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Nonexistence of global solutions to wave Equations with structural damping and nonlinear memory

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Summary

For the following wave equations with structural damping and nonlinear memory source terms

$$u_{tt} + (-\Delta)^{\frac{\alpha}{2}} u + (-\Delta)^{\frac{\beta}{2}} u_t = \int_0^t (t-s)^{\gamma-1} |u(s)|^p \, \mathrm{d}s,$$

and

$$u_{tt} + (-\Delta)^{\frac{\alpha}{2}} u + (-\Delta)^{\frac{\beta}{2}} u_t = \int_0^t (t-s)^{\gamma-1} |u_s(s)|^p \, \mathrm{d}s,$$

posed in $(x,t) \in \mathbb{R}^N \times [0,\infty)$, where $u = u(x,t)$ is real-value unknown function, $p > 1$, $\alpha, \beta \in (0,2]$, $\gamma \in (0,1)$, we prove the nonexistence of global solutions. Moreover, we give an upper bound estimate of the life span of solutions.

KEYWORDS:

Damped wave equation, nonexistence of global solution, life span.

1 | INTRODUCTION

In this paper, we study the nonexistence of global solutions of the problems

$$\begin{cases} u_{tt} + (-\Delta)^{\frac{\alpha}{2}} u + (-\Delta)^{\frac{\beta}{2}} u_t = \displaystyle\int_0^t (t-s)^{\gamma-1} |u(s)|^p \, \mathrm{d}s, & x \in \mathbb{R}^N, \ t > 0, \\ (u, u_t)(x,0) = (u_0, u_1)(x), & x \in \mathbb{R}^N, \end{cases} \tag{1}$$

where $p > 1$, $\alpha, \beta \in (0,2]$, $\gamma \in (0,1)$ and to

$$\begin{cases} u_{tt} + (-\Delta)^{\frac{\alpha}{2}} u + (-\Delta)^{\frac{\beta}{2}} u_t = \displaystyle\int_0^t (t-s)^{\gamma-1} |u_s(s)|^p \, \mathrm{d}s, & x \in \mathbb{R}^N, \ t > 0, \\ (u, u_t)(x,0) = (u_0, u_1)(x), & x \in \mathbb{R}^N, \end{cases} \tag{2}$$

where $p > 1$, $\alpha, \beta \in (0,2]$, $\gamma \in (0,1)$, $(-\Delta)^{\frac{\nu}{2}}$ is the fractional Laplacian operator of order $\nu \in (0,2]$, ($\nu = \alpha$ or β), which accounts of propagation in media with impurities; it is defined by $(-\Delta)^{\frac{\nu}{2}} v(x) = \mathcal{F}^{-1}(|\xi|^\nu \mathcal{F}(v)(\xi))(x)$, where $\mathcal{F}$ denotes the Fourier and $\mathcal{F}^{-1}$ its inverse.

Recently, the class of fractional differential equations of various types plays important roles and tools not only in mathematics but also in physics, control systems, dynamical systems and engineering.

First, we recal some previous results for (1) and (2). There are many results about nonexistence of global solutions of these type of equations (see[3],[8],[17],[6],[12],[13],[9],[7]). Based on the paper of Kirane and Laskri[11], Berbiche and Hakem[2] studied the nonexistence of global solution of the following wave equation

$$\begin{cases} u_{tt} - \Delta u + (-\Delta)^{\frac{\beta}{2}} \mathbb{D}^{\alpha}_{0|t} u = |u|^p, \quad x \in \mathbb{R}^N,\ t > 0, \\ (u, u_t)(x, 0) = (u_0, u_1)(x), \quad x \in \mathbb{R}^N, \end{cases} \tag{3}$$

where $\beta \in (0, 2]$, $\mathbb{D}^{\alpha}_{0|t}$ is fractional derivative of order $\alpha \in (0, 1)$; they proved that for any $1 < p < 1 + 2\alpha/(2 + \alpha N - 2\alpha)$, the solution of problem (3) cannot be global.
In[5], D'Abbicco and Albert considered the following semi-linear evolution equation

$$\begin{cases} u_{tt} + (-\Delta)^{\sigma} u + (-\Delta)^{\delta} u_t = |u_t|^p, \quad x \in \mathbb{R}^N,\ t > 0, \\ (u, u_t)(x, 0) = (u_0, u_1)(x), \quad x \in \mathbb{R}^N, \end{cases} \tag{4}$$

where $\sigma, \delta \in \mathbb{N} \setminus \{0\}$; when $2\delta < \sigma$, they obtained that for any $0 < p < 1 + 2\delta/N$, the problem (4) does not admit a global weak solution. Moreover, they gave an upper estimate of the life span of solutions under some condition on the initial data.
In[4], D'Abbicco considered the equation

$$\begin{cases} u_{tt} - \Delta u + \mu(-\Delta)^{\frac{1}{2}} u_t = \displaystyle\int_0^t (t - s)^{-\gamma} |u(s)|^p \, \mathrm{d}s, \quad x \in \mathbb{R}^N,\ t > 0, \\ (u, u_t)(x, 0) = (u_0, u_1)(x), \quad x \in \mathbb{R}^N, \end{cases} \tag{5}$$

where $\mu > 0$, $\gamma \in (0, 1)$; he obtained some nonexistence results of global solutions for various values of μ, γ and p.

