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RESEARCH ARTICLE

Analytical solution of integro-differential equations describing the process of intense boiling of a superheated liquid

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In this article, an approximate analytical solution of an integro-differential system of equations is constructed, which describes the process of intense boiling of a superheated liquid. The kinetic and balance equations for the bubble-size distribution function and liquid temperature are solved analytically using the Laplace transform and saddle-point methods with allowance for an arbitrary dependence of the bubble growth rate on temperature. The rate of bubble appearance therewith is considered in accordance with the Dering-Volmer and Frenkel-Zeldovich-Kagan nucleation theories. It is shown that the initial distribution function decreases with increasing the dimensionless size of bubbles and shifts to their greater values with time.

KEYWORDS:
integro-differential model, phase transitions, intense boiling, applied mathematical modeling

1 | INTRODUCTION

It is well known that the phase and structural transformations, which are widely encountered in various natural phenomena and numerous technological processes, completely determine the dynamics of the system, and are responsible for its properties, structure and the final result (state of the system) after the phase transition.[1–10] As this takes place, the dynamics of a metastable system is determined by the intensity of the formation of nuclei of a new phase (for example, centers of crystallization or vaporization, the nucleation of which depends on the intensity of fluctuations). We also especially note that the dynamic behavior of a system is often sensitive to changes in its parameters responsible for the formation of different evolutionary scenarios (for example, self-oscillations and instability).[11–19] This, in particular, explains the need to develop approximate analytical methods for analyzing such systems.

Mathematical models of such processes of structural-phase transformations consist of a kinetic equation for the particle size distribution function and a balance equation (for temperature, concentration of a dissolved impurity, etc.).[20–26] Such a system of equations is integro-differential and depends on the kinetics of the evolution of nuclei. Note that the rate of growth/reduction of nuclei, generally speaking, is a solution to a separate problem with moving boundaries.[27–29] Therefore, there are no general methods for solving such a closed model of a process with moving boundaries of phase transformations. The solution to each problem is a separate study, which often requires the development of special mathematical methods.

This study is devoted to the development of an analytical solution to an integro-differential model of intense boiling of a liquid. For simplicity of the model, we assume that the liquid is homogeneous in all spatial directions, and its temperature depends only on the time variable t. For simplicity of the model, we also assume that the properties of the liquid and the radius of critical nuclei are constant throughout the volume, and the bulk concentration of bubbles is assumed to be insignificant. The solution of the integro-differential model under consideration is based on the Laplace integral transform method and the saddle point technique used to approximate the Laplace-type integral. The applied method was previously used for the analytical description of phase transformations in supercooled and supersaturated liquids.[30–34]

This article is organized as follows. The system of integro-differential equations supplemented with the corresponding initial and boundary conditions is given in Section 2. Its analytical solutions are presented in Section 3. Our main conclusions are discussed in Section 4.

2 | THE MODEL

Let us introduce the bubble-size distribution function $f(r,t)$ and current temperature $T(t)$ of the liquid, which describe the time-dependent state of intense boiling. Here r and t designate the bubble radius and time. The distribution function satisfies the kinetic equation

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial r}\left(\frac{dr}{dt} f\right) + \gamma f = 0,\ r > r_*,\ t > 0, \tag{1}$$

where the number of withdrawing bubbles is considered to be proportional to the concentration of bubbles with a constant coefficient γ. Here dr/dt represents the growth rate of bubbles and r_* is the critical radius of nucleating bubbles capable to further growth. The flux of such bubbles is equal to the rate J of their appearance, i.e.

$$\frac{dr}{dt} f = J\left(\frac{T - T_0}{T_0}\right),\ r = r_*, \tag{2}$$

where T_0 stands for the boiling temperature.

The temperature balance equation depends on the bubble-size distribution function and heat exchange with external medium of temperature T_m and takes the form

$$\rho c \frac{dT}{dt} = \alpha(T_m - T) - 4\pi\rho' L \int_{r_*}^{\infty} r^2 \frac{dr}{dt} f(r,t) dr,\ t > 0. \tag{3}$$

Here ρ' is the density of two-phase system, c is its thermal capacity, α is the heat exchange coefficient, and L is the latent heat of phase transition.

