A New Analyzing Method for Hyperbolic Telegraph Equation

Imtiaz Ahmad[1], Ahmed E. Abouelregal[2], Hijaz Ahmad[3]*, Phatiphat Thounthong[4] and M. Abdel-Aty[*5]

[1]Department of Mathematics, University of Swabi, Swabi 23430, Khyber Pakhtunkhwa, Pakistan

[2]Department of Mathematics, College of Science and Arts, Jouf University, Al-Qurayyat, Saudi Arabia

[3]Department of Basic Sciences, University of Engineering and Technology Peshawar, Pakistan

[4]Renewable Energy Research Centre, Department of Teacher Training in Electrical Engineering, Faculty of Technical Education, King Mongkut's University of Technology North Bangkok, 1518 Pracharat 1 Road, Bangsue, Bangkok 10800 Thailand

[5]Department of Mathematics, Faculty of Science, Sohag University, Sohag, Egypt

Abstract We proposed an efficient local differential quadrature method which is based on the radial basis function to the numerical solution of the two-dimensional second-order hyperbolic telegraph equations. The explicit time integration technique is utilized to semi-discretize the model in the time direction whereas the space derivatives of the model is discretized by the proposed local meshless procedure based on multiquadric radial basis function. Numerical experiments on five test problems are performed with the proposed numerical scheme for rectangular and non-rectangular computational domains. The results obtained show that the proposed scheme solutions are converging extremely faster comparable to different existing protocols.

Keywords: Meshless differential quadrature method; Radial basis function; Hyperbolic telegraph equation; Irregular domain.

1 Introduction

The telegraph equation, which has been used to describe phenomena in various fields, belongs to the hyperbolic partial differential equation scope. For instance, the two-dimensional (2D) second-order hyperbolic telegraph equations can model different real world phenomena in sciences and engineering and furthermore has many applications in different fields [1]. The generalized 2D second-order hyperbolic telegraph equation have the following form

$$\frac{\partial^2 W(\overline{\mathbf{z}},t)}{\partial t^2} + 2\alpha\frac{\partial W(\overline{\mathbf{z}},t)}{\partial t} + \beta^2 W(\overline{\mathbf{z}},t) - \delta\left(\frac{\partial^2 W(\overline{\mathbf{z}},t)}{\partial x^2} + \frac{\partial^2 W(\overline{\mathbf{z}},t)}{\partial y^2}\right) = F(\overline{\mathbf{z}},t), \quad \overline{\mathbf{z}} \in \Omega, \quad t > 0, \tag{1}$$

with initial-boundary conditions

$$W(\overline{\mathbf{z}},0) = g_1(\overline{\mathbf{z}}), \quad \frac{\partial W(\overline{\mathbf{z}},0)}{\partial t} = g_2(\overline{\mathbf{z}}), \quad \mathcal{C}(\overline{\mathbf{z}},t) = g_3(\overline{\mathbf{z}},t), \quad \overline{\mathbf{z}} \in \partial\Omega, \tag{2}$$

where $\alpha > 0$, β and δ are known coefficients and $F(\overline{\mathbf{z}},t)$ is the source function.

It is well-known that it is difficult to get the analytical solutions for relatively complex problems [2,3]. Thus the approximate numerical approximations to the telegraph equation is a better choice. Different numerical techniques have been created and compared to deal with the hyperbolic telegraph equations during the past two decades. In the literature, a three-level implicit unconditionally stable numerical method for hyperbolic equation [4], multi-dimensional telegraphic equations [5]. The authors in [6] proposed a Taylor

*Corresponding authors; Email: hijaz555@gmail.com (H. Ahmad) and amisaty@gmail.com (M. Abdel-Aty)

matrix method for the solution of 2D linear hyperbolic equation. In [7], a variational iteration method is used for multi space telegraph equations. The numerical polynomial differential quadrature (DQ) method has been developed for 2D hyperbolic telegraph equation [8], modified and modified extended cubic B-Spline DQ methods [9, 10]. Hafez [11] proposed a spectral method for the numerical solutions of one- and two-dimensional linear telegraph equations. Recently, different approaches for the numerical solutions have been discussed [12, 13].

More recently, meshless techniques have seen the exploration blast in science and engineering. For this branches, Dehghan and his coworkers proposed several meshless methods to investigate 2D hyperbolic telegraph equation which include the implicit collocation method [11], the thin plate splines radial basis functions (RBFs) [14], the local Petrov-Galerkin method [15] and the boundary knot method [16]. Recently, the 2D telegraph equation in regular and irregular domains are investigated by RBF with finite difference scheme [17], the pseudospectral RBFs method [18] and the RBFs with Crank-Nicolson finite difference scheme [19]. By using the Houbolt method, the 2D hyperbolic telegraph equation are solved by the singular boundary method [20] and by a hybrid meshless method [21]. Reutskiy et al. [22] proposed a cubic B-spline method based on finite difference and meshless approaches for solving two-dimensionals generalized telegraph equations in irregular single and multi-connected domains.

