,t)|^{p_1}\right\}. \tag{84}$$

After fixing k we choose a rather small ε, so that

$$y(0)=-E(0)+\varepsilon k\int_0^l u_0(x)u_1(x)dx+\varepsilon\int_0^l xu_{0x}(x)u_1(x)dx>0.$$

Hence

$$y(t)\geq y(0)>0. \tag{85}$$

On the other hand, applying Hölder's inequality and taking into account inequality (33), for sufficiently small $\varepsilon>0$, we obtain

$$y(t)\leq-E(t)+\frac{\varepsilon l^2}{2}(k+1)\int_0^l\left|u_x(x,t)\right|^2dx+\frac{\varepsilon l}{2}(k+1)\int_0^l\left|u_t(x,t)\right|^2dx\leq$$

$$\leq a_1\int_0^l\frac{|u(x,t)|^{p(x)}}{p(x)}dx\leq\frac{a_1}{p_1}\rho(u(\cdot,t)). \tag{86}$$

Next, we define

$$z(t)=y^{1-\alpha_1}(t)+\theta\int_0^l u(x,t)u_t(x,t)dx, \tag{87}$$

$$\alpha_1=\frac{p_1-1}{2p_1}, \tag{88}$$

where θ is chosen rather small to satisfy

$$z(0)>0. \tag{89}$$

Differentiating $z(t)$ and using (1) - (4), we have

$$z'(t)=(1-\alpha_1)y^{-\alpha_1}(t)y'(t)-2\theta E(t)+2\theta\int_0^l\left|u_t(x,t)\right|^2dx+$$

$$+\theta\int_0^l\frac{p(x)-2}{p(x)}|u(x,t)|^{p(x)}dx-\theta\left|u_t(l,t)\right|^{r-1}u(l,t). \tag{90}$$

Using Young's inequality with the exponents $\frac{1}{\alpha_1}$, $\frac{1}{1-\alpha_1}$, get

$$\left|u_t(l,t)\right|^{r-1}|u(l,t)|=\left[\frac{\left|u_t(l,t)\right|^{r-1}|u(l,t)|}{\delta}\right]\delta\leq$$

$$\leq(1-\alpha_1)\delta^{-\frac{1}{1-\alpha_1}}\left[\left|u_t(l,t)\right|^{r-1}|u(l,t)|\right]^{\frac{1}{1-\alpha_1}}+\alpha_1\delta^{\frac{1}{\alpha_1}}.$$

Let $\delta=\Lambda^{-1}y^{\alpha_1(1-\alpha_1)}(t)$, where $\Lambda>0$ will be defined later, then from (84),(90), we have

$$z'(t)\geq\xi(1-\alpha_1)y^{-\alpha_1}(t)\left\{\left|u_t(l,t)\right|^2+\left|u_t(l,t)\right|^{2(r-1)}+\right.$$

$$+\left|u(l,t)\right|^{p_1}\Big\}-\theta(1-\alpha_1)\Lambda^{\frac{1}{1-\alpha_1}}y^{-\alpha_1}(t)\left[\left|u_t(l,t)\right|^{r-1}u(l,t)\right]^{\frac{1}{1-\alpha_1}}+$$

$$+\theta\alpha_1\Lambda^{-\frac{1}{\alpha_1}}y^{1-\alpha_1}(t)-2\theta E(t)+2\theta\int_0^l\left|u_t(x,t)\right|^2dx. \tag{91}$$

From (86) it follows

$$y^{1-\alpha_1}(t)\le\left(\frac{a_1}{p_1}\right)^{1-\alpha_1}\left[\rho(u(\cdot,t))\right]^{1-\alpha_1}. \tag{92}$$

Further, using Young's inequality with the exponents $2(1-\alpha_1)$, $\frac{2(1-\alpha_1)}{1-2\alpha_1}$, we get

$$\left[\left|u_t(l,t)\right|^{r-1}u(l,t)\right]^{\frac{1}{1-\alpha_1}}\le\frac{1}{2(1-\alpha_1)}\left|u_t(l,t)\right|^{2(r-1)}+\frac{1-2\alpha_1}{2(1-\alpha_1)}\left|u(l,t)\right|^{p_1}. \tag{93}$$

