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Existence of solutions of fractional p-laplacian systems with different critical Sobolev-Hardy exponents

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Summary

In this paper, we investigate a fractional p-Laplacian system with different critical Sobolev-Hardy exponents. By variational methods and the Mountain-Pass lemma, positive minimizers of the related best Sobolev constants and the existence of positive solutions of the system are found.

KEYWORDS:
Fractional p-Laplacian operators, Variational method, Mountain-Pass lemma, Solutions.

1 | INTRODUCTION

In this paper we study the existence of nontrivial solutions of the following fractional p-laplacian involving combined critical non-linearities

$$\begin{cases} (-\Delta)_p^s u = \frac{\mu_1(\mu_1|u|^q+\mu_2|v|^q)^{\frac{p_\alpha^*}{q}-1}|u|^{q-2}u}{|x|^\alpha} + \frac{\lambda\eta}{\eta+\nu}\frac{|u|^{\eta-2}|v|^\nu u}{|x|^\alpha} + \eta\frac{|u|^{d-2}u}{|x|^\beta}, & x\in\Omega, \\ (-\Delta)_p^s v = \frac{\mu_2(\mu_1|u|^q+\mu_2|v|^q)^{\frac{p_\alpha^*}{q}-1}|v|^{q-2}v}{|x|^\alpha} + \frac{\lambda\nu}{\eta+\nu}\frac{|u|^{\eta}|v|^{\nu-2}v}{|x|^\alpha} + \nu\frac{|v|^{d-2}v}{|x|^\beta}, & x\in\Omega, \\ u=v=0, & x\in\mathbb{R}^N\setminus\Omega, \end{cases} \tag{1}$$

where $s\in(0,1)$ is fixed, $N>sp$, $1<p<\infty$, $0<\lambda,\mu_1,\mu_2<\infty$, $1<q<p_\alpha^*$, $1<d<p$, $\eta,\nu>1$ such that $\eta+\nu=p_\alpha^*$, where $p_\alpha^*=\frac{(N-\alpha)p}{N-sp}$ is the fractional critical Sobolev exponent and $(-\Delta)_p^s$ is the fractional p-Laplacian operator which, up to normalization factors, may be defined along $u\in C_0^\infty(\mathbb{R}^N)$ as

$$(-\Delta)_p^s u(x) = 2\lim_{\varepsilon\to 0^+}\int\limits_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}dy$$

for $x\in\mathbb{R}^N$, where $B_\varepsilon(x) := \{y\in\mathbb{R}^N : |x-y|<\varepsilon\}$. As for some recent results on the fractional p-Laplacian, we refer to for example[1,2,3] and the references therein.

In the last years, the fractional and nonlocal problems have been investigated by many researchers, for example,[4,5,6,7,8,9], also[10,11,12,13] for fractional p-Laplacian case,[14,15,16,18,19,20,21] for the existence of solutions to fractional Laplacian system. Giacomoni and Mishra[14] by using the idea of Nehari manifold technique and a compactness result based on the classical idea of the Brezis-Lieb lemma show the existence and multiplicity of positive solutions of the following fractional Kirchhoff system

$$\begin{cases} M\left(\int_\Omega |(-\triangle)^{\frac{s}{2}}u|^2dx\right)(-\Delta)^s u = \lambda f(x)|u|^{q-2}u + \frac{2\eta}{\vartheta+\sigma}|u|^{\vartheta-2}|v|^{\sigma}u, & x\in\Omega,\\ M\left(\int_\Omega |(-\triangle)^{\frac{s}{2}}v|^2dx\right)(-\Delta)^s v = \mu g(x)|v|^{q-2}u + \frac{2\sigma\eta}{\vartheta+\sigma}|u|^{\vartheta}|v|^{\sigma-2}v, & x\in\Omega,\\ u=v=0, & x\in\mathbb{R}^N\setminus\Omega \end{cases} \tag{2}$$

where $\lambda,\mu>0$, $1<q<2$ and $\vartheta,\sigma\geq 2$ with $\vartheta+\sigma=2_s^*$, $M(t)=a+bt$ with $a,b>0$ and f,g are sign-changing continuous functions. He et al.[15] considered the problem (2), where $M,f,g\equiv 1$. They proved the multiplicity of solutions using the idea of a Nehari manifold and harmonic extension for suitable choice of $\lambda,\mu>0$. Also, Chen and Deng in[16] obtained the multiplicity of solutions to problem (2) with the fractional p-Laplacian operator $(-\Delta)_p^s$, with $1<q<p$ and $p<\vartheta+\sigma<p_s^*=\frac{np}{n-ps}$. In[17], Guo et al. studied the following problem:

$$\begin{cases} (-\Delta)^s u-\lambda_1 u=\mu_1|u|^{2_s^*-2}u+\frac{\vartheta\gamma}{2_s^*}|u|^{\vartheta-2}|v|^{\sigma}u, & x\in\Omega,\\ (-\Delta)^s v-\lambda_2 v=\mu_2|v|^{2_s^*-2}u+\frac{\sigma\gamma}{2_s^*}|u|^{\vartheta}|v|^{\sigma-2}v, & x\in\Omega,\\ u=v=0, & x\in\mathbb{R}^N\setminus\Omega \end{cases} \tag{3}$$

where $\lambda_1,\lambda_2,\mu_1,\mu_2,\gamma>0$. Using variational methods and critical point theory, the authors proved the existence of the ground state solution to system (3). We also mention[18,19,20] in which the author dealt with the existence, multiplicity and concentration of positive solutions for fractional systems in the whole of $\mathbb{R}^N$ with subcritical and critical nonlinearities.

Notice that as $s\to 1^-$, problem $(-\Delta)_p^s u=f(x,u)$ in Ω reduces to the following problems

$$-\Delta_p u=f(x,u)\quad\text{in }\ \Omega, \tag{4}$$

where $\Omega\subset\mathbb{R}^N$ is a smooth domain. Recently, Kang[22] studied the following quasilinear system

$$\begin{cases} -\Delta_p u-\mu_1\frac{|u|^{p-2}u}{|x|^p}=(|u|^q+|v|^q)^{\frac{p^*}{q}-1}|u|^{q-2}u+\frac{1}{p}Q_s'(u,v), & \text{in }\Omega,\\ -\Delta_p v-\mu_1\frac{|v|^{p-2}v}{|x|^p}=(|u|^q+|v|^q)^{\frac{p_\alpha^*}{q}-1}|v|^{q-2}v+\frac{1}{p}Q_t'(u,v), & \text{in }\Omega,\\ u=v=0 & \text{on }\partial\Omega, \end{cases} \tag{5}$$

where $\Omega\subset\mathbb{R}^N$ is an open bounded domain with smooth boundary such that, $p^*=\frac{Np}{N-p}$ is the critical Sobolev exponent, Q_s',Q_t' are partial derivatives of the homogeneous C^1-function $Q(s,t)$:

$$Q(s,t)=a_1|s|^p+a_2p|s|^{p-2}st+a_3p|t|^{p-2}st+a_4|t|^p,\ \ (s,t)\in\mathbb{R}^2,\ p\geq 2.$$

He obtained positive minimizers of the related best Sobolev constants and the existence of positive solutions of the system. By the use of variational methods and asymptotic properties of solutions at the singular point are established by the Moser iteration method, the author in[23] studied the existence of positive solutions to (5) with $p=2$.