The method used in all cited papers is based on the articles of Mitidieri and Pohozahev[12], Pohozahev and Tesei[14], Pohozahev and Véron[15], Zhang[17], Kirane et al.[10]. It consists in a judicious choice of the test function in the weak formulation of the sought solution of problems (1) and (2).

The remainder of this paper is organized as follows: Section 2 is devoted to some preliminaries and announce our main results. The proof of the main results will be given in Section 3.

2 | PRELIMINARIES AND MAIN RESULTS

We recall some properties of the fractional derivatives and fractional integrals (see for example[16]). For $T > 0$, $0 < \gamma < 1$, the left-handed and right-handed Riemann-Liouville fractional integrals $\mathbb{I}^{\gamma}_{0|t} f(t)$ and $\mathbb{I}^{\gamma}_{t|T} f(t)$ for integrable function f are defined by

$$\mathbb{I}^{\gamma}_{0|t} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t - s)^{1-\gamma}} \, \mathrm{d}s \quad , \quad \mathbb{I}^{\gamma}_{t|T} f(t) = \frac{1}{\Gamma(\gamma)} \int_t^T \frac{f(s)}{(s - t)^{1-\gamma}} \, \mathrm{d}s, \tag{6}$$

where Γ denotes the Gamma function.

Moreover, if $f \in L^p(0, T)$, $g \in L^q(0, T)$, and $p,\ q \geq 1$, $q = p/(p - 1)$, then

$$\int_0^T \left(\mathbb{I}^{\gamma}_{0|t} f\right)(t) g(t) \, \mathrm{d}t = \int_0^T \left(\mathbb{I}^{\gamma}_{t|T} g\right)(t) f(t) \, \mathrm{d}t. \tag{7}$$

If $AC[0, T]$ is the space of all functions which are absolutely continuous on $[0, T]$ with $0 < T < \infty$, then, for $f \in AC[0, T]$, the left-handed and right-handed Riemann-Liouville fractional derivatives $\mathbb{D}^{\gamma}_{0|t} f(t)$ and $\mathbb{D}^{\gamma}_{t|T} f(t)$ of order $\gamma \in (0, 1)$ are defined by

$$\mathbb{D}^{\gamma}_{0|t} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} \mathbb{I}^{1-\gamma}_{0|t} f(t) \quad , \quad \mathbb{D}^{\gamma}_{t|T} f(t) = -\frac{1}{\Gamma(1 - \gamma)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^T \frac{f(s)}{(s - t)^{\gamma}} \, \mathrm{d}s.$$

Furthermore, for every $f,\ g \in C([0, T])$ such that $\mathbb{D}^{\gamma}_{0|t} f(t)$, $\mathbb{D}^{\gamma}_{t|T} g(t)$ exist and are continuous, for all $t \in [0, T]$, $0 < \gamma < 1$, we have the formula of integration by parts

$$\int_0^T \left(\mathbb{D}^{\gamma}_{0|t} f\right)(t) g(t) \, \mathrm{d}t = \int_0^T \left(\mathbb{D}^{\gamma}_{t|T} g\right)(t) f(t) \, \mathrm{d}t. \tag{8}$$

Note also that, for all $f \in AC^{n+1}[0,T]$, $n \geq 0$, we have

$$(-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n} \mathbb{D}^{\gamma}_{t|T} f(t) = \mathbb{D}^{\gamma+n}_{t|T} f(t), \tag{9}$$

where

$$AC^{n+1}[0,T] = \{f \ : \ [0,T] \to \mathbb{R} \text{ and } \frac{\mathrm{d}^n f}{\mathrm{d}t^n} \in AC[0,T]\}.$$

Moreover, for all $1 \leq q \leq \infty$, the following formula holds

$$\mathbb{D}^{\gamma}_{0|t} \mathbb{I}^{\gamma}_{0|t} = Id. \tag{10}$$

Definition 1. (See[19]) Let $0 < \nu \leq 2$. Let $\mathcal{S}$ be the Schwartz space of rapidly decaying C^∞ functions in $\mathbb{R}^N$. Then, the fractional Laplacian $(-\Delta)^{\frac{\nu}{2}}$ in $\mathbb{R}^N$ is a non-local operator given by

$$(-\Delta)^{\frac{\nu}{2}} \quad : \quad v \in \mathcal{S} \mapsto (-\Delta)^{\frac{\nu}{2}} v(x) = C_{N,\nu}\, p.v \int_{\mathbb{R}^N} \frac{v(x) - v(y)}{|x-y|^{N+\nu}} dy, \tag{11}$$

where $p.v.$ stands for Cauchy's principal value, $C_{N,\nu} = \frac{4^{\frac{\nu}{2}} \Gamma(\frac{N}{2} + \frac{\nu}{2})}{\pi^{\frac{N}{2}} \Gamma(-\frac{\nu}{2})}$.