Taking this into account (92) and (93) in the (91), we have

$$z'(t)\ge(1-\alpha_1)y^{-\alpha_1}(t)\left\{\xi\left|u_t(l,t)\right|^2+\left(\xi-\theta\frac{1}{2(1-\alpha_1)}\Lambda^{\frac{1}{1-\alpha_1}}\right)\left|u_t(l,t)\right|^{2(r-1)}+\right.$$

$$\left.+\left(\xi-\theta\frac{1-2\alpha_1}{2(1-\alpha_1)}\Lambda^{\frac{1}{1-\alpha_1}}\right)\left|u(l,t)\right|^{p_1}\right\}+2\theta\int_0^l\left|u_t(x,t)\right|^2dx+$$

$$+\theta\int_0^l\left\{\frac{p(x)-2}{p(x)}-\alpha_1\left(\frac{a_1}{p_1}\right)^{1-\alpha_1}\Lambda^{-\frac{1}{\alpha_1}}\right\}\left|u(x,t)\right|^{p(x)}dx. \tag{94}$$

Let's choose a rather large $\Lambda>0$ and a rather small $\theta>0$ so that

$$\frac{p(x)-2}{p(x)}-\alpha_1\left(\frac{a_1}{p_1}\right)^{1-\alpha_1}\Lambda^{-\frac{1}{\alpha_1}}\ge\theta_0>0,\quad \xi-\theta\frac{1}{2(1-\alpha_1)}\Lambda^{\frac{1}{1-\alpha_1}}>0.$$

Taking into account these inequalities, it follows from (93) that

$$z'(t)\ge m\left\{\int_0^l\left|u_t(x,t)\right|^2dx+\int_0^l\left|u(x,t)\right|^{p(x)}dx\right\}, \tag{95}$$

where $m=\min\left\{2\theta,\theta_0\right\}$.

Applying Hölder's inequality, we obtain

$$\left[\int_0^l u(x,t)u_t(x,t)dx\right]^{\frac{1}{1-\alpha_1}}\le$$

$$\le\left\|u(x,t)\right\|_2^{\frac{1}{1-\alpha_1}}\left\|u_t(x,t)\right\|_2^{\frac{1}{1-\alpha_1}}\le c_{20}\left\|u(x,t)\right\|_2^{\frac{1}{1-\alpha_1}}\left\|u_t(x,t)\right\|_2^{\frac{1}{1-\alpha_1}}.$$

On the other hand, applying Young's inequality with the exponents $\frac{2(1-\alpha_1)}{1-2\alpha_1}$, we have

$$\left[\int_0^l u(x,t)u_t(x,t)dx\right]^{\frac{1}{1-\alpha_1}}\le$$

$$\le c_{21}\left\{\left\|u(x,t)\right\|_{p_1}^{\frac{2}{1-2\alpha_1}}+\left\|u_t(x,t)\right\|_2^2\right\}\le c_{21}\left\{\left\|u(x,t)\right\|_{p_1}^{p_1}+\left\|u_t(x,t)\right\|_2^2\right\}\le$$

$$\le c_{22}\left\{\rho_{p(\cdot)}(u(x,t))+\left\|u_t(x,t)\right\|_2^2\right\}. \tag{96}$$

Taking into account (86), and (96), it follows from (95) that

$$z^{\frac{1}{1-\alpha_1}}(t)\le c_{23}\left\{\int_0^l\left|u_t(x,t)\right|^2dx+\int_0^l\left|u(x,t)\right|^{p(x)}dx\right\}. \tag{97}$$

It follows from (95) and (97) that $z'(t) \geq c_{24} z^{\frac{1}{1-\alpha_1}}(t)$. Thus,

$$z(t) \geq \frac{z_0}{\left[z_0^{\frac{\alpha_1}{1-\alpha_1}} - \frac{c_{24}\alpha_1}{1-\alpha_1} t \right]^{\frac{1-\alpha_1}{\alpha_1}}}.$$

Hence, $\lim_{t \to T'-0} z(t) = +\infty$, where $T' = \frac{1-\alpha_1}{c_{24}\alpha_1} z_0^{\frac{\alpha_1}{1-\alpha_1}}$.

7 | PROOF OF LEMMAS

To prove the lemmas, we use some ideas from[6,7,18,19].

Proof of Lemma 1. Introduce the notation $\Omega_- = \{x : |v(x)| \leq 1\}$, $\Omega_+ = \{x : |v(x)| > 1\}$ and get

$$\rho_{p(\cdot)}(v) \geq \int_{\Omega_-} |v(x)|^{p_2}\, dx + \int_{\Omega_+} |v(x)|^{p_1}\, dx. \tag{98}$$

On the other hand, applying Hölder's inequality, we have

$$\int_{\Omega_-} |v(x)|^{p_2}\, dx \geq l^{\frac{p_1-p_2}{p_1}} \left(\int_{\Omega_-} |v(x)|^{p_1}\, dx \right)^{\frac{p_2}{p_1}}. \tag{99}$$

From (98) and (99) it follows

$$\rho_{p(\cdot)}(v) \geq \int_{\Omega_+} |v(x)|^{p_1}\, dx$$

and

$$l^{\frac{p_2-p_1}{p_2}} \left(\rho_{p(\cdot)}(v) \right)^{\frac{p_1}{p_2}} \geq \int_{\Omega_-} |v(x)|^{p_1}\, dx.$$

Adding these inequalities, we obtain (26).