The initial distribution function and system temperature should be regarded as known

$$f(r,t) = f_0(r),\ T(t) = T(0),\ t = 0. \tag{4}$$

For the sake of definitness, let us assume that the growth rate of bubbles is given by

$$\frac{dr}{dt} = \mu w\left(\frac{T - T_0}{T_0}\right),\ w\left(\frac{T - T_0}{T_0}\right) = \left(\frac{T - T_0}{T_0}\right)^n, \tag{5}$$

where μ is a constant coefficient and w represents the dimensionless function of rescaled temperature $(T - T_0)/T_0$. Let us especially note that we use here the power law with the constant exponent n frequently met in applications. So, for example, if the bubble growth is limited by the inertia of liquid, one obtains $n = 1/2$.[35] In addition, if the growth of bubbles is controlled by the heat supply rate of evaporation, one can get $n = 1$.[35]

The rate J of bubble appearance satisfies the Dering-Volmer and Frenkel-Zeldovich-Kagan nucleation models assuming the following law[35]

$$J(u) = J_0 R(u),\ R[u(\tau)] = \exp\left[-\frac{\kappa}{u^2(\tau)[u(\tau) + 1]}\right],\ \kappa = \frac{16\pi\sigma^3\Psi}{3L^2\rho''^2 k T_0}, \tag{6}$$

where $u = (T - T_0)/T_0$ stands for the rescaled temperature, J_0 represents the constant factor, σ is the surface tension, Ψ is the constant coefficient, ρ'' is the density of vapour, and k is the Boltzmann constant.

Model (1)-(6) is a closed system of integro-differential equations, boundary and initial conditions for studying the evolution of the process of intense boiling with polydisperse bubbles.

For the convenience of solving the formulated nonlinear model, we introduce dimensionless variables and parameters as follows

$$y=\frac{(r-r_*)\gamma}{\mu w(u_0)},\ \tau=\gamma t,\ \Phi(y,\tau)=l_0^4 f(r,t),\ \Phi_0(y)=l_0^4 f_0(r),\ l_0=\frac{\mu}{\gamma},\ y_*=\frac{r_*\gamma}{\mu w(u_0)},$$
$$u=\frac{T-T_0}{T_0},\ u_m=\frac{T_m-T_0}{T_0},\ u(0)=\frac{T(0)-T_0}{T_0},\ a=\frac{\gamma\rho c}{\alpha},\ \nu=\frac{J_0 l_0^4}{\mu},\ h=\frac{4\pi\rho' L w^3(u_0)}{\rho c T_0}. \tag{7}$$

Here u_0 represents a characteristic system temperature introduced as

$$\int\limits_0^\tau w[u(\tau_1)]d\tau_1=w(u_0)\tau. \tag{8}$$

Using the dimensionless variables and parameters (7) we rewrite the model (1)-(6) in the form of

$$\frac{\partial\Phi}{\partial\tau}+\frac{w(u)}{w(u_0)}\frac{\partial\Phi}{\partial y}+\Phi=0,\ y>0,\ \tau>0, \tag{9}$$

$$\Phi(y,\tau)=Y[u(\tau)],\ y=0;\ Y[u(\tau)]=\frac{\nu R[u(\tau)]}{w[u(\tau)]}, \tag{10}$$

$$\frac{du}{d\tau}=\frac{u_m-u}{a}-hw(u)\Lambda[u(\tau),\tau],\ \tau>0;\ \Lambda[u(\tau),\tau]=\int\limits_0^\infty (y_*+y)^2\Phi(y,\tau)dy, \tag{11}$$

$$\Phi(y,\tau)=\Phi_0(y),\ u(\tau)=u(0),\ \tau=0. \tag{12}$$

As this takes place, the growth rate of bubbles takes the form

$$\frac{dy}{d\tau}=\frac{w(u)}{w(u_0)}. \tag{13}$$

Let us consider below the method of analytical solution of the dimensionless integro-differential model (9)-(13).

3 | ANALYTICAL SOLUTIONS

Applying the Laplace transform with respect to the spatial variable y to the kinetic equation (9) and the corresponding initial condition (12) and keeping in mind the boundary condition (10), we come to

$$\frac{d\Phi_s}{d\tau}+\frac{w(u)}{w(u_0)}\left[s\Phi_s-Y[u(\tau)]\right]+\Phi_s=0,\ \tau>0;\ \Phi_s=\Phi_{0s},\ \tau=0, \tag{14}$$

where $\Phi_s=\Phi_s(s,\tau)$ and s is the Laplace transform parameter.