Kang and Yu[24] studied the following system of elliptic equations

$$\begin{cases} -\Delta u-\lambda_1(x)\frac{|u|}{|x|^2}=\nu_1(\nu_1|u|^q+\nu_2|v|^q)^{\frac{2^*}{q}-1}|u|^{q-2}u, & \text{in }\mathbb{R}^N,\\ -\Delta v-\lambda_2(x)\frac{|v|}{|x|^2}=\nu_2(\nu_1|u|^q+\nu_2|v|^q)^{\frac{p_\alpha^*}{q}-1}|v|^{q-2}v, & \text{in }\mathbb{R}^N,\\ u,v\in D^{1,2}(\mathbb{R}^N), \end{cases} \tag{6}$$

where $D^{1,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to $\|u\|:=(\int_{\mathbb{R}^N}|\nabla u|^2dx)^{\frac{1}{2}}$, $2^*=\frac{2N}{N-2}$, $\nu_1,\nu_2>0$, $1<q\leq 2^*$ and $\lambda_1(x),\lambda_2(x)\in C(\mathbb{R}^N)$. By using analytic techniques and variational arguments, the authors established the existence of minimizers to Rayleigh quotients and ground state solutions to systems (6).

The paper is organized into three sections. In Section 2, we recall some basic definitions of fractional Sobolev space and we give some useful auxiliary lemmas. Also, we state the main results of this paper and we give the proof of main results in this Section.

2 | PRELIMINARY LEMMAS

Let $0 < s < 1 < p < \infty$ be real numbers. The fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined by

$$W^{s,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : \ [u]^p_{s,p} := \iint\limits_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dxdy < \infty \right\}.$$

Set $Q = \mathbb{R}^{2N} \setminus \mathcal{O}$, where $\mathcal{O} = (\mathbb{R}^N \setminus \Omega) \times (\mathbb{R}^N \setminus \Omega)$. We denote the set $W^{s,p}(\Omega)$ by

$$W^{s,p}(\Omega) = \left\{ u : \mathbb{R}^N \to \mathbb{R} : \ u|_\Omega \in L^p(\Omega), \ \iint\limits_{Q} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dx \, dy < \infty \right\},$$

where $u|_\Omega$ represents the restriction to Ω of function $u(x)$. Also, we denote by $W^{s,p}_0(\Omega)$ the following linear subspace of $W^{s,p}(\Omega)$

$$W^{s,p}_0(\Omega) = \left\{ g \in W^{s,p}(\Omega) : \ g = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}.$$

The linear space $W^{s,p}(\Omega)$ is endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \|u\|_{L^2(\Omega)} + \left(\iint\limits_{Q} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dx \, dy \right)^{\frac{1}{p}}.$$

Also, we know that $W^{s,p}_0(\Omega)$, endowed with the norm

$$\|u\|_{W_0} = \left(\iint\limits_{Q} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dx \, dy \right)^{\frac{1}{p}} \quad \text{for all } v \in W^{s,p}_0(\Omega), \tag{7}$$

is a uniformly convex Banach space and a reflexive Banach space (see, Remark 2.1 and Lemma 2.4 of[26]).

The embeddings $W^{s,p}_0(\Omega) \hookrightarrow L^r(\Omega, \frac{dx}{|x|^\alpha})$ is continuous for $r \in [1, p^*_\alpha]$ and compactly for $r \in [1, p^*_\alpha)$; see[27] for details.

Now, we define the space $\mathbf{W} = W^{s,p}_0(\Omega) \times W^{s,p}_0(\Omega)$ with the norm

$$\|(u, v)\|^p = \|u\|^p_{W_0} + \|v\|^p_{W_0}.$$

For any $\varepsilon > 0$ and $U \in D^{s,p}(\mathbb{R}^N)$, we know that

$$U_\varepsilon(x) = \frac{1}{\varepsilon^{\frac{N-sp}{p}}} U\left(\frac{|x|}{\varepsilon}\right), \tag{8}$$

is a solution to

$$(-\triangle)^s_p U = S_* U^{p^*_s - 1} \quad \text{in } \mathbb{R}^N,$$

where $D^{s,p}(\mathbb{R}^N)$ is the fractional Beppo-Levi space, that is the completion of $C^\infty_0(\mathbb{R}^N)$ with respect to $[\cdot]_{s,p}$.

Moreover, as showed in[29], $U \in L^\infty(\mathbb{R}^N) \bigcap C^0(\mathbb{R}^N)$ is a positive, radially symmetric, decreasing function, and there exist constants $c_1, c_2 > 0$ and $\theta > 1$ such that for all $r \geq 1$,

$$\frac{c_1}{r^{\frac{N-sp}{p-1}}} \leq U(r) \leq \frac{c_2}{r^{\frac{N-sp}{p-1}}} \quad \text{and} \quad \frac{U(\theta r)}{U(r)} \leq \frac{1}{2}. \tag{9}$$

If $\theta = \theta(N, s, p)$ is the above constant, then for $\varepsilon, \delta > 0$, as in[28], set

$$m_{\varepsilon,\delta} = \frac{U_\varepsilon(\delta)}{U_\varepsilon(\delta) - U_\varepsilon(\theta\delta)},$$

and define $u_{\varepsilon,\delta}(r) = G_{\varepsilon,\delta}(U_\varepsilon(r))$, where

$$G_{\varepsilon,\delta}(U_\varepsilon(r)) := \int_0^{U_\varepsilon(r)} (\Phi'_{\varepsilon,\delta}(t))^{\frac{1}{p}} dt,$$

and

$$\Phi_{\varepsilon,\delta}(t) = \begin{cases} 0, & \text{if } 0 \le t \le U_\varepsilon(\theta\delta), \\ m^p_{\varepsilon,\delta}(t - U_\varepsilon(\theta\delta)), & \text{if } U_\varepsilon(\theta\delta) \le t \le U_\varepsilon(\delta), \\ t + U_\varepsilon(\delta)(m^{p-1}_{\varepsilon,\delta} - 1), & \text{if } t \ge U_\varepsilon(\delta). \end{cases}$$

