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RESEARCH ARTICLE

Inertial algorithms with adaptive stepsizes for split variational inclusion problems and their applications to signal recovery problem

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With the help of the Meir-Keeler contraction method and the Mann-type method, two adaptive inertial iterative schemes are introduced for finding solutions of the split variational inclusion problem in Hilbert spaces. The strong convergence of the suggested algorithms are guaranteed by a new stepsize criterion that does not require calculation of the bounded linear operator norm. Some numerical experiments and applications in signal recovery problems are given to demonstrate the efficiency of the proposed algorithms.

KEYWORDS:
split variational inclusion problem, inertial method, adaptive stepsize, Meir-Keeler contraction, Mann-type method, signal recovery

1 | INTRODUCTION

In finite-dimensional Hilbert spaces, Censor and Elfving[1] first proposed the split feasibility problem (shortly, SFP) in 1994, which is to solve the inverse problem in phase retrievals and medical image reconstruction.[2] At the same time, SFP was also applied to image restoration, computer tomograph, radiation therapy treatment planning, signal recovery and so on.[3,4,5,6] Because of its crucial application background, SFP has become a hot research field in the past 20 years and has been generalized in various ways. To solve the split feasibility problem, it is necessary to mention that the fixed point equation and the CQ algorithm proposed by Byrne[2] in finite-dimensional Hilbert spaces. On the basis of this work, many results of weak convergence and strong convergence were proved in Hilbert spaces and Banach spaces.[7,8,9,10,11] Most of these iterative algorithms ususlly choose a fixed stepsize or a stepsize sequence associated with the norm of the bounded linear operator. To get rid of such limitations, López et al.[12] used the idea of the gradient-projection algorithm to construct an adaptive stepsize sequence that does not depend on the norm of the bounded linear operator and proposed a modified CQ algorithm with such a stepsize sequence. It turn out that such a stepsize sequence is useful in practice and has been applied to many generalizations of SFP, such as the split common fixed point problem, the split equality problem and the multiple-set split feasibility problem, see[10,13,14,15,16] and the references therein.

Censor et al.[17] introduced more general problems, i.e., the split inverse problems. By the selection of the inverse problem, many generalization of SFP is generated. The split variational inclusion problem (shortly, SVIP) as one of them, including the split variational inequality problem, the split equilibrium problem and the split feasibility problem. Based on resolvent mappings of maximal monotone mappings, many strong convergent results are guaranteed under different iterative methods, such as the Halpern iterative method,[18] the viscosity iterative method[19,20,21] and the Mann-type iterative method.[22,23] Further, in terms of the convergence rate of the algorithm, many related work have been done in various mathematical problems, among which the most important and recognized is the inertial technique method. This method was first proposed by Polyak[24] and was called the heavy-ball method in a second-order time dynamic system. The acceleration of the algorithm is realized by setting the inertial extrapolation step, which contains the values of the previous two iterations in the next iteration. More recently, this technique has also been applied to various aspects, such as variational inclusion problems, fixed point problems, variational inequalities, split feasibilities and equilibrium problems.[25,26,27,28,29,30,31,32]

Along with the existing research results, many interesting insights have emerged in the improvement study of SVIP. On the basis of the adaptive stepsize criterion and the inertial technique method, we will introduce a new Meir-Keeler contraction algorithm and a new Mann-type algorithm for solving SVIP, as well as obtain strong convergence properties of the iterative sequences generated by the proposed algorithms under certain parameter constraints. More precisely, our innovation and major contributions in this paper are as follows:

(i) Based on the stepsize selection in[12], our algorithm does not involve the norm of the bounded linear operator. This improves the existing results[7,9,18,19,21,22] and is easier to implement in practical applications;

(ii) The inertial technique method is taken into account in both algorithms and effectively accelerates the convergence rate of the algorithms. This also promote many previous corresponding results;[18,19,20,23]

(iii) The Meir-Keeler contraction method and the Mann-type method are inserted to ensure the strong convergence property of the both algorithms. Meanwhile, the familiar viscosity algorithm and the Halpern algorithm are also special cases of the Meir-Keeler contraction algorithm;

(iv) The proposed algorithms are applied to the split variational inequality problem, the split feasibility problem and the split equilibrium problem;

(v) In the numerical experiments, the convergence rate of our algorithms are faster than that of existing algorithms.[19,21,22] Furthermore, our algorithms are also effectively applied to signal recovery problem.

An outline of this paper is as follows. Sect. 2 introduces the split variational inclusion problem and its some special split forms. Meanwhile, some notations and lemmas for later proofs are given in this section. In Sect. 3, the inertial Meir-Keeler contraction algorithm and the inertial Mann-type algorithm are proposed, and their convergence theorems are built under mild conditions. In Sect. 4, some theoretical applications on the split variational inequality problem, the split feasibility problem and the split equilibrium problem are presented by our main results. Finally, in Sect. 5, some practical examples, especially for signal recovery problems, are given to show the convergence behavior of the proposed algorithms over the existing ones.

2 | STATE OF PROBLEM AND PRELIMINARIES

2.1 | State of problem

For the sake of simplicity, the notations $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}$ represent Hilbert spaces, $0_{\mathcal{H}_1}$ and $0_{\mathcal{H}_2}$ represent the zero elements of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The symbols C_1, Q_1 and C denote nonempty closed convex subsets of $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}$, respectively. $\mathcal{F}(T)$ stands for the fixed point set of a mapping T. Let $B_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $B_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ be two maximal monotone mappings, $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. The split variational inclusion problem is to find $z^* \in \mathcal{H}_1$ such that

$$0_{\mathcal{H}_1} \in B_1(z^*) \text{ and } 0_{\mathcal{H}_2} \in B_2(Az^*). \tag{SVIP}$$

The solution set of this problem is represented by Γ, i.e., $\Gamma := \{z^* \in \mathcal{H}_1 \mid 0_{\mathcal{H}_1} \in B_1(z^*),\ 0_{\mathcal{H}_2} \in B_2(Az^*)\}$. On the other hand, let $F : \mathcal{H} \to \mathcal{H}$ be a α-inverse strongly monotone operator. The classical variational inequality problem is to find $z^* \in C$ such that

$$\langle F(z^*), z - z^* \rangle \geq 0,\ \forall z \in C. \tag{1}$$

Meanwhile, the normal cone $N_C(z)$ of C at $z \in C$ is defined by

$$N_C(z) := \{v \in \mathcal{H} \mid \langle v, y - z\rangle \leq 0,\ \forall y \in C\}.$$

The set valued mapping S_F related to $N_C(z)$ is defined by

$$S_F(z) := \begin{cases} F(z) + N_C(z), & z \in C, \\ \emptyset, & \text{otherwise}. \end{cases} \tag{2}$$

In fact, such a mapping S_F is maximal monotone and z^* is a solution of (1) if and only if $0_{\mathcal{H}} \in S_F(z^*)$, where $0_{\mathcal{H}}$ is a zero element in $\mathcal{H}$. So, let $F_1 : \mathcal{H}_1 \to \mathcal{H}_1$ and $F_2 : \mathcal{H}_2 \to \mathcal{H}_2$ be α-inverse strongly monotone operators, $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. The split variational inequality problem (shortly, SVIP*) is to find $z^* \in C_1$ such that

$$\langle F_1(z^*), z - z^* \rangle \geq 0,\ \forall z \in C_1 \text{ and } \langle F_2(Az^*), u - Az^* \rangle \geq 0,\ \forall u \in Q_1. \tag{SVIP*}$$

From the above methods, set $B_1 := S_{F_1}$ and $B_2 := S_{F_2}$, where S_{F_1} and S_{F_2} are constructed as (2), SVIP is equivalent to the split variational inequality problem.

In addition, suppose that $G : C \times C \to \mathbb{R}$ is a bifunction that satisfies the following conditions:

(A1) $G(z, z) = 0, \forall z \in C$;

(A2) $G(z, y) + G(y, z) \leq 0, \forall z, y \in C$;

(A3) $\limsup_{t \to 0} G(tz + (1 - t)x, y) \leq G(x, y), \forall x, y, z \in C$;

(A4) For each $z \in C$, the function $y \mapsto G(z, y)$ is convex and lower semi-continuous.

The famous equilibrium problem is to find $z^* \in C$ such that

$$G(z^*, y) \geq 0,\ \forall y \in C.$$

Firstly, Blum and Oettli[33] gave the existence of the following inequality for the bifunction G, that is, for any $r > 0$ and $z \in \mathcal{H}$, there exists $x \in C$ such that

$$G(x, y) + \frac{1}{r}\langle y - x, x - z \rangle \geq 0, \forall y \in C.$$

Further, Combettes and Hirstoaga[34] defined a mapping $T_r^G : \mathcal{H} \to \mathcal{H}$ as follows:

$$T_r^G(z) := \{x \in C \mid G(x, y) + \frac{1}{r}\langle y - x, x - z \rangle \geq 0,\ \forall y \in C\},\ \forall r > 0,\ z \in \mathcal{H}. \tag{3}$$

Then, T_r^G is a single-valued and firmly nonexpansive mapping; $\mathcal{F}(T_r^G) = EP(G)$ is a nonempty closed and convex set, where $EP(G)$ denotes the solution set of the equilibrium problem. Meanwhile, T_r^G is also called the resolvent mapping of G for any $r > 0$. For explanation, Takahashi et al.[35] introduced a set-valued mapping $\mathcal{K}_G$ from $\mathcal{H}$ into itself, i.e.,

$$\mathcal{K}_G(z) := \begin{cases} \{x \in H \mid G(z, y) \geq \langle y - z, x \rangle, \forall y \in C\}, & z \in C, \\ \emptyset, & z \notin C. \end{cases} \tag{4}$$

Then, $EP(G) = \mathcal{K}_G^{-1}(0)$ and $\mathcal{K}_G$ is maximal monotone with the effective domain of $\mathcal{K}_G \subset C$. Hence, for any $z \in \mathcal{H}$ and $r > 0$, the resolvent T_r^G of G coincides with the resolvent of $\mathcal{K}_G$, i.e.,

$$T_r^G(z) = \left(I + r\mathcal{K}_G\right)^{-1}(z). \tag{5}$$

Moreover, the equilibrium problem is extended to the split equilibrium problem (shortly, SEP): Let $G_1 : C_1 \times C_1 \to \mathbb{R}$ and $G_2 : Q_1 \times Q_1 \to \mathbb{R}$ be two bifunctions satisfying (A1)-(A4). SEP is to find a point z^* such that

$$G_1(z^*, x) \geq 0,\ \forall x \in C_1 \text{ and } G_2(Az^*, y) \geq 0,\ \forall y \in Q_1. \tag{SEP}$$

Using the above conclusions, $\mathcal{K}_{G_1}$, $\mathcal{K}_{G_2}$, $T_r^{G_1}$ and $T_r^{G_2}$ are also generated in the same way. In the case that $B_1 = \mathcal{K}_{G_1}$ and $B_2 = \mathcal{K}_{G_2}$, SVIP is equivalent to SEP.