Taking into account expression (8), we arrive at the solution to equation (14) in the form of

$$\Phi_s(s,\tau)=\exp\left[-(s+1)\tau\right]\left\{\Phi_{0s}+\int\limits_0^\tau \frac{w[u(\tau_1)]Y[u(\tau_1)]}{w(u_0)}\exp\left[(s+1)\tau_1\right]d\tau_1\right\}. \tag{15}$$

Now applying the inverse Laplace transform to (15), we get

$$\Phi(y,\tau)=\begin{cases}\Phi_0(y-\tau)\exp(-\tau),\ y\ge\tau\\ \dfrac{\nu R\left[u(\tau-y)\right]}{w(u_0)}\exp(-y),\ y<\tau\end{cases}, \tag{16}$$

where $\Phi_0(0)=\nu R[u(0)]/w(u_0)$. This expression determines the bubble-size distribution function in dimensionless form.

Now combining expressions (11) and (16), we obtain

$$\Lambda[u(\tau),\tau] = N[u(\tau),\tau] + M(\tau),$$
$$N[u(\tau),\tau] = \frac{\nu}{w(u_0)}\int_0^{\tau}(y_* + y)^2 R[u(\tau-y)]\exp(-y)\,dy,\ \ M(\tau) = \int_{\tau}^{\infty}(y_*+y)^2\Phi_0(y-\tau)\exp(-\tau)\,dy. \tag{17}$$

As is easily seen, the temperature dynamics is defined by the integro-differential equation (11), where its right-hand side is given by the function $N[u(\tau),\tau]$, and $M(\tau)$ represents the known dependence.

To approximately evaluate the integral $N[u(\tau),\tau]$ let us introduce the new variable $\xi = \tau - y$. In this case, $N[u(\tau),\tau]$ can be written in the form of

$$N[u(\tau),\tau] = \frac{\nu}{w(u_0)}\int_0^{\tau}\tilde{f}(\xi,\tau)\exp\{\kappa S[u(\xi)]\}\,d\xi,$$
$$\tilde{f}(\xi,\tau) = \left(y_* + \tau - \xi\right)^2\exp\left[-(\tau-\xi)\right],\ \ S(u) = -\frac{1}{u^2(u+1)}. \tag{18}$$

The Laplace-type integral (18) can be evaluated using the saddle-point technique.[36,37] Namely, the function $S[u(\xi)]$ attains the maximum value at maximal $u(\xi)$, i.e. at the upper limit of integration $\xi = \tau$. Taking this into account let us write out the main contribution of the Laplace-type integral (18), which takes the form[36,37]

$$N[u(\tau),\tau] = \frac{\nu y_*^2\exp\{\kappa S[u(\tau)]\}}{\kappa w(u_0)S'[u(\tau)]}, \tag{19}$$

where κ is considered large enough, and

$$S'[u(\tau)] = \frac{3u(\tau)+2}{u^3(\tau)\,[u(\tau)+1]^2}u'(\tau). \tag{20}$$

To calculate $u'(\tau)$ we use equation (11), where its right-hand side Λ is given by expression (17). As a result, we have from (19)

$$N[u(\tau),\tau] \approx \frac{\nu y_*^2\exp\{\kappa S[u(\tau)]\}\,u^3(\tau)\,[u(\tau)+1]^2}{\kappa w(u_0)\,[3u(\tau)+2]\,\{P[u(\tau),\tau] - hw(u)N[u(\tau),\tau]\}}, \tag{21}$$

where

$$P[u(\tau),\tau] = \frac{u_m - u(\tau)}{a} - hw(u)M(\tau).$$

Now expressing $N[u(\tau),\tau]$ from (21), we come to the following approximation

$$N[u(\tau),\tau] \approx \frac{P[u(\tau),\tau] \pm \sqrt{P^2[u(\tau),\tau] - 4hw(u)Q[u(\tau)]}}{2hw(u)}, \tag{22}$$

where

$$Q[u(\tau)] = \frac{\nu y_*^2\exp\{\kappa S[u(\tau)]\}\,u^3(\tau)\,[u(\tau)+1]^2}{\kappa w(u_0)\,[3u(\tau)+2]}.$$

