WAVELETS FOR NONUNIFORM NON-STATIONARY MRA ON $H^s(\mathbb{K})$

ASHISH PATHAK[1] AND GURU P SINGH[2]
DEPARTMENT OF MATHEMATICS
INSTITUTE OF SCIENCE
BANARAS HINDU UNIVERSITY
VARANASI-221005, INDIA.

Abstract. In this paper, we are defined the nonuniform non-stationary multiresolution analysis (NUNSMRA) on Sobolev space over local fields ($H^s(\mathbb{K})$) and with help of NUNSMRA orthonormal wavelets are constructed.

1. Introduction

The definition of multiresolution analysis on local fields with positive characteristic is given by Jian, Li and Jin [1] and they have constructed orthonormal wavelets associated to the multiresolution analysis. Their theory have been extended by Behra and jahan in [5]. A nonuniform multiresolution analysis and Generalized nonuniform multiresolution analyses on $L^2(\mathbb{K})$ are given by F. A. Shah [2] and Shukla et.al [12, 13, 14]. Recently, Pathak, Singh and Kumar [[6], [7], [8], [9]] considered non-stationary MRA on Sobolev space over local fields of positive characteristic ($H^s(\mathbb{K})$) in which one single scaling function cannot generate orthonormal functions at each level of dilation. In this paper, we construct Non-stationary MRA on $H^s(\mathbb{K})$ in which translation set is nonuniform and no longer a group, but is the union of T_1 and a translate of T_1,

2010 *Mathematics Subject Classification.* 42C40; 46E35;11F87.
Key words and phrases. Wavelets , Nonuniform Multiresolution analysis, Local fields, Sobolev space, Fourier Transform.

[1] E-mail: ashishpathak@bhu.ac.in, [2] E-mail: gurusinghab9092@gmail.com.

where $T_1 = \{u(m) : m \in \mathbb{N}_0\}(\mathbb{N}_0 = \{0, 1, 2, 3, ...\})$ is a complete list of (distinct) coset representation of $\mathfrak{D}$ in $\mathbb{K}^+$. That is, this set is of the form $T = \{0, \frac{r}{M}\} + T_1$, where $M \geq 1$ is an integer and r is an odd integer such that $g.c.d.(r, M) = 1$. We call this a nonuniform non-stationary MRA on $H^s(\mathbb{K})$. By this generalization, our main aim is to develop nonuniform non-stationary MRA on Sobolev space and to construct associated orthonormal wavelets in which scaling functions depend on level and translation set is nonuniform.

This paper is organised as follows. Section 2, contains notation and definitions of local fields and Sobolev space over local fields. We define nonuniform non-stationary multiresolution analysis on $H^s(\mathbb{K})$ and constructed corresponding wavelets in Section 3 and its subsection.

2. Preliminaries

The following list of notation and definitions are given below will be used throughout the paper.

- Throughout this paper $\mathbb{K}$ denotes the local field of positive characteristic.
- dx is the normalized Haar measure for $\mathbb{K}^+$.
- $|\alpha|$ is the valuation of $\alpha \in \mathbb{K}$ and it is non-archimedian norm.
- Let $\mathfrak{p}$ be a prime element in $\mathbb{K}$.
- For $k \in \mathbb{Z}$, $\mathfrak{P}^k = \{x \in \mathbb{K} : |x| \leq q^{-k}\}$ is a compact subgroup of $\mathbb{K}^+$, where $q = p^c$, p is a prime number and c is a positive integer.
- $\mathfrak{P}^0 = \mathfrak{D}$ is called ring of integres in $\mathbb{K}$.
- $|\mathfrak{P}^k| = q^{-k}$ and $|\mathfrak{D}| = 1$.
- χ be a fixed character on $\mathbb{K}^+$ that is trivial on $\mathbb{D}$ but is non trivial on $\mathfrak{P}^{-1}$. For $y \in \mathbb{K}$, $\chi_y(x) = \chi(yx)$, $x \in \mathbb{K}$.
- The "natural"order on the sequence is denoted by $\{u(k) \in \mathbb{K}\}_{k=0}^{\infty}$ and is described as follows.

  $\mathfrak{D}/\mathfrak{P} \cong GF(q) = \tau$, $q = p^s$, p is a prime, $s \in \mathbb{N}$ and $\Omega : \mathbb{D} \to \tau$ the canonical homomorphism of $\mathfrak{D}$ on to τ. $\tau = GF(q)$ is a vector space over $GF(p) \subset \tau$. We consider a set $\{1 = \epsilon_0, \epsilon_1, ..., \epsilon_{s-1}\} \subset \mathfrak{D}^* = \mathfrak{D} \backslash \mathfrak{P}$ in such a way that $\{\Omega(\epsilon_k)\}_{k=0}^{s-1}$ is a basis of $GF(q)$ over $GF(p)$.

For k, $0 \leq k < q$, $k = a_0 + a_1 p + ... + a_{s-1} p^{s-1}$, $0 \leq a_i < p$, $i = 0, 1, ..., s-1$, we define

$$u(k) = (a_0 + a_1\epsilon_1 + ... + a_{s-1}\epsilon_{c-1})\mathfrak{p}^{-1} \ \ (0 \leq k < q).$$