In addition, let $g : \mathcal{H} \to (-\infty, +\infty)$ be a proper convex lower semicontinuous function. The subdifferential ∂g of g is defined by $\partial g(x) = \{z \in \mathcal{H} \mid g(y) - g(x) - \langle z, y - x \rangle \geq 0,\ \forall y \in \mathcal{H}\}, \forall x \in \mathcal{H}$. Suppose that i_{C_1} and i_{Q_1} are indicator functions of C_1 and Q_1, respectively, that is,

$$i_{C_1}(x) := \begin{cases} 0, & x \in C_1, \\ \infty, & x \notin C_1; \end{cases} \qquad i_{Q_1}(y) := \begin{cases} 0, & y \in Q_1, \\ \infty, & y \notin Q_1. \end{cases}$$

So, i_{C_1} and i_{Q_1} are proper convex lower semicontinuous functions on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, the subdifferentials ∂i_{C_1} and ∂i_{Q_1} are two maximal monotone mappings. Meanwhile,

$$\begin{aligned} \partial i_{C_1}(x) &= \{p \in \mathcal{H}_1 \mid i_{C_1}(z) - i_{C_1}(x) - \langle p, z - x \rangle \geq 0,\ \forall z \in \mathcal{H}_1\}, \\ &= \{p \in \mathcal{H}_1 \mid \langle p, z - x \rangle \leq 0,\ \forall z \in C_1\} = N_{C_1}(x); \\ \partial i_{Q_1}(y) &= \{q \in \mathcal{H}_2 \mid i_{Q_1}(u) - i_{Q_1}(y) - \langle q, u - y \rangle \geq 0,\ \forall u \in \mathcal{H}_2\} \\ &= \{q \in \mathcal{H}_2 \mid \langle q, u - y \rangle \leq 0,\ \forall u \in Q_1\} = N_{Q_1}(y). \end{aligned}$$

Further, for $\gamma > 0$, we define the following resolvent operators $J_\gamma^{\partial i_{C_1}}$ and $J_\gamma^{\partial i_{Q_1}}$ with respect to ∂i_{C_1} and ∂i_{Q_1}, respectively,

$$J_\gamma^{\partial i_{C_1}}(x) = (I + \gamma \partial i_{C_1})^{-1}(x),\ x \in \mathcal{H}_1,\ J_\gamma^{\partial i_{Q_1}}(y) = (I + \gamma \partial i_{Q_1})^{-1}(y),\ y \in \mathcal{H}_2.$$

Furthermore, for any $\gamma > 0$,

$$\begin{aligned} u = J_\gamma^{\partial i_{C_1}}(x) \Leftrightarrow x \in u + \gamma \partial i_{C_1}(u) &\Leftrightarrow x - u \in \gamma \partial i_{C_1}(u) \\ &\Leftrightarrow \langle x - u, z - u\rangle \le 0,\ \forall z \in C_1 \\ &\Leftrightarrow u = P_{C_1}(x), \end{aligned}$$

where P_{C_1} is a metric projection from $\mathcal{H}_1$ onto C_1. In the same way, $v = J_\gamma^{\partial i_{Q_1}}(y) \Leftrightarrow v = P_{Q_1}(y)$, where P_{Q_1} is also a metric projection from $\mathcal{H}_2$ onto Q_1. So, for any $\gamma > 0$, we have $J_\gamma^{\partial i_{C_1}} = P_{C_1}$ and $J_\gamma^{\partial i_{Q_1}} = P_{Q_1}$. In other words, when $B_1 = \partial i_{C_1}$ and $B_2 = \partial i_{Q_1}$, SVIP is equivalent to the following split feasibility problem, which is to find z^* such that

$$z^* \in C_1 \text{ and } Az^* \in Q_1. \tag{SFP}$$

2.2 | Preliminaries

Let $\mathcal{H}$ be a Hilbert space and C be a nonempty closed convex subset of $\mathcal{H}$. The symbols $\to$ and $\rightharpoonup$ denote strong convergence and weak convergence, respectively. $\mathcal{F}(T)$ denotes the fixed point set of a mapping T.

Definition 1. For any $z, y \in \mathcal{H}$, a mapping $T : \mathcal{H} \to \mathcal{H}$ is said to be

(i) contraction, if there exists a constant $\mu \in [0, 1)$ such that

$$\|T(z) - T(y)\| \le \mu\|z - y\|.$$

(ii) L-Lipschitz continuous with $L > 0$, if

$$\|T(z) - T(y)\| \le L\|z - y\|.$$

(iii) nonexpansive, if

$$\|T(z) - T(y)\| \le \|z - y\|.$$

(iv) firmly nonexpansive, if

$$\|T(z) - T(y)\|^2 \le \langle T(z) - T(y), z - y\rangle.$$

(v) η-inverse strongly monotone, if

$$\eta\|T(z) - T(y)\|^2 \le \langle T(z) - T(y), z - y\rangle.$$

In addition, for any $z, y \in \mathcal{H}$ and $\beta \in \mathbb{R}$, the following properties hold

(1) $\|z + y\|^2 = \|z\|^2 + \|y\|^2 + 2\langle z, y\rangle \le \|z\|^2 + 2\langle y, z + y\rangle$;

(2) $\|\beta z + (1 - \beta)y\|^2 = \beta\|z\|^2 + (1 - \beta)\|y\|^2 - \beta(1 - \beta)\|z - y\|^2$.

Remark 1. If T is a firmly nonexpansive mapping, then it is also nonexpansive and $I - T$ is also a firmly nonexpansive mapping.

Definition 2. The metric projection of $\mathcal{H}$ onto C, denoted by $P_C(z)$, is defined by

$$P_C(z) := \operatorname{argmin}_{y \in C} \|z - y\|,\ \forall z \in \mathcal{H}.$$

This definition has the following important consequences.

Lemma 1 (Bauschke and Combettes[36])**.** For any $z \in \mathcal{H}$ and $y \in C$, the metric projection P_C from $\mathcal{H}$ onto C possesses the following equivalent properties:

$$\langle P_C(z) - z, P_C(z) - y\rangle \le 0 \ \Leftrightarrow \|y - P_C(z)\|^2 + \|z - P_C(z)\|^2 \le \|z - y\|^2.$$

Definition 3 (Meir and Keeler[37])**.** Let $(\mathcal{X}, d)$ be a metric space. $h : \mathcal{X} \to \mathcal{X}$ is a Meir-Keeler contraction mapping if and only if for each $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$\varepsilon \le d(z, y) < \varepsilon + \delta \Rightarrow d(h(z), h(y)) < \varepsilon,\ \forall z, y \in \mathcal{X}.$$

Lemma 2 (Suzuki[38]). Let $\mathcal{K}$ be a convex subset of a Banach space $\mathcal{B}$, $h : \mathcal{K} \to \mathcal{K}$ be a Meir-Keeler contraction mapping. For each $\varepsilon > 0$, there exists a number $\delta \in (0, 1)$ such that $\|z - y\| \geq \varepsilon$ implies $\|h(z) - h(y)\| \leq \delta\|z - y\|, \forall z, y \in \mathcal{K}$.

Definition 4. Let $B : \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued mapping with domain $\mathcal{D}(B) := \{z \in \mathcal{H} \mid B(z) \neq \emptyset\}$ and graph $\mathcal{G}(B) := \{(z, w) \in \mathcal{H} \times \mathcal{H} \mid z \in \mathcal{D}(B), w \in B(z)\}$. Recall that $B : \mathcal{H} \to 2^{\mathcal{H}}$ is monotone mapping if and only if $\langle z - y, w - v\rangle \geq 0$, $\forall w \in B(z),\ v \in B(y)$. Further, a monotone mapping $B : \mathcal{H} \to 2^{\mathcal{H}}$ is maximal, that is, the graph $\mathcal{G}(B)$ is not properly contained in the graph of any other monotone mapping.

Lemma 3 (Marino and Xu,[39] Chuang[40]). The resolvent mapping J_γ^B of a maximal monotone mapping B with $\gamma > 0$ is defined as $J_\gamma^B(z) = (I + \gamma B)^{-1}(z), \forall z \in \mathcal{H}$. The following properties hold.

(1) J_γ^B is a single-valued and firmly nonexpansive mapping;

(2) $\mathcal{F}(J_\gamma^B) \Leftrightarrow B^{-1}(0) := \{z \in \mathcal{D}(B) \mid 0 \in B(z)\}$.

Remark 2. Because of the nature of the resolvent mapping J_γ^B, the solution set Γ of the split variational inclusion problem is equivalent to the set $\{z^* \in \mathcal{H}_1 \mid z^* \in \mathcal{F}(J_\gamma^{B_1}),\ \ Az^* \in \mathcal{F}(J_\gamma^{B_2})\}$, where $J_\gamma^{B_1}$ (resp. $J_\gamma^{B_2}$) is the resolvent mapping of a maximal monotone mapping B_1 (resp. B_2).

Lemma 4 (Cui and Su[41]). Let $B : \mathcal{D}(B) \subset \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone mapping. For any $0 < \beta \leq \gamma$, then

$$\|x - J_\beta^B(x)\| \leq 2\|x - J_\gamma^B(x)\|,\ \forall x \in \mathcal{H}.$$

Lemma 5 (Zhou and Qin[42]). Let $T : C \to C$ be a nonexpansive mapping with $\mathcal{F}(T) \neq \emptyset$. $I - T$ is demiclosed at zero, that is, for any sequence $\{z_n\}$ in C, satisfying $z_n \rightharpoonup z$ and $z_n - T(z_n) \to 0$, then $z \in \mathcal{F}(T)$.

