Four-body quantum system in the presence of solvable potential

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Abstract

In this article, a four-particle quantum system is introduced with energy discrete spectrum including a harmonic potential and a three-body interaction potential. By defining the Jacobi coordinates for each particle, separately, one coordinate is eliminated as a transition in the energy spectrum. Then the system is studied in polar coordinates and by using the variables separation method, the Schrödinger equation of the system is transformed into three separate differential equations. Therefore, energy eigenvalues and wave eigenfunctions are calculated in each dimension. Also, The wave eigenfunctions figures are investigated in one and three dimensions.

Keywords: Four-body quantum system; Schrödinger equation; Harmonic potential; Three-body interaction; Energy eigenvalues; Wave eigenfunctions.

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1 Introduction

Almost five decades ago, Calogero Quantum Mechanics introduced many-body models. For three identical particles, he considered the harmonic oscillator and inverse square potentials with pairwise interactions and solved the corresponding Schrödinger equation [1]. This was also brought to the attention of Wolfes, for the first time examining the three-particle system Calogero in the presence of three-body interaction [2]. Callogero and Marchioro studied new aspects of the dispersion of the three-particle system with two-body and three-body interactions, they examined the problem from classical and quantum perspective [3]. Perelomov and Olshanetsky, while giving an overview of many-body systems, showed that quantum models are related to the root system of Lie algebras, they studied many-body models from algebraic viewpoints and their relation to symmetric spaces [4-5]. In Reference [6], Levai has investigated the shape-invariant solvable potentials, which can be used to obtain complete solution and wavefunction. It can also be mentioned in Articles [7-8] that have investigated the shape-invariant solvable potentials for the Dirac equation.

In this article we tend to use Calogero's method to solve four-particle in one dimension in equation, two-body Harmonical potential and three body interaction.

This paper is organized as follows:

In section 2, first the intended four particle quantum model is introduced then using Jacobi conversions, mass center coordinates is deleted as a transition in spectrum and the system is ported to spherical coordinates which contains three scale of freedom. In the end, using the method of separating variables, Schrödinger equation is divided into a radial section and two angular sections.

In section 3, first by considering a variable change, first order differential sentence is removed in intended equation then using article [6-8] and existing potentials in article, a wave function related to the equation situation is obtained which consists of Laguerre polynomial.

In section 4, like the previous section we calculate a function related to existing equations one attached to θ and the other one to ϕ, using article [6-7], which consists of Jacobi polynomial.

In the last section we present an overall conclusion.

2 Four-body Hamiltonian system with an interaction

A four-particle quantum model written below as a solvable model is introduced [4-5]:
$(\hbar = 2m = 1)$

$$H = \left((-1/2)\sum_{i=1}^{4}(\frac{\partial^2}{\partial x_i^2}) + (1/8)(\omega^2 \sum_{i<j}^{4}(x_i - x_j)^2) + \frac{g_1^2}{((x_1+x_2-2x_3)^2)} \right), \tag{2.1}$$

This system consists of three-body interaction potential and simple coordinate oscillator in which the second order sentence of simple coordinate oscillator creates a discrete spectrum. It's important to note the fact that the three-body potential is a particular mode. To justify the choice of interaction type, we can assume the Coupling coefficient of fourth particle is quite little. Such assumption was made by Calogero[1], it means a three particle system with considering the Coupling coefficient zero in two other interactions and keeping the two particle interactions between two particles and solved the problem analytically. Another justification for the three particle potential is to consider the fourth particle in a long distance and create a solvable model. In order to solve the eigenvalue equation of this Hamiltonian, first we use Jacobi conversions and port the system to the spherical coordinates and using the change of variables method, we investigate Schrödinger equation in radial and angular section [1-2-3]. We consider Jacobi conversions written bellow as four particle coordinates:

$$x_1 - x_2 = \sqrt{2}X_1 \tag{2.2}$$

$$x_1 + x_2 - 2x_3 = \sqrt{6}X_2 \tag{2.3}$$

$$x_1 + x_2 + x_3 - 3x_4 = \sqrt{12}X_3 \tag{2.4}$$

$$x_1 + x_2 + x_3 + x_4 = 2X \tag{2.5}$$

By applying above transformations, Hamiltonian is transformed into the following relation:

$H = H(X_1, X_2, X_3) - \frac{\partial^2}{\partial X^2}$

We can remove mass center coordinate as a transition in spectrum, therefore using the following transformations we port the system to the spherical coordinate:

$$X_1 = rSin(\theta)Cos(\phi) \tag{2.6}$$

$$X_2 = rSin(\theta)Sin(\phi) \tag{2.7}$$

$$X_3 = rCos(\theta) \tag{2.8}$$

$0 \leq r < \infty \quad , \quad 0 \leq \theta \leq 2\pi \quad , \quad 0 \leq \phi \leq \pi$

Oscillator sentence is converted as follows:

$$\sum_{i<j}^{4}(x_i - x_j)^2 = 4r^2 \tag{2.9}$$

The kinetic sentence is converted to Laplace operator in spherical coordinate and (2.1) is written as follows after making essential conversions and removing the mass center:

$$H = \left(\tfrac{-1}{2r^2}\tfrac{\partial}{\partial r}(r^2\tfrac{\partial}{\partial r}) - \tfrac{1}{2r^2 Sin(\theta)}\tfrac{\partial}{\partial \theta}(Sin(\theta)\tfrac{\partial}{\partial \theta}) - \tfrac{1}{2r^2 Sin^2(\theta)}\tfrac{\partial^2}{\partial \phi^2} + \tfrac{1}{2}\omega^2 r^2 + \tfrac{g_1^2}{6r^2 Sin^2(\theta) Sin^2(\phi)} \right), \tag{2.10}$$

Now considering the changes and (2.10) , we can review Schrödinger equation:

$(H - E)\Psi = 0$

$$(\frac{-1}{2r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{2}\omega^2 r^2 + \frac{K}{2r^2} - E)\Psi = 0 \tag{2.11}$$

$$K = \frac{-1}{Sin(\theta)}\frac{\partial}{\partial \theta}(Sin(\theta)\frac{\partial}{\partial \theta}) + \frac{F}{Sin^2(\theta)} \tag{2.12}$$

$$F = -\frac{\partial^2}{\partial \phi^2} + \frac{g_1^2}{3Sin^2(\phi)} \tag{2.13}$$

Therefore by separating variables, Schrödinger equation is solvable and it's divided into a radial and two angular parts:

$$\Psi(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi) \tag{2.14}$$

3 Analyzing radial coordinates of Schrödinger equation

First, the following relation is defined [1]:

$$K\Theta_l(\theta) = k_l^2\Theta_l(\theta) \qquad\qquad l = 0, 1, 2, ... \tag{3.1}$$

k_l^2 is isolation constant.

Therefore (2.11) Schrödinger equation is converted to the radial section of this equation:

$$(\frac{-1}{2r^2}\frac{d}{dr}(r^2\frac{d}{dr}) + \frac{1}{2}\omega^2 r^2 + \frac{k_l^2}{2r^2} - E)R(r) = 0 \tag{3.2}$$

Considering the relation, $R(r) = \frac{U(r)}{r}$, first order differential sentence is removed in (3.2) [7]:

$$\frac{d^2U(r)}{dr^2} + (2E - \omega^2 r^2 - \frac{k_l^2}{r^2})U(r) = 0 \tag{3.3}$$

Considering [6-7-8], following differential equation is comparable to equation (3.3):

$$\frac{d^2U(r)}{dr^2} + (2n\omega + (l + \frac{3}{2})\omega - \frac{1}{4}\omega^2 r^2 - \frac{l(l+1)}{r^2})U(r) = 0 \tag{3.4}$$

Comparing (3.3) and (3.4), relations between parameters is found as:

$$k_l^2 = l(l+1) \tag{3.5}$$

$$E = \frac{1}{2}(4)^{\frac{1}{2}}(2n + l + \frac{3}{2})\omega \tag{3.6}$$

Based on [6-7-8], wavefunction resulting from equation (3.4), is written as followed:

$$U_{n,l}(r) = g^{(l+1)/2} exp(\frac{-g}{2})L_n^{(l+1/2)}(g(r)) \tag{3.7}$$

$$g(r) = \tfrac{1}{2}\omega r^2$$

Considering wavefunction (3.7), we can obtain the related function with equation (3.3):

$$U_{n,l}(r) = (\frac{1}{2}\omega r^2)^{(l+1)/2} exp(\frac{-1}{4}(4)^{\frac{1}{2}}\omega r^2)L_n^{(l+1/2)}(\frac{1}{2}(4)^{\frac{1}{2}}\omega r^2) \tag{3.8}$$

In accordance with $R(r) = \frac{U(r)}{r}$, radial section of Schrödinger equation is written as follows:

$$R(r) = (\frac{1}{2}\omega)^{(l+1)/2} r^l exp(\frac{-1}{2}\omega r^2) L_n^{(l+1/2)}(\omega r^2) \tag{3.9}$$

figures (1) and (2) are related to this function.

