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Legendrian warped product immersions of Sasakian space forms for characterizing spheres via differential equations

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Summary

The goal of this paper is to investigate the geometry of the warping function on a n-dimensional compact Legenderian warped product submanifold M^n of Sasakian space form with free boundary. We establish sharp estimates to the squared norm of the second fundamental form and the Laplacian of the warping function. Besides, we provide some triviality results for M^n by using the Ricci curvature along the gradient of the warping function. Taking the clue from the Bochner formula and second-order ordinary differential equation, we find the characterization for the base of M^n via the first non-zero eigenvalue of the warping function and proved that it is isometric to Euclidean space $\mathbb{R}^p$ or Euclidean sphere $\mathbb{S}^p$ under some extrinsic conditions.

KEYWORDS:
Ordinary differential equations, Warped products, Sasakian space form, Ricci curvature

1 | INTRODUCTION AND MAIN MOTIVATIONS

An outstanding topic in Riemannian geometry is to find the relation between extrinsic and intrinsic invariant on given warped product manifolds. A promising way for that proposal is to study warping function which arises as solutions of the Euler-Lagrange equations and partial differential equations for conditions on curvature functionals. Since there are so many Riemannian invariants on a manifold, one can regard, philosophy, the finding of Riemannian invariants as an approach to searching for the best relationship between intrinsic and extrinsic invariant for the given Riemannian manifolds. In this scenario, Chen[8,9] given the inequality for the second fundamental form as main intrinsic invariant and holomorphic constant sectional curvature and Laplace of the warping function as a main extrinsic invariant for CR-warped products in complex space form and complex projective spaces. He was also proven the complete classification which satisfied the equality case of this inequality. Several great successes in warped product submanifolds continue to be achieved for different ambient space forms in[1,24,27] through the work of Chen[10,15]. Thus, the classifying the second fundamental form of inequality related to the warping functions on warped product submanifolds or good knowledge of their geometry are important issues in Riemannian geometry. To proceed, let us recall the definition of warped product manifold considered initially by Bishop and O'Neill[4].

Let (N_1, g_1) and (N_2, g_2) are two Riemannian manifolds and $f : N_1 \to (0, \infty)$, is a positive differentiable function on N_1. Consider the product manifold $N_1 \times N_2$ with its canonical projections $\gamma_1 : N_1 \times N_2 \to N_1$, $\gamma_2 : N_1 \times N_2 \to N_2$ and the projection maps defined by $\gamma_1(x, y) = x$, and $\gamma_2(x, y) = y$ for every $p = (x, y) \in N_1 \times N_2$. The warped product $M^n = N_1 \times_f N_2$ is the product manifold $N_1 \times N_2$ equipped with the Riemannian structure such that

$$||X||^2 = ||\gamma_1 * (X)||^2 + f^2(\gamma_1(x))||\gamma_2 * (X)||^2, \tag{1.1}$$

[0]**Abbreviations:** ANA, anti-nuclear antibodies; APC, antigen-presenting cells; IRF, interferon regulatory factor

for any tangent vector $X \in \mathfrak{X}(T_xM)$, where $*$ is the symbol of tangent maps and we have $g = g_1 + f^2 g_2$. Thus, the function f is called a warping function on M^n. The following lemma is a direct consequence of warped product manifolds:

Lemma 1.1. [4] Let $M = N_1 \times_f N_2$ be a warped product manifold, then we have

(i) $\nabla_X Y \in \mathfrak{X}(TN_1)$,

(ii) $\nabla_Z X = \nabla_X Z = (X \ln f)Z$,

(iii) $\nabla_Z W = \nabla'_Z W - g(Z, W)\nabla \ln f$,

for any $X, Y \in \mathfrak{X}(TN_1)$ and $Z, W \in \mathfrak{X}(TN_2)$, where ∇ and ∇' denote the Levi-Civita connection on M^n and $N_2^{n_2}$, respectively. Further, $\nabla \ln f$ is the gradient of $\ln f$ which is defined as:

$$g(\nabla \ln f, X) = X(\ln f). \tag{1.2}$$

The following remarks are consequences of Lemma 1.1:-

Remark 1.1. A warped product manifold $M = N_1 \times_f N_2$ is said to be *trivial* or simply a Riemannian product manifold if the warping function f is constant function along N_1.