Lemma 6 (He and Yang[43]). Let $\{a_n\}$ and $\{c_n\}$ be two sequences of nonnegative real numbers such that

$$a_{n+1} \leq (1 - k_n)a_n + k_n b_n,\ n \geq 1,$$

$$a_{n+1} \leq a_n - c_n + d_n, n \geq 1,$$

where $\{k_n\}$, $\{b_n\}$ and $\{d_n\}$ are real sequences with $0 < k_n < 1$. If $\sum_{n=1}^{\infty} k_n = \infty$, $\lim_{n\to\infty} d_n = 0$, and $\lim_{k\to\infty} c_{n_k} = 0$ implies $\lim_{k\to\infty}\ b_{n_k} \leq 0$, where $\{n_k\}$ is any subsequence of $\{n\}$. The sequence $\{a_n\}$ converges to 0 as $n \to \infty$.

3 | TWO ADAPTIVE INERTIAL ITERATIVE ALGORITHMS

In this section, we assume that $\mathcal{H}_1$, $\mathcal{H}_2$ are two Hilbert spaces and $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator with adjoint operator A^*, $B_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $B_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ are two set-valued maximal monotone mappings, $h : \mathcal{H}_1 \to \mathcal{H}_1$ is a Meir-Keeler contraction mapping. For solving SVIP, we introduce the following Algorithm 1 and Algorithm 2 using the Meir-Keeler contraction method and the Mann-type method, respectively.

Remark 3. We have the following observations from Algorithms 1 and 2.

(1) The value of $\|z_n - z_{n-1}\|$ is known in each iteration of Algorithms 1 and 2. Hence, the sequence $\{\alpha_n\}$ is constructed as follows:

$$\alpha_n = \begin{cases} \min\left\{\alpha, \dfrac{\rho_n}{\|z_n - z_{n-1}\|}\right\}, & z_n \neq z_{n-1}, \\ \alpha, & \text{otherwise}, \end{cases}$$

where $\{\rho_n\}$ is a positive sequence with $\rho_n = o(\tau_n)$ and $\alpha \in [0, 1)$.

(2) Furthermore, the sequence $\{\tau_n\}$ can be chosen by $\tau_n = n^{-p}$ with $0 < p \leq 1$. Thus, the above positive sequence $\{\rho_n\}$ can be obtained $\rho_n = n^{-q}$ with $q > p$, for more detail, see[10].

(3) In Algorithms 1 and 2, the coefficient $\sigma_n \in [a, b] \subset (0, 2)$ is satisfied to avoid the limit of σ_n being equal to 0 or 1. In fact, it is permissible to set σ_n in the following cases:

(i) $0 < \liminf_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \sigma_n < 2$;

Algorithm 1 Inertial Meir-Keeler contraction algorithm (IMKCA)

Suppose $\gamma_n > 0$, $\alpha_n \in [0, \alpha] \subset [0, 1)$, $\tau_n \in (0, 1)$ and $\sigma_n \in [a, b] \subset (0, 2)$. Take any $z_0, z_1 \in \mathcal{H}_1$ and compute

$$\begin{cases} t_n = z_n + \alpha_n(z_n - z_{n-1}), \\ w_n = J_{\gamma_n}^{B_1}(t_n), \\ u_n = w_n - \lambda_n A^*(I - J_{\gamma_n}^{B_2})Aw_n. \end{cases} \tag{6}$$

If $t_n = w_n = u_n$, then stop. Otherwise, calculate

$$z_{n+1} = \tau_n h(u_n) + (1 - \tau_n)u_n.$$

Here the stepsize λ_n is defined by

$$\lambda_n = \begin{cases} \frac{\sigma_n \|(I - J_{\gamma_n}^{B_2})Aw_n\|^2}{\|A^*(I - J_{\gamma_n}^{B_2})Aw_n\|^2}, & Aw_n \notin B_2^{-1}(0), \\ 0, & \text{otherwise}, \end{cases} \tag{7}$$

and

$$\sum_{n=1}^{\infty} \tau_n = \infty,\ \lim_{n\to\infty} \tau_n = 0,\ \lim_{n\to\infty} \frac{\alpha_n}{\tau_n}\|z_n - z_{n-1}\| = 0,\ \inf_n\{\gamma_n\} \geq \gamma > 0.$$

Algorithm 2 Inertial Mann-type algorithm (IMTA)

Suppose $\gamma_n > 0$, $\alpha_n \in [0, \alpha] \subset [0, 1)$, $\tau_n \in (0, 1)$ and $\sigma_n \in [a, b] \subset (0, 2)$. Take any initial guesses $z_0, z_1 \in \mathcal{H}_1$ and t_n, w_n, u_n are generated in the same way as (6). If $t_n = w_n = u_n$, then stop. Otherwise, calculate

$$z_{n+1} = (1 - \theta_n - \tau_n)w_n + \theta_n u_n.$$

Here λ_n is defined as (7) and

$$\sum_{n=1}^{\infty} \tau_n = \infty,\ \lim_{n\to\infty} \tau_n = 0,\ \lim_{n\to\infty} \frac{\alpha_n}{\tau_n}\|z_n - z_{n-1}\| = 0,\ \{\theta_n\} \subset [c, d] \subset (0, 1 - \tau_n),\ \inf_n\{\gamma_n\} \geq \gamma > 0.$$

(ii) $\inf_n \sigma_n \left(2 - \sigma_n\right) > 0$.

Lemma 7. If $t_n = w_n = u_n$ in Algorithms 1 and 2, then t_n is a solution of SVIP, i.e., $t_n \in \Gamma$.

Proof. From Lemmas 3 and 4, and the definition of t_n, w_n and u_n, for any $z^* \in \Gamma$,

$$\begin{aligned} 0 = \langle t_n - u_n, t_n - z^* \rangle &= \langle t_n - w_n, t_n - z^* \rangle + \langle w_n - u_n, t_n - z^* \rangle \\ &= \langle t_n - J_{\gamma_n}^{B_1}(t_n), t_n - z^* \rangle + \lambda_n \langle A^*(I - J_{\gamma_n}^{B_2})Aw_n, t_n - z^* \rangle \\ &= \langle t_n - J_{\gamma_n}^{B_1}(t_n), t_n - z^* \rangle + \lambda_n \langle (I - J_{\gamma_n}^{B_2})At_n, At_n - Az^* \rangle \\ &\geq \|t_n - J_{\gamma_n}^{B_1}(t_n)\|^2 + \|(I - J_{\gamma_n}^{B_2})At_n\|^2 \\ &\leq \frac{1}{2}(\|t_n - J_{\gamma}^{B_1}(t_n)\|^2 + \|(I - J_{\gamma}^{B_2})At_n\|^2). \end{aligned}$$

Thus $\|t_n - J_\gamma^{B_1} t_n\| = \|(I - J_\gamma^{B_2})At_n\| = 0$. From Remark 2, we obtain that t_n is a solution of SVIP. □

Lemma 8. For any $z_1 \in \mathcal{H}_1$, $\gamma_n > 0$ and $\lambda_n > 0$, set $u_n = z_n - \lambda_n A^*(I - J_{\gamma_n}^{B_2})Az_n$. Then,

$$\|u_n - z^*\|^2 \leq \|z_n - z^*\|^2 - \lambda_n \left(2\|(I - J_{\gamma_n}^{B_2})Az_n\|^2 - \lambda_n\|A^*(I - J_{\gamma_n}^{B_2})Az_n\|^2\right),\ \forall z^* \in \Gamma.$$

Proof. For any $z^* \in \Gamma$, i.e., $z^* \in B_1^{-1}(0)$ and $Az^* \in B_2^{-1}(0)$. According to the firmly nonexpansive mappings $J_{\gamma_n}^{B_1}$, $J_{\gamma_n}^{B_2}$ and $I - J_{\gamma_n}^{B_2}$, we have

$$\begin{aligned}
\|u_n - z^*\|^2 &= \|z_n - z^*\|^2 + \lambda_n^2 \|A^*(I - J_{\gamma_n}^{B_2})Az_n\|^2 - 2\lambda_n \left\langle z_n - z^*, A^*(I - J_{\gamma_n}^{B_2})Az_n \right\rangle \\
&= \|z_n - z^*\|^2 + \lambda_n^2 \|A^*(I - J_{\gamma_n}^{B_2})Az_n\|^2 \\
&\quad - 2\lambda_n \left\langle Az_n - Az^*, (I - J_{\gamma_n}^{B_2})Az_n - (I - J_{\gamma_n}^{B_2})Az^* \right\rangle \\
&\le \|z_n - z^*\|^2 + \lambda_n^2 \|A^*(I - J_{\gamma_n}^{B_2})Az_n\|^2 - 2\lambda_n \|(I - J_{\gamma_n}^{B_2})Az_n\|^2 \\
&= \|z_n - z^*\|^2 - \lambda_n \left(2\|(I - J_{\gamma_n}^{B_2})Az_n\|^2 - \lambda_n \|A^*(I - J_{\gamma_n}^{B_2})Az_n\|^2 \right).
\end{aligned}$$

□

Theorem 1. If the solution set Γ is nonempty, the sequence $\{z_n\}$ generated by Algorithm 1 converges to $z^* \in \Gamma$ in norm and $z^* = P_\Gamma h(z^*)$, i.e., $\langle h(z^*) - z^*, \bar{z} - z^* \rangle \le 0, \ \forall\, \bar{z} \in \Gamma$.