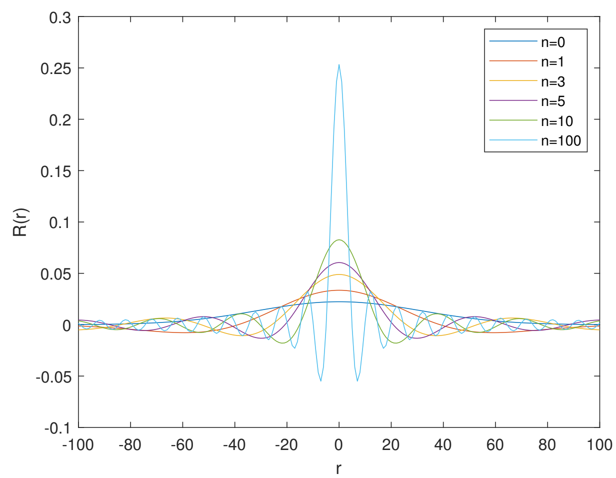

Figure 1: $R(r)$ versus r with $l = 0, \omega = 10^{-3}$.

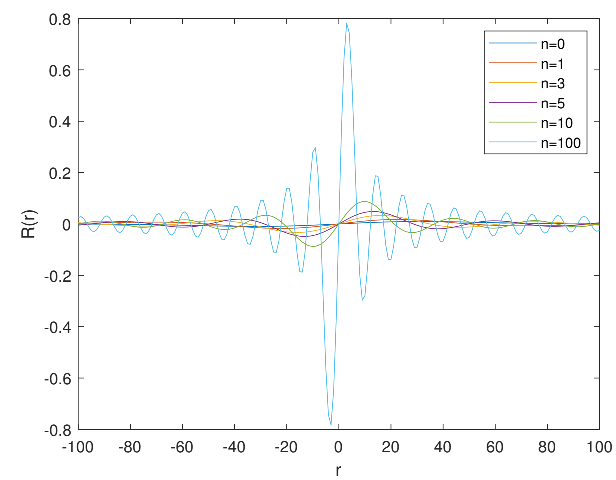

Figure 2: $R(r)$ versus r with $l = 1, \omega = 10^{-3}$.

4 Investigating angular coordinates of Schrödinger equation

First, angular part related to θ is investigated:

$$K\Theta(\theta) = \left(\tfrac{-1}{Sin(\theta)} \tfrac{d}{d\theta}(Sin(\theta)\tfrac{d\Theta(\theta)}{d\theta}) + \tfrac{F}{Sin^2(\theta)}\Theta(\theta) = k_l^2\Theta(\theta) \right), \tag{4.1}$$

Following relation is defined [1] :

$$F\Phi_m(\phi) = f_m^2\Phi_m(\phi) \qquad\qquad m = 0, 1, 2, ... \tag{4.2}$$

f_m^2 is isolation constant.

Considering (4.2), equation (4.1) is written as followed:

$$\frac{-1}{Sin(\theta)}\frac{d}{d\theta}(Sin(\theta)\frac{d\Theta(\theta)}{d\theta}) + (\frac{f_m^2}{Sin^2(\theta)} - k_l^2)\Theta(\theta) = 0 \tag{4.3}$$

Considering the relation $\Theta(\theta) = \frac{H(\theta)}{Sin^{1/2}(\theta)}$, first order differential sentence in (4.3) will be deleted [7]:

$$\frac{d^2H(\theta)}{d\theta^2} + ((\frac{1}{4} - f_m^2)csc^2(\theta) + (k_l^2 + \frac{1}{4}))H(\theta) = 0 \tag{4.4}$$

Considering [6-7], following differential equation is compared with (4.4):

$$\frac{d^2H(x)}{dx^2} + (-(\lambda^2 + s^2 - s)csc^2(x) + \lambda(2s-1)csc(x)cot(x) + (s+n)^2)H(x) = 0 \tag{4.5}$$