Lemma 1. (See[18]) Let $\langle x \rangle = (1 + (|x| - 1)^4)^{\frac{1}{4}}$ for all $x \in \mathbb{R}^N$. let $0 < \nu \leq 2$ and $\phi \ : \ \mathbb{R}^N \to \mathbb{R}$ be the function

$$\varphi(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ \langle x \rangle^{-N-\nu} & \text{if } |x| \geq 1. \end{cases} \tag{12}$$

Then, $\varphi \in C^2(\mathbb{R}^N)$ and the following estimate holds

$$|(-\Delta)^{\frac{\nu}{2}} \varphi(x)| \leq C\varphi(x), \text{for all } x \in \mathbb{R}^N. \tag{13}$$

Lemma 2. (See[20]) Let $0 < \nu \leq 2$ and ϕ be a smooth function satisfying $\partial_x^2 \phi \in L^\infty(\mathbb{R}^N)$. For any $T > 0$, let ϕ_T be the function

$$\phi_T(x) = \phi\left(\frac{x}{T}\right), \qquad \text{for all } x \in \mathbb{R}^N.$$

Then $(-\Delta)^{\frac{\nu}{2}} \phi_T$ satisfies

$$(-\Delta)^{\frac{\nu}{2}} \phi_T(x) = T^{-\nu} (-\Delta)^{\frac{\nu}{2}} \phi\left(\frac{x}{T}\right). \tag{14}$$

Lemma 3. Let $0 < \nu \leq 2$, $T > 0$, and $p > 1$. Then, the following estimate holds

$$\int_{\mathbb{R}^N} \varphi_T^{-\frac{p}{p-1}}(x) |(-\Delta)^{\frac{\nu}{2}} \varphi_T(x)|^{\frac{p}{p-1}} \,\mathrm{d}x \leq C T^{N - \frac{p\nu}{p-1}}, \tag{15}$$

where $\varphi_T(x) = \varphi\left(\frac{x}{T}\right)$ and φ is given in (12).

Lemma 4. (See[21]) Let $f \in L^1(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} f(x)\,\mathrm{d}x > 0$. Then there exists a test function $0 \leq \varphi \leq 1$ such that

$$\int_{\mathbb{R}^N} f\,\varphi \,\mathrm{d}x > 0.$$

Definition 2. Let $p > 1$. We say that $u \in L^1_{Loc}([0,T) \times \mathbb{R}^N$ is a local weak solution of problem (1), or that $u \in L^1_{Loc}([0,T) \times \mathbb{R}^N)$ with $u_t \in L^p_{Loc}([0,T) \times \mathbb{R}^N)$, is a local weak solution of problem (2), if for any function $\zeta \in C([0,T]; H^\alpha(\mathbb{R}^N)) \cap C^1([0,T]; H^\beta(\mathbb{R}^N)) \cap C^2([0,T]; L^2(\mathbb{R}^N))$, with $\operatorname{supp}_x \zeta \subset \mathbb{R}^N$ such that $\zeta(\cdot, T) = 0$ and $\zeta_t(\cdot, T) = 0$ it holds:

$$\begin{aligned} \mathcal{U} = & \int_0^T \int_{\mathbb{R}^N} u\left(\zeta_{tt}(x,t) + (-\Delta)^{\frac{\alpha}{2}} \zeta(x,t) - (-\Delta)^{\frac{\beta}{2}} \zeta_t(x,t)\right) \mathrm{d}x\mathrm{d}t \\ & - \int_{\mathbb{R}^N} u_1(x)\zeta(x,0)\,\mathrm{d}x + \int_{\mathbb{R}^N} u_0(x)(\zeta_t(x,0) - (-\Delta)^{\frac{\beta}{2}} \zeta(x,0))\,\mathrm{d}x, \end{aligned} \tag{16}$$

where
$$\mathcal{U} = c_\gamma \int_0^T \int_{\mathbb{R}^N} \mathbb{I}^\gamma_{0|t} |u|^p \zeta(x,t)\,\mathrm{d}x\mathrm{d}t,$$
for problem (1), and
$$\mathcal{U} = c_\gamma \int_0^T \int_{\mathbb{R}^N} \mathbb{I}^\gamma_{0|t} |u_t|^p \zeta(x,t)\,\mathrm{d}x\mathrm{d}t.$$
for problem (2), and $c_\gamma = \Gamma(\gamma)$.