Remark 1.2. If $M = N_1 \times_f N_2$ is a warped product manifold, then N_1 is totally geodesic and N_2 is totally umbilical submanifold of M^n, respectively.

From [(3.3) in[7]], the following relation is obtained.

$$\sum_{\alpha=1}^{p}\sum_{\beta=1}^{q} K(e_\alpha \wedge e_\beta) = \frac{q\Delta f}{f}. \tag{1.3}$$

According to Nash's theorem[29], we know that in a sufficiently high co-dimension, every Riemannian manifold is isometrically immersed in some Euclidean spaces. In particular, every warped product $N_1^p \times_f N_2^q$ can be immersed as a Riemannian submanifold in some Euclidean space. Moreover, a constant curvature (c) on every Riemannian manifold can be expressed locally as a warped product whose warping function satisfies $\Delta f = cf$. Based on these concepts, many geometers have obtained geometric obstruction for CR-warped products in different ambient space forms (for instance[14,17,24]). Some applications are also derived on compact Riemannian submanifold considering equality cases with empty boundaries. Chen[8] developed a novel technique to find the relationship between extrinsic and intrinsic invariant for warped product submanifolds of Kaehler geometry and space forms geometry by using the Coddazi equation. From this point of view, by using the Gauss equation instead of the Coddazi equation in the sense of[9]. In the first part of the paper, in the spirit of[1,9,24] and motivated by the historical development on the study of the warped function of warped product submanifold, we going to provide a sharp estimate to the squared norm of second fundamental form in terms of warping function and holomorphic constant sectional curvature c, we now announce our first result.

Theorem 1.1. Let $\Psi : M^n = N_1^p \times_f N_2^q \longrightarrow \widetilde{M}^{2+1}(c)$ be an isometric immersion from a Legendrian warped product submanifold $N_1^p \times_f N_2^q$ into a Sasakian space form $\widetilde{M}^{2n+1}(c)$. If N_1^p is minimal in $\widetilde{M}^{2n+1}(c)$, then the following equality is satisfied

$$\|h(u, v)\|^2 = q\Big\{ ||\nabla\mu||^2 + \Big(\frac{c+3}{4}\Big)p - \Delta\mu \Big\}, \tag{1.4}$$

where ∇ and Δ are gradient and the Laplacian of the warping function $\mu = \ln f$ on N_1^p. Moreover, $u = \{e_i\}_{1\leq i\leq p}$ and $v = \{e_j^*\}_{1\leq j\leq q}$ are vector fields on N_1^p and N_2^q, respectively.

An immediate consequence of Theorem 1.1, we consider the warping function $\ln f$ is a harmonic function, then

Corollary 1.1. Let $\Psi : M^n = N_1^p \times_f N_2^q \longrightarrow \widetilde{M}^{2n+1}(c)$ be an isometric immersion from a compact Legendrian warped product submanifold $N_1^p \times_f N_2^q$ into a Sasakian space form $\widetilde{M}^{2n+1}(c)$ such that N_1^p is minimal in $\widetilde{M}^{2n+1}(c)$. If $\ln f$ is a harmonic function, then we have

$$\|h(u, v)\|^2 = q||\nabla\mu||^2 + \Big(\frac{c+3}{4}\Big)pq. \tag{1.5}$$

where ∇ is the gradient $\mu = \ln f$ on N_1^p.

Boundary estimates are classical objects of study in Geometry and Physics. Another goal of our equality (1.4) is to provide potential applications to the gradient Ricci curvature by considering that a Riemannian manifold is compact, and taking into account the Green Theorem (see[36] for more detail). As a consequence, we give the following:

Theorem 1.2. Assume that $\Psi : M^n = N_1^p \times_f N_2^q$ be an isometric immersion of a compact Legendrian warped product submanifold $N_1^p \times_f N_2^q$ into a Sasakian space form $\widetilde{M}^{2n+1}(c)$ such that N_1^p is minimal in $\widetilde{M}^{2n+1}(c)$. If the following equality is satisfied for the warped product submanifold M^n

$$\|h(u,v)\|^2 = q\left\{ \left(\frac{c+3}{4}\right)p + \int_M \mathcal{R}ic(\nabla\mu,)\mathrm{dV} \right\}. \tag{1.6}$$

Then, Legendrian warped product submanifold $N_1^p \times_f N_2^q$ into a Sasakian space form, is simply a Riemannian product of N_1^p and N_2^q.