Comparing (4.4) and (4.5), relations between parameters is found as:

$$f_m^2 - \frac{1}{4} = \lambda^2 + s^2 - s \qquad \Rightarrow \qquad f_m = \lambda \tag{4.6}$$

$$0 = \lambda(2s-1) \qquad \Rightarrow \qquad s = \frac{1}{2} \tag{4.7}$$

$$k_l^2 + \frac{1}{4} = (s+n)^2 = (n + \frac{1}{2})^2 \qquad \Rightarrow \qquad k_l^2 = n(n+1) \tag{4.8}$$

Wavefunction resulting from equation (4.5), is written as followes [6-7]:

$$H(x) = (1-g)^{\frac{(s-\lambda)}{2}}(1+g)^{\frac{(s+\lambda)}{2}}P_n^{(-\lambda+s-1/2,\lambda+s-1/2)}(g(x)) \tag{4.9}$$

$$g(x) = Cos(x)$$

Considering wavefunction (4.9), a function related to (4.4) is resulted:

$$H(\theta) = (1 - Cos(\theta))^{\frac{(s-\lambda)}{2}}(1 + Cos(\theta))^{\frac{(s+\lambda)}{2}} P_n^{(-\lambda+s-1/2,\lambda+s-1/2)}(Cos(\theta)) \tag{4.10}$$

In accordance with $\Theta(\theta) = \frac{H(\theta)}{Sin^{1/2}(\theta)}$, the wavefunction related to angular section (θ) is as followed:

$$\Theta(\theta) = 2^{s-1}(Sin(\theta))^{s-\lambda-1/2}(Cos(\theta))^{s+\lambda-1/4} P_n^{(-\lambda+s-1/2,\lambda+s-1/2)}(Cos(\theta)) \tag{4.11}$$

Figures (3) and (4) is related to this function.

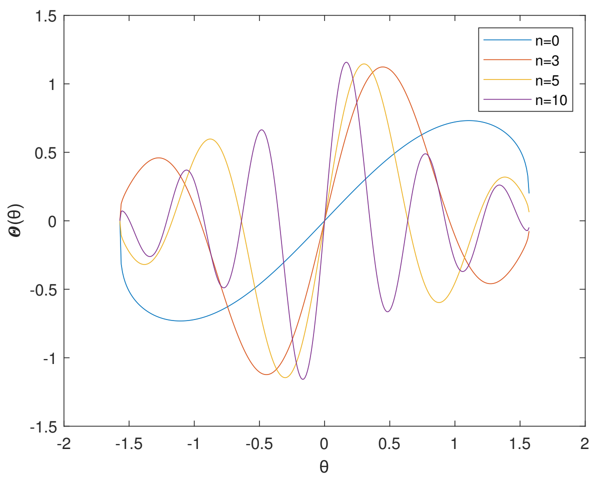

Figure 3: $\Theta(\theta)$ versus θ with $s = 1, \lambda = -0.5$.

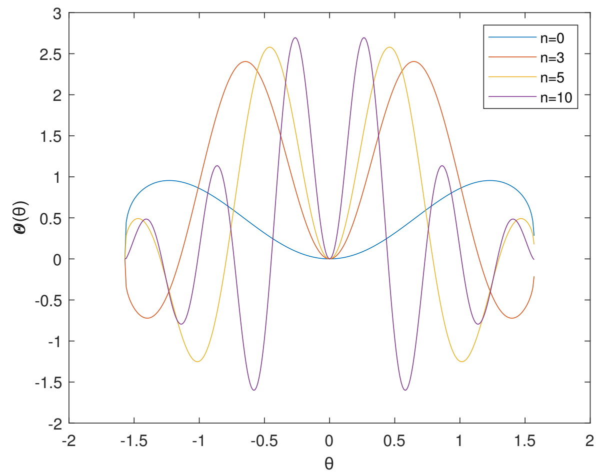

Figure 4: $\Theta(\theta)$ versus θ with $s = 1.5, \lambda = -1$.