Based on the above-mentioned investigations, we propose a local meshless differential quadrature method (LMM) for two-dimensional hyperbolic telegraph equations. The local meshless procedures produces sparse matrix which dose not have the typical ill-conditioning, additionally, this sparse matrix can be solved efficiently and accurately, detail can be found in [23–26]. In current work, the space derivatives of the model equation are approximated by the proposed meshless methodology utilizing multiquadric (MQ) radial basis functions (RBFs) whereas explicit scheme is utilize for time derivatives. To check the performance of the method on both rectangular and non-rectangular domains are considered in numerical examinations.
The rest of this paper is organized as follows. Section 2, present the methodology of the propose procedure. We present numerical results and discussion for the model equations using the proposed procedure. Several test problems are presented to validate the accuracy of the proposed algorithm in Section 3. Some conclusions are given in Section 4 with some additional remarks.

2 Proposed Methodology

The LMM [27, 28] is extended to the two-dimensional hyperbolic telegraph model equation. The derivatives of $W(\overline{\mathbf{z}}, t)$ are approximated at the centers $\overline{\mathbf{z}}_h$ by the neighborhood of $\overline{\mathbf{z}}_h$, $\{\overline{\mathbf{z}}_{h1}, \overline{\mathbf{z}}_{h2}, \overline{\mathbf{z}}_{h3}, ..., \overline{\mathbf{z}}_{hn_h}\} \subset \{\overline{\mathbf{z}}_1, \overline{\mathbf{z}}_2, \ldots, \overline{\mathbf{z}}_{N^n}\}, n_h \ll N^n$, where $h = 1, 2, \ldots, N^n$. In case of one-dimensional (1D), $\overline{\mathbf{z}} = x$ and for two-dimensional (2D), $\overline{\mathbf{z}} = (x, y)$ and for three-dimensional (3D) case, $\overline{\mathbf{z}} = (x, y, z)$. Now in 1D case, we have

$$W^{(m)}(x_h) \approx \sum_{k=1}^{n_h} \lambda_k^{(m)} W(x_{hk}),\ h = 1, 2, \ldots, N. \tag{3}$$

By substituting radial basis function (RBF) $\psi(\|x - x_p\|)$ in Equation (3), we have

$$\psi^{(m)}(\|x_h - x_p\|) = \sum_{k=1}^{n_h} \lambda_{hk}^{(m)} \psi(\|x_{hk} - x_p\|),\ p = h1, h2, \ldots, hn_h, \tag{4}$$

where $\psi(\|x_{hk} - x_p\|) = \sqrt{1 + (c\|x_{hk} - x_p\|)^2}$ in case of multiquadric RBFs respectively.

Matrix form of Equation (4) can be written as

$$\boldsymbol{\psi}_{n_h}^{(m)} = \mathbf{A}_{n_h} \boldsymbol{\lambda}_{n_h}^{(m)}, \tag{5}$$

From Equation (5), we obtain

$$\boldsymbol{\lambda}_{n_h}^{(m)} = \mathbf{A}_{n_h}^{-1} \boldsymbol{\psi}_{n_h}^{(m)}. \tag{6}$$

Equation (3) and (6) implies

$$W^{(m)}(x_h) = (\boldsymbol{\lambda}_{n_h}^{(m)})^T \mathbf{W}_{n_h}, \quad \text{where} \quad \mathbf{W}_{n_h} = [W(x_{h1}), W(x_{h2}), \ldots, W(x_{hn_h})]^T.$$

For 2D case, the derivatives of $W(x, y, t)$ with respect to x and y are approximated in the similar way as follows

$$W_x^{(m)}(x_h, y_h) \approx \sum_{k=1}^{n_h} \gamma_k^{(m)} W(x_{hk}, y_{hk}),\ h = 1, 2, \ldots, N^2, \tag{7}$$

$$W_y^{(m)}(x_h, y_h) \approx \sum_{k=1}^{n_h} \eta_k^{(m)} W(x_{hk}, y_{hk}),\ h = 1, 2, \ldots, N^2. \tag{8}$$

For corresponding coefficients $\gamma_k^{(m)}$ and $\eta_k^{(m)}$ $(k = 1, 2, \ldots, n_h\ )$, we continue as

$$\boldsymbol{\gamma}_{n_h}^{(m)} = \mathbf{A}_{n_h}^{-1} \mathbf{\Phi}_{n_h}^{(m)}, \tag{9}$$

$$\boldsymbol{\eta}_{n_h}^{(m)} = \mathbf{A}_{n_h}^{-1} \mathbf{\Phi}_{n_h}^{(m)}. \tag{10}$$