For $k = b_0 + b_1 q + ... + b_r q^r$, $0 \leq b_i < q$, $k \geq 0$, we set

$$u(k) = u(b_0) + \mathfrak{p}^{-1}u(b_1) + ... + \mathfrak{p}^{-r}u(b_r).$$

- Note that for $k, l \geq 0$, $u(k+l) \neq u(k) + u(l)$. However, it is true that for all $r, l \geq 0$, $u(rq^l) = \mathfrak{p}^{-l}u(r)$, and for $r, l \geq 0$, $0 \leq t < q^l$, $u(rq^l + t) = u(rq^l) + u(t) = \mathfrak{p}^{-l}u(r) + u(t)$.
- For $k \in \mathbb{N}_0$, we denote $\chi_{u(k)}$ by χ_k.
- $\mathscr{S}(\mathbb{K})$ is the space of all finite linear combinations of characteristic function of balls of $\mathbb{K}$. Also $\mathscr{S}(\mathbb{K})$ is dense in $L^p(\mathbb{K}), 1 \leq p < \infty$.
- $\mathscr{S}'(\mathbb{K})$ is the space of distributions.
- $\hat{f}(\xi)$ is the Fourier transform of $f \in \mathscr{S}(\mathbb{K})$ and is defined by

$$\hat{f}(\xi) = \int_{\mathbb{K}} f(x)\overline{\chi_\xi(x)}dx, \ \xi \in \mathbb{K},$$

and the inverse transform by

$$f(x) = \int_{\mathbb{K}} \hat{f}(\xi)\chi_x(\xi)d\xi, \ x \in \mathbb{K}.$$

- Let $s \in \mathbb{R}$, we denote Sobolev space over local fields by $H^s(\mathbb{K})$ is the space of all functions in $\mathscr{S}'(\mathbb{K})$ such that

$$\hat{\gamma}^{\frac{s}{2}}(\xi)\hat{f}(\xi) \in L^2(\mathbb{K}), \ \text{where } \hat{\gamma}^s(\xi) = (max(1, |\xi|))^s.$$

- The inner product in $H^s(\mathbb{K})$ is denoted by

$$\langle f,g\rangle = \langle f,g\rangle_{H^s(\mathbb{K})} = \int_{\mathbb{K}} \hat{\gamma}^s(\xi)\hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi.$$

- The space $\mathscr{S}(\mathbb{K})$ is also dense in $H^s(\mathbb{K})$.

For more details refer to [[1], [4]].

3. Nonuniform Non-stationary Multiresolution Analysis on $H^s(\mathbb{K})$

Let $T_1 = \{u(m) : m \in \mathbb{N}_0\}$ is a complete list of (distinct) coset representation of $\mathfrak{D}$ in $\mathbb{K}^+$. Let $M \geq 1$ is an integer and r is an odd integer with $1 \leq r \leq qM - 1$ such that $g.c.d.(r, M) = 1$, define

$$T = \{0, \frac{r}{M}\} + T_1.$$

It is easily verified that T is not a group on $\mathbb{K}$, but is the union of T_1 and a translate of T_1. Now, we first establish two important theorems which leads to define nonuniform non-stationary MRA on $H^s(\mathbb{K})$ as follows.

Theorem 3.1. *For $s \in \mathbb{R}$, let $\varphi^{(j)}$ in $H^s(\mathbb{K})$, then the distribution $\{(qM)^{\frac{j}{2}}\varphi^{(j)}((\mathfrak{p}^{-1}M)^j.- u(\lambda)), \lambda \in T\}$ are orthonormal in $H^s(\mathbb{K})$ if and only if*

$$\sum_{l\in\mathbb{N}_0} \hat{\gamma}^s((\mathfrak{p}^{-1}M)^j(\xi+u(l)))|\hat{\varphi}^{(j)}(\xi+u(l))|^2 = 1, \tag{3.1}$$

and

$$\sum_{l\in\mathbb{N}_0} \hat{\gamma}^s((\mathfrak{p}^{-1}M)^j(\xi+u(l)))|\hat{\varphi}^{(j)}(\xi+u(l))|^2\overline{\chi}_l(\frac{r}{M}) = 0 \quad for\ a.e.\ \xi \in \mathbb{K}. \tag{3.2}$$

Proof. We have

$$\begin{aligned}
\delta_{\lambda,0} &= < (qM)^{\frac{j}{2}}\varphi^{(j)}((\mathfrak{p}^{-1}M)^j.-\lambda), (qM)^{\frac{j}{2}}\varphi^{(j)}((\mathfrak{p}^{-1}M)^j.) > \\
&= \int_{\mathbb{K}} \hat{\gamma}^s(\xi)(qM)^{-\frac{j}{2}}\hat{\varphi}^{(j)}((\mathfrak{p}^{-1}M)^{-j}\xi)\bar{\chi}_\lambda((\mathfrak{p}^{-1}M)^{-j}\xi)(qM)^{-\frac{j}{2}}\overline{\hat{\varphi}^{(j)}((\mathfrak{p}^{-1}M)^{-j}\xi)}d\xi \\
&= \int_{\mathbb{K}} \hat{\gamma}^s((\mathfrak{p}^{-1}M)^j\xi)|\hat{\varphi}^{(j)}(\xi)|^2\bar{\chi}_\lambda(\xi)d\xi.
\end{aligned}$$

Splitting the integral, we get

$$\delta_{\lambda,0} = \int_{\mathfrak{D}} \sum_{l=0}^{\infty} \hat{\gamma}^s((\mathfrak{p}^{-1}M)^j(\xi+u(l)))|\hat{\varphi}^{(j)}(\xi+u(l))|^2 \bar{\chi}_\lambda(\xi+u(l))d\xi. \tag{3.3}$$

On taking $\lambda \in \mathbb{N}_0$, we have

$$\begin{aligned} &\langle (qM)^{\frac{j}{2}}\varphi^{(j)}((\mathfrak{p}^{-1}M)^j.-\lambda), (qM)^{\frac{j}{2}}\varphi^{(j)}((\mathfrak{p}^{-1}M)^j.)\rangle \\ = &\int_{\mathfrak{D}} \sum_{l=0}^{\infty} \hat{\gamma}^s((\mathfrak{p}^{-1}M)^j(\xi+u(l)))|\hat{\varphi}^{(j)}(\xi+u(l))|^2 \bar{\chi}_\lambda(\xi+u(l))d\xi. \end{aligned} \tag{3.4}$$