Also, we can define the best fractional Sobolev constant:

$$S_\alpha := \inf_{u \in W_0^{s,p}(\Omega)\setminus\{0\}} \frac{\|u\|^p_{W_0}}{(\int_\Omega \frac{|u(x)|^{p^*_\alpha}}{|x|^\alpha} dx)^{\frac{p}{p^*_\alpha}}} > 0. \tag{10}$$

So, S_α is attained by a family of functions U_ε that is

$$[U_\varepsilon]^p_{s,p} = S_\alpha \|U_\varepsilon\|^p_{L^p(\mathbb{R}^N)}. \tag{11}$$

For any $\vartheta, \sigma > 1$ and $\vartheta + \sigma = p^*_s$, by the Young inequality, the following best constant are well defined:

$$S(\alpha, \vartheta, \sigma) := \inf_{(u,v) \in \mathbf{W}\setminus\{(0,0)\}} \frac{\|u\|^p_{W_0} + \|v\|^p_{W_0}}{\left(\int_\Omega \frac{|u|^\vartheta |v|^\sigma}{|x|^\alpha} dx\right)^{\frac{p}{p^*_s}}}, \tag{12}$$

$$S(\alpha, \lambda, \mu_1, \mu_2, \vartheta, \sigma) := \inf_{(u,v) \in \mathbf{W}\setminus\{(0,0)\}} \frac{\|u\|^p_{W_0} + \|v\|^p_{W_0}}{\left(\int_\Omega \frac{(\mu_1|u|^q + \mu_2|v|^q)^{\frac{p^*_\alpha}{q}} + \lambda|u|^\vartheta|v|^\sigma}{|x|^\alpha} dx\right)^{\frac{p}{p^*_\alpha}}}. \tag{13}$$

In this paper, we choose the positive constat $\widetilde{R}_0$ such that $\Omega \subset B_{\widetilde{R}_0}(0)$, where $B_{\widetilde{R}_0}(0) = \{x \in \mathbb{R}^N : |x| < \widetilde{R}_0\}$. By Hölder and (10), for all $u \in X_0$, we obtain

$$\begin{aligned} \int_\Omega \frac{|u|^q}{|x|^\beta} &\le \Big(\int_{B_{\widetilde{R}_0}(0)} |x|^{-\beta}\Big)^{\frac{p^*_\beta - q}{p^*_\beta}} \Big(\int_\Omega \frac{|u|^{p^*_\beta}}{|x|^\beta}\Big)^{\frac{q}{p^*_\beta}} \\ &\le \Big(N\omega_N \int_0^{\widetilde{R}_0} r^{-\beta+N-1} dr\Big)^{\frac{p^*_\beta - q}{p^*_\beta}} (S_\beta)^{-\frac{q}{p}} \|u\|^q \\ &\le \mathcal{D}_0 (S_\beta)^{-\frac{q}{p}} \|u\|^q_{W_0}, \end{aligned} \tag{14}$$

where $\omega_N = \frac{2\pi^{\frac{N}{2}}}{N\Gamma(\frac{N}{2})}$ and $\mathcal{D}_0 := \left(\frac{N\omega_N \widetilde{R}_0^{N-\beta}}{N-\beta}\right)^{\frac{p^*_\beta - q}{p^*_\beta}}$.

To obtain our results we need the following facts. Here, we recall a recent result on the extremal functions of S_α [30].

For $0 < \alpha < sp < N$, there exists a minimizer for S_α; see[30] Theorem 1.1 for more details. Now, by similar method as in[31], we fix a radially symmetric decreasing minimizer $U_\alpha = U_\alpha(r)$ for S_α, multiplying U_α by a positive constant if necessary, we assume that

$$(-\triangle)^s_p U_\alpha = \frac{U_\alpha^{p^*_\alpha - 1}}{|x|^\alpha} \quad \text{in } \mathbb{R}^N. \tag{15}$$

Lemma 1. ([30]) There exist constants $c_1, c_2 > 0$ and $\kappa > 1$ such that for all $r \ge 1$,

$$\frac{c_1}{r^{\frac{N-sp}{p-1}}} \le U_\alpha(r) \le \frac{c_2}{r^{\frac{N-sp}{p-1}}} \quad \text{and} \quad \frac{U_\alpha(\kappa r)}{U_\alpha(r)} \le \frac{1}{2}. \tag{16}$$

If κ is the above constant, then for $\delta \ge \varepsilon > 0$, we set

$$m_{\varepsilon,\delta} = \frac{U_{\alpha,\varepsilon}(\delta)}{U_{\alpha,\varepsilon}(\delta) - U_{\alpha,\varepsilon}(\kappa\delta)},$$

and

$$g_{\varepsilon,\delta}(t)=\begin{cases}0, & \text{if } 0\le t\le U_{\alpha,\varepsilon}(\kappa\delta),\\ m^p_{\varepsilon,\delta}(t-U_{\alpha,\varepsilon}(\kappa\delta)), & \text{if } U_{\alpha,\varepsilon}(\kappa\delta)\le t\le U_{\alpha,\varepsilon}(\delta),\\ t+U_{\alpha,\varepsilon}(\delta)(m^{p-1}_{\varepsilon,\delta}-1), & \text{if } t\ge U_{\alpha,\varepsilon}(\delta),\end{cases}$$

and define $u_{\alpha,\varepsilon,\delta}(r)=G_{\varepsilon,\delta}(U_{\alpha,\varepsilon}(r))$, where

$$G_{\varepsilon,\delta}(U_{\alpha,\varepsilon}(r)) := \int_0^{U_{\alpha,\varepsilon}(r)} (g'_{\varepsilon,\delta}(t))^{\frac{1}{p}}\,dt.$$

Also, $u_{\alpha,\varepsilon,\delta}$ satisfies

$$u_{\alpha,\varepsilon,\delta}(r)=\begin{cases}U_{\alpha,\varepsilon}(r), & \text{if } r\le\delta,\\ 0, & \text{if } r\ge\kappa\delta.\end{cases}\tag{17}$$

To obtain our results we need the following lemmas.