Now, we are in position to announce our results.

Theorem 1. Let $p > 1$, $0 < \alpha, \beta \leq 2$, $0 < \gamma < 1$, and assume that $u_0 \equiv 0$, whereas $u_1 \in L^1$ verifies
$$\int_{\mathbb{R}^N} u_1(x)\,\mathrm{d}x > 0. \tag{17}$$
Then there exists no global weak solutions to problem (1) for any
$$p \leq 1 + \frac{\gamma\theta + \alpha}{\left(N - \theta\gamma - \min\{\beta, \frac{\alpha}{2}\}\right)_+}. \tag{18}$$
Moreover, if $[0, T_\varepsilon)$ is the life span of u, then, for the initial data $u_1(x) = \varepsilon g(x)$, $g \in L^1$ and verifies (17), there exists a constant $C > 0$ such that
$$T_\varepsilon \leq C\varepsilon^{-\frac{\theta}{(\gamma\theta+\alpha)q+\gamma\theta-N}}, \quad \theta = \alpha - \min\{\beta, \frac{\alpha}{2}\},\ N < q(\gamma\theta + \alpha) - \gamma\theta,$$
where $q = p/(p-1)$.

Theorem 2. Let $p > 1$, $0 < \alpha, \beta \leq 2$, $0 < \gamma < 1$, and assume that $u_0 \equiv 0$, whereas $u_1 \in L^1$ verifies (17). Then there exists no global weak solutions to problem (2) for any
$$p \leq 1 + \frac{\min\{\beta, \frac{\alpha}{2}\}}{N}. \tag{19}$$
Moreover, denote $[0, T_\varepsilon)$ the life span of u. Then, for the initial data $u_1(x) = \varepsilon g(x)$, $g \in L^1$ and verifies (17), there exists a constant $C > 0$ such that
$$T_\varepsilon \leq C\varepsilon^{-\frac{\theta}{(1+\gamma)\theta q-\gamma\theta-N}}, \quad \theta = \min\{\beta, \frac{\alpha}{2}\},\ N < q\theta(1 + \gamma) - \gamma\theta,$$
where $q = p/(p-1)$.

Theorem 3. Let $p > 1$, $0 < \alpha, \beta \leq 2$, $0 < \gamma < 1$, and assume that $u_0 \equiv 0$, whereas $u_1 \in L^1_{Loc}$ verifies
$$u_1(x) \geq \varepsilon(1 + |x|)^{-\mu}, \quad \varepsilon > 0 \text{ and } N < \mu < q(\gamma\theta + \alpha) - \gamma\theta. \tag{20}$$
Then, there exists no global weak solutions to problem (1) for any
$$p \leq 1 + \frac{\theta\gamma + \alpha}{\left(N - \theta\gamma - \min\{\beta, \frac{\alpha}{2}\}\right)_+}. \tag{21}$$
Moreover, denote $[0, T_\varepsilon)$ the life span of u. Then, there exists a constant $C > 0$, independent of ε, such that
$$T_\varepsilon \leq C\varepsilon^{-\frac{\theta}{(\gamma\theta+\alpha)q-\gamma\theta-\mu}}, \quad \theta = \alpha - \min\{\beta, \frac{\alpha}{2}\},.$$
where $q = p/(p-1)$.

Theorem 4. Let $p > 1$, $0 < \alpha, \beta \leq 2$, $0 < \gamma < 1$, and assume that $u_0 \equiv 0$, whereas $u_1 \in L^1_{Loc}$ verifies
$$u_1(x) \geq \varepsilon(1 + |x|)^{-\mu}, \quad \varepsilon > 0 \text{ and } N < \mu < q(1 + \gamma)\theta - \gamma\theta. \tag{22}$$
Then, there exists no global weak solutions to problem (2) for any
$$p \leq 1 + \frac{\min\{\beta, \frac{\alpha}{2}\}}{N}. \tag{23}$$
Moreover, there exists $C > 0$ such that
$$T_\varepsilon \leq C\varepsilon^{-\frac{\theta}{q\theta(1+\gamma)-\gamma\theta-\mu}}, \quad \theta = \min\{\beta, \frac{\alpha}{2}\},$$
where $q = p/(p-1)$.