If $\lambda = \frac{r}{M} + u(m), m \in \mathbb{N}_0$, then

$$\begin{aligned} &\langle (qM)^{\frac{j}{2}}\varphi^{(j)}((\mathfrak{p}^{-1}M)^j.-\lambda), (qM)^{\frac{j}{2}}\varphi^{(j)}((\mathfrak{p}^{-1}M)^j.)\rangle \\ &= \textstyle\int_{\mathfrak{D}} \sum_{l=0}^{\infty} \hat{\gamma}^s((\mathfrak{p}^{-1}M)^j(\xi+u(l)))|\hat{\varphi}^{(j)}(\xi+u(l))|^2 \bar{\chi}_\lambda(\xi+u(l))\bar{\chi}(\frac{r}{M}u(l)) \\ &\qquad \times \bar{\chi}(\tfrac{r}{M}\xi)\bar{\chi}(\tfrac{r}{M}u(m))d\xi. \end{aligned} \tag{3.5}$$

Hence the distribution $\{(qM)^{\frac{j}{2}}\varphi^{(j)}((\mathfrak{p}^{-1}M)^j.-u(\lambda)), \lambda \in T\}$ is orthonormal if and only if equalities (3.1) and (3.2) hold. □

Theorem 3.2. *Suppose $\varphi^{(j)}, j \in \mathbb{Z}$ be functions of $H^s(\mathbb{K})$ such that, for every j, the distributions*

$$\varphi^{(j)}_{j,\lambda}(.) = (qM)^{\frac{j}{2}}\varphi^{(j)}((\mathfrak{p}^{-1}M)^j.-\lambda), \lambda \in T, \tag{3.6}$$

are orthonormal in $H^s(\mathbb{K})$ and $V_j = \overline{\{\varphi_{j,\lambda}(\xi) : \lambda \in T\}}$. Under the condition,

$$\lim_{j\to\infty} |\hat{\varphi}^{(j)}((\mathfrak{p}^{-1}M)^{-j}\xi)| = \hat{\gamma}^{-\frac{s}{2}}(\xi), \tag{3.7}$$

$\overline{\cup_{j\in\mathbb{Z}}V_j} = H^s(\mathbb{K})$ *and* $\cap_{j\in\mathbb{Z}}V_j = \{0\}$.

Proof. Let $\eta \in (\overline{\cup_{j\in\mathbb{Z}}V_j})^\perp$. If π_j is the orthogonal projection from $H^s(\mathbb{K})$ onto V_j. Then,

$$\pi_j\eta = 0 \text{ for all } j \in \mathbb{Z}. \tag{3.8}$$

Let $\epsilon > 0$ and condition (3.7) holds. Since $\mathscr{S}(\mathbb{K})$ is dense in $H^s(\mathbb{K})$, there exists $\sigma \in \mathscr{S}(\mathbb{K})$ such that

$$||\eta - \sigma||_{H^s(\mathbb{K})} < \epsilon. \tag{3.9}$$

So for all $j \in \mathbb{Z}$,

$$||\pi_j \sigma||_{H^s(\mathbb{K})} = ||\pi_j(\eta - \sigma)||_{H^s(\mathbb{K})} \leq ||(\eta - \sigma)||_{H^s(\mathbb{K})} < \epsilon. \tag{3.10}$$

Then by definition of π_j,

$$\begin{aligned} ||\pi_j \sigma||^2_{H^s(\mathbb{K})} &= \sum_{\lambda \in T} | < \sigma, \varphi^{(j)}_{j,\lambda} >_{H^s(\mathbb{K})} |^2 \\ &= \sum_{\lambda \in T_1} | < \sigma, \varphi^{(j)}_{j,\lambda} >_{H^s(\mathbb{K})} |^2 + \sum_{\lambda \in \frac{r}{M} + T_1} | < \sigma, \varphi^{(j)}_{j,\lambda} >_{H^s(\mathbb{K})} |^2. \end{aligned} \tag{3.11}$$

Now,

$$\begin{aligned} &\sum_{\lambda \in T_1} | < \sigma, \varphi^{(j)}_{j,\lambda} >_{H^s(\mathbb{K})} |^2 \\ = &\sum_{\lambda \in T_1} (qM)^{-j} \int_{\mathbb{K}} \hat{\gamma}^s(\xi) \hat{\sigma}(\xi) \overline{\hat{\varphi}^{(j)}((\mathfrak{p}^{-1}M)^{-j}\xi)} \chi_\lambda((\mathfrak{p}^{-1}M)^{-j}\xi) d\xi \\ &\quad \times \int_{\mathbb{K}} \hat{\gamma}^s(\xi) \overline{\hat{\sigma}(\xi)} \hat{\varphi}^{(j)}((\mathfrak{p}^{-1}M)^{-j}\xi) \overline{\chi_\lambda((\mathfrak{p}^{-1}M)^{-j}\xi)} d\xi \\ = &\sum_{\lambda \in T_1} (qM)^{j} \int_{\mathbb{K}} \hat{\gamma}^s((\mathfrak{p}^{-1}M)^{j}\xi) \hat{\sigma}((\mathfrak{p}^{-1}M)^{j}\xi) \overline{\hat{\varphi}^{(j)}(\xi)} \chi_\lambda(\xi) d\xi \\ &\quad \times \int_{\mathbb{K}} \hat{\gamma}^s((\mathfrak{p}^{-1}M)^{j}\xi) \overline{\hat{\sigma}((\mathfrak{p}^{-1}M)^{j}\xi)} \hat{\varphi}^{(j)}(\xi) \overline{\chi_\lambda(\xi)} d\xi \\ = &\sum_{\lambda \in T_1} (qM)^{j} \int_{\mathbb{K}} \{ \sum_{l=0}^{\infty} \int_{\mathfrak{D}} \hat{\gamma}^s((\mathfrak{p}^{-1}M)^{j}(\xi + u(l))) \hat{\sigma}((\mathfrak{p}^{-1}M)^{j}(\xi + u(l))) \overline{\hat{\varphi}^{(j)}((\xi + u(l)))} \\ &\quad \times \chi_\lambda(\xi) \} d\xi \times \hat{\gamma}^s((\mathfrak{p}^{-1}M)^{j}\xi) \overline{\hat{\sigma}((\mathfrak{p}^{-1}M)^{j}\xi)} \hat{\varphi}^{(j)}(\xi) \overline{\chi_\lambda(\xi)} d\xi, \end{aligned}$$