Now expression (22) defines $N[u(\tau),\tau]$ and the right-hand side of temperature equation (11) represents a function of $u(\tau)$ and τ. By this is meant that the temperature dynamics can be found from the following Cauchy problem

$$\frac{du}{d\tau} = \frac{u_m - u}{a} - hw(u)\Lambda[u(\tau),\tau],$$
$$u(\tau) = u(0),\ \ \tau = 0, \tag{23}$$

where $\Lambda[u(\tau),\tau]$ and $M(\tau)$ are defined by expressions (17), and $N[u(\tau),\tau]$ is given by (22).

Determining the temperature from (23), we completely know the bubble-size distribution function (16) at $y < \tau$. Thus, the approximate analytical solution of the integro-differential problem on intense boiling is described by solutions (16), (17), (22) and (23).

To illustrate the analytical solutions obtained let us introduce the normal initial distribution function

$$f_0(r) = \frac{A}{\sqrt{2\pi}\sigma}\exp\left[-\frac{1}{2}\left(\frac{r-\mu_f}{\sigma}\right)^2\right], \tag{24}$$

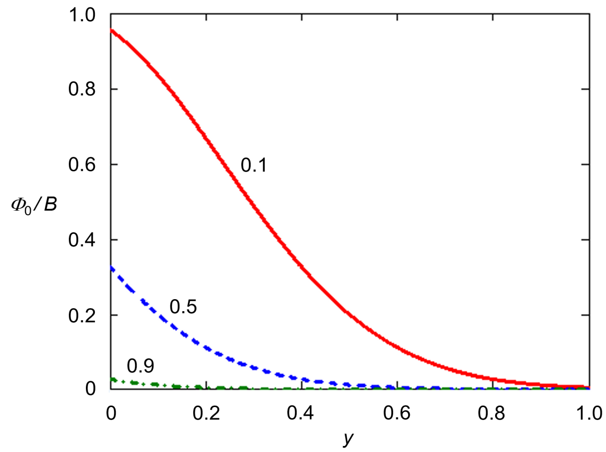

FIGURE 1 The rescaled initial distribution function $\Phi_0(y)/B$ versus the dimensionless size y of bubbles at different values of y_* (numbers at the curves), $\chi = 3$, $\mu_* = 0$.

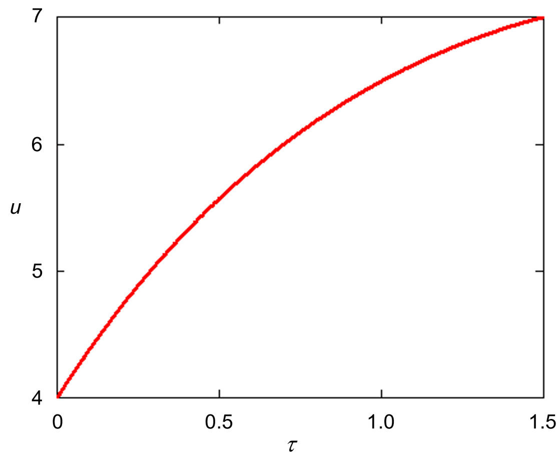

FIGURE 2 The dimensionless temperature $u(\tau)$ as a function of dimensionless time. The system parameters are in the text and the caption to Fig. 1 .

where μ stands for the mean of the distribution, σ represents the standard deviation, and A is a constant. Rewriting (24) in dimensionless form, we get

$$\Phi_0(y) = B \exp\left\{-\frac{1}{2}\left[\chi\left(y + y_* - \mu_*\right)\right]^2\right\}, \tag{25}$$

where

$$B = \frac{l_0^4 A}{\sqrt{2\pi\sigma}}, \ \chi = \frac{l_0 w(u_0)}{\sigma}, \ \mu_* = \frac{\gamma \mu_f}{w(u_0)\mu}.$$

The initial distribution function (25) at different values of y_* is shown in Fig. 1 . As is easily see, the smaller y_*, the larger the initial distribution function, since bubbles are easier to be born.