3 | PROOF OF MAIN RESULTS

Proof of Theorem (1). Let us assume that u is global weak solution to problem (1). Using Definition (2), and recalling that $u_0 \equiv 0$, we obtain

$$c_\gamma \int_0^{T^\theta}\int_{\mathbb{R}^N} \mathbb{I}^\gamma_{0|t}|u|^p\zeta(x,t)\,\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^N} u_1(x)\zeta(x,0)\,\mathrm{d}x$$
$$= \int_0^{T^\theta}\int_{\mathbb{R}^N} u\left(\zeta_{tt}(x,t) + (-\Delta)^{\frac{\alpha}{2}}\zeta(x,t) - (-\Delta)^{\frac{\beta}{2}}\zeta_t(x,t)\right)\,\mathrm{d}x\mathrm{d}t, \tag{24}$$

for all test function $\zeta \in C([0,T];H^\alpha(\mathbb{R}^N)) \cap C^1([0,T];H^\beta(\mathbb{R}^N)) \cap C^2([0,T];L^2(\mathbb{R}^N))$, with $\operatorname{supp}_x \zeta \subset\subset \mathbb{R}^N$, $\zeta(\cdot,T^\theta) = 0$ and $\zeta_t(\cdot,T^\theta) = 0$.
Now, we take

$$\zeta(x,t) = \mathbb{D}^\gamma_{t|T^\theta}\zeta(x,t) = \varphi_T(x)\,\mathbb{D}^\gamma_{t|T^\theta}\psi(t), \quad t \geq 0,\ T > 0,\ \gamma \in (0,1),$$

where

$$\varphi_T(x) = \varphi\left(\frac{|x|}{T}\right) \quad , \quad \psi(t) = \left(1 - \frac{t}{T^\theta}\right)^m_+,\ m \gg 1$$

and φ is given in (12), $\theta > 0$ which we will fixed later. Then, we can write

$$c_\gamma \int_0^{T^\theta}\int_{\mathbb{R}^N} |u|^p\zeta(x,t)\,\mathrm{d}x\mathrm{d}t + \int_0^{T^\theta}\int_{\mathbb{R}^N} u_1(x)\varphi_T\,\mathbb{D}^\gamma_{t|T^\theta}\psi\,\mathrm{d}x\mathrm{d}t$$
$$= \int_0^{T^\theta}\int_{\mathbb{R}^N} u\left(\varphi_T\,\mathbb{D}^{\gamma+2}_{t|T^\theta}\psi + (-\Delta)^{\frac{\alpha}{2}}\varphi_T\,\mathbb{D}^\gamma_{t|T^\theta}\psi - (-\Delta)^{\frac{\beta}{2}}\varphi_T\,\mathbb{D}^{\gamma+1}_{t|T^\theta}\psi\right)\mathrm{d}x\mathrm{d}t \tag{25}$$
$$\leq \int_0^{T^\theta}\int_{\mathbb{R}^N} |u|\left(\varphi_T|\mathbb{D}^{\gamma+2}_{t|T^\theta}\psi| + |(-\Delta)^{\frac{\alpha}{2}}\varphi_T||\mathbb{D}^\gamma_{t|T^\theta}\psi| + |(-\Delta)^{\frac{\beta}{2}}\varphi_T||\mathbb{D}^{\gamma+1}_{t|T^\theta}\psi|\right)\mathrm{d}x\mathrm{d}t,$$

where we have used the formula of integration by parts (7), (8), and the formula (10).
Then, in order to estimate the right hand side of (25), we use Young's inequality; we get

$$\int_0^{T^\theta}\int_{\mathbb{R}^N} |u|\left(\varphi_T\,|\mathbb{D}^{\gamma+2}_{t|T^\theta}\psi| + |(-\Delta)^{\frac{\alpha}{2}}\varphi_T||\mathbb{D}^\gamma_{t|T^\theta}\psi| + |(-\Delta)^{\frac{\beta}{2}}\varphi_T|\,||\mathbb{D}^{\gamma+1}_{t|T^\theta}\psi|\right)\mathrm{d}x\mathrm{d}t$$
$$\leq C\mathcal{U}_T + C\int_0^{T^\theta}\int_{\mathbb{R}^N} \zeta^{-\frac{q}{p}}\left(\varphi_T|\mathbb{D}^{\gamma+2}_{t|T^\theta}\psi| + |(-\Delta)^{\frac{\alpha}{2}}\varphi_T||\mathbb{D}^\gamma_{t|T^\theta}\psi| + |(-\Delta)^{\frac{\beta}{2}}\varphi_T||\mathbb{D}^{\gamma+1}_{t|T^\theta}\psi|\right)^q\mathrm{d}x\mathrm{d}t,$$

where $q = p/(p-1)$, and

$$\mathcal{U}_T = \int_0^{T^\theta}\int_{\mathbb{R}^N} |u|^p\zeta(x,t)\,\mathrm{d}x\mathrm{d}t.$$