and

$$\begin{aligned}
&\sum_{\lambda\in\frac{r}{M}+T_1} |<h,\varphi^{(j)}_{j,\lambda}>_{H^s(\mathbb{K})}|^2\\
=&\sum_{\lambda\in T_1}(qM)^j\int_{\mathbb{K}}\{\sum_{l=0}^{\infty}\int_{\mathfrak{D}}\hat{\gamma}^s((\mathfrak{p}^{-1}M)^j(\xi+u(l)))\hat{\sigma}((\mathfrak{p}^{-1}M)^j(\xi+u(l)))\overline{\hat{\varphi}^{(j)}((\xi+u(l)))}\\
&\qquad\times\chi(\frac{r}{M}\xi)\chi(\frac{r}{M}u(l))\chi_\lambda(\xi)\}d\xi\times\hat{\gamma}^s((\mathfrak{p}^{-1}M)^j\xi)\overline{\hat{\sigma}((\mathfrak{p}^{-1}M)^j\xi)}\hat{\varphi}^{(j)}(\xi)\overline{\chi(\frac{r}{M}\xi)}\overline{\chi_\lambda(\xi)}d\xi
\end{aligned}$$

By the convergence theorem of Fourier Series on $\mathfrak{D}$, we obtain

$$\begin{aligned}
||\pi_j\sigma||^2_{H^s(\mathbb{K})} &= \int_{\mathbb{K}}\hat{\gamma}^s(\xi)\hat{\sigma}(\xi)\overline{\hat{\varphi}^{(j)}}((\mathfrak{p}^{-1}M)^{-j}\xi)\\
&\quad\times\left\{\sum_{l=0}^{\infty}\gamma^s(\xi+(\mathfrak{p}^{-1}M)^j u(l))\overline{\hat{\sigma}}(\xi+(\mathfrak{p}^{-1}M)^j u(l))\hat{\varphi}^{(j)}((\mathfrak{p}^{-1}M)^{-j}\xi+u(l))\right\}d\xi\\
&\quad+\int_{\mathbb{K}}\hat{\gamma}^s(\xi)\hat{\sigma}(\xi)\overline{\hat{\varphi}^{(j)}}((\mathfrak{p}^{-1}M)^{-j}\xi)\chi(\frac{r}{M}(\mathfrak{p}^{-1}M)^{-j}\xi)\\
&\quad\times\{\sum_{l=0}^{\infty}\gamma^s(\xi+(\mathfrak{p}^{-1}M)^j u(l))\overline{\hat{\sigma}}(\xi+(\mathfrak{p}^{-1}M)^j u(l))\hat{\varphi}^{(j)}((\mathfrak{p}^{-1}M)^{-j}\xi+u(l))\\
&\qquad\times\bar{\chi}(\frac{r}{M}(\mathfrak{p}^{-1}M)^{-j}\xi)\bar{\chi}(\frac{r}{M}u(l))\}d\xi\\
&\le \int_{\mathbb{K}}\hat{\gamma}^{2s}(\xi)|\hat{\varphi}^{(j)}((\mathfrak{p}^{-1}M)^{-j}\xi)|^2|\hat{\sigma}(\xi)|^2d\xi\\
&\quad+\int_{\mathbb{K}}\sum_{l=1}^{\infty}\hat{\gamma}^s(\xi)\hat{\gamma}^s(\xi+(\mathfrak{p}^{-1}M)^j u(l))\hat{\sigma}(\xi)\\
&\qquad\times\overline{\hat{\varphi}^{(j)}}((\mathfrak{p}^{-1}M)^{-j}\xi)\overline{\hat{\sigma}}(\xi+(\mathfrak{p}^{-1}M)^j u(l))\hat{\varphi}^{(j)}((\mathfrak{p}^{-1}M)^{-j}\xi+u(l))d\xi\\
&= G_1+G_2(say). \qquad (3.12)
\end{aligned}$$

Now by using (3.1) and Cauchy-Schwarz's inequality, we have

$$\begin{aligned} |G_2| &\leq \int_{\mathbb{K}} \hat{\gamma}^{\frac{s}{2}}(\xi)|\hat{\sigma}(\xi)| \sum_{l=1}^{\infty} \hat{\gamma}^{\frac{s}{2}}(\xi + (\mathfrak{p}^{-1}M)^j u(l))|\hat{\sigma}(\xi + (\mathfrak{p}^{-1}M)^j u(l))|d\xi \\ &\leq \sum_{l=1}^{\infty} ||\hat{\gamma}^{\frac{s}{2}}(.)\hat{\sigma}(.)||_{L^2(\mathbb{K})}||\hat{\gamma}^{\frac{s}{2}}(. + (\mathfrak{p}^{-1}M)^j u(l)\sigma(. + (\mathfrak{p}^{-1}M)^j u(l))||_{L^2(\mathbb{K})}. \end{aligned}$$

Since $\hat{\sigma} \in \mathscr{S}(\mathbb{K})$ there exists l for which $\hat{\sigma}(\xi) \neq 0$ for $\xi \in \mathfrak{C}^{-l} = \{x \in \mathbb{K} : |x| \leq (qM)^l\}$, so $|\xi| \leq (qM)^l$. For $j > l$ and for any $l \in \mathbb{N}$, we have

$$|(\mathfrak{p}^{-1}M)^j u(l)| = (qM)^j |u(l)| \geq (qM)^j > (qM)^l.$$

So, we have $|\xi| \neq |(\mathfrak{p}^{-1}M)^j u(l)|$. Hence

$$|\xi + (\mathfrak{p}^{-1}M)^j u(l)| = max(|\xi|, |(\mathfrak{p}^{-1}M)^j u(l)|) \geq (qM)^j > (qM)^l.$$

That is, $\hat{\sigma}(\xi + (\mathfrak{p}^{-1}M)^j u(l)) = 0, \ \forall\ j > l$. This shows that $\lim_{j\to\infty} |G_2| = 0$.