Lemma 2. (See[31,32]) There exists $\widetilde{C}>0$ such that for any $0<2\varepsilon\le\delta<\kappa^{-1}\delta_\Omega$ the following estimates hold:

$$\iint_{\mathbb{R}^{2N}}\frac{|u_{\alpha,\varepsilon,\delta}(x)-u_{\alpha,\varepsilon,\delta}(y)|^p}{|x-y|^{N+ps}}dxdy\le S_\alpha^{\frac{N-\alpha}{sp-\alpha}}+\widetilde{C}\left(\frac{\varepsilon}{\delta}\right)^{\frac{N-sp}{p-1}},\tag{18}$$

$$\int_{\mathbb{R}^N}\frac{u^{p^*_\alpha}_{\alpha,\varepsilon,\delta}}{|x|^\alpha}dx\ge S_\alpha^{\frac{N-\alpha}{sp-\alpha}}-\widetilde{C}\left(\frac{\varepsilon}{\delta}\right)^{\frac{N-\alpha}{p-1}}.\tag{19}$$

Lemma 3. (See[33] Lemma 2.3) For any $1<d<p^*_\sigma$, there exists a constant $\widetilde{C}_d=\widetilde{C}_q(N,p,s)>0$ such that

$$\int_\Omega\frac{|u_{\alpha,\varepsilon,\delta}(x)|^d}{|x|^\beta}dx\ge\begin{cases}\widetilde{C}_d\varepsilon^{N-\frac{N-sp}{p}d-\beta}, & \text{if } d>\frac{(N-\beta)(p-1)}{N-sp},\\ \widetilde{C}_d\varepsilon^{N-\frac{N-sp}{p}d-\beta}|\ln\varepsilon|, & \text{if } d=\frac{(N-\beta)(p-1)}{N-sp},\\ \widetilde{C}_d\varepsilon^{(N-sp)(\frac{d}{p-1}-\frac{d}{p})}, & \text{if } d<\frac{(N-\beta)(p-1)}{N-sp}.\end{cases}\tag{20}$$

For any $\vartheta,\sigma>1$ with $\vartheta+\sigma=p^*_\alpha$, we define

$$g(\varsigma):=\frac{1+\varsigma^p}{((\mu_1+\mu_2\varsigma^q)^{\frac{p^*_\alpha}{q}}+\lambda\varsigma^\sigma)^{\frac{p}{p^*_\alpha}}},\quad \varsigma\ge0,\tag{21}$$

$$g(\varsigma_{\min}):=\min_{\varsigma\ge0}g(\varsigma)>0,\tag{22}$$

where $\varsigma_{\min}>0$ is a minimal point of $g(\varsigma)$ and so a root of the equation

$$(\mu_1+\mu_2\varsigma^q)^{\frac{p^*_\alpha}{q}-1}(\mu_1\varsigma^{p-1}-\mu_1\tau^{q-1})+\frac{\lambda}{p^*_\alpha}\vartheta\tau^{p+\sigma-1}-\frac{\lambda}{p^*_\alpha}\sigma\tau^{\sigma-1}=0,\quad \tau\ge0.\tag{23}$$

Now, we state the main results of this paper.

Theorem 1. Let $1<q<p$ for $N>sp$. Assume that $s\in(0,1)$, $1<q<p$, $N>sp$, $0<\lambda,\mu_1,\mu_2<\infty$, $\eta,\nu>1$ and $\eta+\nu=p^*_\alpha$. Then, for all $\delta>0$ there exists $\varepsilon_\delta>0$ and $\Lambda_0>0$ such that for all $\varepsilon\in(0,\varepsilon_\delta)$ and $0<\eta^{\frac{p}{p-d}}+\nu^{\frac{p}{p-d}}<\Lambda_0$, problem (1) has at least one positive solution in **W**.

The proofs of Theorem 1 is obtained by applying variational arguments inspired by[22,23,24,25].

Theorem 2. Suppose $N>sp$, $s\in(0,1)$ and $0<\lambda<\infty$, then:
(i) $S(\alpha,\lambda,\mu_1,\mu_2,\vartheta,\sigma)=g(\varsigma_{\min})S_\alpha$;
(ii) $S(\alpha,\lambda,\mu_1,\mu_2,\vartheta,\sigma)$ has the minimizers $(U_\alpha(x),\varsigma_{\min}U_\alpha(x))$, $\forall\epsilon>0$, where $U_\alpha(x)$ are defined as in (15).

Proof. We know that $\lim_{\varsigma\to 0^+} g(\varsigma) = \frac{1}{\mu_1^{\frac{p}{q}}}$, $\lim_{\varsigma\to+\infty} g(\varsigma) = \frac{1}{\mu_2^{\frac{p}{q}}}$. So $\min_{t\geq 0} g(t)$ must be achieved at finite $\varsigma_{\min} \geq 0$. Furthermore, direct calculation shows that there exists a positive constant C such that

$$0 < C \leq g(\varsigma_{\min}) = \min_{\varsigma\geq 0} g(\varsigma) \leq \min\left\{\frac{1}{\mu_1^{\frac{p}{q}}}, \frac{1}{\mu_2^{\frac{p}{q}}}\right\}.$$

From the fact that $g'(\varsigma_{\min}) = 0$ we deduce that $\varsigma_{\min}$ is a root of (23).

To continue,, by the same as that in[25], Let $\{w_n\} \subset D^{s,p}(\mathbb{R}^N)$ is a minimizing sequence for S_*. Let $e_1, e_2 > 0$ be chosen later. Set $u_n = e_1 w_n$ and $v_n = e_2 w_n$ in (13), so one can get

$$\frac{e_1^p + e_2^p}{\left((\mu_1 e_1^q + \mu_2 e_2^q)^{\frac{p_\alpha^*}{q}} + \lambda e_1^\vartheta e_2^\sigma\right)^{\frac{p}{p_\alpha^*}}} \frac{\|w_n\|_{W_0}^p}{(\int_\Omega \frac{|w_n(x)|^{p_\alpha^*}}{|x|^\alpha} dx)^{p/p_\alpha^*}} \geq S(\alpha, \lambda, \mu_1, \mu_2, \vartheta, \sigma). \tag{24}$$

Note that

$$g\left(\frac{e_2}{e_1}\right) = \frac{e_1^p + e_2^p}{\left((\mu_1 e_1^q + \mu_2 e_2^q)^{\frac{p_\alpha^*}{q}} + \lambda e_1^\vartheta e_2^\sigma\right)^{\frac{p}{p_\alpha^*}}}. \tag{25}$$