Figure 2 demonstrates the solution of Cauchy's problem (23) where $N[u(\tau), \tau]$ is taken from (22). The temperature increases with time from its initial value $u(0) = 5$ up to the maximal dimensionless temperature $u_m = 8$. The other dimensionless parameters are estimated as follows $\kappa = 3$, $h = 0.01$, $n = 1$, $B = 1$, $a = 1$, and

$$\frac{\nu}{w(u_0)} = \Phi_0(0) \exp\left\{\frac{\kappa}{u^2(0)\left[u(0) + 1\right]}\right\}.$$

Figure 3 shows the evolutionary behavior of the bubble-size distribution function (16). At all times, this function decreases with increasing the bubble size. However, the form of this dependence changes with time. This is due to the competition of

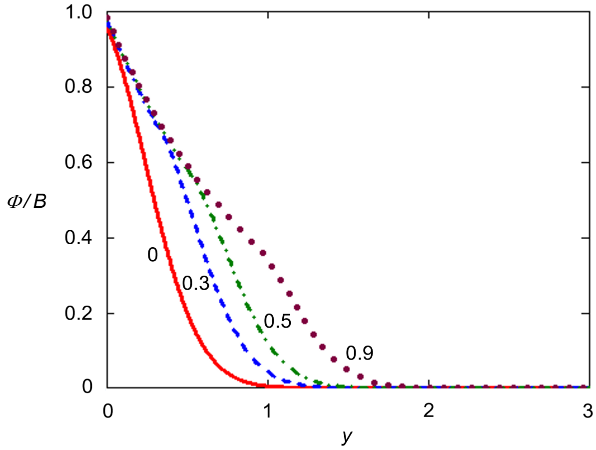

FIGURE 3 The rescaled distribution function $\Phi(y, \tau)/B$ versus the dimensionless size y of bubbles at different values of dimensionless time τ (numbers at the curves). The system parameters are in the text and the caption to Fig. 1 and $y_* = 0.1$.

two processes: the nucleation of new bubbles and the enlargement of the existing ones. Namely, the number of small bubbles increases due to their inflow (temperature rise), and the number of large bubbles increases due to the enlargement of existing bubbles.

4 | CONCLUSION

In summary, in the present study, a system of integro-differential equations is formulated and solved, which describes the process of intense boiling of a liquid. The kinetic equation for the bubble-size distribution function takes into account the process of bubble withdrawal. As this takes place, the rate of bubble appearance is used accordingly to the Dering-Volmer and Frenkel-Zeldovich-Kagan nucleation theories. The subsequent growth of nucleated bubbles is considered as an arbitrary function of the relative superheating of the liquid. An important circumstance is a fact that the distribution function and temperature of the fluid are found using the saddle point method to calculate the Laplace-type integral. In this case, for simplicity of presentation of the theory and demonstration of the main idea of the proposed method, we limited ourselves only to the main term in the expansion (see expressions (18) and (19)). Note that to improve the theory being developed, one can easily take into account terms of a higher order of smallness by analogy with the previously developed theory for the crystallization of supercooled liquids.[38,39]

The considered model of the boiling process of a superheated liquid can be generalized in future studies to take into account the “diffusion” of the distribution function in the space of bubble sizes,[40] the dependence of the growth rate of bubbles on their size, a more general law for the rate of bubble nucleation, the possible presence of an impurity dissolved in the liquid, as well as the other processes and phenomena. Such generalizations can be made by analogy with the theories of crystallization and dissolution in metastable liquids.[41–50]

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Author contributions

The authors contributed equally to the present research article.

Conflict of interest

The authors declare no potential conflict of interests.