Now, by using (3.10) and (3.12), we have

$$\int_{\mathbb{K}} \hat{\gamma}^{2s}(\xi)|\hat{\varphi}^{(j)}((\mathfrak{p}^{-1}M)^{-j}\xi)|^2|\hat{\sigma}(\xi)|^2 d\xi < \epsilon^2 - G_2.$$

By using the condition (3.7) and Dominated convergence theorem, we get

$$||\sigma||_{H^s(\mathbb{K})} < \epsilon.$$

Hence

$$||\eta||_{H^s(\mathbb{K})} < 2\epsilon.$$

Since ϵ was arbitrary, we get that $\eta = 0$ a.e. $\xi \in \mathbb{K}$.

Now, we prove the last part of the theorem. Let $g \in \cap_{j\in\mathbb{Z}} V_j$ and we know that $\mathscr{S}(\mathbb{K})$ is dense in $H^s(\mathbb{K})$. So, we have

$$||g||_{H^s(\mathbb{K})} - ||\pi_j\phi||_{H^s(\mathbb{K})} \leq ||g - \pi_j\phi||_{H^s(\mathbb{K})} \leq ||f - \phi||_{H^s} < \epsilon, \tag{3.13}$$

where

$$\phi(\xi) = (\hat{\gamma}^{-\frac{s}{2}}(\xi)\sigma(\xi))^{\vee}, \text{ and } \sigma(\xi) \in \mathscr{S}(\mathbb{K}).$$

Using equation (3.12), (3.13) and Cauchy-Schwarz's inequality, we get

$$\begin{aligned} ||\pi_j\phi||^2_{H^s(\mathbb{K})} \;\leq\; & \sum_{k=0}^{\infty}\Big(\int_{\mathbb{K}} \hat{\gamma}^s(\xi)|\sigma(\xi)|^2|\hat{\varphi}^{(j)}\big((\mathfrak{p}^{-1}M)^{-j}\xi)|^2 d\xi\Big)^{\frac{1}{2}} \\ & \times\Big(\int_{\mathbb{K}} \hat{\gamma}^s(\xi+(\mathfrak{p}^{-1}M)^j u(k))|\sigma(\xi+(\mathfrak{p}^{-1}M)^j u(k)))|^2|\hat{\varphi}^{(j)}\big((\mathfrak{p}^{-1}M)^{-j}\xi+u(k))|^2 d\xi\Big)^{\frac{1}{2}}. \end{aligned}$$

Since $\sigma \in \mathscr{S}(\mathbb{K})$, so there exists a characteristic funtion $\eta_i(\xi-\xi_0)$ of the set $\xi_0+\mathfrak{E}^i$, where i is some integers. Now σ can be written as $\sigma(\xi) = (qM)^{\frac{i}{2}}\eta_i(\xi-\xi_0)$. If $\xi+(\mathfrak{p}^{-1}M)^j u(k) \in \xi_0+\mathfrak{E}^r$, then $|(\mathfrak{p}^{-1}M)^j u(k)| \leq (qM)^{-i}$, hence $|u(k)| \leq (qM)^{-i-j}$. Then summation index k is bounded by $(qM)^{-i-j}$. So using this, we get

$$\begin{aligned} ||\pi_j\phi||^2_{H^s(\mathbb{K})} &\leq (qM)^{-i-j}\Big(\int_{\mathbb{K}} \hat{\gamma}^s(\xi)|\sigma(\xi)|^2|\hat{\varphi}^{(j)}\big((\mathfrak{p}^{-1}M)^{-j}\xi)|^2 d\xi\Big)^{\frac{1}{2}} \\ &\leq (qM)^{-i-j}\int_{\xi_0+\mathfrak{E}^r} \hat{\gamma}^s(\xi)|\hat{\varphi}^{(j)}\big((\mathfrak{p}^{-1}M)^{-j}\xi)|^2 d\xi \\ &= (qM)^{-i}\int_{(\mathfrak{p}^{-1}M)^j\xi_0+\mathfrak{E}^{-j+i}} \hat{\gamma}^s((\mathfrak{p}^{-1}M)^j\xi)|\hat{\varphi}^{(j)}(\xi)|^2 d\xi. \end{aligned}$$

Suppose that $\xi_0 \neq 0$. For any $\epsilon > 0$, choose $J < 0$ enough small satisfies the following two inequalities : $(qM)^J < |\xi_0| = (qM)^{\rho}$ such that $J+\rho < 0$, and $\int_{\mathfrak{E}^{-J-\rho}} \hat{\gamma}^s((\mathfrak{p}^{-1}M)^J\xi)|\hat{\varphi}^{(J)}(\xi)|^2 d\xi < \epsilon$.
We have,

$$(\mathfrak{p}^{-1}M)^j\xi_0+\mathfrak{E}^{-j+i} \subset \mathfrak{E}^{-J-\rho} \text{ for all } j \leq J. \tag{3.14}$$