Proof. Obviously, the solution set Γ is closed and convex. Hence, P_Γ is well defined. Choose $z^* \in \Gamma$ and $z^* = P_\Gamma h(z^*)$, that is, $z^* \in B_1^{-1}(0)$ and $Az^* \in B_2^{-1}(0)$. If for any $\varepsilon > 0$, $\|x_n - x^*\| \le \varepsilon$, this implies that $\{x_n\}$ is a bounded sequence. On the contrary, $\|x_n - x^*\| \ge \varepsilon$, there exists a number $\delta \in (0,1)$ by Lemma 2 such that $\|h(x_n) - h(x^*)\| \le \delta \|x_n - x^*\|$. Take

$$\Delta_n = 2\|(I - J_{\gamma_n}^{B_2})Aw_n\|^2 - \lambda_n \|A^*(I - J_{\gamma_n}^{B_2})Aw_n\|^2.$$

By Lemma 8 and the choice of the stepsize $\{\lambda_n\}$, we have $\Delta_n \ge 0$ and

$$\|u_n - z^*\|^2 = \|w_n - z^*\|^2 - \lambda_n \Delta_n \le \|w_n - z^*\|^2. \tag{8}$$

Further, it follows from (6) and (8) that

$$\begin{aligned}
\|z_{n+1} - z^*\| &\le \tau_n \|h(u_n) - z^*\| + (1 - \tau_n)\|u_n - z^*\| \\
&\le \tau_n \|h(u_n) - h(z^*)\| + \tau_n \|h(z^*) - z^*\| + (1 - \tau_n)\|u_n - z^*\| \\
&\le (1 - \tau_n(1 - \delta))\|u_n - z^*\| + \tau_n \|h(z^*) - z^*\| \\
&\le (1 - \tau_n(1 - \delta))\|w_n - z^*\| + \tau_n \|h(z^*) - z^*\| \\
&\le (1 - \tau_n(1 - \delta))\|z_n - z^*\| + \tau_n \|h(z^*) - z^*\| + (1 - \tau_n(1 - \delta))\alpha_n \|z_n - z_{n-1}\| \\
&\le (1 - \tau_n(1 - \delta))\|z_n - z^*\| + \tau_n(1 - \delta) \left(\frac{\|h(z^*) - z^*\|}{1 - \delta} + \frac{\alpha_n \|z_n - z_{n-1}\|}{\tau_n(1 - \delta)} \right).
\end{aligned}$$

Using the conditions $\lim_{n\to\infty} \frac{\alpha_n}{\tau_n} \|z_n - z_{n-1}\| = 0$ and $\delta \in (0,1)$, we have $\lim_{n\to\infty} \frac{\alpha_n \|z_n - z_{n-1}\|}{\tau_n(1 - \delta)} = 0$. That is to say, there exists a non-negative constant K such that

$$K/2 = \max\left\{ \frac{\|h(z^*) - z^*\|}{1 - \delta}, \frac{\alpha_n \|z_n - z_{n-1}\|}{\tau_n(1 - \delta)} \right\}.$$

Hence,

$$\begin{aligned}
\|z_{n+1} - z^*\| &\le (1 - \tau_n(1 - \delta))\|z_n - z^*\| + \tau_n(1 - \delta)K \\
&\le \max\{\|z_n - z^*\|, K\} \le \cdots \le \max\{\|z_0 - z^*\|, K\}.
\end{aligned}$$

This shows that $\{z_n\}$ is bounded. Similarly, $\{t_n\}$, $\{w_n\}$ and $\{u_n\}$ are also bounded. In addition, from the firmly nonexpansive mapping $J_{\gamma_n}^{B_1}$, we obtain

$$\begin{aligned}
\|w_n - z^*\|^2 &\le 2\langle w_n - z^*, t_n - z^* \rangle - \|w_n - z^*\|^2 \\
&= \|t_n - z^*\|^2 - \|w_n - t_n\|^2 \\
&\le \|z_n - z^*\|^2 + 2\alpha_n \langle t_n - z^*, z_n - z_{n-1} \rangle - \|w_n - t_n\|^2 \\
&\le \|z_n - z^*\|^2 + 2\alpha_n \|t_n - z^*\| \|z_n - z_{n-1}\| - \|w_n - t_n\|^2.
\end{aligned} \tag{9}$$

By combining (8) and (9), we get

$$\begin{aligned}\|z_{n+1}-z^*\|^2 &\le \|\tau_n(h(u_n)-h(z^*))+(1-\tau_n)(u_n-z^*)\|^2+2\tau_n\langle h(z^*)-z^*,z_{n+1}-z^*\rangle\\&\le \tau_n\|h(u_n)-h(z^*)\|^2+(1-\tau_n)\|u_n-z^*\|^2+2\tau_n\langle h(z^*)-z^*,z_{n+1}-z^*\rangle\\&\le (1-\tau_n(1-\delta^2))\|u_n-z^*\|^2+2\tau_n\langle h(z^*)-z^*,z_{n+1}-z^*\rangle\\&\le (1-\tau_n(1-\delta^2))\|w_n-z^*\|^2-(1-\tau_n(1-\delta^2))\lambda_n\Delta_n+2\tau_n\langle h(z^*)-z^*,z_{n+1}-z^*\rangle\\&\le (1-\tau_n(1-\delta^2))\|z_n-z^*\|^2+2(1-\tau_n(1-\delta^2))\alpha_n\|t_n-z^*\|\|z_n-z_{n-1}\|\\&\quad+2\tau_n\langle h(z^*)-z^*,z_{n+1}-z^*\rangle-(1-\tau_n(1-\delta^2))(\lambda_n\Delta_n+\|w_n-t_n\|^2).\end{aligned}$$

From the above inequality, we have

$$a_{n+1}\le(1-k_n)a_n+k_nb_n \text{ and } a_{n+1}\le a_n-c_n+d_n,\ n\ge 1,$$

where

$$\begin{aligned}&a_n=\|z_n-z^*\|^2,\ k_n=\tau_n(1-\delta^2),\ c_n=(1-\tau_n(1-\delta^2))(\lambda_n\Delta_n+\|w_n-t_n\|^2),\\&b_n=\frac{2(1-k_n)\alpha_n\|t_n-z^*\|\|z_n-z_{n-1}\|+2\tau_n\langle h(z^*)-z^*,z_{n+1}-z^*\rangle}{\tau_n(1-\delta^2)},\\&d_n=2(1-k_n)\alpha_n\|t_n-z^*\|\|z_n-z_{n-1}\|+2\tau_n\langle h(z^*)-z^*,z_{n+1}-z^*\rangle.\end{aligned}$$

Since $\sum_{n=1}^{\infty}\tau_n=\infty$, $\lim_{n\to\infty}\tau_n=0$, $\lim_{n\to\infty}\frac{\alpha_n}{\tau_n}\|z_n-z_{n-1}\|=0$ and $\{t_n\}$ is bounded, we see that $\sum_{n=1}^{\infty}k_n=\infty$ and $\lim_{n\to\infty}d_n=0$. Besides, suppose that $\{c_{n_k}\}$ is a subsequence of $\{c_n\}$ such that $\lim_{k\to\infty}c_{n_k}=0$. If $Aw_{n_k}\notin B_2^{-1}(0)$, we have $\lim_{k\to\infty}\|(I-J^{B_2}_{\gamma_{n_k}})Aw_{n_k}\|=0$ and $\lim_{k\to\infty}\|w_{n_k}-t_{n_k}\|=0$. Further,

$$\|w_{n_k}-z_{n_k}\|\le\|w_{n_k}-t_{n_k}\|+\|t_{n_k}-z_{n_k}\|\le\|w_{n_k}-t_{n_k}\|+\alpha_{n_k}\|z_{n_k}-z_{n_k-1}\|\to 0,\ \text{as } k\to\infty;$$

$$\|(I-J^{B_1}_{\gamma_{n_k}})z_{n_k}\|\le\|z_{n_k}-w_{n_k}\|+\|w_{n_k}-J^{B_1}_{\gamma_{n_k}}z_{n_k}\|\le\|z_{n_k}-w_{n_k}\|+\alpha_{n_k}\|z_{n_k}-z_{n_k-1}\|\to 0,\ \text{as } k\to\infty.$$

By Lemma 4 and $\inf_n\{\gamma_n\}\ge\gamma>0$, we have

$$\|(I-J^{B_2}_{\gamma})Aw_{n_k}\|\le 2\|(I-J^{B_2}_{\gamma_{n_k}})Aw_{n_k}\|\to 0,\ \|(I-J^{B_1}_{\gamma})z_{n_k}\|\le 2\|(I-J^{B_1}_{\gamma_{n_k}})z_{n_k}\|\to 0.$$

On the other hand, from the boundedness of $\{z_{n_k}\}$, there exists a subsequence $\{z_{n_{k_i}}\}$ of $\{z_{n_k}\}$ such that $z_{n_{k_i}}\rightharpoonup\hat z$ and $\limsup_{k\to\infty}\langle h(z^*)-z^*,z_{n_k}-z^*\rangle=\lim_{i\to\infty}\langle h(z^*)-z^*,z_{n_{k_i}}-z^*\rangle$. By virtue of $\lim_{k\to\infty}\|w_{n_k}-z_{n_k}\|=0$ and the bounded linear operator A, we obtain $w_{n_{k_i}}\rightharpoonup\hat z$ and $Aw_{n_{k_i}}\rightharpoonup A\hat z$. It follows from Lemma 5 that $\hat z\in\mathcal{F}(J^{B_1}_{\gamma})$ and $A\hat z\in\mathcal{F}(J^{B_2}_{\gamma})$, i.e., $\hat z\in\Gamma$. Meanwhile, if $Aw_{n_k}\in B_2^{-1}(0)$, one can also get the same result. According to the property of metric projection, $\lim_{i\to\infty}\langle h(z^*)-z^*,z_{n_{k_i}}-z^*\rangle=\langle h(z^*)-z^*,\hat z-z^*\rangle\le 0$. Besides,

$$\begin{aligned}\|z_{n_k+1}-z_{n_k}\|&\le\|z_{n_k+1}-u_{n_k}\|+\|u_{n_k}-w_{n_k}\|+\|w_{n_k}-z_{n_k}\|\\&\le\tau_{n_k}\|h(u_{n_k})-u_{n_k}\|+\lambda_{n_k}\|A\|\|(I-J^{B_2}_{\gamma_{n_k}})Aw_{n_k}\|+\|w_{n_k}-z_{n_k}\|\to 0,\ \text{as } k\to\infty.\end{aligned}$$

Hence, $\limsup_{k\to\infty}\langle h(z^*)-z^*,z_{n_k+1}-z^*\rangle\le 0$ and

$$\lim_{n\to\infty}\frac{(1-k_n)\alpha_n\|t_n-z^*\|\|z_n-z_{n-1}\|}{\tau_n(1-\delta^2)}\le\lim_{n\to\infty}\frac{\alpha_n\|t_n-z^*\|\|z_n-z_{n-1}\|}{\tau_n(1-\delta^2)}=0.$$

This means that $\limsup_{k\to\infty}b_{n_k}\le 0$. It follows from Lemma 6 that $\lim_{n\to\infty}\|z_n-z^*\|=0$, i.e., the iterative sequence $\{z_n\}$ converges to z^* in norm and $z^*=P_\Gamma h(z^*)$. □

Theorem 2. If the solution set Γ is nonempty, the sequence $\{z_n\}$ generated by Algorithm 2 converges in norm to $z^*\in\Gamma$ and $z^*=P_\Gamma(0)$, i.e., the minimum-norm element of Γ.