Proof of Lemma 2. Lemma 2 is a corollary of the definition of $\rho_{p(\cdot)}(\cdot)$.

Proof of Lemma 3. The proof of Lemma 3 is given in the section 3 where the lemma is stated.

Proof of Lemma 4. To prove this lemma, we use some ideas from the proof of Lemma 3 given in[30].

If $\rho(u) > 1$, then it is obvious that

$$(\rho(u))^{\varkappa} \leq \rho(u). \tag{100}$$

Now consider the case $\rho(u) \leq 1$. Then $\rho_{p(\cdot)}(u) \leq 1$ and $|u(l,t)|^q \leq 1$.

Let's suppose that $\|u\|_{C[0,l]} \leq 1$. Then by Lemma 2 and the embedding theorem, we have

$$(\rho_{p(\cdot)}(u))^{\varkappa} \leq L^{\varkappa} \|u\|_{C[0,l]}^{\varkappa p_1}.$$

On the other hand, $\varkappa p_1 > 2$ so

$$(\rho_{p(\cdot)}(u))^{\varkappa} \leq l^{\varkappa} \|u\|_{C[0,l]}^2 \leq l^{\varkappa+1} \left\|u_x\right\|_2^2. \tag{101}$$

As $\varkappa q > 2$ and $|u(l,t)|^q \leq 1$, we have

$$(|u(l,t)|^q)^{\varkappa} \leq |u(l,t)|^{q\varkappa} \leq \|u\|_{C[0,l]}^{q\varkappa} \leq \|u\|_{C[0,l]}^2 \leq l \left\|u_x\right\|_2^2. \tag{102}$$

If $\|u\|_{C[0,l]} > 1$, then by Lemma 2 $\rho_{p(\cdot)}(u) \leq l \|u\|_{C[0,l]}^{p_2}$. On the other hand, $\rho_{p(\cdot)}(u) \leq 1$, hence

$$(\rho_{p(\cdot)}(u))^{\varkappa} \leq (\rho_{p(\cdot)}(u))^{\varkappa \frac{p_1}{p_2}}. \tag{103}$$

From (100), (101), and (103), we obtain

$$(\rho_{p(\cdot)}(u))^{\varkappa} \leq \left(l \|u\|_{C[0,l]}^{p_2} \right)^{\varkappa \frac{p_1}{p_2}} = l^{\varkappa \frac{p_1}{p_2}} \|u\|_{C[0,l]}^{\varkappa p_1} \leq$$

$$\leq l^{\varkappa \frac{p_1}{p_2}} \|u\|^2_{C[0,l]} \leq l^{\varkappa \frac{p_1}{p_2}+1} \left\|u_x\right\|^2_2 . \tag{104}$$

From (102) and (104) we have

$$(\rho(u))^{\varkappa} \leq \left\{\rho_{p(\cdot)}(u) + |u(l,t)|^q\right\}^{\varkappa} \leq c_{25} \left\|u_x\right\|^2_2 . \tag{105}$$

Thus, from (100) and (105) it follows the inequality

$$(\rho(u))^{\varkappa} \leq c_{26} \left[\left\|u_x\right\|^2_2 + \rho(u)\right] . \tag{106}$$

Inequality (38) is proved in a similar way. The proof is complete.

Proof of Lemma 5. Applying Hölder's inequality and Lemma 3, we have

$$\int_0^l |u(x,t)|^r\,dx \leq l^{\frac{p_1-r}{p_1}} \|u(\cdot,t)\|^r_{p_1} \leq c_{27} l^{\frac{p_1-r}{p_1}} \left(\rho_{p(\cdot)}(u(\cdot,t))\right)^{\frac{r}{p_1}} . \tag{107}$$

Further, integrating by parts, we obtain

$$|u(l,t)|^r \leq \frac{1}{l}\int_0^l |u(x,t)|^r\,dx + r\int_0^l |u(x,t)|^{r-1}\left|u_x(x,t)\right| dx. \tag{108}$$