Since $|(\mathfrak{p}^{-1}M)^j\xi_0| = (qM)^j(qM)^{\rho} \leq (qM)^J(qM)^{\rho}$ and $\mathfrak{E}^{-j+i} \subset \mathfrak{E}^{-J-\rho}$.
Hence, $||\pi_j\phi||_{H^s(\mathbb{K})} \to 0$ as $j \to -\infty$. Therefore there exists j such that

$$||\pi_j\phi||_{H^s(\mathbb{K})} < \epsilon.$$

So,

$$||g||_{H^s(\mathbb{K})} < 2\epsilon.$$

Since ϵ was arbitrary we get $g = 0$ a.e. Hence, $\cap_{j\in\mathbb{Z}} V_j = \{0\}$. □

3.1. **Construction of Wavelets.** By theorem 3.1, we have seen that one single scaling function can't generate orthonormal basis at each level of dilation, that is, scaling function depends on level which leads to non-stationary MRA. Since translation set is nonuniform, hence, in this section, with the help of theorem 3.2, we define NUNSMRA on Sobolev space $H^s(\mathbb{K})$ as follows :

Definition 3.3. A nonuniform non-stationary multiresolution analysis of $H^s(\mathbb{K})$ for an integer M and an odd integer r with $1 \leq r \leq qM - 1$ such that g.c.d.(r, M) = 1, consists of a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $H^s(\mathbb{K})$, satisfying
(1) $V_j \subset V_{j+1}$;
(2) $\overline{\cup_{j\in\mathbb{Z}} V_j} = H^s(\mathbb{K})$;
(3) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$;
(4) there exist a function $\varphi^{(j)} \in H^s(\mathbb{K})$ such that $\{(qM)^{\frac{j}{2}}\varphi^{(j)}((\mathfrak{p}^{-1}M)^j. - \lambda)\}_{\lambda\in T}$, forms an orthonormal basis of V_j. The function $\varphi^{(j)}$ is called the scaling function of the MRA.

With the help of MRA $\{V_j\}_{j\in\mathbb{Z}}$, we can define another sequence $\{W_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $H^s(\mathbb{K})$ by

$$V_{j+1} = W_j \oplus V_j \text{ and } W_k \perp W_n \;\; if \;\; k \neq n.$$

It follows that for $j > J$,

$$V_j = V_J \oplus \bigoplus_{l=0}^{j-J-1} W_{j-l},$$

where all these subspaces are orthogonal. By virtue of (2) and (3) in the Definition 3.3, this implies

$$H^s(\mathbb{K}) = \bigoplus_{l\in\mathbb{Z}} W_l, \tag{3.15}$$

a decomposition of $H^s(\mathbb{K})$ into mutually orthogonal subspaces.

Let $\varphi^{(j)}$ be a scaling function of the given MRA. Since $\varphi^{(j)} \in V_j \subset V_{j+1}$, and $\{\varphi^{(j+1)}_{j+1,\lambda}\}_{\lambda\in T}$ is an orthonormal basis in V_{j+1}, we have

$$\varphi^{(j)}(.) = \sum_{\lambda\in T} c^{(j)}_\lambda \varphi^{(j+1)}_{j+1,\lambda}(.),$$

where

$$c^{(j)}_\lambda \;\; = \;\; <\varphi^{(j)}, \varphi^{(j+1)}_{j+1,\lambda}>, \;\text{ with } \sum_{\lambda\in T} |c^{(j)}_\lambda|^2 < \infty.$$

Now,

$$\varphi^{(j)}((\mathfrak{p}^{-1}M)^j x) = (qM)^{\frac{j+1}{2}} \sum_{\lambda\in T} c^{(j)}_\lambda \varphi^{(j+1)}((\mathfrak{p}^{-1}M)^{j+1}x - \lambda).$$

Hence,

$$\begin{aligned}
\hat{\varphi}^{(j)}(\xi) &= (qM)^{\frac{j-1}{2}} \sum_{\lambda\in T} c^{(j)}_\lambda \overline{\chi_\lambda}((\mathfrak{p}^{-1}M)^{-1}\xi)\hat{\varphi}^{(j+1)}((\mathfrak{p}^{-1}M)^{-1}\xi) \\
&= m^{(j+1)}_0((\mathfrak{p}^{-1}M)^{-1}\xi)\hat{\varphi}^{(j+1)}((\mathfrak{p}^{-1}M)^{-1}\xi), \qquad (3.16)
\end{aligned}$$

where

$$\begin{aligned}
m^{(j+1)}_0(\xi) &= (qM)^{\frac{j-1}{2}} \sum_{\lambda\in T} c^{(j)}_\lambda \overline{\chi_\lambda(\xi)} \qquad (3.17) \\
&= m^{(j+1)}_{0,1}(\xi) + m^{(j+1)}_{0,2}(\xi)\bar{\chi}((\frac{r}{M})\xi). \qquad (3.18)
\end{aligned}$$

The function $m^{(j+1)}_{0,1}$ and $m^{(j+1)}_{0,2}$ are L^2-periodic functions.

The condition $V_j \subset V_{j+1}$ for every $j \in \mathbb{Z}$ is equivalent to the existence of L^2-periodic functions $m^{(j+1)}_{0,1}$ and $m^{(j+1)}_{0,2}$ such that the scale relation (3.16) hold.