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References

1. Mullin JW. *Crystallization*. London, UK: Butterworths; 1972.
2. Skripov VP. *Metastable Liquids*. New York: Wiley; 1974.
3. Buyevich YuA, Natalukha IA. Unsteady processes of combined polymerization and crystallization in continuous apparatuses. *Chem Eng Sci.* 1994;49:3241–3247.
4. Aseev DL, Alexandrov DV. Unidirectional solidification with a mushy layer. The influence of weak convection. *Acta Mater.* 2006;54:2401–2406.
5. Alexandrov DV, Nizovtseva IG, Malygin AP, Huang H-N, Lee D. Unidirectional solidification of binary melts from a cooled boundary: Analytical solutions of a nonlinear diffusion-limited problem. *J Phys: Condens Matter.* 2008;20:114105.
6. Alexandrova IV, Alexandrov DV, Aseev DL, Bulitcheva SV. Mushy layer formation during solidification of binary alloys from a cooled wall: the role of boundary conditions. *Acta Phys Polonica A*. 2009;115:791–794.
7. Alexandrov DV, Galenko PK, Herlach DM. Selection criterion for the growing dendritic tip in a non-isothermal binary system under forced convective flow. *J Cryst Growth.* 2010;312:2122–2127.
8. Ivanov AA, Alexandrova IV, Alexandrov DV. Phase transformations in metastable liquids combined with polymerization. *Phil Trans R Soc A*. 2019;377:20180215.
9. Vollmer U, Raisch J. H_∞-control of a continuous crystallizer. *Control Eng Pract.* 2001;9:837–845.
10. Dubrovskii VG. *Nucleation Theory and Growth of Nanostructures*. Berlin: Springer; 2014.
11. Buyevich YuA, Korolyova NA, Natalukha IA. Modelling of unsteady combustion regimes for polydisperse fuels – I. Instability and auto-oscillations. *Int J Heat Mass Trans.* 1993;36:2223–2231.
12. Buyevich YuA, Korolyova NA, Natalukha IA. Modelling of unsteady combustion regimes for polydisperse fuels – II. Parametrically controlled combustion. *Int J Heat Mass Trans.* 1993;36:2233–2238.
13. Buyevich YuA, Mansurov VV, Natalukha IA. Instability and unsteady processes of the bulk continuous crystallization – I. Linear stability analysis. *Chem Eng Sci.* 1991;46:2573–2578.
14. Buyevich YuA, Mansurov VV, Natalukha IA. Instability and unsteady processes of the bulk continuous crystallization – II. Non-linear periodic regimes. *Chem Eng Sci.* 1991;46:2579–2588.
15. Alexandrov DV, Ivanov AO. Dynamic stability analysis of the solidification of binary melts in the presence of a mushy region: changeover of instability. *J Cryst Growth.* 2000;210:797–810.
16. Alexandrov DV, Malygin AP. Flow-induced morphological instability and solidification with the slurry and mushy layers in the presence of convection. *Int J Heat Mass Trans.* 2012;55:3196–3204.
17. Alexandrov DV, Galenko PK, Toropova LV. Thermo-solutal and kinetic modes of stable dendritic growth with different symmetries of crystalline anisotropy in the presence of convection. *Phil Trans R Soc A*. 2018;376:20170215.

18. Alexandrov DV, Malygin AP. Coupled convective and morphological instability of the inner core boundary of the Earth. *Phys Earth Planet Inter.* 2011;189:134–141.

19. Alexandrov DV, Malygin AP. Convective instability of directional crystallization in a forced flow: The role of brine channels in a mushy layer on nonlinear dynamics of binary systems. *Int J Heat Mass Trans.* 2011;54:1144–1149.

20. Buyevich YA, Goldobin YM, Yasnikov GP. Evolution of a particulate system governed by exchange with its environment. *Int J Heat Mass Trans.* 1994;37:3003–3014.

21. Buyevich YuA, Mansurov VV. Kinetics of the intermediate stage of phase transition in batch crystallization. *J Cryst Growth.* 1990;377:861–867.

22. Aseev DL, Alexandrov DV. Directional solidification of binary melts with a non-equilibrium mushy layer. *Int J Heat Mass Trans.* 2006;49:4903–4909.

23. Alexandrov DV. Nucleation and crystal growth kinetics during solidification: The role of crystallite withdrawal rate and external heat and mass sources. *Chem Eng Sci.* 2014;117:156–160.

24. Makoveeva EV, Alexandrov DV. Effects of nonlinear growth rates of spherical crystals and their withdrawal rate from a crystallizer on the particle-size distribution function. *Phil Trans R Soc A.* 2019;377:20180210.

25. Makoveeva EV, Alexandrov DV. Effects of external heat/mass sources and withdrawal rates of crystals from a metastable liquid on the evolution of particulate assemblages. *Eur Phys J Special Topics.* 2019;228:25–34.

26. Alexandrova IV, Alexandrov DV. Dynamics of particulate assemblages in metastable liquids: a test of theory with nucleation and growth kinetics. *Phil Trans R Soc A.* 2020;378:20190245.