Applying Hölder's inequality with the exponents $\alpha+1$ and $\frac{\alpha+1}{\alpha}$, where $\frac{2}{p_1} < \alpha < 1$, we have

$$\int_0^l |u(x,t)|^{r-1}\left|u_x(x,t)\right| dx \leq$$

$$\leq \frac{\alpha+1}{\alpha}\int_0^l |u(x,t)|^{(r-1)\frac{\alpha+1}{\alpha}}\,dx + (\alpha+1)\int_0^l \left|u_x(x,t)\right|^{\alpha+1} dx. \tag{109}$$

Choosing the Hölder's exponents $\eta_1 = \frac{p_1\alpha}{(r-1)(\alpha+1)}, \eta_1' = \frac{p_1\alpha}{p_1\alpha-(r-1)(\alpha+1)}$, we obtain the inequality

$$\int_0^l |u(x,t)|^{(r-1)\frac{\alpha+1}{\alpha}}\,dx \leq l^{\frac{p_1\alpha-(r-1)(\alpha+1)}{p_1\alpha}}\left(\int_0^l |u(x,t)|^{p_1}\,dx\right)^{\frac{(r-1)(\alpha+1)}{p_1\alpha}} . \tag{110}$$

Similarly, applying Hölder's inequality with the exponents $\eta_2 = \frac{2}{\alpha+1},\ \eta_2' = \frac{2}{1-\alpha}$, we get

$$\int_0^l \left|u_x(x,t)\right|^{\alpha+1} dx \leq l^{\frac{1-\alpha}{2}}\left(\int_0^l \left|u_x(x,t)\right|^2 dx\right)^{\frac{\alpha+1}{2}} . \tag{111}$$

Taking into account (32) - (34), hence we have

$$\int_0^l \left|u_x(x,t)\right|^{\alpha+1} dx \leq c_{28}(\rho(u))^{\frac{\alpha+1}{2}} . \tag{112}$$

By virtue of (109) - (110), it follows from (112) that

$$|u(l,t)|^r \leq c_{29}\left[(\rho(u))^{\frac{r-1}{p_1}} + (\rho(u))^{\frac{\alpha+1}{2}} + (\rho(u))^{\frac{(r-1)(\alpha+1)}{\alpha p_1}}\right].$$

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9 | CONFLICT OF INTEREST

The authors declare that this work does not have any conflicts of interest.

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References

1. Ruzicka M.. *Electrorheological Fluids: Modeling and Mathematical Theory*. Berlin: Lecture Notes in Math., Springer; 1748.
2. Zhikov V.V.. Questions of convergence duality and averaging for functionals of the calculus of variations. *Math. USSR-Izv.* 1984;23(2):243–276.
3. Zhikov VV. Meyer-type estimates for solving the nonlinear Stokes system. *Differtial Equations.*. 1997;33(1):108–115.
4. Antontsev S. Wave equation with $p(x,t)-$Laplacian and damping term: blow-up of solutions. *C.R. Mecanique.* 2011;339(12):751–755.
5. Antontsev S. Wave equation with $p(x,t)-$Laplacian and damping term: existence and blow-up. *Differtial Equations and Applications.* 2011;3(4):503–525.
6. Messaoudi SA, Talahmeh AA. A blow-up result for a nonlinear wave equation with variable-exponent nonlinearities. *Math. Meth. Appl. Sci.*. 2017;96:1509–1515.
7. Sun L, Ren Y, Gao W. Lower and upper bounds for the blow-up time for nonlinear wave equation with variable sources. *Comput. Math. Appl.*. 2016;71(1):267–277.
8. Ball J. Remarks on blow - up and nonexistence theorems for nonlinear evolutions equations. *Quart. J. Math. Oxford.* 1977;28(2):473–486.
9. Kalantarov VK, Ladyzhenskaya OA. The occurrence of collapse for quasilinear equations of parabolic and hyperbolic types. *Jour. Soviet. Math.*. 1978;10(1):53–70.
10. Levine HA. Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + F(u)$,. *Trans. Amer. Math. Soc.*. 1974;192:1–21.
11. Levine HA. Some additional remarks on the nonexistence of global solutions to nonlinear wave equations. *SIAM Jour. Math. Anal.*. 1974;5(1):138–146.
12. Georgiev V, Todorova G. Existence of a solution of the wave equation with nonlinear damping and source term. *Jour. Differ. Equ.*. 1994;109(2):295–308.
13. Levine HA, Park SR, Serrin J. Global existence and global nonexistence of solutions of the Cauchy problem for a nonlinearly damped wave equation. *Journal of Math. Anal. Appl.*. 1998;228(1):181–205.
14. Levine HA, Serrin J. Global nonexistence theorems for quasilinear evolution equations with dissipation. *Arch. Rat. Mech. Anal.*. 1997;137(4):341–361.
15. Korpusov MO. Blow-up of the solution of a nonlinear system of equations with positive energy. *Theoret. and Math. Phys.*. 2012;171(3):725—738.
16. Bilgin BA, Kalantarov VK. Blow up of solutions to the initial boundary value problem for quasilinear strongly damped wave equations. *J. Math. Anal. Appl.*. 2013;403(1):89–94.