A direct consequence of Theorem 1.2 as corollary, we give the following

Corollary 1.2. Assume that $\Psi : M^n = N_1^p \times_f N_2^q$ be an isometric immersion of a compact Legendrian warped product submanifold $N_1^p \times_f N_2^q$ into a Sasakian space form $\widetilde{M}^{2n+1}(c)$ such that N_1^p is minimal in $\widetilde{M}^{2n+1}(c)$. If M^n is Ricci flat and the following equality is satisfied

$$\|h(u,v)\|^2 = \left(\frac{c+3}{4}\right)pq. \tag{1.7}$$

Then, Legendrian warped product submanifold $N_1^p \times_f N_2^q$ into a Sasakian space form is simply a Riemannian product of N_1^p and N_2^q.

The next observation is devoted to the work of Obata[31] which is characterized by specific Riemannian manifolds by second-order ordinary differential equations. He derived the necessary and sufficient condition for an n-dimensional complete and connected Riemannian manifold (M^n, g) to be isometric to the n-sphere $\mathbb{S}^n(c)$ if there exists a non-constant smooth function φ on M^n that satisfies the second-order differential equation $H_\phi = -c\phi g$, where H_ϕ is stand for Hessian of ϕ and c is a constant sectional curvature. A great amount of investigations has been devoted to this subject and therefore, characterization of spaces, the Euclidean space $\mathbb{R}^n$, the Euclidean sphere $\mathbb{S}^n$ and the complex projective space $\mathbb{C}P^n$, are important topics in geometric analysis. For example, in[25], Lichnerowicz has shown that on a compact manifold (M^n, g) with $Ric \geq n-1$, the first non-zero eigenvalue μ_1 of the Laplacian is not less than n, while $\mu_1 = n$, then (M^n, g) is isometric to the standard sphere. This means that the Obata's rigidity theorem can be used to characterize the equality case of Lichnerowicz's eigenvalue estimates. By using the techniques of a conformal vector field which plays an important role in obtaining characterizations of spaces but also have an important role in the general theory of Relativity as well as in Mechanics, Deshmukh-Al-Solamy[18] proved that an n-dimensional compact connected Riemannian manifold whose Ricci curvature satisfies the bound $0 < Ric \leq (n-1)(2 - \frac{nc}{\mu_1})c$ for a constant c and μ_1 is the first non-zero eigenvalue of the Laplace operator, then M^n is isometric to $\mathbb{S}^n(c)$ if M^n admitted a non-zero conformal gradient vector field. They also proved that if M^n is Einstein manifold with Einstein constant $\mu = (n-1)c$, then M^n is isometric to $\mathbb{S}^n(c)$ with $c > 0$ if it is admitted conformal gradient vector field. Taking into account the Obata equation[31], Barros, et.al[3] proved that a compact gradient almost Ricci soliton $(M^n, g, \nabla\phi, \lambda)$, whose Ricci tensor is Codazzi and has constant sectional curvature, is isometric to a Euclidean sphere and ϕ is a height function in this case. Similar results have been obtained in[17,18,20,22].

After these observations, we are ready to state our next result which is characterized version of Theorem 1.1, utilizing an ordinary differential equation. To be precise, we have obtained the following result.

Theorem 1.3. Let $\Psi : M^n = N_1^p \times_f N_2^q$ be an isometric immersion of a Legendrian warped product submanifold $N_1^p \times_f N_2^q$ into a Sasakian space form $\widetilde{M}^{2n+1}(c)$ with base N_1^p is minimal in $\widetilde{M}^{2n+1}(c)$, connected and compact. Then N_1^p is isometric to the sphere $\mathbb{S}^p(\frac{\lambda_1}{p})$ with constant sectional curvature is equal to $\frac{\lambda_1}{p}$ if and only if the following equality is satisfied

$$\left\|Hess(\mu)\right\|^2 = \frac{2\lambda_1}{3pq}\left\{ \left\|h(u,v)\right\|^2 - \left(\frac{c+3}{4}\right)pq \right\}, \tag{1.8}$$

where $\lambda_1 > 0$ is a positive eigenvalue associated to the eigenfunction $\mu = \ln f$ and $Hess(\mu)$ is a Hessian tensor of the function μ.