The above technique can be rehashed for three-dimensional case, etc. Using the above procedure to the model Equation (1) in space, convert it to the second order ordinary differential equation (ODE) which is further reduce to first order ODE by substituting $W_t(\overline{\mathbf{z}}, t) = V(\overline{\mathbf{z}}, t)$ as follows

$$W_t = V, \quad V_t = F(\overline{\mathbf{z}}, t) - 2\alpha V - \beta^2 W + \delta \left(W_{xx} + W_{yy}\right), \tag{11}$$

with initial and boundary conditions

$$W(\overline{\mathbf{z}}, 0) = f_1(\overline{\mathbf{z}}), \quad V(\overline{\mathbf{z}}, 0) = f_2(\overline{\mathbf{z}}), \quad \overline{\mathbf{z}} \in \Omega, \tag{12}$$

$$W(\overline{\mathbf{z}}, t) = f_3(t), \quad V(\overline{\mathbf{z}}, t) = f_4(t), \quad \overline{\mathbf{z}} \in \partial\Omega, \quad t > 0. \tag{13}$$

Now, Employing the proposed procedure to Equations (11)-(13) in space at each nodal point, we get the accompanying type of an initial value problem

$$\begin{aligned} &\frac{d\mathbb{W}}{dt} = \mathbb{V}, \quad \frac{d\mathbb{V}}{dt} = \mathbf{h}(t) - 2\alpha\mathbb{V} + \mathcal{A}\mathbb{W}, \\ &\mathbb{W}(0) = \mathbf{f_1}, \ \mathbb{V}(0) = \mathbf{f_2}, \end{aligned} \tag{14}$$

where $\mathcal{A}$ is the sparse coefficient matrix of order $N^n \times N^n$. The vectors $\mathbf{f_1}$ and $\mathbf{f_2}$ denote the corresponding initial conditions and $\mathbf{h}$ is the boundary condition of the problem. Orders of the vectors $\mathbf{f_1}$, $\mathbf{f_2}$ and $\mathbf{h}$ are $N^n \times 1$.

3 Numerical Results

In this section, five test problems are considered to validate the local meshless method (LMM). For fair comparison with the other numerical methods, we use the maximum absolute error ($Max(\varepsilon)$) and root mean square error (RMS). In this computational process, the LMM using local stencil five with uniform set of distributed points. Multiquadric (MQ) radial basis function is used with shape parameter value $c = 0.1$, computational domain $[0, 1]$ with $N^2 = 30$ and time step size $dt = 0.0001$ in all computational experiments unless mentioned explicitly. The $Max(\varepsilon)$ and RMS errors are defined as

$$Max(\varepsilon) = \max(|\widehat{\mathbb{W}} - W|), \quad RMS = \sqrt{\frac{\sum_{i=1}^{N} \left(\widehat{\mathbb{W}}_i - W_i\right)^2}{N}}, \tag{15}$$

where $\widehat{\mathbb{W}}$, W are the exact and the approximate solutions respectively.

Test Problem 1. In the first place, we consider the model Equation (1) with $\alpha = \beta = \delta = 1$ having analytical/exact solution

$$W(\overline{\mathbf{z}}, t) = \cos(t)\sin(x)\sin(y), \quad \overline{\mathbf{z}} = (x, y) \in \Omega, \tag{16}$$

Table 1: Numerical results for Test Problem 1.

$Max(\varepsilon)$							
	t	LMM	[8]	[9]	[10]	[17]	[18]
	0.5	4.2516e-06	$\cdots$	$\cdots$	$\cdots$	3.11e-05	4.59e-05
	1	8.8869e-07	$\cdots$	2.27e-03	4.57e-06	1.01e-05	2.51e-05
	2	1.7723e-07	$\cdots$	2.87e-03	5.61e-06	8.25e-06	2.41e-05
RMS							
	0.5	1.6001e-06	3.24e-06	$\cdots$	$\cdots$	7.82e-06	7.15e-05
	1	3.4794e-07	4.27e-06	5.98e-03	2.47e-05	2.07e-05	5.45e-05
	2	6.8431e-08	3.94e-06	8.50e-03	3.83e-05	3.01e-06	8.13E-05

Table 2: Numerical results with time $t = 1$ for Test Problem 1.