Proof. Firstly, since the solution set Γ is closed and convex, then there exists the minimum-norm element of Γ. Take $z^* \in \Gamma$ and $z^* = P_\Gamma(0)$, it follows from (8) that

$$\begin{aligned}\|z_{n+1} - z^*\| &= \|(1-\theta_n-\tau_n)(w_n - z^*) + \theta_n(u_n - z^*) - \tau_n z^*\| \\ &\le (1-\theta_n-\tau_n)\|w_n - z^*\| + \theta_n\|u_n - z^*\| + \tau_n\|z^*\| \\ &\le (1-\tau_n)\|w_n - z^*\| + \tau_n\|z^*\| \\ &\le (1-\tau_n)\|z_n - z^*\| + \tau_n\|z^*\| + (1-\tau_n)\alpha_n\|z_n - z_{n-1}\| \\ &\le (1-\tau_n)\|z_n - z^*\| + \tau_n\Big(\|z^*\| + \frac{\alpha_n\|z_n - z_{n-1}\|}{\tau_n}\Big).\end{aligned}$$

Similar to the proof of Theorem 1, we also obtain that the sequences $\{z_n\}$, $\{t_n\}$, $\{w_n\}$ and $\{u_n\}$ are bounded. Further, by (8) and (9), we have

$$\begin{aligned}\|z_{n+1} - z^*\|^2 &= \|(1-\theta_n-\tau_n)(w_n - z^*) + \theta_n(u_n - z^*) - \tau_n z^*\|^2 \\ &\le \|(1-\theta_n-\tau_n)(w_n - z^*) + \theta_n(u_n - z^*)\|^2 - 2\tau_n\langle z^*, z_{n+1} - z^*\rangle \\ &\le (1-\theta_n-\tau_n)^2\|w_n - z^*\|^2 + \theta_n^2\|u_n - z^*\|^2 - 2\tau_n\langle z^*, z_{n+1} - z^*\rangle \\ &\quad + 2(1-\theta_n-\tau_n)\theta_n\|w_n - z^*\|\|u_n - z^*\| \\ &\le (1-\theta_n-\tau_n)(1-\tau_n)\|w_n - z^*\|^2 + (1-\tau_n)\theta_n\|u_n - z^*\|^2 - 2\tau_n\langle z^*, z_{n+1} - z^*\rangle \\ &\le (1-\tau_n)^2\|w_n - z^*\|^2 - (1-\tau_n)\theta_n\lambda_n\Delta_n + 2\tau_n\langle z^*, z^* - z_{n+1}\rangle \\ &\le (1-\tau_n(1-\theta_n))\|z_n - z^*\|^2 + 2\alpha_n(1-\tau_n)^2\|t_n - z^*\|\|z_n - z_{n-1}\| \\ &\quad - (1-\tau_n)^2\|w_n - t_n\|^2 - (1-\tau_n)\theta_n\lambda_n\Delta_n + 2\tau_n\langle z^*, z^* - z_{n+1}\rangle.\end{aligned}$$

Thus, for each $n \ge 1$, we get

$$a_{n+1} \le (1-k_n)a_n + k_n b_n \text{ and } a_{n+1} \le a_n - c_n + d_n,$$

where

$$\begin{aligned}a_n &= \|z_n - z^*\|^2,\ k_n = \tau_n(1-\theta_n),\ c_n = (1-\tau_n)^2\|w_n - t_n\|^2 + (1-\tau_n)\theta_n\lambda_n\Delta_n, \\ b_n &= \frac{2\alpha_n(1-\tau_n)^2\|t_n - z^*\|\|z_n - z_{n-1}\| + 2\tau_n\langle z^*, z^* - z_{n+1}\rangle}{\tau_n(1-\theta_n)}, \\ d_n &= 2\alpha_n(1-\tau_n)^2\|t_n - z^*\|\|z_n - z_{n-1}\| + 2\tau_n\langle z^*, z^* - z_{n+1}\rangle.\end{aligned}$$

Based on the above derivation and the same method in Theorem 1, it follows from Lemma 6 that $\lim_{n\to\infty}\|z_n - z^*\| = 0$, i.e., the iterative sequence $\{z_n\}$ converges to $z^* = P_\Gamma(0)$ in norm. Meanwhile, from the property of the metric projection mapping, we have

$$\langle z^*, z^* - \bar{z}\rangle \le 0 \Leftrightarrow \|z^*\|^2 \le \|z^*\|\|\bar{z}\| \Leftrightarrow \|z^*\| \le \|\bar{z}\|,\ \forall \bar{z} \in \Gamma,$$

that is, z^* is the minimum-norm element of Γ. □

Remark 4. From the definition of Meir-Keeler contraction mapping, we can easily know that the contraction mapping is a special case. Thus, the viscosity algorithm with the contraction mapping and the Halpern algorithm are special cases of Algorithm 1.

Corollary 1. Let $\mathcal{H}_1$, $\mathcal{H}_2$, A, A^*, B_1 and B_2 be the same as Theorem 1 and γ_n, α_n, τ_n, σ_n, λ_n be the same as Algorithm 1. Let $h : \mathcal{H}_1 \to \mathcal{H}_1$ be a contraction mapping with coefficient $\mu \in [0,1)$. Take any $z_0, z_1 \in \mathcal{H}_1$, the sequence $\{z_n\}$ generated by the following iterative algorithm: t_n, w_n, u_n are generated in the same manner as (6). If $t_n = w_n = u_n$, then stop. Otherwise, calculate

$$z_{n+1} = \tau_n h(u_n) + (1-\tau_n)u_n, n \ge 1.$$

If the solution set Γ is nonempty, the sequence $\{z_n\}$ converges in norm to $z^* \in \Gamma$ and $z^* = P_\Gamma h(z^*)$, i.e., $\langle h(z^*) - z^*, \bar{z} - z^*\rangle \le 0$, $\forall\, \bar{z} \in \Gamma$.

Corollary 2. Let $\mathcal{H}_1$, $\mathcal{H}_2$, A, A^*, B_1 and B_2 be the same as Theorem 1 and γ_n, α_n, τ_n, σ_n, λ_n be the same as Algorithm 1. Let u be a fixed point in $\mathcal{H}_1$. Take any initial points $z_0, z_1 \in \mathcal{H}_1$, the sequence $\{z_n\}$ generated by the following algorithm: t_n, w_n, u_n are generated in the same manner as (6). If $t_n = w_n = u_n$, then stop. Otherwise, calculate

$$z_{n+1} = \tau_n u + (1-\tau_n)u_n, n \ge 1.$$

If the solution set Γ is nonempty, the sequence $\{z_n\}$ converges in norm to $z^* \in \Gamma$ and $z^* = P_\Gamma(u)$, that is, $\langle u - z^*, \bar{z} - z^* \rangle \leq 0$, $\forall\, \bar{z} \in \Gamma$.

Remark 5. (1) When Hilbert spaces $\mathcal{H}_1 = \mathcal{H}_2$, the split variational inclusion problem becomes the simultaneous variational inclusion problem: to find a point $z^* \in \mathcal{H}_1$ such that

$$0_{\mathcal{H}_1} \in B_1^{-1}(z^*) \text{ and } 0_{\mathcal{H}_1} \in B_2^{-1}(Az^*),$$

where B_1 and B_2 are maximal monotone mappings from $\mathcal{H}_1$ onto $2^{\mathcal{H}_1}$. Hence, the results of Theorems 1 and 2 can be applied to this problem. More importantly, the simultaneous variational inclusion problem involves the simultaneous variational inequality problem and the simultaneous equilibrium problem.

(2) When the bounded linear operator $A = I$ (I is identity mapping on $\mathcal{H}_1$) and $\mathcal{H}_1 = \mathcal{H}_2$, the split variational inclusion problem becomes the common solution of the variational inclusion problem (shortly, CSVIP). Using the conclusion of this paper, the iterative sequences generated by the Algorithms 1 and 2 with $\mathcal{H}_1 = \mathcal{H}_2$ and $A = I$ converges strongly to approximate solution of CSVIP. On the other hand, the common solution of the variational inequality problem, the common solution of the equilibrium problem and the convex feasibility problem can all be special cases of CSVIP.

4 | THEORETICAL APPLICATIONS

According to the transformation of SVIP, the split variational inequality problem, the split equilibrium problem and the split feasibility problem can be represented indirectly by appropriate settings in Subsection 2.1. Therefore, by our Theorems 1 and 2, the following results are easy to obtain and prove.

4.1 | The split variational inequality problem

Theorem 3. Let $\mathcal{H}_1, \mathcal{H}_2, C_1, Q_1, A, A^*, F_1, F_2$ be the same as SVIP* and $h, \gamma_n, \alpha_n, \tau_n, \sigma_n$ be the same as Algorithm 1. Take any initial points $z_0, z_1 \in \mathcal{H}_1$, the sequence $\{z_n\}$ generated by the following algorithm:

$$\begin{cases} t_n = z_n + \alpha_n(z_n - z_{n-1}), \\ w_n = J_{\gamma_n}^{S_{F_1}}(t_n), \\ u_n = w_n - \lambda_n A^*(I - J_{\gamma_n}^{S_{F_2}})Aw_n. \end{cases} \tag{10}$$

If $t_n = w_n = u_n$, then stop. Otherwise, go on to calculate

$$z_{n+1} = \tau_n h(u_n) + (1 - \tau_n)u_n, n \geq 1.$$

Here S_{F_1} and S_{F_2} are defined as (2),

$$\lambda_n = \begin{cases} \dfrac{\sigma_n \|(I - J_{\gamma_n}^{S_{F_2}})Aw_n\|^2}{\|A^*(I - J_{\gamma_n}^{S_{F_2}})Aw_n\|^2}, & Aw_n \notin S_{F_2}^{-1}(0), \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

If the solution set Φ of SVIP* is nonempty, the sequence $\{z_n\}$ converges in norm to $z^* \in \Phi$ and $z^* = P_\Phi h(z^*)$, i.e., $\langle h(z^*) - z^*, \bar{z} - z^* \rangle \leq 0$, $\forall\, \bar{z} \in \Phi$.