27. Alexandrov DV. Nucleation and evolution of spherical crystals with allowance for their unsteady-state growth rates. *J Phys A: Math Theor.* 2018;51:075102.

28. Alexandrov DV, Nizovtseva IG, Alexandrova IV. On the theory of nucleation and nonstationary evolution of a polydisperse ensemble of crystals. *Int J Heat Mass Trans.* 2019;128:46–53.

29. Alexandrov DV, Alexandrova IV. On the theory of the unsteady-state growth of spherical crystals in metastable liquids. *Phil Trans R Soc A.* 2019;377:20180209.

30. Alexandrov DV, Ivanov AA. Solidification of a ternary melt from a cooled boundary, or nonlinear dynamics of mushy layers. *Int J Heat Mass Trans.* 2009;52:4807–4811.

31. Alexandrov DV. Nucleation and growth of crystals at the intermediate stage of phase transformations in binary melts. *Phil Mag Lett.* 2014;94:786–793.

32. Alexandrov DV, Ivanov AA, Alexandrova IV. Analytical solutions of mushy layer equations describing directional solidification in the presence of nucleation. *Phil Trans R Soc A.* 2018;376:20170217.

33. Makoveeva EV, Alexandrov DV. A complete analytical solution of the Fokker-Planck and balance equations for nucleation and growth of crystals. *Phil Trans R Soc A.* 2018;376:20170327.

34. Makoveeva EV, Alexandrov DV. The Gibbs-Thomson effect in the evolution of particulate assemblages in a metastable liquid. *Phys Lett A.* 2020;384:126259.

35. Buyevich YuA, Natalukha IA. Self-oscillating regimes of nucleate, transition and film boiling. *Int J Heat Mass Trans.* 1996;39:2363–2373.

36. Fedoruk MV. *Saddle-Point Method.* Moscow: Nauka; 1977.

37. Alexandrov DV. Nonlinear dynamics of polydisperse assemblages of particles evolving in metastable media. *Eur Phys J Special Topics.* 2020;229:383–404.

38. Lifshitz EM, Pitaevskii LP. *Physical Kinetics.* Oxford, UK: Pergamon; 1981.

39. Alexandrov DV, Malygin AP. Transient nucleation kinetics of crystal growth at the intermediate stage of bulk phase transitions. *J Phys A: Math Theor.* 2013;46:455101.

40. Alexandrov DV. Nucleation and crystal growth in binary systems. *J Phys A: Math Theor.* 2014;47:125102.

41. Shneidman VA. Transient nucleation with monotonically changing barrier. *Phys Rev E.* 2010;82:031603.

42. Shneidman VA. Time-dependent distributions in self-quenching nucleation. *Phys Rev E.* 2011;84:031602.

43. Alexandrov DV, Nizovtseva IG. On the theory of crystal growth in metastable systems with biomedical applications: Protein and insulin crystallization. *Phil Trans R Soc A.* 2019;377:20180214.

44. Ivanov AA, Alexandrov DV, Alexandrova IV. Dissolution of polydisperse ensembles of crystals in channels with a forced flow. *Phil Trans R Soc A.* 2020;378:20190246.

45. Alexandrov DV, Alexandrova IV. From nucleation and coarsening to coalescence in metastable liquids. *Phil. Trans. R. Soc. A.* 2020;378:20190247.

46. Alyab'eva AV, Buyevich YuA, Mansurov VV. Evolution of a particulate assemblage due to coalescence combined with coagulation. *J Phys II France.* 1994;4:951–957.

47. Slezov VV. *Kinetics of First-order Phase Transitions*. Weinheim, Germany: Wiley, VCH; 2009.

48. Barlow DA. Theory of the intermediate stage of crystal growth with applications to insulin crystallization. *J Cryst Growth.* 2017;470:8–14.

49. Nikishina MA, Alexandrov DV. Kinetics of the intermediate stage of phase transition with elliptical crystals. *Eur Phys J Special Topics.* 2020;229:2937–2949.

50. Makoveeva EV, Alexandrov DV. An analytical solution to the nonlinear evolutionary equations for nucleation and growth of particles. *Phil Mag Lett.* 2018;98:199–208.

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