	$N^2 = 10$		$N^2 = 20$		$N^2 = 30$	
dt	$Max(\varepsilon)$	RMS	$Max(\varepsilon)$	RMS	$Max(\varepsilon)$	RMS
0.01	2.5775e-04	1.1509e-04	2.5593e-04	1.2922e-04	2.5576e-04	1.3297e-04
0.001	1.0891e-05	3.6808e-06	2.2312e-05	1.1151e-05	2.4397e-05	1.2806e-05
0.0002	1.5146e-05	7.4161e-06	1.3030e-06	5.4194e-07	3.4410e-06	1.6766e-06
0.0001	1.7396e-05	8.6833e-06	2.7746e-06	1.2885e-06	8.8869e-07	3.4794e-07

where

$$F(\bar{\mathbf{z}}, t) = 2\left(\cos(t) - \sin(t)\right)\sin(x)\sin(y). \tag{17}$$

The numerical results of the LMM in term of RMS and $Max(\varepsilon)$ errors are listed in Table 1 for different times $t = 0.5$, $t = 1$, $t = 2$. We have also compared our results with the ones reported in [8–10, 17, 18]. However, all the RMS and $Max(\varepsilon)$ of the LMM perform better than the other numerical methods and the aftereffects of the LMM are in amazing agreement with the exact solutions too. Table 2 shows the numerical results determined by taking different values of N, time step size dt and final time $t = 1$. It very well may be seen from the table that the accuracy increase by increasing N and decreasing dt to some extend. Figure 1 shows the numerical results in term of absolute error for $t = 5$ and $t = 10$ which is superb.

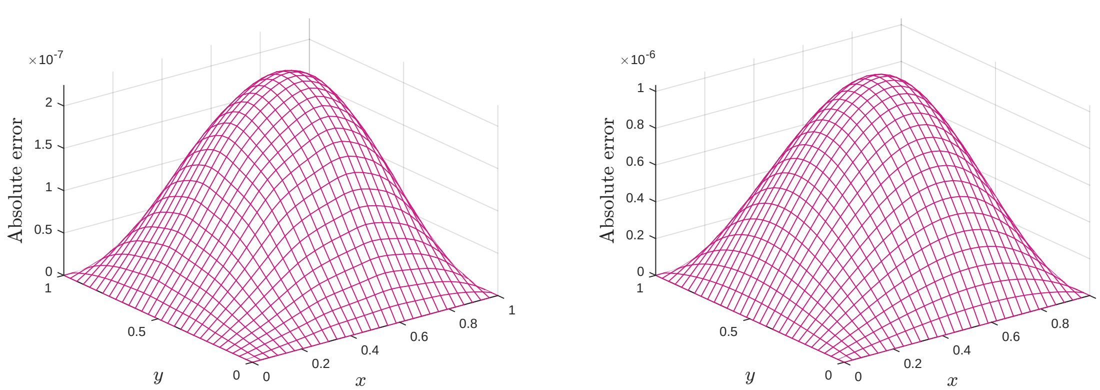


Figure 1: Absolute error for $t = 5$ (left) and $t = 10$ (right) for Test Problem 1.

Table 3: Numerical results for Test Problem 2.

	t	LMM	[9]	[10]	[17]
$Max(\varepsilon)$					
	0.5	5.4956e-05	9.51e-03	2.36e-04	1.16e-03
	1	4.3474e-05	7.47e-03	1.78e-04	9.25e-04
	2	1.5112e-05	1.04e-03	2.39e-05	2.75e-04
RMS					
	0.5	2.3254e-05	8.42e-05	$\cdots$	7.10e-04
	1	2.5324e-05	1.29e-04	$\cdots$	4.36e-04
	2	6.9656e-06	3.10e-05	$\cdots$	7.23e-05