Choose e_1 and e_2 in (24) such that $\frac{e_2}{e_1} = \varsigma_{\min}$. Passing to the limit as $n \to \infty$ one has

$$g(\varsigma_{\min}) S_\alpha \geq S(\alpha, \lambda, \mu_1, \mu_2, \vartheta, \sigma). \tag{26}$$

On the other hand, Let $\{(u_n, v_n)\}$ be a minimizing sequence of $S(\alpha, \lambda, \mu_1, \mu_2, \vartheta, \sigma)$ and define $s_n = \frac{v_n}{u_n}$, then we get

$$\begin{aligned} \frac{\|u_n\|_{W_0}^p + \|v_n\|_{W_0}^p}{\left(\int_\Omega \frac{(\mu_1|u|^q+\mu_2|v|^q)^{\frac{p_\alpha^*}{q}} + \lambda|u|^\vartheta|v|^\sigma}{|x|^\alpha} dx\right)^{\frac{p}{p_\alpha^*}}} &= \frac{(1+s_n^p)\|u_n\|_{W_0}^p}{\left((\mu_1 + \mu_2 s_n^q)^{\frac{p_\alpha^*}{q}} + \lambda s_n^\sigma\right)^{\frac{p}{p_\alpha^*}} \left(\int_\Omega \frac{|u_n(x)|^{p_\alpha^*}}{|x|^\alpha} dx\right)^{p/p_\alpha^*}} \\ &\geq g(s_n) S_\alpha \\ &\geq g(\varsigma_{\min}) S_\alpha. \end{aligned}$$

Now, as $n \to \infty$ one can get

$$S(\alpha, \lambda, \mu_1, \mu_2, \vartheta, \sigma) \geq g(\varsigma_{\min}) S_\alpha. \tag{27}$$

Hence (25) and (27) imply that

$$S(\alpha, \lambda, \mu_1, \mu_2, \vartheta, \sigma) = g(\varsigma_{\min}) S_\alpha. \tag{28}$$

(ii) In view of (13), (17) and (28) we have the desired conclusion.

The corresponding energy functional of problem (1) is defined by

$$\begin{aligned} J_{\eta,\nu}(u, v) &= \frac{1}{p}\|(u, v)\|^p - \frac{1}{d} \int_\Omega \left(\eta \frac{|u|^d}{|x|^\beta} + \nu \frac{|v|^d}{|x|^\beta}\right) dx \\ &\quad - \frac{1}{p_\alpha^*}\left[\int_\Omega \frac{(\mu_1|u|^q + \mu_2|v|^q)^{\frac{p_\alpha^*}{q}}}{|x|^\alpha} dx + \lambda \int_\Omega \frac{|u|^\vartheta|v|^\sigma}{|x|^\alpha} dx\right] \\ &:= \frac{1}{p}\|(u, v)\|^p - \frac{1}{d} L_{\eta,\nu}(u, v) - \frac{1}{p_\alpha^*} K(u, v), \end{aligned}$$

for each $(u, v) \in \mathbf{W}$. It is easy to check that $J \in C^1(\mathbf{W}, \mathbb{R})$.

Now, we recall that a sequence $\{(u_n, v_n)\}$ is a Palais-Smale sequence at the level c ($(\mathrm{PS})_c$ sequence in short) for the functional J if $J(u_n, v_n) \to c$ and $J'(u_n, v_n) \to 0$. If any $(\mathrm{PS})_c$ sequence $\{(u_n, v_n)\}$ has a convergent subsequence, we say that J satisfies the $(\mathrm{PS})_c$ condition.

Lemma 4. Suppose that $\{(u_n, v_n)\} \subset \mathbf{W}$ is a $(\mathrm{PS})_c$-sequence of $J_{\eta,\nu}$. Then $\{(u_n, v_n)\}$ is bounded in $\mathbf{W}$.

Proof. By contradiction assume that $\|(u_n, v_n)\| \to +\infty$. Set $(\widetilde{u}_n, \widetilde{v}_n) = \left(\frac{u_n}{\|(u_n,v_n)\|}, \frac{v_n}{\|(u_n,v_n)\|}\right)$. Since $\|(\widetilde{u}_n, \widetilde{v}_n)\| = 1$, then $(\widetilde{u}_n, \widetilde{v}_n) \rightharpoonup (\widetilde{u}, \widetilde{v})$ in $\mathbf{W}$, and this implies that $\widetilde{u}_n \to \widetilde{u}$ and $\widetilde{v}_n \to \widetilde{v}$ strongly in $L^d(\Omega, \frac{dx}{|x|^\beta})$ for any $d \in [1, p^*_\beta)$ and $\widetilde{u}_n \to \widetilde{u}$ and $\widetilde{v}_n \to \widetilde{v}$ a.e. in Ω, so

$$L_{\eta,\nu}(\widetilde{u}_n, \widetilde{v}_n) = L_{\eta,\nu}(\widetilde{u}, \widetilde{v}) + o_n(1). \tag{29}$$

Since $\{(u_n, v_n)\}$ is a $(\mathrm{PS})_c$-sequence of $J_{\eta,\nu}$ and $\|(u_n, v_n)\| \to +\infty$, we deduce that

$$\begin{aligned}\frac{1}{p}\|(\widetilde{u}_n, \widetilde{v}_n)\|^p - \frac{1}{p^*_\alpha}\|(u_n, v_n)\|^{p^*_\alpha-2}K(\widetilde{u}_n, \widetilde{v}_n)\\ -\frac{1}{q}\|(u_n, v_n)\|^{q-2}L_{\eta,\nu}(\widetilde{u}_n, \widetilde{v}_n) = o_n(1),\end{aligned} \tag{30}$$

and

$$\begin{aligned}\|(\widetilde{u}_n, \widetilde{v}_n)\|^p - \|(u_n, v_n)\|_s^{p^*_s-2}K(\widetilde{u}_n, \widetilde{v}_n)\\ -\|(u_n, v_n)\|^{q-2}L_{\eta,\nu}(\widetilde{u}_n, \widetilde{v}_n) = o_n(1).\end{aligned} \tag{31}$$

From (29)-(31), $1 < q < p$ and $\|(u_n, v_n)\|_s \to +\infty$ one has

$$\|(\widetilde{u}_n, \widetilde{v}_n)\|^p = \frac{p(p^*_s - q)}{q(p^*_s - 2)}\|(u_n, v_n)\|^{q-2}L_{\eta,\nu}(\widetilde{u}_n, \widetilde{v}_n) + o_n(1) \to 0, \tag{32}$$

as $n \to \infty$, which contradicts $\|(\widetilde{u}_n, \widetilde{v}_n)\| = 1$. Therefore, $\{(u_n, v_n)\}$ is bounded in $\mathbf{W}$.