Now, we calculate angular section related to ϕ, based on equation (4.2), we can rewrite the equation related to this section:

$$F\Phi(\phi) = (\frac{-d^2}{d\phi^2} + \frac{g_1^2}{3Sin^2(\phi)})\Phi(\phi) = f_m^2\Phi(\phi) \qquad \Rightarrow \qquad \frac{d^2\Phi(\phi)}{d\phi^2} + (f_m^2 - \frac{1}{3}g_1^2 csc^2(\phi))\Phi(\phi) = 0 \tag{4.12}$$

Comparing (4.12) and (4.5), relations between parameters is found as:

$$-(\lambda^2 + s^2 - s) = -\frac{1}{3}g_1^2 \qquad \Rightarrow \qquad g_1^2 = 3(\lambda^2 - \frac{1}{4}) \tag{4.13}$$

$$\lambda(2s - 1) = 0 \qquad \Rightarrow \qquad s = \frac{1}{2} \tag{4.14}$$

$$(s + n)^2 = f_m^2 \qquad \Rightarrow \qquad f_m = n + \frac{1}{2} \tag{4.15}$$

Considering function (4.9), wavefunction related to equation (4.12) is obtained:

$$\Phi(\phi) = (1 - Cos(\phi))^{\frac{(s-\lambda)}{2}}(1 + Cos(\phi))^{\frac{(s+\lambda)}{2}} P_n^{(-\lambda+s-1/2,\lambda+s-1/2)}(Cos(\phi)) \tag{4.16}$$

Figures (5) and (6) is related to this function.

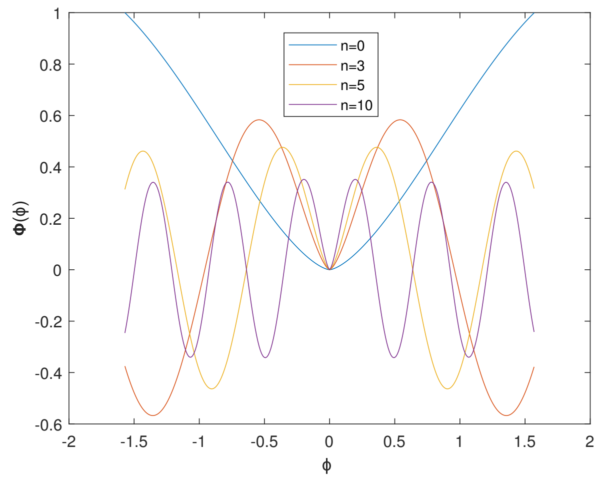

Figure 5: $\Phi(\phi)$ versus ϕ with $s = 1, \lambda = -0.5$.

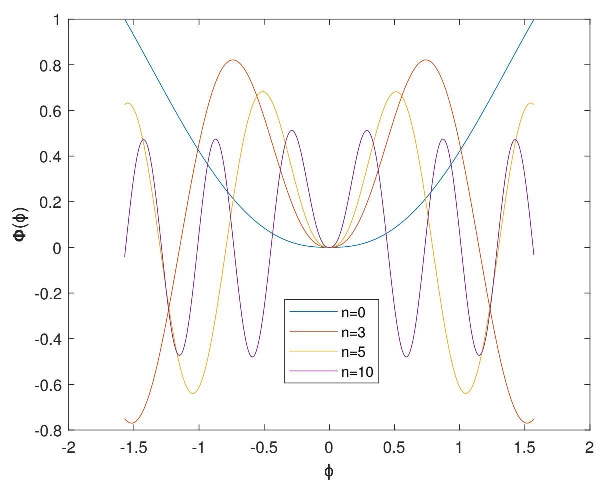

Figure 6: $\Phi(\phi)$ versus ϕ with $s = 1.5, \lambda = -1$.

After examining the probability density of the functions $R(r), \Theta(\theta), \Phi(\phi)$ and as well as their diagrams, it is observed that increasing the probability interval decreases the probability value, that is, by increasing n the probability value decreases and it can be said that the probability of the particle being present for small values of n increases Finds. As we know, the existence of the Laguerre polynomial in the radial section and the Jaccobi polynomial in the angular section cause a decreasing trend in the probability of the particle being present. Considering the resulted functions (3.9), (4.11), (4.16), and also relations between parameters, we can write eigenvalue function and eigenvalue of this four-body problem, as follows:

$$\Psi_{n,l,m}(r,\theta,\phi) = (\frac{1}{2}\omega)^{\frac{\sqrt{k_l^2+1/4}}{2}} r^{(\sqrt{k_l^2+1/4})-1/2} exp(-\frac{1}{2}\omega r^2)$$
$$\frac{\sqrt{2}}{2}\frac{(Cos(\theta))^{(f_m+1/4)}}{(Sin(\theta))^{f_m}}(1-Cos(\phi))^{\frac{\frac{1}{2}-\sqrt{\frac{1}{3}g_1^2+\frac{1}{4}}}{2}}(1+Cos(\phi))^{\frac{\frac{1}{2}+\sqrt{\frac{1}{3}g_1^2+\frac{1}{4}}}{2}}$$
$$L_n^{(\sqrt{k_l^2+1/4})}(\omega r^2)P_n^{(-f_m,f_m)}(Cos(\theta))P_n^{(-\sqrt{\frac{1}{3}g_1^2+\frac{1}{4}},\sqrt{\frac{1}{3}g_1^2+\frac{1}{4}})}(Cos(\phi)) \tag{4.17}$$

$$E = (2n + \sqrt{k_l^2+1/4}+1)\omega \tag{4.18}$$

Figures (7), (8), (9), (10), (11) and (12) is related to this function.

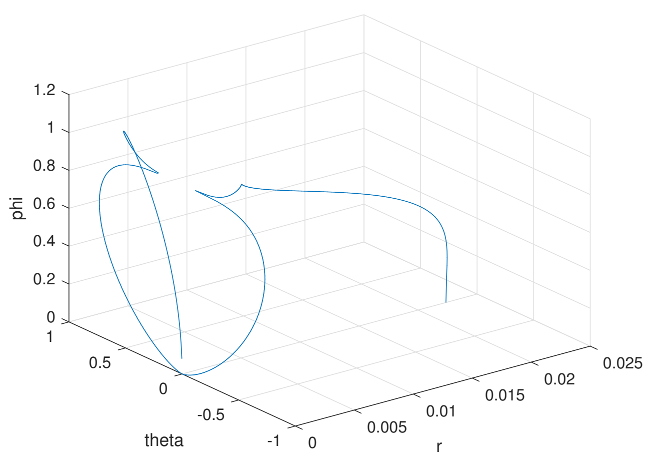

Figure 7: $\Psi(r,\theta,\phi)$ with $n=0, l=0, \omega=10^{-3}, s=1, \lambda=-0.5$.

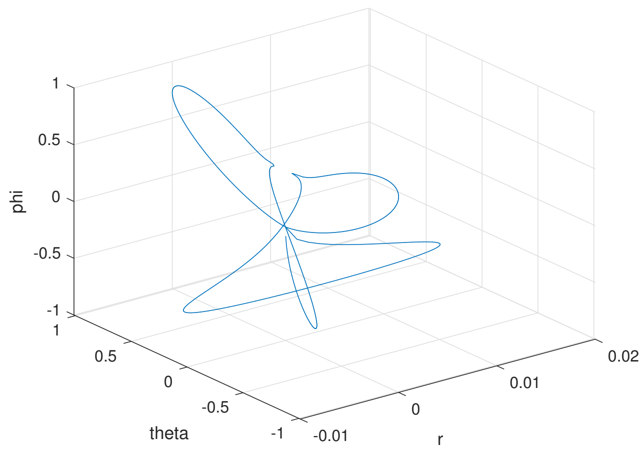

Figure 8: $\Psi(r, \theta, \phi)$ with $n = 1, l = 1, \omega = 10^{-3}, s = 1, \lambda = -0.5$.

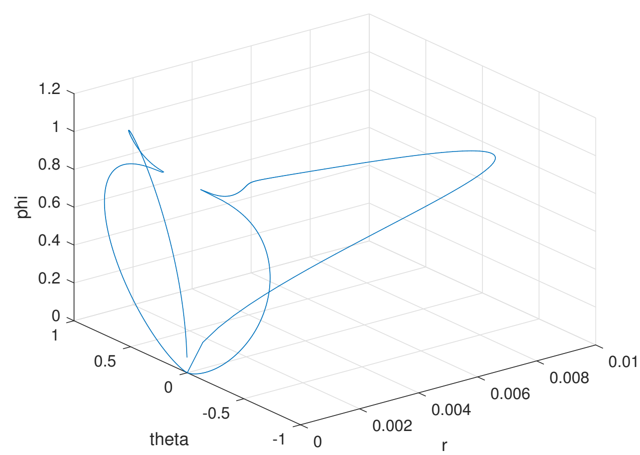

Figure 9: $\Psi(r, \theta, \phi)$ with $n = 0, l = 1, \omega = 10^{-3}, s = 1, \lambda = -0.5$.