Note that, for $t \geq 0$, $T > 0$, $m \gg 1$ and $0 < \gamma < 1$, we have (see [17])

$$\begin{cases} \mathbb{D}^\gamma_{t|T^\theta}\left(1 - \dfrac{t}{T^\theta}\right)^m_+ = C_1 T^{-\gamma\theta}\left(1 - \dfrac{t}{T^\theta}\right)^{m-\gamma}_+, \\ \mathbb{D}^{\gamma+1}_{t|T^\theta}\left(1 - \dfrac{t}{T^\theta}\right)^m_+ = C_2 T^{-(\gamma+1)\theta}\left(1 - \dfrac{t}{T^\theta}\right)^{m-\gamma-1}_+, \\ \mathbb{D}^{\gamma+2}_{t|T^\theta}\left(1 - \dfrac{t}{T^\theta}\right)^m_+ = C_3 T^{-(\gamma+2)\theta}\left(1 - \dfrac{t}{T^\theta}\right)^{m-\gamma-2}_+, \end{cases} \tag{26}$$

where $C_1 = \frac{\Gamma(m+1)}{\Gamma(m-\gamma+1)}$, $C_2 = \frac{\Gamma(m+1)}{\Gamma(m-\gamma)}$ and $C_3 = \frac{\Gamma(m+1)}{\Gamma(m-\gamma-1)}$, and for a constant $C > 0$, we have

$$\int_0^{T^\theta} \mathbb{D}^\gamma_{t|T^\theta} \left(1 - \frac{t}{T^\theta}\right)^m \mathrm{d}t = CT^{\theta(1-\gamma)}. \tag{27}$$

Therefore, considering the scaled variables $\tau = t/T^\theta$, $\xi = x/T$, and using the scaling property (14), inequality (15), and (26), we may estimate

$$\begin{cases} \int_0^{T^\theta}\int_{\mathbb{R}^N} \zeta^{-\frac{q}{p}} |\varphi_T\, \mathbb{D}^{\gamma+2}_{t|T^\theta}\psi|^q \,\mathrm{d}x\mathrm{d}t \leq CT^{-(2+\gamma)\theta q+\theta+N}, \\ \int_0^{T^\theta}\int_{\mathbb{R}^N} \zeta^{-\frac{q}{p}} |(-\Delta)^{\frac{\alpha}{2}}\varphi_T\, \mathbb{D}^{\gamma}_{t|T^\theta}\psi|^q \,\mathrm{d}x\mathrm{d}t \leq CT^{-(\gamma\theta+\alpha)q+\theta+N}, \\ \int_0^{T^\theta}\int_{\mathbb{R}^N} \zeta^{-\frac{q}{p}} |(-\Delta)^{\frac{\beta}{2}}\varphi_T\, \mathbb{D}^{\gamma+1}_{t|T^\theta}\psi|^q \,\mathrm{d}x\mathrm{d}t \leq CT^{-((\gamma+1)\theta+\beta)q+\theta+N}. \end{cases} \tag{28}$$

Next, setting

$$\theta = \max\left\{\alpha-\beta, \frac{\alpha}{2}\right\} = \alpha - \min\left\{\beta, \frac{\alpha}{2}\right\}.$$

Then, expression(25) can be written as follows

$$\mathcal{U}_T + CT^{\theta(1-\gamma)} \int_{\mathbb{R}^N} u_1(x)\varphi_T \,\mathrm{d}x \leq CT^{-(\gamma\theta+\beta)q+\theta+N} = CT^{-\kappa}, \tag{29}$$

where $\kappa = (\gamma\theta+\alpha)q - \theta - N$. We have to distinguish two cases :

The case when $\kappa > 0$: recalling assumption (17), passing to the limit in (29), as T goes to ∞, it follows that

$$0 < \int_0^\infty \int_{\mathbb{R}^n} |u|^p \,\mathrm{d}x\mathrm{d}t \leq 0,$$

which is a contradiction.

The case when $\kappa = 0$: we treat this cas in standard way as above by taking this time

$$\varphi_T(x) = \Phi\left(\frac{|x|}{L^{-1}T}\right)$$

where $1 \leq L \leq T$ is large enough such that when $T \to \infty$ we don't have $L \to \infty$ in the same time. Note that there exists a constant $C > 0$ independent of T and L such that

$$\mathcal{U}_T + CT^{\theta(1-\gamma)} \int_{\mathbb{R}^N} u_1(x)\varphi_T \,\mathrm{d}x \leq CL^{-N} + CL^{\alpha q-N} + CL^{\beta q-N}. \tag{30}$$

Thus, using $\max\{\alpha,\beta\} < N/q$ and taking the limit when T tends to ∞ and then L tends to ∞, we have

$$0 < \int_0^\infty \int_{\mathbb{R}^n} |u|^p \,\mathrm{d}x\mathrm{d}t \leq 0;$$

this contradicts again the assumption.