Lemma 5. $J_{\eta,\nu}$ satisfies the $(PS)_c$ condition with c satisfying

$$0 < c < c_* = \frac{ps - \alpha}{p(N - \alpha)}(S(\alpha, \lambda, \mu_1, \mu_2, \vartheta, \sigma))^{\frac{N-\alpha}{sp-\alpha}}.$$

Proof. Let $\{(u_n, v_n)\} \in \mathbf{W}$ be a $(PS)_c$-sequence for $J_{\eta,\nu}$ with $c < c_*$. It follows from Lemma 4 that $\{(u_n, v_n)\}$ is bounded in $\mathbf{W}$, and then $(u_n, v_n) \rightharpoonup (u, v)$ up to a subsequence, (u, v) is a critical point of $J_{\eta,\nu}$. Furthermore, we may assume

$$\begin{cases} u_n \rightharpoonup u, \quad v_n \rightharpoonup v, \quad \text{weakly in } W_0^{s,p}(\mathbb{R}^N),\\ u_n \rightharpoonup u, \quad v_n \rightharpoonup v, \quad \text{weakly in } L^{p^*_\alpha}(\Omega, \frac{dx}{|x|^\alpha}),\\ u_n \rightharpoonup u, \quad v_n \rightharpoonup v, \quad \text{weakly in } L^d(\Omega, \frac{dx}{|x|^\beta})),\ \forall\, 1 \le d < p^*_\beta,\\ u_n \to u, \quad v_n \to v, \quad \text{a.e on } \Omega,\end{cases}$$

So, one can get $J'_{\eta,\nu}(u, v) = 0$ and by the proof of Lemma 4,

$$L_{\eta,\nu}(u_n, v_n) \to L_{\eta,\nu}(u, v), \quad \text{as} \quad n \to \infty. \tag{33}$$

Let $\overline{u}_n = u_n - u$, $\overline{v}_n = v_n - v$. Then by the Brèzis-Lieb lemma[34], we obtain

$$\|(\overline{u}_n, \overline{v}_n)\|^p \to \|(u_n, v_n)\|^p - \|(u, v)\|^p, \quad \text{as} \quad n \to \infty, \tag{34}$$

$$K(\overline{u}_n, \overline{v}_n) \to K(u_n, v_n) - K(u, v), \quad \text{as} \quad n \to \infty. \tag{35}$$

Since $J_{\eta,\nu}(u_n, v_n) = c + o(1)$, $J'_{\eta,\nu}(u_n, v_n) = o(1)$ and (33)-(35), we can deduce that

$$\frac{1}{p}\|(\overline{u}_n, \overline{v}_n)\|^p - \frac{1}{p^*_\alpha}K(\overline{u}_n, \overline{v}_n) = c - J_{\eta,\nu}(u, v) + o(1), \tag{36}$$

and

$$\|(\overline{u}_n, \overline{v}_n)\|^p - K(\overline{u}_n, \overline{v}_n) = o(1).$$

Hence, we may assume that

$$\|(\overline{u}_n, \overline{v}_n)\|^p \to l, \qquad K(\overline{u}_n, \overline{v}_n) \to l. \tag{37}$$

If $l = 0$, the proof is completed. Suppose that $l > 0$, then from definition of $S(\alpha, \lambda, \mu_1, \mu_2, \vartheta, \sigma)$ and (37), we obtain

$$
\begin{aligned}
S(\alpha,\lambda,\mu_1,\mu_2,\vartheta,\sigma)l^{\frac{p}{p_\alpha^*}} &= S(\alpha,\lambda,\mu_1,\mu_2,\vartheta,\sigma)\lim_{n\to\infty}\left(\int_\Omega \frac{(\mu_1|\overline{u}_n|^q+\mu_2|\overline{v}_n|^q)^{\frac{p_\alpha^*}{q}}+\lambda|\overline{u}_n|^\vartheta|\overline{v}_n|^\sigma}{|x|^\alpha}dx\right)^{\frac{p}{p_\alpha^*}}\\
&\le \lim_{n\to\infty}(\|\overline{u}_n\|_{W_0}^p+\|\overline{v}_n\|_{W_0}^p)\\
&\le \lim_{n\to\infty}\|(\overline{u}_n,\overline{v}_n)\|^p = l,
\end{aligned}
$$

which implies that $l \ge (S(\alpha,\lambda,\mu_1,\mu_2,\vartheta,\sigma))^{\frac{N-\alpha}{sp-\alpha}}$.
In additional, from (36) and (37), we get

$$
c=\frac{ps-\alpha}{p(N-\alpha)}l+J_{\eta,\nu}(u,v)\ge \frac{ps-\alpha}{p(N-\alpha)}(S(\alpha,\lambda,\mu_1,\mu_2,\vartheta,\sigma))^{\frac{N-\alpha}{sp-\alpha}},
$$

which contradicts $c<\frac{ps-\alpha}{p(N-\alpha)}(S(\alpha,\lambda,\mu_1,\mu_2,\vartheta,\sigma))^{\frac{N-\alpha}{sp-\alpha}}$.

Lemma 6. Under the assumption of Theorem 1 we have

$$
\sup_{\tau\ge0} J_{\eta,\nu}(\tau u_{\alpha,\varepsilon,\delta},\tau\varsigma_{\min}u_{\alpha,\varepsilon,\delta})<c_*=\frac{ps-\alpha}{p(N-\alpha)}(S(\alpha,\lambda,\mu_1,\mu_2,\vartheta,\sigma))^{\frac{N-\alpha}{sp-\alpha}}. \tag{38}
$$

Proof. Consider the functions

$$
\begin{aligned}
G(\tau) &= J_{\eta,\nu}(\tau u_{\alpha,\varepsilon,\delta},\tau(\varsigma_{\min}u_{\alpha,\varepsilon,\delta}))\\
&\le \frac{\tau^p}{p}(1+\varsigma_{\min}^p)\|u_{\alpha,\varepsilon,\delta}\|_{W_0}^p-\frac{\tau^q}{q}L_{\eta,\nu}(\tau u_{\alpha,\varepsilon,\delta},\tau(\varsigma_{\min}u_{\alpha,\varepsilon,\delta}))\\
&\quad -\frac{\tau^{p_\alpha^*}}{p_\alpha^*}\left((\mu_1+\mu_2\varsigma_{\min}^q)^{\frac{p_\alpha^*}{q}}+\lambda\varsigma_{\min}^\sigma\right)\int_\Omega\frac{|u_{\alpha,\varepsilon,\delta}|^{p_\alpha^*}}{|x|^\alpha}dx,\\
g_1(\tau) &= \frac{\tau^p}{p}(1+\varsigma_{\min}^p)\|u_{\alpha,\varepsilon,\delta}\|_{W_0}^p\\
&\quad -\frac{\tau^{p_\alpha^*}}{p_\alpha^*}\left((\mu_1+\mu_2\varsigma_{\min}^q)^{\frac{p_\alpha^*}{q}}+\lambda\varsigma_{\min}^\sigma\right)\int_\Omega\frac{|u_{\alpha,\varepsilon,\delta}|^{p_\alpha^*}}{|x|^\alpha}dx.
\end{aligned}
$$