Motivated by the Bochner formula, we give the following result

Theorem 1.4. Let $\Psi : M^n = N_1^p \times_f N_2^q$ be an isometric immersion of a Legendrian warped product $N_1^p \times_f N_2^q$ into a Sasakian space form $\widetilde{M}^{2n+1}(c)$ with base N_1^p is minimal in $\widetilde{M}^{2n+1}(c)$, connected and compact. Then N_1^p is isometric to the sphere $\mathbb{S}^p(\frac{\lambda_1}{p})$ with constant sectional curvature is equal to $\frac{\lambda_1}{p}$ if and only if the following equality is satisfied

$$Ric(\nabla\mu, \nabla\mu) = \lambda_1\Big(\frac{3p+2}{3pq}\Big)\Bigg\{\Big(\frac{c+3}{4}\Big)pq - \Big\|h(u,v)\Big\|^2\Bigg\}. \tag{1.9}$$

where $\lambda_1 > 0$ is a positive eigenvalue associated to the eigenfunction $\mu = \ln f$.

There is another more interesting theorem which a consequence of Theorem 1.4, we find that

Theorem 1.5. Let $\Psi : M^n = N_1^p \times_f N_2^q$ be an isometric immersion of a Legendrian warped product submanifold $N_1^p \times_f N_2^q$ into a Sasakian space form $\widetilde{M}^{2n+1}(c)$ with with base N_1^p is minimal in $\widetilde{M}^{2n+1}(c)$, connected and complete. If N_1^p has non-negative Ricci curvature then its is isometric to the sphere $\mathbb{S}^p(\frac{\lambda_1}{p})$ if and only if the following equality is satisfied

$$\Big\|h(u,v)\Big\|^2 = \Big(\frac{c+3}{4}\Big)pq, \tag{1.10}$$

where $\lambda_1 > 0$.

In[21], Rio, Kupeli, and Unal characterized Euclidean sphere using a standard differential equation which is another version of Obata's differential equation. If a complete Riemannian manifold M^n admits a real-valued non-constant function ϕ such $\Delta\phi + \lambda_1\phi = 0$ such that $\lambda_1 < 0$, then M^n is isometric to a warped product of the Euclidean line and a complete Riemannian manifold whose warping function ψ satisfies the equation that $\frac{d^2\psi}{dt^2} + \lambda_1\psi = 0$. On these concepts, we give the following result.

Theorem 1.6. Let $\Psi : M^n = N_1^p \times_f N_2^q$ be an isometric immersion of a Legendrian warped product submanifold $N_1^p \times_f N_2^q$ into a Sasakian space form $\widetilde{M}^{2n+1}(c)$ with base N_1^p is minimal in $\widetilde{M}^{2n+1}(c)$, complete and has positive Ricci curvature. Then N_1^p is isometric to a warped product of the Euclidean line and a complete Riemannian manifold whose warping function ψ satisfies the equation that $\frac{d^2\psi}{dt^2} + \lambda_1\psi = 0$ if and only if the following equality is satisfied

$$\Big\|h(u,v)\Big\|^2 = \Big(\frac{c+3}{4}\Big)pq, \tag{1.11}$$

where $\lambda_1 < 0$ is a negative eigenvalue associated to the eigenfunction $\mu = \ln f$.

Tashiro[34] also proved more general results similar to the results of Obata and Kanai. The following theorem is also of interest from the viewpoint of this survey in characterizing the Euclidean space via a differential equation.

Theorem 1.7. Let $\Psi : M^n = N_1^p \times_f N_2^q$ be an isometric immersion of a Legendrian warped product submanifold $N_1^p \times_f N_2^q$ into a Sasakian space form $\widetilde{M}^{2n+1}(c)$ with base N_1^p is minimal in $\widetilde{M}^{2n+1}(c)$, complete and has positive Ricci curvature. Then N_1^p is isometric to the Euclidean space $\mathbb{R}^p$ if and only if the following equality is satisfied

$$\Big\|h(u,v)\Big\|^2 = q^2\Bigg\{\frac{\lambda_1}{p} + \Big(\frac{c+3}{4}\Big)p\Bigg\}, \tag{1.12}$$

with $\lambda_1 > 0$ is positive eigenvalue of the non-constant warping function $\mu = \ln f$.