Now, we give an upper estimate of the life span of solution in the case $p < 1 + (\gamma\theta+\alpha)/(N - \gamma\theta - \min\{\beta, \frac{\alpha}{2}\})_+$. Noting that for $g \in L^1$, verifiying (17), there exists $\tilde{T} > 0$ such that

$$\int_{\mathbb{R}^N} g(x)\varphi_T \,\mathrm{d}x \geq c > 0, \qquad \text{for all } T \geq \tilde{T}.$$

Assume that u is a local solution in $[0, T_\varepsilon]$, with $T_\varepsilon \geq \tilde{T}^\theta$. Then, setting $T = T_\varepsilon^{\frac{1}{\theta}}$, we arrive at

$$0 \leq \mathcal{U}_T \leq CT^{-(\gamma\theta+\alpha)q+\theta+N} - c\varepsilon T^{\theta(1-\gamma)} \int_{\mathbb{R}^N} u_1(x)\varphi_T \,\mathrm{d}x \leq CT_\varepsilon^{-\frac{(\gamma\theta+\alpha)q-\theta-N}{\theta}} - c\varepsilon T_\varepsilon^{1-\gamma}.$$

Finally, for some positive constant C, independent of ε, we get

$$T_\varepsilon \leq c\varepsilon^{-\frac{\theta}{(\theta\gamma+\alpha)q-\gamma\theta-N}}.$$

This completes the proof. $\square$

Proof of Theorem (2). The proof is by contradiction. Supposing that u is global weak solution to (2), using Definition (2), recalling that $u_0 \equiv 0$, and taking

$$\zeta(x,t) = \mathbb{D}^{\gamma}_{t|T^\theta}\zeta(x,t) = \varphi_T(x)\,\mathbb{D}^{\gamma}_{t|T^\theta}\psi(t)$$

in (16). Thus, by the formula of integration by parts (7), (8), and the formula (10), we get

$$\begin{aligned} c_\gamma \int_0^{T^\theta}\int_{\mathbb{R}^N} |u_t|^p\zeta(x,t)\,\mathrm{d}x\mathrm{d}t + \int_0^{T^\theta}\int_{\mathbb{R}^N} u_1(x)\varphi_T\,\mathbb{D}^{\gamma}_{t|T^\theta}\psi\,\mathrm{d}x \\ = \int_0^{T^\theta}\int_{\mathbb{R}^N} u_t\left(\varphi_T\,\mathbb{D}^{\gamma+1}_{t|T^\theta}\psi + \mathbf{\Theta}(t)(-\Delta)^{\frac{\alpha}{2}}\varphi_T - (-\Delta)^{\frac{\beta}{2}}\varphi_T\,\mathbb{D}^{\gamma}_{t|T^\theta}\psi\right)\mathrm{d}x\mathrm{d}t \qquad (31)\\ \leq \int_0^{T^\theta}\int_{\mathbb{R}^N} |u_t|\left(\varphi_T\,|\mathbb{D}^{\gamma+1}_{t|T^\theta}\psi| + \mathbf{\Theta}(t)|(-\Delta)^{\frac{\alpha}{2}}\varphi_T| + |(-\Delta)^{\frac{\beta}{2}}\varphi_T||\mathbb{D}^{\gamma}_{t|T^\theta}\psi|\right)\mathrm{d}x\mathrm{d}t, \end{aligned}$$

where $\mathbf{\Theta} \in C^\infty([0,T])$ is test function defined by

$$\mathbf{\Theta}(t) = \int_t^{T^\theta} \mathbb{D}^{\gamma}_{s|T^\theta}\psi(s)\,\mathrm{d}s, \quad \text{and} \quad \mathbf{\Theta}'(t) = -\mathbb{D}^{\gamma}_{t|T^\theta}\psi(t).$$

The difference, with respect to the proof of Theorem(1), is related to the estimate of the term containing $\mathbf{\Theta}(t)$. Applying Young's inequality in the right hand side to (31), we obtain

$$\begin{aligned} &\int_0^{T^\theta}\int_{\mathbb{R}^N} |u_t|\left(\varphi_T\,|\mathbb{D}^{\gamma+1}_{t|T^\theta}\psi| + \mathbf{\Theta}(t)|(-\Delta)^{\frac{\alpha}{2}}\varphi_T| + |(-\Delta)^{\frac{\beta}{2}}\varphi_T||\mathbb{D}^{\gamma}_{t|T^\theta}\psi|\right)\mathrm{d}x\mathrm{d}t \\ &\leq C\mathcal{U}_T + C\int_0^{T^\theta}\int_{\mathbb{R}^N} \zeta^{-\frac{q}{p}}\left(\varphi_T\,|\mathbb{D}^{\gamma+1}_{t|T^\theta}\psi| + \mathbf{\Theta}(t)|(-\Delta)^{\frac{\alpha}{2}}\varphi_T| + |(-\Delta)^{\frac{\beta}{2}}\varphi_T||\mathbb{D}^{\gamma}_{t|T^\theta}\psi|\right)^q\mathrm{d}x\mathrm{d}t, \end{aligned}$$

where $q = p/(p-1)$, and

$$\mathcal{U}_T = \int_0^{T^\theta}\int_{\mathbb{R}^N} |u_t|^p\zeta(x,t)\,\mathrm{d}x\mathrm{d}t.$$