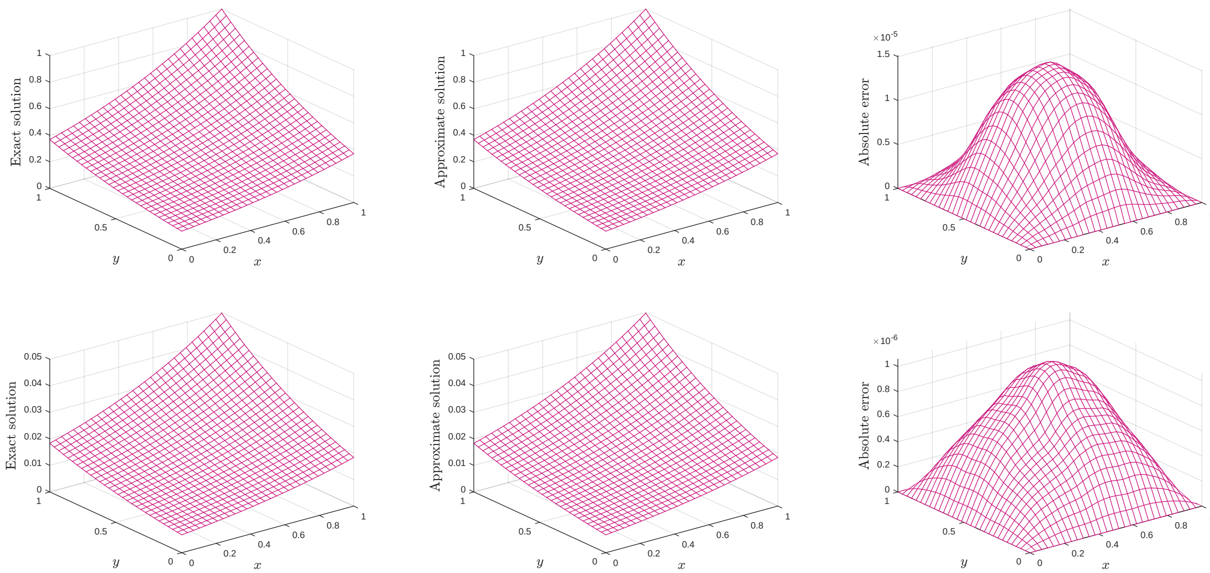


Figure 2: Results of the LMM for $t = 2$ (first row), $t = 5$ (second row) for Test Problem 2.

Test Problem 2. Consider the hyperbolic telegraph Equation (1) with $\alpha = \beta = \delta = 1$ having analytical/exact solution

$$W(\overline{\mathbf{z}}, t) = \exp(x + y - t), \quad \overline{\mathbf{z}} = (x, y) \in \Omega, \tag{18}$$

where

$$F(\overline{\mathbf{z}}, t) = -2\exp(x + y - t). \tag{19}$$

Similar to the previous test problem, the results in term of RMS and $Max(\varepsilon)$ errors are shown in Table 3 and the results of the LMM are compared with the results given in [9,10,17,29–31]. It is observed from the table, that the results of the LMM consistent with the exact solutions and the error norm remains stable around $Max(\varepsilon) \approx 10^{-5}$ for all times tested. Figure 2 shows exact solutions, approximate solutions and absolute error norm, which are self explanatory.

Test Problem 3. Here, we consider the hyperbolic telegraph equation of the form (1) with $\alpha = \beta = \delta = 1$ having analytical/exact solution

$$W(\overline{\mathbf{z}}, t) = \ln(1 + x + y + t), \quad \overline{\mathbf{z}} = (x, y) \in \Omega, \tag{20}$$

where

$$F(\overline{\mathbf{z}}, t) = 1/(1 + x + y + t) + \ln(1 + x + y + t) + 1/(1 + x + y + t)^2. \tag{21}$$

Table 4: Numerical results for Test Problem 3.

	t	LMM	[9]	[10]	[17]
$Max(\varepsilon)$					
	0.5	8.1136e-06	2.47e-03	8.18e-05	3.87e-05
	1	3.8719e-06	3.31e-03	9.35e-05	2.55e-05
	2	1.9407e-06	1.14e-03	4.24e-05	1.98e-05
	3	2.3478e-07	4.36e-04	1.79e-05	2.09e-05
	5	3.1310e-07	3.48e-04	1.08e-05	2.55e-05
RMS					
	0.5	3.1807e-06	1.11e-03	4.50e-05	1.13e-05
	1	2.1918e-06	1.33e-03	5.81e-05	1.76e-05
	2	8.9983e-07	3.20e-04	1.89e-05	8.93e-06
	3	9.3378e-08	1.30e-04	6.58e-06	1.14e-05
	5	1.5839e-07	8.42e-05	3.65e-06	1.42e-05

Table 5: Numerical results for $\alpha = 10$, $\beta = 5$ and $\delta = 1$ for Test Problem 4.

	t	LMM	[8]	[9]	[18]
$Max(\varepsilon)$					
	0.5	8.6758e-06	$\cdots$	2.47e-04	7.13e-05
	1	7.1653e-06	$\cdots$	3.31e-04	3.73e-06
	2	3.0534e-06	$\cdots$	1.14e-05	2.51e-05
RMS					
	0.5	3.5821e-06	3.30e-05	1.11e-04	3.01e-05
	1	3.2568e-06	3.23e-05	1.33e-04	1.80e-05
	2	1.4538e-06	3.12e-05	3.20e-04	1.20e-05

Table 4 contains a comparison of numerical results in term of RMS and $Max(\varepsilon)$ error for the LMM to those obtained by the other numerical methods [9, 10, 17] for different times $t = 0.5$, $t = 1$, $t = 2$, $t = 3$ and $t = 5$. It is observed from the table, that the results of the LMM are more accurate than the other numerical methods. Numerical results in the form of absolute error is shown in Figure 3 for different N. It tends to be seen that the accuracy increases with the increase in N.