Furthermore, if $f \in W_j$, this is equivalent to $f \in V_{j+1}$ and $f \perp V_j$.Since $f \in V_{j+1}$, we have

$$\begin{aligned} f(x) &= \sum_{\lambda \in T} f_\lambda^{(j)} \ \varphi_{j+1,\lambda}^{(j+1)}(x) \\ &= (qM)^{\frac{j+1}{2}} \sum_{k \in \mathbb{N}_0} f_\lambda^{(j)} \varphi^{(j+1)}((\mathfrak{p}^{-1}M)^{j+1}x - \lambda), \end{aligned}$$

where $f_\lambda^{(j)} =< f,\ \varphi_{j+1,\lambda}^{(j+1)} >_{H^s(\mathbb{K})}$.

$$\begin{aligned} \hat{f}(\xi) &= (qM)^{-\frac{j+1}{2}} \sum_{\lambda \in T} f_\lambda^{(j)} \overline{\chi}_\lambda((\mathfrak{p}^{-1}M)^{-j-1}\xi)\hat{\varphi}^{(j+1)}((\mathfrak{p}^{-1}M)^{-j-1}\xi) \\ &= m_f^{(j+1)}((\mathfrak{p}^{-1}M)^{-j-1}\xi)\hat{\varphi}^{(j+1)}((\mathfrak{p}^{-1}M)^{-j-1}\xi), \end{aligned} \tag{3.19}$$

where

$$m_f^{(j+1)}(\xi) = (qM)^{-\frac{j+1}{2}} \sum_{\lambda \in T} f_\lambda^{(j)} \overline{\chi}_k(\xi) \tag{3.20}$$

$$= m_{f,1}^{(j+1)}(\xi) + m_{f,2}^{(j+1)}(\xi)\bar{\chi}((\frac{r}{M})\xi). \tag{3.21}$$

Note that $m_{f,1}^{(j+1)}$ and $m_{f,2}^{(j+1)}$ are L^2-periodic functions.

Theorem 3.4. *If $\varphi^{(j)}$ are scaling functions of the given MRA $\{V_j\}$, then $m_0^{(j)}(\xi)$ in (3.17) satisfies*

$$\sum_{t=0}^{qM-1} \Big[|m_{0,1}^{(j)}((\mathfrak{p}^{-1}M)^{-1}\xi + (\mathfrak{p}^{-1}M)^{-1}u(t))|^2 + |m_{0,2}^{(j)}((\mathfrak{p}^{-1}M)^{-1}\xi + (\mathfrak{p}^{-1}M)^{-1}u(t))|^2\Big] = 1, \tag{3.22}$$

and

$$\sum_{t=0}^{qM-1} \Big[|m_{0,1}^{(j)}((\mathfrak{p}^{-1}M)^{-1}\xi + (\mathfrak{p}^{-1}M)^{-1}u(t))|^2 + |m_{0,2}^{(j)}((\mathfrak{p}^{-1}M)^{-1}\xi + (\mathfrak{p}^{-1}M)^{-1}u(t))|^2\Big] \times \bar{\chi}(\frac{r}{M}u(t)) = 0 \quad a.e.\ \xi \in \mathbb{K}. \tag{3.23}$$

Proof. Substituting (3.16) into (3.1), we get

$$\sum_{l=0}^{\infty}\hat{\gamma}^s((\mathfrak{p}^{-1}M)^j(\xi+u(l))|m_0^{(j)}((\mathfrak{p}^{-1}M)^{-1}(\xi+u(l)))|^2|\hat{\varphi}^{(j)}((\mathfrak{p}^{-1}M)^{-1}(\xi+u(l)))|^2=1.$$

Splitting the sum, we obtain

$$\begin{aligned}
1 &= \sum_{t=0}^{qM-1}\sum_{l=0}^{\infty}[|m_{0,1}^{(j)}((\mathfrak{p}^{-1}M)^{-1}\xi+(\mathfrak{p}^{-1}M)^{-1}u(t)+u(l)))|^2\\
&\quad +|m_{0,2}^{(j)}((\mathfrak{p}^{-1}M)^{-1}\xi+(\mathfrak{p}^{-1}M)^{-1}u(t)+u(l)))|^2]\\
&\qquad \times\hat{\gamma}^s((\mathfrak{p}^{-1}M)^{j+1})(\mathfrak{p}^{-1}M)^{-1}\xi+(\mathfrak{p}^{-1}M)^{-1}u(t)+u(l)))\\
&\qquad \times|\hat{\varphi}^{(j)}((\mathfrak{p}^{-1}M)^{-1}\xi+(\mathfrak{p}^{-1}M)^{-1}u(t)+u(l)))|^2\\
&= \sum_{t=0}^{qM-1}[|m_{0,1}^{(j)}((\mathfrak{p}^{-1}M)^{-1}\xi+(\mathfrak{p}^{-1}M)^{-1}u(t))|^2\\
&\quad +|m_{0,2}^{(j)}((\mathfrak{p}^{-1}M)^{-1}\xi+(\mathfrak{p}^{-1}M)^{-1}u(t))|^2]\\
&\qquad \times\sum_{l=0}^{\infty}\hat{\gamma}^s((\mathfrak{p}^{-1}M)^{j+1})(\mathfrak{p}^{-1}M)^{-1}\xi+(\mathfrak{p}^{-1}M)^{-1}u(t)+u(l)))\\
&\qquad \times|\hat{\varphi}^{(j)}((\mathfrak{p}^{-1}M)^{-1}\xi+(\mathfrak{p}^{-1}M)^{-1}u(t)+u(l)))|^2.
\end{aligned}$$

Applying (3.1), we obtain (3.22). In the similar manner by substituting (3.16) into (3.2), we obtain (3.23). □