Next, introducing the scaled variables $\tau = t/T^\theta$, $\xi = x/T$, and using the scaling property (14), inequality (15),(26), and the fact that $\mathbf{\Theta}(t) \leq \mathbf{\Theta}(0)$, we have the following estimates

$$\left\{\begin{aligned} &\int_0^{T^\theta}\int_{\mathbb{R}^N} \zeta^{-\frac{q}{p}}|\varphi_T\,\mathbb{D}^{\gamma+1}_{t|T^\theta}\psi|^q\,\mathrm{d}x\mathrm{d}t \leq CT^{-(1+\gamma)\theta q+\theta+N}, \\ &\int_0^{T^\theta}\int_{\mathbb{R}^N} \zeta^{-\frac{q}{p}}|(-\Delta)^{\frac{\beta}{2}}\varphi_T\,\mathbb{D}^{\gamma}_{t|T^\theta}\psi|^q\,\mathrm{d}x\mathrm{d}t \leq CT^{-(\gamma\theta+\beta)q+\theta+N}, \\ &\int_0^{T^\theta}\int_{\mathbb{R}^N} \zeta^{-\frac{q}{p}}|\mathbf{\Theta}(t)\,(-\Delta)^{\frac{\alpha}{2}}\varphi_T|^q\,\mathrm{d}x\mathrm{d}t \leq CT^{(1-\gamma)\theta q-\alpha q+\theta+N}. \end{aligned}\right. \qquad (32)$$

Whereupon

$$\mathcal{U}_T + CT^{\theta(1-\gamma)}\int_{\mathbb{R}^N} u_1(x)\varphi_T\,\mathrm{d}x \leq CT^{-(1+\gamma)\theta q+\theta+N} = CT^{-\kappa},$$

where $\kappa = (1+\gamma)\theta q - \theta - N$ and $\theta = \min\{\frac{\alpha}{2}, \beta\}$. Using the assumption (17), in the case $\kappa > 0$, and passing to the limit when T goes to ∞, we get

$$0 < \int_0^\infty \int_{\mathbb{R}^n} |u|^p \,\mathrm{d}x\mathrm{d}t \leq 0,$$

which is a contradiction.
The case $\kappa = 0$ is treated as in Theorem (1). Then, the solution of problem (2) cannot be global.
Proceeding as in the proof of Theorem (1), we give an upper bound estimate of the life span of solution of (2) as follows

$$T_\varepsilon \leq C\varepsilon^{-\frac{\theta}{(1+\gamma)\theta q - \gamma\theta - N}}, \quad \theta = \min\{\frac{\alpha}{2}, \beta\}, \; N < q\theta(1+\gamma) - \gamma\theta.$$

□

Proof of Theorem (3). We repeat the same calculation as in the proof of Theorem (1); we arrive at

$$\mathcal{U}_T + CT^{\theta(1-\gamma)} \int_{\mathbb{R}^N} u_1(x)\varphi_T \,\mathrm{d}x \leq CT^{-(\gamma\theta+\alpha)q+\theta+N}. \tag{33}$$

On the other hand, using the assumption (22) on the initial data u_1, and by the scaled variable $\xi = x/T$, we obtain

$$\int_0^{T^\theta} \int_{\mathbb{R}^N} u_1(x)\varphi_T \,\mathbb{D}^\gamma_{t|T^\theta}\psi \,\mathrm{d}x\mathrm{d}t \geq c\varepsilon T^{\theta(1-\gamma)} \int_{\mathbb{R}^N} (1+|x|)^{-\mu} \,\mathrm{d}x \geq c\varepsilon T^{\theta(1-\gamma)+N-\mu}. \tag{34}$$

From (33) and (34), it follows that

$$0 \leq \mathcal{U}_T \leq CT^{-(\gamma\theta+\alpha)q+\theta+N} - c\varepsilon T^{\theta(1-\gamma)+N-\mu}, \tag{35}$$

Fixing $T = T_\varepsilon^{\frac{1}{\theta}}$, where T_ε is the maximal existence time of solution. Then we have

$$T_\varepsilon \leq c\varepsilon^{-\frac{\theta}{(\gamma\theta+\alpha)q-\theta\gamma-\mu}}.$$

This concludes the proof. □

Proof of Theorem (4). The proof is completely analogous to the proof of Theorem (3), but here we obtain

$$T_\varepsilon \leq C\varepsilon^{-\frac{\theta}{(1+\gamma)\theta q-\gamma\theta-\mu}}.$$

The conclusion follows. □

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