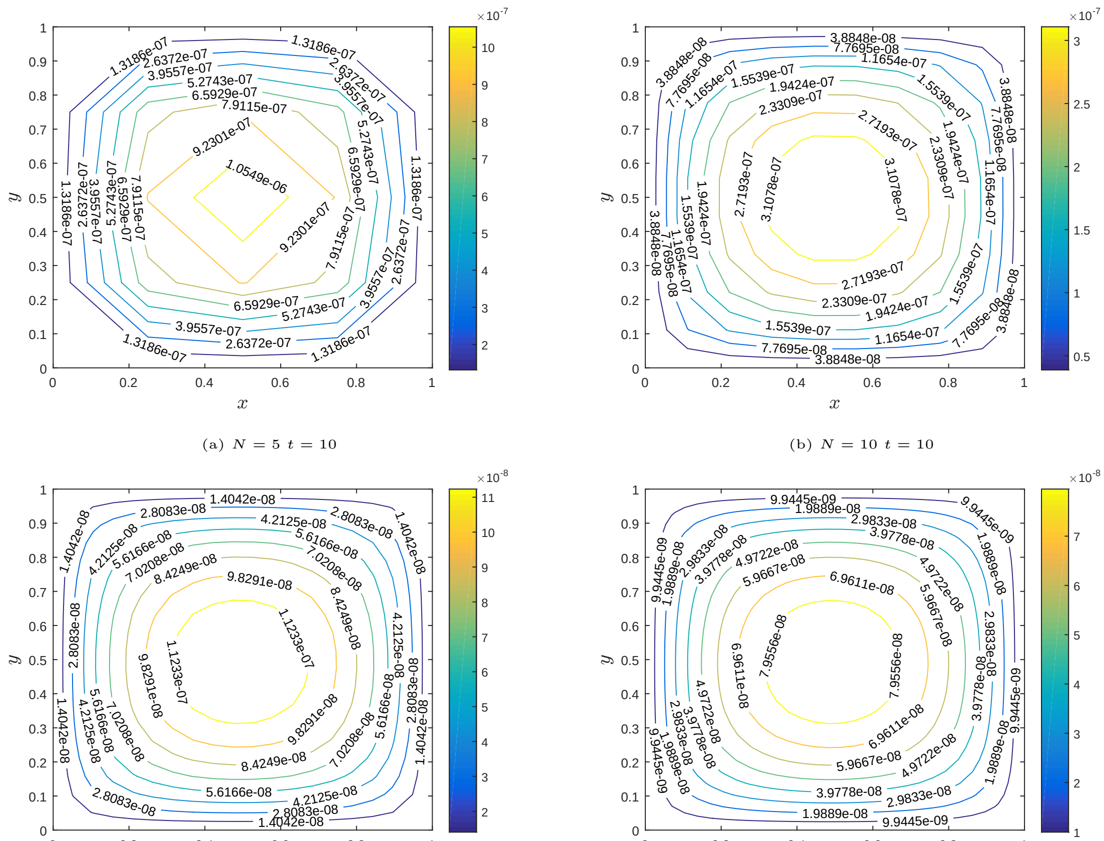
Test Problem 4. Let's consider the equation of the form (1) having analytical/exact solution

$$W(\bar{\mathbf{z}}, t) = \exp(-t)\sinh(x)\sinh(y), \quad \bar{\mathbf{z}} = (x, y) \in \Omega, \tag{22}$$

where

$$F(\bar{\mathbf{z}}, t) = (-2\alpha + \beta^2 - 1)\exp(-t)\sinh(x)\sinh(y). \tag{23}$$

Tables 5-6 present the RMS and $Max(\varepsilon)$ error norms for $\alpha = 10$, $\beta = 5$ and $\alpha = 10$, $\beta = 0$ respectively. We have compared our results with the the revealed results in [8, 9, 18]. From these tables, we can see that numerical results of the LMM for times $t = 0.5$, $t = 1$ and $t = 2$ are better than the other numerical methods. The LMM is also checked on non-rectangular domain as appeared in Figures 4-5 up to final time $t = 5$. It is obvious from these figures that the LMM gives good numerical results irrespective of the domain.