Then by definition of $S(\alpha,\lambda,\mu_1,\mu_2,\vartheta,\sigma)$, we obtain that

$$
\begin{aligned}
\sup_{\tau\ge0} g_1(\tau) &\le \left(\frac1p-\frac{1}{p_\alpha^*}\right)\left(\frac{(1+\varsigma_{\min}^p)\|u_{\alpha,\varepsilon,\delta}\|_{W_0}^p}{\left(\left((\mu_1+\mu_2\varsigma_{\min}^q)^{\frac{p_\alpha^*}{q}}+\lambda\varsigma_{\min}^\sigma\right)\int_\Omega\frac{|u_{\alpha,\varepsilon,\delta}|^{p_\alpha^*}}{|x|^\alpha}dx\right)^{\frac{p}{p_\alpha^*}}}\right)^{\frac{p_\alpha^*}{p_\alpha^*-p}}\\
&\le \frac{ps-\alpha}{p(N-\alpha)}\left(g(\tau_{\min})\frac{\|u_{\alpha,\varepsilon,\delta}\|_{W_0^{s,p}}^p}{\left(\int_\Omega\frac{|u_{\alpha,\varepsilon,\delta}|^{p_\alpha^*}}{|x|^\alpha}dx\right)^{\frac{p}{p_\alpha^*}}}\right)^{\frac{N-\alpha}{sp-\alpha}}\\
&\le \frac{ps-\alpha}{p(N-\alpha)}\left(g(\varsigma_{\min})S_\alpha\right)^{\frac{N-\alpha}{sp-\alpha}}+\widetilde{C}\left(\frac{\varepsilon}{\delta}\right)^{\frac{N-sp}{p-1}}\\
&\le \frac{ps-\alpha}{p(N-\alpha)}\left(S(\alpha,\lambda,\mu_1,\mu_2,\vartheta,\sigma)\right)^{\frac{N-\alpha}{sp-\alpha}}+\widetilde{C}\left(\frac{\varepsilon}{\delta}\right)^{\frac{N-sp}{p-1}}
\end{aligned} \tag{39}
$$

where the following facts has been used:

$$
\sup_{t\ge0}\left(\frac{t^p}{p}A-\frac{t^{p_\alpha^*}}{p_\alpha^*}B\right)=\frac{ps-\alpha}{p(N-\alpha)}\left(\frac{A}{B^{\frac{N-sp}{N-\alpha}}}\right)^{\frac{N-\alpha}{sp-\alpha}},\quad A,B>0.
$$

Since

$$
G(\tau)=J_{\eta,\nu}(\tau u_{\alpha,\varepsilon,\delta},\tau(\varsigma_{\min}u_{\alpha,\varepsilon,\delta}))\le\frac{\tau^p}{p}\|(u_{\alpha,\varepsilon,\delta},\varsigma_{\min}u_{\alpha,\varepsilon,\delta})\|_s^p \quad\text{for all}\quad \tau\ge0 \quad\text{and}\quad \lambda,\mu_1,\mu_2>0,
$$

this implies that there exists $\tau_0 \in (0,1)$ satisfying

$$\sup_{0\le\tau\le\tau_0} G(\tau) < c_*. \tag{40}$$

Using the definitions of $J_{\eta,\nu}(u,v)$ and $(u_{\alpha,\varepsilon,\delta}, \varsigma_{\min} u_{\alpha,\varepsilon,\delta})$, and by (39), we have

$$\begin{aligned}
\sup_{\tau\ge\tau_0} G(\tau) &= \sup_{\tau\ge\tau_0}\left(g_1(\tau) - \frac{t^d}{d} L_{\eta,\nu}(\tau u_{\alpha,\varepsilon,\delta}, \tau(\varsigma_{\min} u_{\alpha,\varepsilon,\delta}))\right)\\
&\le \frac{ps-\alpha}{p(N-\alpha)}\Big(S(\alpha,\lambda,\mu_1,\mu_2,\vartheta,\sigma)\Big)^{\frac{N-\alpha}{sp-\alpha}} + \widetilde{C}\left(\frac{\varepsilon}{\delta}\right)^{\frac{N-sp}{p-1}}\\
&\quad -\frac{\tau^d}{d}\left(\eta + \nu\varsigma_{\min}^d\right)\int_\Omega \frac{|u_{\alpha,\varepsilon,\delta}|^d}{|x|^\beta}dx.
\end{aligned} \tag{41}$$

(i) If $1 \le d < \frac{(N-\beta)(p-1)}{N-sp}$, then by (43) one gets

$$\int_\Omega \frac{|u_{\alpha,\varepsilon,\delta}|^d}{|x|^\beta} dx \ge \widetilde{C}_d \varepsilon^{\frac{(N-sp)d}{p(p-1)}} \tag{42}$$

and since $\frac{N-sp}{p-1} > \frac{(N-sp)d}{p(p-1)}$, then in view of (40) and (41) we get

$$\sup_{\tau\ge 0} G(\tau) < c_*.$$

(ii) If $\frac{(N-\beta)(p-1)}{N-sp} \le d < p$, by (43), we have

$$\int_{\mathbb{R}^N} \frac{|u_{\alpha,\varepsilon,\delta}(x)|^d}{|x|^\beta} dx \ge \begin{cases} \widetilde{C}_d \varepsilon^{N-\frac{N-sp}{p}d-\beta}, & \text{if } d > \frac{(N-\beta)(p-1)}{N-sp},\\ \widetilde{C}_d \varepsilon^{N-\frac{N-sp}{p}d-\beta} |\ln\varepsilon|, & \text{if } d = \frac{(N-\beta)(p-1)}{N-sp}.\end{cases}$$

By straightforward calculation, it holds $-\frac{d(N-sp)}{p} + N - \beta \le \frac{N-\beta}{p} < \frac{N-sp}{p-1}$. Hence (40) and (41) imply that

$$\sup_{\tau\ge 0} G(\tau) < c_*.$$

Therefore, we have the desired conclusion.
In order to prove the main result, we will use the following lemma.