Proof. Set $B_1 = S_{F_1}$ and $B_2 = S_{F_2}$ in Theorem 1, we get the proof. □

Theorem 4. Let $\mathcal{H}_1, \mathcal{H}_2, C_1, Q_1, A, A^*, F_1, F_2$ be the same as SVIP* and $\gamma_n, \alpha_n, \tau_n, \sigma_n, \theta_n$ be the same as Algorithm 2. Take any initial points $z_0, z_1 \in \mathcal{H}_1$, the sequence $\{z_n\}$ generated by the following algorithm: t_n, w_n, u_n are generated in the same manner as (10). If $t_n = w_n = u_n$, then stop. Otherwise, go on to calculate

$$z_{n+1} = (1 - \theta_n - \tau_n)w_n + \theta_n u_n, n \geq 1.$$

Here S_{F_1} and S_{F_2} are defined as (2), λ_n is defined as (11). If the solution set Φ of SVIP* is nonempty, the sequence $\{z_n\}$ converges in norm to $z^* \in \Phi$ and $z^* = P_\Phi(0)$, i.e., the minimum-norm element of Φ.

4.2 | The split feasibility problem

Theorem 5. Let $\mathcal{H}_1, \mathcal{H}_2, C_1, Q_1, A, A^*$ be the same as SFP and $h, \alpha_n, \tau_n, \sigma_n$ be the same as Algorithm 1. Take any initial points $z_0, z_1 \in \mathcal{H}_1$, the sequence $\{z_n\}$ generated by the following algorithm:

$$\begin{cases} t_n = z_n + \alpha_n(z_n - z_{n-1}), \\ w_n = P_{C_1}(t_n), \\ u_n = w_n - \lambda_n A^*(I - P_{Q_1})Aw_n. \end{cases} \tag{12}$$

If $t_n = w_n = u_n$, then stop. Otherwise, go on to calculate

$$z_{n+1} = \tau_n h(u_n) + (1 - \tau_n)u_n, n \geq 1.$$

Here

$$\lambda_n = \begin{cases} \dfrac{\sigma_n \|(I - P_{Q_1})Aw_n\|^2}{\|A^*(I - P_{Q_1})Aw_n\|^2}, & Aw_n \notin Q_1, \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

If the solution set Υ of SFP is nonempty, the sequence $\{z_n\}$ converges in norm to $z^* \in \Upsilon$ and $z^* = P_\Upsilon h(z^*)$, i.e., $\langle h(z^*) - z^*, \bar{z} - z^* \rangle \leq 0, \ \forall \ \bar{z} \in \Upsilon$.

Proof. Using metric projection mappings P_{C_1}, P_{Q_1} and Theorem 1, we have the proof. □

Theorem 6. Let $\mathcal{H}_1, \mathcal{H}_2, C_1, Q_1, A$ and A^* be the same as above and $\alpha_n, \tau_n, \sigma_n, \theta_n$ be the same as Algorithm 2. Take any initial points $z_0, z_1 \in \mathcal{H}_1$, the sequence $\{z_n\}$ generated by the following algorithm: t_n, w_n, u_n are generated in the same manner as (12). If $t_n = w_n = u_n$, then stop. Otherwise, go on to calculate

$$z_{n+1} = (1 - \theta_n - \tau_n)w_n + \theta_n u_n, n \geq 1.$$

Here λ_n is defined as (13). If the solution set Υ of SFP is nonempty, the sequence $\{z_n\}$ converges in norm to $z^* \in \Upsilon$ and $z^* = P_\Upsilon(0)$, i.e., the minimum-norm element of Υ.

4.3 | The split equilibrium problem

From the definition of the split equilibrium problem, the symbol Λ represents the solution set of the split equilibrium problem, i.e., $\Lambda = \{z^* \in C_1 \mid z^* \in EP(G_1) \text{ and } Az^* \in EP(G_2)\}$.

Theorem 7. Let $\mathcal{H}_1, \mathcal{H}_2, C_1, Q_1, G_1, G_2, A, A^*$ be the same as SEP and $h, \alpha_n, \tau_n, \sigma_n$ be the same as Algorithm 1. Take any initial points $z_0, z_1 \in \mathcal{H}_1$ and $r > 0$, the sequence $\{z_n\}$ generated by the following algorithm:

$$\begin{cases} t_n = z_n + \alpha_n(z_n - z_{n-1}), \\ w_n = T_r^{G_1}(t_n), \\ u_n = w_n - \lambda_n A^*(I - T_r^{G_2})Aw_n. \end{cases} \tag{14}$$

If $t_n = w_n = u_n$, then stop. Otherwise, calculate

$$z_{n+1} = \tau_n h(u_n) + (1 - \tau_n)u_n, n \geq 1.$$

Here

$$\lambda_n = \begin{cases} \dfrac{\sigma_n \|(I - T_r^{G_2})Aw_n\|^2}{\|A^*(I - T_r^{G_2})Aw_n\|^2}, & Aw_n \notin EP(G_2), \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

If the solution set Λ is nonempty, the sequence $\{z_n\}$ converges in norm to $z^* \in \Lambda$ and $z^* = P_\Lambda h(z^*)$, i.e., $\langle h(z^*) - z^*, \bar{z} - z^* \rangle \leq 0, \ \forall \ \bar{z} \in \Lambda$.

Proof. Set $B_1 = \mathcal{K}_{G_1}$, $B_2 = \mathcal{K}_{G_2}$ and Theorem 1, we get the proof. □

Theorem 8. Let $\mathcal{H}_1, \mathcal{H}_2, C_1, Q_1, G_1, G_2, A$ and A^* be the same as SEP and $\alpha_n, \tau_n, \sigma_n, \theta_n$ be the same as Algorithm 2. Take any initial points $z_0, z_1 \in \mathcal{H}_1$ and $r > 0$, the sequence $\{z_n\}$ generated by the following algorithm: t_n, w_n, u_n are generated in the same manner as (14). If $t_n = w_n = u_n$, then stop. Otherwise, calculate

$$z_{n+1} = (1 - \theta_n - \tau_n)w_n + \theta_n u_n, n \geq 1.$$

Here λ_n is defined as (15). If the solution set Λ is nonempty, the sequence $\{z_n\}$ converges in norm to $z^* \in \Lambda$ and $z^* = P_\Lambda(0)$, i.e., the minimum-norm element of Λ.

Remark 6. The split variational inequality problem, the split feasibility problem and the split equilibrium problem are often used in practical applications, which includes image restoration, computer tomograph, signal recovery, etc. At the same time, many generalizations and meaningful results have been obtained thanks to these problems.

5 | NUMERICAL EXAMPLES

This section provides some numerical examples to illustrate the convergence behavior of the iterative sequence generated by our Algorithms 1 and 2. Meanwhile, these numerical results also confirm the conclusion of Theorems 1 and 2 on the split variational inclusion problem. All the programs were implemented in Matlab 2018a on a Intel(R) Core(TM) i5-8250U CPU @1.60 GHz computer with RAM 8.00 GB. Firstly, let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator with the adjoint operator A^*. Let $B_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $B_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ be two set-valued maximal monotone mappings. In such an environment, the following results for SVIP have been given as comparative algorithms.

Theorem 9. (Kazmi and Rizvi[19] Algorithm (3.1)) Let $h : \mathcal{H}_1 \to \mathcal{H}_1$ be a contraction mapping with coefficient $\mu \in (0, 1)$. For any initial point $z_1 \in \mathcal{H}_1$, $\tau_n \in (0, 1)$ and $\gamma > 0$, the iterative sequence $\{z_n\}$ is generated by the following iterative scheme

$$z_{n+1} = \tau_n h(z_n) + (1 - \tau_n)J_\gamma^{B_1}\left(z_n - \lambda A^*(I - J_\gamma^{B_2})Az_n\right),\ n \geq 1, \tag{KR}$$

where L is the spectral radius of the operator A^*A, $0 < \lambda < 1/L$, $\lim_{n\to\infty} \tau_n = 0$ and $\sum_{n=1}^{\infty} \tau_n = \infty$ and $\sum_{n=1}^{\infty} \left|\tau_n - \tau_{n-1}\right| < \infty$. The iterative sequence $\{z_n\}$ converges strongly to a point $z^* = P_\Gamma h(z^*)$, which is a solution of SVIP.

Theorem 10. (Long er al.[21] Algorithm (49)) Let $h : \mathcal{H}_1 \to \mathcal{H}_1$ be a contraction mapping with coefficient $\mu \in [0, 1)$. For any initial points $z_0, z_1 \in \mathcal{H}_1$ and $\gamma > 0$, the iterative sequence $\{z_n\}$ is generated by the following iterative scheme

$$\begin{cases} w_n = z_n + \alpha_n(z_n - z_{n-1}), \\ u_n = J_\gamma^{B_1}\left(w_n - \lambda_n A^*(I - J_\gamma^{B_2})Aw_n\right), \\ z_{n+1} = \tau_n h(z_n) + (1 - \tau_n)u_n, n \geq 1, \end{cases} \tag{LTD}$$

where $\{\tau_n\}$ is a sequence in $(0, 1)$ such that $\lim_{n\to\infty} \tau_n = 0$ and $\sum_{n=1}^{\infty} \tau_n = \infty$, $0 < a \leq \lambda_n \leq b < 1/\|A\|^2$, $0 \leq \alpha_n \leq \alpha$ and $\lim_{n\to\infty} \frac{\alpha_n\|z_n - z_{n-1}\|}{\tau_n} = 0$. The iterative sequence $\{z_n\}$ converges strongly to a point $z^* = P_\Gamma h(z^*)$.