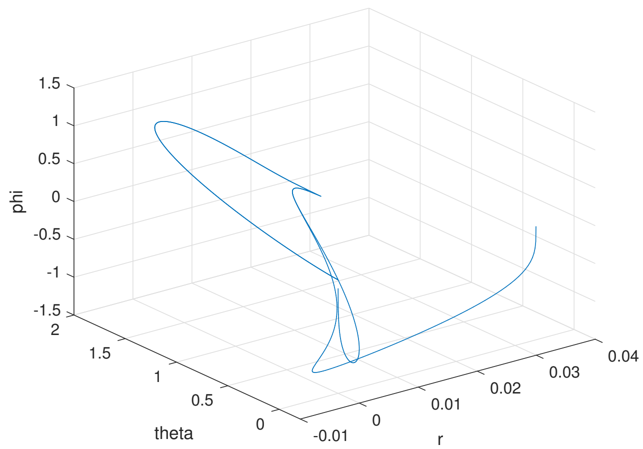

Figure 10: $\Psi(r, \theta, \phi)$ with $n = 1, l = 0, \omega = 10^{-3}, s = 1.5, \lambda = -1$.

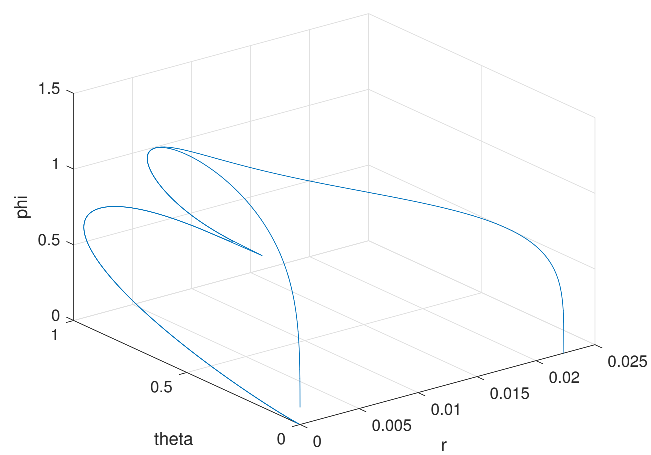

Figure 11: $\Psi(r, \theta, \phi)$ with $n = 0, l = 0, \omega = 10^{-3}, s = 1.5, \lambda = -1$.

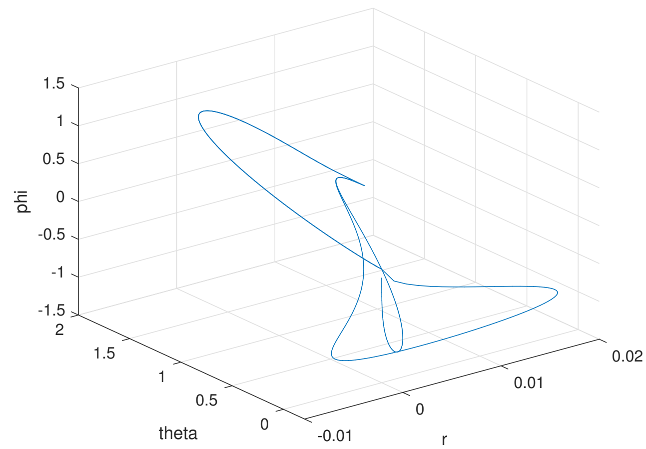

Figure 12: $\Psi(r, \theta, \phi)$ with $n = 1, l = 1, \omega = 10^{-3}, s = 1.5, \lambda = -1$.

5 Conclusion

In this article, Schrödinger equation was investigated to study four-body problem with Harmonical potential and three-body interaction potential and eigenvalue and eigenfunction of this problem was determined using severation of variable method. Then we plot graphs of the radial and angular functions as well as the function $\Psi(r, \theta, \phi)$. Considering the diagrams, it can be seen that the functions are integral squared. The type of generalization that appears natural is an increase in the number of particles and/or dimensions of the space.

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