(a) $N = 5$ $t = 10$

(b) $N = 10$ $t = 10$

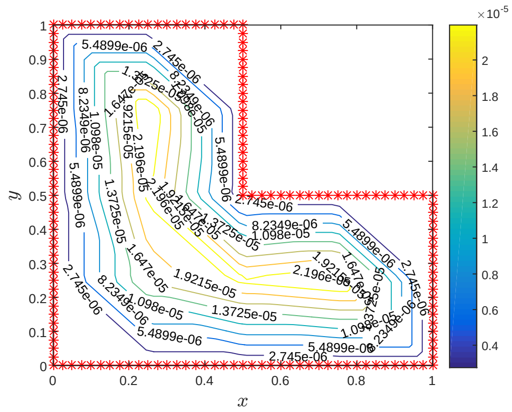
(a) $N = 20$ $t = 10$

(b) $N = 30$ $t = 10$

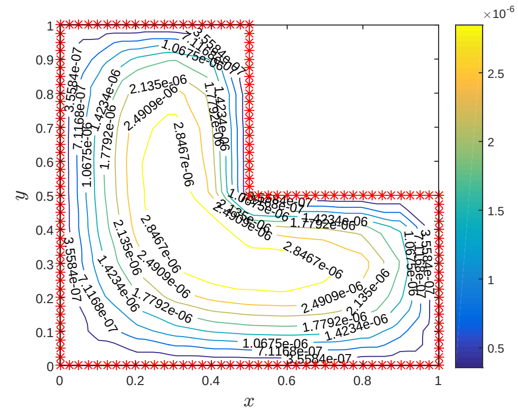
Figure 3: Numerical results for Test Problem 3.

Table 6: Numerical results for $\alpha = 10$, $\beta = 0$ and $\delta = 1$ for Test Problem 4.

$Max(\varepsilon)$					
	t	LMM	[8]	[9]	[18]
	0.5	1.0771e-05	$\cdots$	4.23e-4	9.52e-5
	1	1.1722e-05	$\cdots$	2.58e-04	9.74e-05
	2	7.8558e-06	$\cdots$	9.58e-05	8.45e-05
RMS					
	0.5	4.4207e-06	3.31e-5	3.47e-4	3.91e-5
	1	5.3563e-06	3.34e-05	3.91e-04	4.35e-05
	2	3.8287e-06	3.41e-05	4.27e-04	3.07e-05

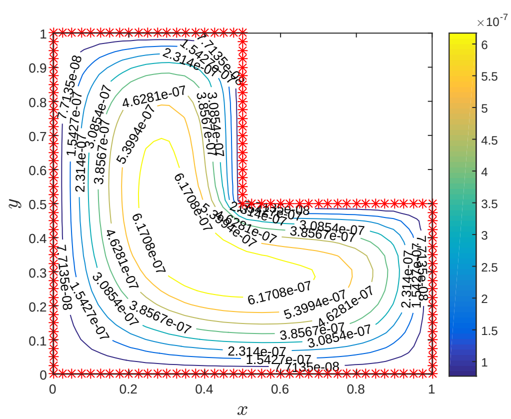


(a) $N = 21$ $t = 1$

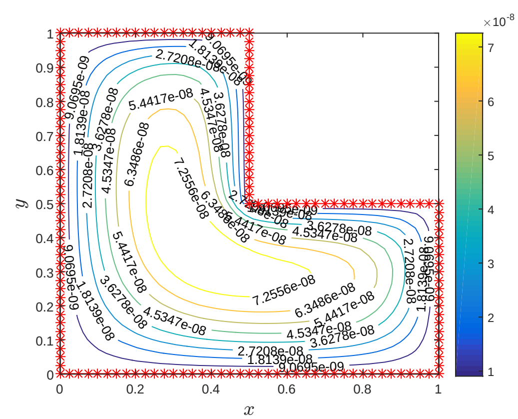


(b) $N = 96$ $t = 2$

Figure 4: Computational domain ("$*$" red line) and the absolute error for $\alpha = 10$, $\beta = 0$ and $\delta = 1$ for Test Problem 4.



(a) $N = 341$ $t = 3$



(b) $N = 736$ $t = 5$

Figure 5: Computational domain ("$*$" red line) and the absolute error for $\alpha = 10$, $\beta = 0$ and $\delta = 1$ for Test Problem 4.

Test Problem 5. Let's consider the model equation of the form (1) having analytical/exact solution

$$W(\bar{\mathbf{z}}, t) = \cos(t)\sinh(x)\sinh(y), \quad \bar{\mathbf{z}} = (x, y) \in \Omega, \tag{24}$$

where

$$F(\bar{\mathbf{z}}, t) = (-3\cos(t) - 2\alpha\sin(t) + \beta^2\cos(t))\sinh(x)\sinh(y). \tag{25}$$

Figure 6 shows the comparative results of the proposed method and the results reported in [8,9]. As can be seen from this figure, numerical results for times $t = 0.5$, $t = 1$ and $t = 2$ are better than the other numerical methods. In this test problem again we have testified the LMM on non-rectangular domains as shown in Figures 7-8 up to final time $t = 5$. These results indicate that the LMM gives good numerical results irrespective of the domains.

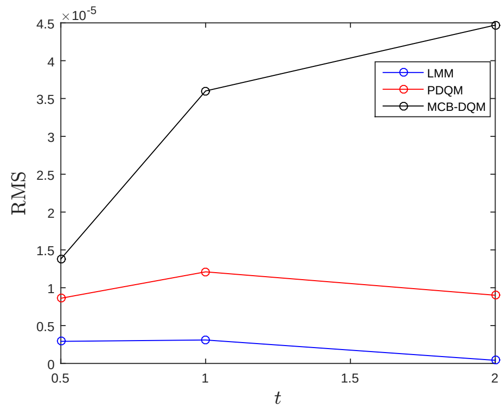


Figure 6: Numerical results of the proposed LMM, PDQM [8] and MCB-DQM [9] for $\alpha = 10$, $\beta = 5$ and $\delta = 1$ for Test Problem 5.