Lemma 7. (Mountain-Pass lemma, see[35]) Let X be a real Banach space and $J \in C^1(X,\mathbb{R})$. Suppose that
(i) $J(0) = 0$.
(ii) There are constants $\rho, \beta > 0$ such that $J(u) \ge \beta$ for all $u \in X$, with $\|u\|_X = \rho$.
(iii) There is an $e \in X$ such that $\limsup_{t\to\infty} J(te) < 0$.
Take $t_0 > 0$ such that $\|t_0 e\|_X > \rho$ and $J(t_0 e) < 0$. Set

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = t_0 e\},$$

$$c = \inf_{\gamma\in\Gamma}\max_{s\in[0,1]} J(\gamma(s)) \ge \beta.$$

Then there exists a Palais-Smale sequence at level c for J.

Now, we complete the proof of Theorem 1.
Proof of Theorem 1. In view of (14), we get

$$\begin{aligned}
L_{\eta,\nu}(u,v) &= \int_\Omega \left(\eta\frac{|u|^d}{|x|^\beta} + \nu\frac{|v|^d}{|x|^\beta}\right) dx\\
&\le \mathcal{D}_0 (S_\beta)^{-\frac{q}{p}} \left(\eta^{\frac{p}{p-d}} + \nu^{\frac{p}{p-d}}\right)^{\frac{p-d}{p}} \|(u,v)\|^q.
\end{aligned} \tag{43}$$

Also, by (10), we have

$$\begin{aligned} K(u,v) &= \int_\Omega \frac{(\mu_1|u|^q+\mu_2|v|^q)^{\frac{p_\alpha^*}{q}}}{|x|^\alpha}dx + \lambda\int_\Omega \frac{|u|^\vartheta|v|^\sigma}{|x|^\alpha}dx \\ &\le 2^{\frac{p_\alpha^*}{q}} \max\left\{\mu_1^{\frac{p_\alpha^*}{q}} + \mu_2^{\frac{p_\alpha^*}{q}}\right\}\left(\int_\Omega \frac{|u|^{p_\alpha^*}}{|x|^\alpha}dx + \int_\Omega \frac{|v|^{p_\alpha^*}}{|x|^\alpha}dx\right) \\ &\quad +\lambda\Big(\frac{\alpha}{p_\alpha^*}\int_\Omega \frac{|u|^{p_\alpha^*}}{|x|^\alpha}dx + \frac{\beta}{p_\alpha^*}\int_\Omega \frac{|v|^{p_\alpha^*}}{|x|^\alpha}dx\Big) \\ &\le \left[2^{\frac{p_\alpha^*}{q}} \max\left\{\mu_1^{\frac{p_\alpha^*}{q}} + \mu_2^{\frac{p_\alpha^*}{q}}\right\} + \lambda\right](S_\alpha)^{-\frac{p_\alpha^*}{p}}\|(u,v)\|^{p_\alpha^*} \end{aligned} \tag{44}$$

Let

$$\begin{aligned} r &:= \|(u,v)\| \\ F(r) &:= \frac{1}{p}r^p - \frac{1}{p_\alpha^*}\left[2^{\frac{p_\alpha^*}{q}} \max\left\{\mu_1^{\frac{p_\alpha^*}{q}} + \mu_2^{\frac{p_\alpha^*}{q}}\right\} + \lambda\right](S_\alpha)^{-\frac{p_\alpha^*}{p}} r^{p_\alpha^*} \\ \Upsilon(r) &:= \frac{1}{q}\mathcal{D}_0(S_\beta)^{-\frac{q}{p}}\left(\eta^{\frac{p}{p-d}} + \nu^{\frac{p}{p-d}}\right)^{\frac{p-d}{p}}\|(u,v)\|^q. \end{aligned}$$

In view of (43) and (44), one can get

$$J_{\eta,\nu}(u,v) \ge F(r) - \Upsilon(r),$$

Note that $p < p_\alpha^*$, it is easy to see that there exists $\sigma > 0$ such that $F(r)$ achieves its maximum at σ and $f(\sigma) > 0$. Hence, there exists $\Lambda_0 > 0$, such that for $0 < \eta^{\frac{p}{p-d}} + \nu^{\frac{p}{p-d}} < \Lambda_0$ and $\lambda, \mu_1, \mu_2 > 0$,

$$\inf_{\|(u,v)\|=\sigma} J_{\eta,\nu}(u,v) \ge F(\sigma) - \Upsilon(\sigma) > 0 = J_{\eta,\nu}(0,0). \tag{45}$$

Set $c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_{\eta,\nu}(\gamma(t))$, where

$$\Gamma := \left\{\gamma \in C([0,1],\mathbf{W}) \mid \gamma(0) = (0,0),\ J_{\eta,\nu}(\gamma(1)) < 0,\ \|\gamma(1)\| > \sigma\right\}.$$

If $c < c_*$, so from Lemma 5, $(PS)_c$ condition holds, and the conclusion follows by Lemma 7. If $c = c_*$, then by Lemma 6, $\gamma(t) = (tu_{\alpha,\varepsilon,\delta}, t\varsigma_{\min}u_{\alpha,\varepsilon,\delta})$, with $0 \le t < 1$, is a path in Γ such that $\max_{t\in[0,1]} J_{\eta,\nu}(\gamma(t)) = c$, Hence, either $G'(\bar t) = J_{\eta,\nu}(\bar t u_{\alpha,\varepsilon,\delta}, \bar t(\varsigma_{\min}u_{\alpha,\varepsilon,\delta})) = 0$ and we are done, or $\gamma(t)$ can be deformed to a path $\overline{\gamma}(t)$ with $\max_{t\in[0,1]} J_{\eta,\nu}(\overline{\gamma}(t)) < c$ and it is a contradiction. So, we conclude that there exists a nontrivial solution $(u,v) \in \mathbf{W}\setminus\{(0,0\}$ of problem (1). Replacing u, v in the term on the right hand side of the equations in (1) by $u^+ = \max\{u,0\}$, $v^+ = \max\{v,0\}$ respectively and by the above argument, we have a nonnegative solution $(u_*, v_*) \ne (0,0)$ of (1) with $J(u_*, v_*) > 0$. Therefore from the strong maximum principle, we know that $u_* > 0$, $v_* > 0$. ■

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