Theorem 11. (Anh et al.[22] Algorithm (4)) For any initial points $z_0, z_1 \in \mathcal{H}_1$ and $\gamma > 0$, the iterative sequence $\{z_n\}$ is generated by the following iterative scheme

$$\begin{cases} w_n = z_n + \alpha_n(z_n - z_{n-1}), \\ u_n = J_\gamma^{B_1}\left(w_n - \lambda_n A^*(I - J_\gamma^{B_2})Aw_n\right), \\ z_{n+1} = (1 - \theta_n - \tau_n)z_n + \theta_n u_n, n \geq 1, \end{cases} \tag{ATD}$$

where $\{\tau_n\}$ is a sequence in $(0, 1)$ with $\lim_{n\to\infty} \tau_n = 0$, $\sum_{n=1}^{\infty} \tau_n = \infty$, $0 < a \leq \lambda_n \leq b < 1/\|A\|^2$, $0 \leq \alpha_n < \alpha$ and $\lim_{n\to\infty} \frac{\alpha_n\|z_n - z_{n-1}\|}{\tau_n} = 0$, $0 < c < \theta_n < d < 1 - \tau_n$. The iterative sequence $\{z_n\}$ converges strongly to a point $z^* = \operatorname{argmin}_{z\in\Gamma} \|z\|$.

Example 5.1. Assume that $A, A_1, A_2 : \mathbb{R}^m \to \mathbb{R}^m$ are created from a normal distribution with mean zero and unit variance. Let $B_1 : \mathbb{R}^m \to \mathbb{R}^m$ and $B_2 : \mathbb{R}^m \to \mathbb{R}^m$ be defined by $B_1(z) = A_1^* A_1 z$ and $B_2(y) = A_2^* A_2 y$, respectively. Consider the problem of finding a point $\bar{z} = \left(\bar{z}_1, \ldots, \bar{z}_m\right)^\mathsf{T} \in \mathbb{R}^m$ such that $B_1(\bar{z}) = (0, \ldots, 0)^\mathsf{T}$ and $B_2(A\bar{z}) = (0, \ldots, 0)^\mathsf{T}$. It is easy to see that the solution of the problem mentioned above is $z^* = (0, \ldots, 0)^\mathsf{T}$. The parameters of all algorithms are set as follows. For all algorithms, take $\gamma_n = \gamma = 1$, $\tau_n = 1/(n+1)$, $\theta_n = 0.5(1 - \tau_n)$ and $h(z) = 0.5z$. Set inertial parameters $\alpha = 0.5$ and $\rho_n = 1/(n+1)^2$ for Algorithm (LTD), Algorithm (ATD) and the proposed Algorithms 1 and 2. Select $\sigma_n = 1.5$ for the proposed Algorithms 1 and 2. Choose

$\lambda_n = \lambda = 0.5/\|A^*A\|$ for Algorithm (KR), Algorithm (LTD) and Algorithm (ATD). The process starts with the initial values $z_0 = z_1 = rand(n,1)$. $D_n = \|z_n - z^*\|$ is used to measure the iteration error of all the algorithms. The stopping condition is either $D_n < \epsilon$, or maximum number of iterations which is set to 299. Table 1 and Figure 1 describe the numerical behavior of all algorithms in different dimensions with the same stopping criterion $\epsilon = 10^{-7}$.

TABLE 1 Numerical results of Example 5.1

Algorithms	$m = 50$		$m = 100$		$m = 150$		$m = 200$	
	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)
Our Alg. 3.1	23	0.0082	17	0.0187	19	0.0304	19	0.0454
Our Alg. 3.2	26	0.0082	24	0.0226	23	0.0375	27	0.0628
KR Alg.	147	0.0654	98	0.1473	129	0.3786	136	0.6908
LTD Alg.	61	0.0283	35	0.0567	53	0.1535	55	0.2962
ATD Alg.	154	0.0652	108	0.1591	136	0.3997	146	0.7465

Example 5.2. Consider $\mathcal{H}_1 = \mathcal{H}_2 = L^2([0,1])$ with the inner product $\langle z, y\rangle := \int_0^1 z(t)y(t)\,\mathrm{d}t$ and the induced norm $\|z\| := \left(\int_0^1 |z(t)|^2\,\mathrm{d}t\right)^{1/2}$, for any $z, y \in L^2([0,1])$. Select the following nonempty closed and convex subsets C_1 and Q_1 in $L^2([0,1])$:

$$C_1 = \left\{ z \in L_2([0,1]) \mid \int_0^1 z(t)\,\mathrm{d}t \le 1 \right\},$$

$$Q_1 = \left\{ y \in L_2([0,1]) \mid \int_0^1 |y(t) - \sin(t)|^2\,\mathrm{d}t \le 16 \right\}.$$

Let $A : L^2([0,1]) \to L^2([0,1])$ be the Volterra integration operator, which is given by $(Az)(t) = \int_0^t z(s)\,\mathrm{d}s$, $\forall t \in [0,1]$, $z \in \mathcal{H}_1$. Then A is a bounded linear operator and its operator norm is $\|A\| = \frac{2}{\pi}$. Moreover, the adjoint operator A^* of A is defined by $(A^*z)(t) = \int_t^1 z(s)\,\mathrm{d}s$. This example is to solve SFP that is a special case of SVIP. Then, $z(t) = 0$ is a solution of SFP and thus the solution set of the problem is nonempty. On the other hand, it is known that projections on sets C_1 and Q_1 have display formulas, that is,

$$P_{C_1}(z) = \begin{cases} 1 - a + z\,, & a > 1\,; \\ z\,, & a \le 1\,. \end{cases} \quad \text{and } P_{Q_1}(y) = \begin{cases} \sin(\cdot) + \frac{4(y - \sin(\cdot))}{\sqrt{b}}\,, & b > 16\,; \\ y\,, & b \le 16\,, \end{cases}$$

where $a := \int_0^1 z(t)\,\mathrm{d}t$ and $b := \int_0^1 |y(t) - \sin(t)|^2\,\mathrm{d}t$.

We use symbolic computation in MATLAB to implement these algorithms for generating the sequences of iterates and use $E_n = \|(I - P_{C_1})z_n\|^2 + \|A^*(I - P_{Q_1})Az_n\|^2 < 10^{-5}$ for stopping criterion. All parameters in the five algorithms are selected as set in Example 5.1. Table 2 shows the numerical behavior of all algorithms with four different initial values (the inertial values $z_0 = z_1$ in Algorithms 1, 2, LTD and ATD).

Example 5.3. Compressed sensing is an effective method to recover a clean signal from a polluted signal. This requires us to solve the following underdetermined system problems:

$$\mathbf{y} = \mathbf{A}\mathbf{z} + \epsilon\,,$$

where $\mathbf{y} \in \mathbb{R}^M$ is the observed noise data, $\mathbf{A} \in \mathbb{R}^{M\times N}$ is a bounded linear observation operator, $\mathbf{z} \in \mathbb{R}^N$ with k ($k \ll N$) non-zero elements is the original and clean data that needs to be restored, and ϵ is the noise observation encountered during data transmission. An important consideration of this problem is that the signal $\mathbf{z}$ is sparse, that is, the number of non-zero elements in the signal $\mathbf{z}$ is much smaller than the dimension of the signal $\mathbf{z}$. A successful model used to solve the above problem can be translated into the following convex constraint minimization problem:

$$\min_{\mathbf{z}\in\mathbb{R}^N} \frac{1}{2}\|\mathbf{y} - \mathbf{A}\mathbf{z}\|^2 \quad \text{subject to} \quad \|\mathbf{z}\|_1 \le t\,, \tag{LASSO}$$

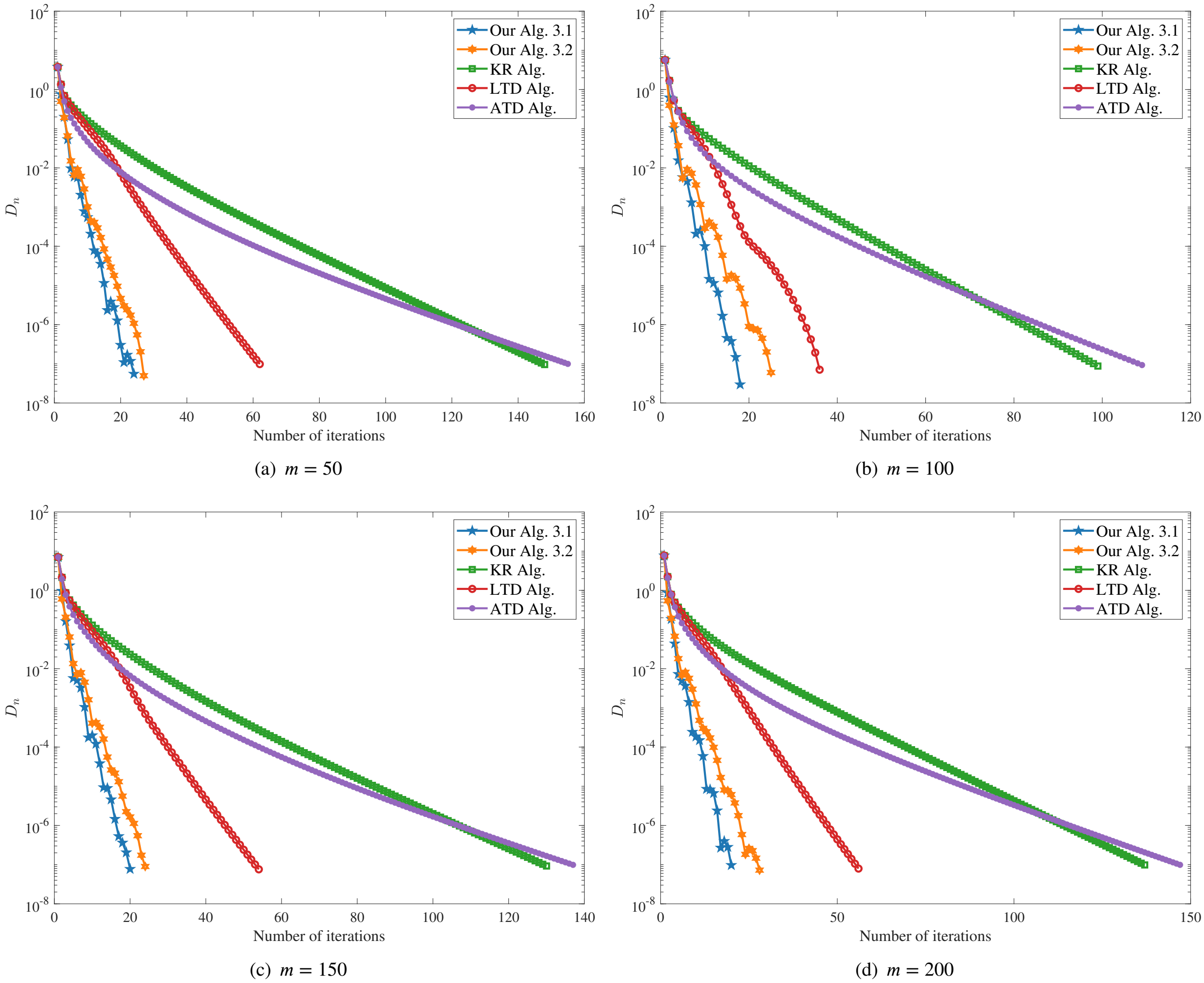


FIGURE 1 Numerical behavior of all algorithms with different dimensions in Example 5.1