4 Conclusions

In the current research, an efficient and accurate computational technique named local meshless differential quadrature method based on radial basis functions to investigate for the two-dimensional second-order hyperbolic telegraph equations. The time derivative part is discretized explicitly. Test problems have been considered on rectangular and non-rectangular domains to check the accuracy of the proposed scheme. From the numerical results, we can conclude that the proposed meshless method is flexible and accurate as compared to other numerical methods. In light of the current work, we can say that the proposed technique is powerful and effective to find hyperbolic PDEs, so it can be also applied for a large-scale of complex problems that occur in natural sciences and engineering.

Conflicts of Interest

No conflict of interest.

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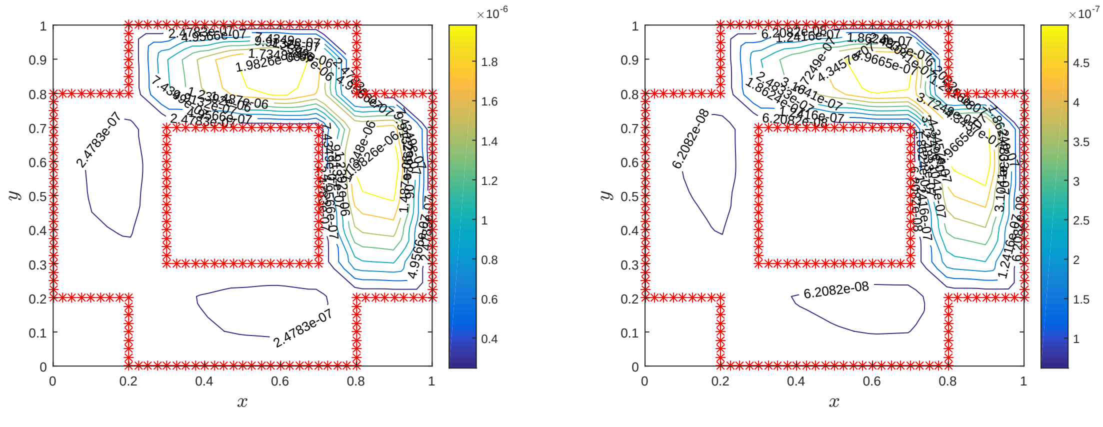
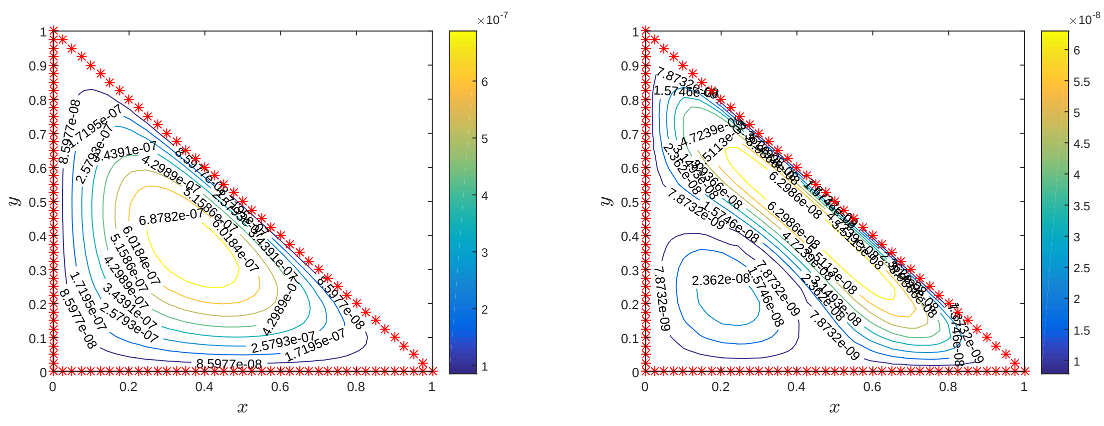


(a) $N = 96$ $t = 1$ (b) $N = 96$ $t = 5$

Figure 7: Computational domain ("∗" red line) and the absolute error for $\alpha = 10$, $\beta = 5$ and $\delta = 1$ for Test Problem 5.



(a) $N = 231$ $t = 1$ (b) $N = 231$ $t = 5$

Figure 8: Computational domain ("∗" red line) and the absolute error for $\alpha = 10$, $\beta = 5$ and $\delta = 1$ for Test Problem 5.

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