Theorem 3.5. *If $f\in W_j$, then $m_f^{(j)}(\xi)$ defined in (3.20) satisfies*

$$\begin{aligned}
&\sum_{t=0}^{qM-1}[m_{f,1}^{(j+1)}((\mathfrak{p}^{-1}M)^{-1}\xi+\mathfrak{p}u(t))\overline{m_{0,1}^{(j+1)}((\mathfrak{p}^{-1}M)^{-1}\xi+\mathfrak{p}u(t))}\\
&+m_{f,2}^{(j+1)}((\mathfrak{p}^{-1}M)^{-1}\xi+\mathfrak{p}u(t))\overline{m_{0,2}^{(j+1)}((\mathfrak{p}^{-1}M)^{-1}\xi+\mathfrak{p}u(t))}]=0,
\end{aligned}\tag{3.24}$$

and

$$\begin{aligned}
&\sum_{t=0}^{qM-1}[m_{f,1}^{(j+1)}((\mathfrak{p}^{-1}M)^{-1}\xi+\mathfrak{p}u(t))\overline{m_{0,1}^{(j+1)}((\mathfrak{p}^{-1}M)^{-1}\xi+\mathfrak{p}u(t))}\\
&+m_{f,2}^{(j+1)}((\mathfrak{p}^{-1}M)^{-1}\xi+\mathfrak{p}u(t))\overline{m_{0,2}^{(j+1)}((\mathfrak{p}^{-1}M)^{-1}\xi+\mathfrak{p}u(t))}]\bar{\chi}(\frac{r}{M}u(t))=0.
\end{aligned}\tag{3.25}$$

Proof. If $f \in W_j \implies f \perp \varphi^{(j)}_{j,\lambda}$. Therefore $< f, \varphi^{(j)}_{j,\lambda}(\xi) >= 0$. So, by using (3.21), above result is followed. $\square$

Theorem 3.6. *There are integral-periodic functions $m_i^{(j)}$, where $i \in L = \{1, 2, 3, ..., qM-1\}$, $j \in \mathbb{Z}$, such that*

$$M^{(j)}(\xi) = [m_s^{(j)}((\mathfrak{p}^{-1}M)^{-1}\xi + (\mathfrak{p}^{-1}M)^{-1}u(t))]_{s,t=0}^{qM-1}, \ j \in \mathbb{Z}, \tag{3.26}$$

is unitary then there exists an orthonormal wavelet basis $\{\psi^{(j)}_{i,j,\lambda}\}_{j\in\mathbb{Z},\lambda\in T,i\in L}$ for $H^s(\mathbb{K})$, where

$$\hat{\psi}_i^{(j)}((\mathfrak{p}^{-1}M)^{-j}\xi) = m_i^{(j+1)}((\mathfrak{p}^{-1}M)^{-j-1}\xi)\hat{\varphi}^{(j+1)}((\mathfrak{p}^{-1}M)^{-j-1}\xi), \ j \in \mathbb{Z}, \ i \in L, \tag{3.27}$$

with $i \in L$ and $m_0^{(j)}$ as defined by (3.17).

Proof. It can be easily proved that $\{(qM)^{\frac{j}{2}}\psi_i^{(j)}((\mathfrak{p}^{-1}M)^j.-\lambda)\}_{i\in L,\lambda\in T}$ is an orthonormal basis for W_j. So, $\{\psi^{(j)}_{i,j,\lambda}\}_{j\in\mathbb{Z},\lambda\in T,i\in L}$ is an orthonormal basis for $H^s(\mathbb{K})$. $\square$

Acknowledgements

The work of second author is supported by the CSIR grant no : 09/013(0647)/2016 - EMR - 1, New Delhi.

References

[1] H. Jiang, D. Li and N. Jin, "Multiresolution analysis on local fields", J. Math. Anal. Appl. 294 (2004), 523-532.

[2] Shah, Firdous A. "Nonuniform multiresolution analysis on local fields of positive characteristic" Complex Analysis and Operator Theory 9.7 (2015): 1589-1608.

[3] D. Ramakrishnan and R. J. Valenza, "Fourier Analysis on Number Fields", Graduate Texts in Mathematics 186, Springer-Verlag, New York, 1999.

[4] M. H. Taibleson, "Fourier Analysis on Local Fields", Mathematical Notes 15, Princeton University Press, Princeton, NJ, 1975.

[5] Biswaranjan Behera and Qaiser Jahan, "Multiresolution analysis on local fields and characterization of scaling functions", Adv. Pure Appl. Math. 3 (2012), 181-202.

[6] Ashish Pathak and Guru P. Singh, “Wavelets in Sobolev space over loacl fields of positive characteristic”, Int. Jou. of Wavelets Multuresolut Inf. Process., 16(4)(2018).

[7] Ashish Pathak, Dileep Kumar and Guru P. Singh, “The Necessary Condition and Sufficient conditions for Wavelet Frames in Sobolev Space over Local field of Positive characteristic”, Bol. Soc. Paran. Mat(2019).

[8] Ashish Pathak and Dileep Kumar, “Characterization of Multiwavelets and MRA Wavelets in $H^s(\mathbb{F})$”, Int. J. Appl. Comput. Math., 5(6), 143(2019).

[9] Ashish Pathak and Guru P. Singh, “Biorthogonal Wavelets in Sobolev Space Over Local Fields of Positive Characteristic”, Int. J. Appl. Comput. Math. 6(2), 25(2020).

[10] Ashish Pathak and Guru P. Singh, “Multilevel Wavelet Packet on Sobolev space over Local field of Positive Characteristic”.(Communicated)

[11] Ashish Pathak and Dileep Kumar, “Multuresolution Analysis on Sobolev Space over Local fields of Positive Characteristic and Characterization of scaling function”.(communicated)

[12] Shiva Mittal, , Niraj K. Shukla, Generalized nonuniform multiresolution analyses. Colloq. Math. 153 (2018), 1, 121-147

[13] Shukla, Niraj K.; Maury, Saurabh Chandra; Mittal, Shiva Semi-orthogonal Parseval wavelets associated with GMRAs on local fields of positive characteristic. Mediterr. J. Math. 16 (2019), no. 5, Art. 120, 20.

[14] Atlouba, Nadya A. S.; Mittal, Shiva; Shukla, Niraj K. A characterization of nonuniform multi-wavelets using dimension function. Results Math. 72 (2017), 3, 1239-1255.