TABLE 2 Numerical results of Example 5.2

Algorithms	$z_1 = 200\log(t)$		$z_1 = 1000\sin(t)$		$z_1 = 2000t^2$		$z_1 = 1000(t^3 + 2t)$	
	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)
Our Alg. 1	4	1.9061	7	4.7476	14	6.5722	14	8.6165
Our Alg. 2	5	2.3435	7	4.4679	11	5.0096	12	6.8348
KR Alg.	20	4.5199	28	6.1316	41	6.7691	45	7.8257
LTD Alg.	20	25.1634	28	24.8726	41	20.0938	46	31.1601
ATD Alg.	10	7.2614	15	12.3174	24	11.5068	29	18.3746

where t is a positive constant. It should be pointed out that this problem is related to the least absolute shrinkage and selection operator problem. Note that the problem (LASSO) described above can be regarded as a special case of the split feasibility problem when $C_1 = \left\{\mathbf{z} \in \mathbb{R}^N \mid \|\mathbf{z}\|_1 \leq t\right\}$ and $Q_1 = \{\mathbf{y}\}$.

We now consider using the proposed iterative schemes to solve (LASSO) and compare them with some known algorithms in the literature. In our numerical experiments, the matrix $\mathbf{A} \in \mathbb{R}^{M\times N}$ is created from a standard normal distribution with zero mean and unit variance and then orthonormalizing the rows. The clean signal $\mathbf{z} \in \mathbb{R}^N$ contains k ($k \ll N$) randomly generated

± 1 spikes. The observation $\mathbf{y}$ is formed by $\mathbf{y} = \mathbf{A}\mathbf{z} + \epsilon$ with white Gaussian noise ϵ of variance 10^{-4}. The recovery process starts with the initial signals $\mathbf{z_0} = \mathbf{z_1} = \mathbf{0}$ and ends after 2000 iterations. We use the mean squared error MSE $= (1/N)\,\|\mathbf{z}^* - \mathbf{z}\|^2$ ($\mathbf{z}^*$ is an estimated signal of $\mathbf{z}$) to measure the restoration accuracy of all algorithms. In our test, we set $M = 256$, $N = 512$ and $k = 50$. The parameters of all algorithms are the same as those set in Example 5.1. The numerical results are shown in the following figures. More precisely, Figure 2 displays the original signal and the contaminated signal. The recovery results of the suggested algorithms are shown in Figure 3 . Last, Figure 4 gives the numerical behavior of the MSE of all algorithms in the iteration process.

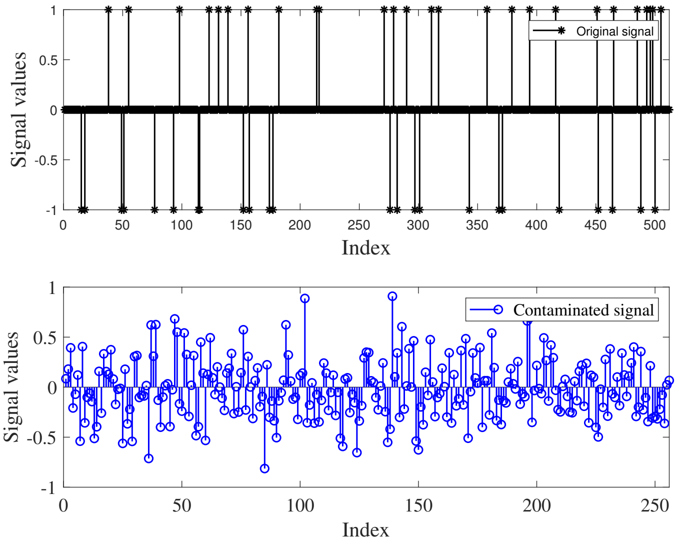
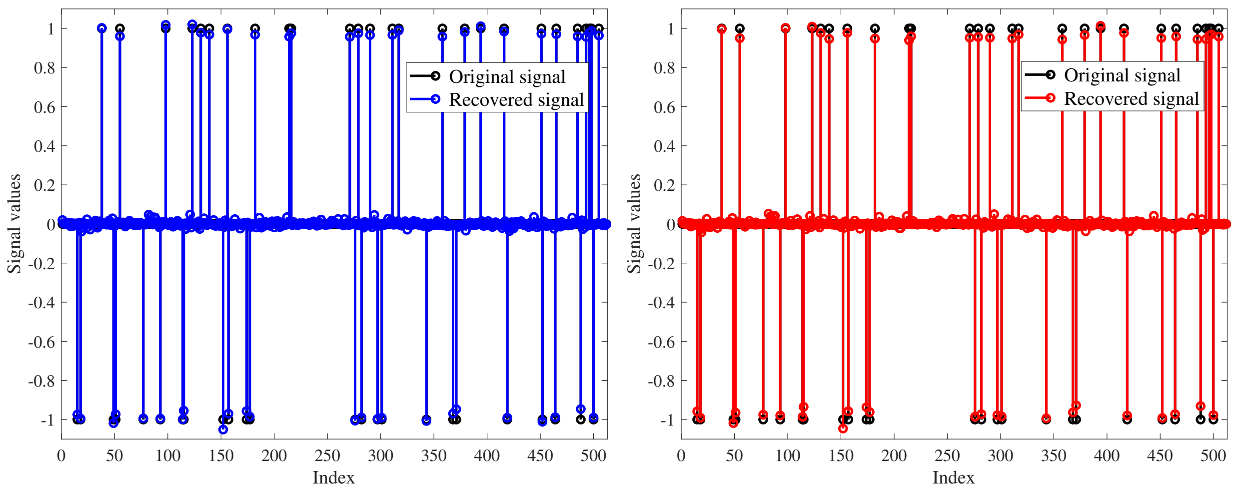


FIGURE 2 Original signal and contaminated signal



(a) Recovered by our Algorithm 1

(b) Recovered by our Algorithm 2

FIGURE 3 The original signal and the signal recovered by our algorithms

Remark 7. From the above Examples 5.1–5.3 and the test results, we can see that all our programming have been implemented. Meanwhile, some interesting observations for our algorithm are as follows:

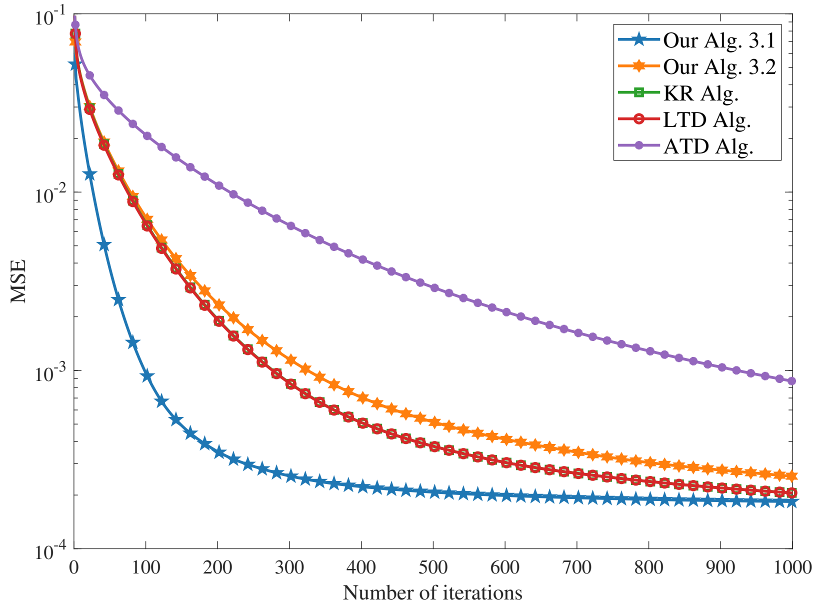


FIGURE 4 The discrepancy of mean squared error (MSE) of all algorithms

(1) The proposed algorithms can work well and converge quickly. Hence, our iterative schemes are efficient.

(2) The algorithms presented in this paper are better than some known ones in the literature, and these results have nothing to do with the choice of initial values and the size of the dimension. Therefore, our algorithms are robust.

(3) Note that the described algorithms can work adaptively, while the algorithms compared need to know the prior knowledge of the operator norm to work. Thus, our algorithms are more useful.

6 | CONCLUSIONS

In this paper, two inertial strong convergent iterative algorithms are given to approximate the solution of the split variational inclusion problem in real Hilbert spaces. The proposed algorithms add a new adaptive stepsize to overcome the shortcomings of existing fixed stepsize algorithms that require prior information about the operator norm. The strong convergence of the iterative sequences formed by our algorithms is demonstrated under appropriate conditions. In addition, three theoretical applications of our main results are given. Numerical experiments including signal recovery show that the suggested algorithms improve and extend some existing ones in the literature.

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