17. Aliev AB, Yusifova GI. Nonexistence of global solutions of Cauchy problems for systems of semilinear hyperbolic equations with positive initial energy. *Electronic Journal of Differential Equations.* 2017;2017(211):1-10.

18. Rauch Jeffrey. *Hyperbolic Partial Differential Equations and Geometric Optics*. Rhode Island, Graduate Studies in Math.: American Mathematical Society Providence; 2012.

19. Aliev AB. A mixed problem with dissipation for second order quasilinear hyperbolic equations. *Doklady Akademii Nauk.* 1986;288(6):1289-1292.

20. Aliev AB, Khanmamedov AK. Energy estimates for solutions of the mixed problem for linear second – order hyperbolic equations. *Mathematical Notes.* 1996;59(4):345-349.

21. Chueshov I, Eller M, Lasiecka I. On the attractor for a semilinear wave equation with critical exponent and nonlinear boundary dissipation. *Communications in Partial Differential Equations.* 2002;27(9):1901–1951.

22. Trigiani R. Wave Equation on a Bounded Domain with Boundary Dissipation: An Operator Approach. *Journal of Mathematical Analysis and Applications.* 1964;137:438–461.

23. Lasiecka I, Tataru D. Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping. *Differential and Integral Equations.* 1993;6(3):507–533.

24. Cavalcanti MM, Domingos Cavalcanti VN, Martinez P. Existence and decay rate estimates for the wave equation with nonlinear boundary damping and source term. *Trans. Moscow Math. Soc.*. 2004;203(1):119-158.

25. Bociu L, Rammaha M, Toundykov D. On a wave equation with supercritical interior and boundary sources and damping terms. *Mathematische Nachrichten.* 2011;284(16):2032–2064.

26. Vitillaro E. Global existence for the wave equation with nonlinear boundary damping and source terms. *Journal of Differential Equations.* 2002;186(1):259–298.

27. Cavalcanti MM, Domingos Cavalcanti VN, Lasiecka I. Well - posedness and optimal decay rates for the wave equation with nonlinear boundary damping – source interaction. *Journal of Differential Equations.* 2007;236(2):407–459.

28. Feng H., Li S., Zhi X.. Blow-up solutions for a nonlinear wave equation with boundary damping and interior source. *Nonlinear Analysis.* 2012;75(4):2273–2280.

29. Liu Wenjun, Sun Yun, Li Gang. Blow-up of solutions for a nonlinear wave equation with nonnegative initial energy. *Electronic Journal of Differential Equations.* 2013;2013(115):1–8.

30. Aliev AB, Shafieva GKh. Blow-up of solutions of a mixed problem for systems of wave equations with boundary dissipation and with an interior nonlinear focusing source of variable growth order. *Differential Equations.* 2021;57(3):291–303.

31. Showalter RE. *Monotone Operators in Banach Spaces and Nonlinear Partial Differential Equations*. USA: AMS: Providence; 1997.

32. Lions JL. *Quelques Methodes deResolution des Problemes Aux Limites Non Lineaires*. Paris, Dunod: Etudes Mathematiques; 1969.

33. Lars D, Harjulehto P, Hasto P, Ruzicka M. *Lebesgue and Sobolev Spaces with Variable Exponents*. Springer Berlin, Heidelberg: Lect. Notes Math.; 2011.

34. Almeida A, Samko S. Embeddings of variable Hajłasz–Sobolev spaces into Holder spaces of variable order. *J. Math. Anal. Appl.*. 2009;353(2):489–496.

35. Fan X, Zhao D. On the spaces $L_{p(x)}(\Omega)$ and $W_{m,p(x)}(\Omega)$. *J. Math. Anal. Appl.*. 2001;263(2):424–446.

36. Kováčik Ondrej, Rákosník Jiří. On spaces $L^{p(x)}$ and $W^{m,p(x)}$. *Czechoslovak Mathematical Journal.* 1991;41(4):592–618.

37. Efendiev MA. *Attractors for Degenerate Parabolic type Equations.* USA: Providence, Rhode Island: American Mathematical Society: Mathematical Surveys and Monographs; 2013.

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