The present paper, we considered only the non-trivial Legendrian warped product submanifold of the type $M^n = N_1^p \times_f N_2^q$ to be isometrically immersed into a Sasakian space form. Then, we will consider connected, compact Riemannian submanifolds whose boundaries are non-empty, and provided some new necessary and sufficient conditions for Legendrian warped product submanifolds, which can be reduced to a Riemannian product manifold.

2 | PRELIMINARIES

An $(2m+1)$-dimensional manifold $\widetilde{M}$ endowed with almost contact structure (ϕ, ξ, η, g) is called an almost contact metric manifold when satisfies the following properties:

$$\varphi^2 = -I + \eta \otimes \xi, \;\; \eta(\xi) = 1, \;\; \varphi(\xi) = 0, \;\; \eta \circ \varphi = 0, \tag{2.1}$$

$$g(\varphi U, \varphi V) = g(U, V) - \eta(U)\eta(V), \;\; and \;\; \eta(U) = g(U, \xi), \tag{2.2}$$

for any $U, V \in \mathfrak{X}(T\widetilde{M})$, where φ, g, ξ and η are called $(1,1)$-tensor fields, a structure vector field and dual 1-form, respectively. Furthermore, an almost contact metric manifold is known to be a *Sasakian manifold* (cf.[?]) if

$$(\widetilde{\nabla}_U \varphi)V = g(U,V)\xi - \eta(V)U, \quad \widetilde{\nabla}_U \xi = -\varphi U, \tag{2.3}$$

for any vector fields U, V on $\widetilde{M}$, where $\widetilde{\nabla}$ denotes the Riemannian connection with respect to g, and we shall use the symbol $\mathfrak{X}(T\widetilde{M})$ to denote Lie algebra of vector fields on a manifold $\widetilde{M}$.

Assume that M^n be isometrically immersed into an almost Hermitian manifold $\widetilde{M}^{2m+1}$ with induced metric g. If ∇ and $\nabla^{\perp}$ are the induced Riemannian connections on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M^n, respectively, then the Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_U V = \nabla_U V + h(U,V), \tag{2.4}$$

$$\widetilde{\nabla}_U N = -A_N U + \nabla^{\perp}_U N, \tag{2.5}$$

for each $U, V \in \mathfrak{X}(TM)$ and $N \in \mathfrak{X}(T^{\perp}M)$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N), respectively, for the immersion of M^n into $\widetilde{M}^{2m+1}$. They are related as follows: $g(h(U,V),N) = g(A_N U, V)$, where g denotes the Riemannian metric on $\widetilde{M}^{2m+1}$ as well as the metric induced on M^n. The Gauss equation for a submanifold M^n is defined as:-

$$\widetilde{R}\big(X,Y,Z,W\big) = R(X,Y,Z,W) + g\big(h(X,Z),h(Y,W)\big) - g\big(h(X,W),h(Y,Z)\big), \tag{2.6}$$

for any $X, Y, Z, W \in \mathfrak{X}(TM)$, where $\widetilde{R}$ and R are the curvature tensors on $\widetilde{M}^m$ and M^n, respectively. A Sasakian manifold is said to be Sasakian space form with constant ϕ-sectional curvature c if and only if the Riemannian curvature tensor $\widetilde{R}$ can be written as (see[1,27]);

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) = \frac{c+3}{4}&\Big\{ g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \Big\}\\ +\frac{c-1}{4}&\Big\{ \eta(X)\eta(Z)g(Y,W) + \eta(W)\eta(Y)g(X,Z)\\ &- \eta(Y)\eta(Z)g(X,W) - \eta(X)g(Y,Z)\eta(W)\\ &+ g(\varphi Y,Z)g(\varphi X,W)\\ &- g(\varphi X,Z)g(\varphi Y,W) + 2g(X,\varphi Y)g(\varphi Z,W) \Big\}.\end{aligned} \tag{2.7}$$

Moreover, $\mathbb{R}^{2m+1}$ and $\mathbb{S}^{2m+1}$ with standard Sasakian structures can be given as typical examples of Sasakian space forms. An n-dimensional Riemannian submanifold M^n of $\widetilde{M}^{2m+1}(\epsilon)$ is referred to as totally real if the standard almost contact structure ϕ of $\widetilde{M}^{2m+1}(\epsilon)$ maps any tangent space of N^n into its corresponding normal space (see[1,27,32]). Now, let M^n be an isometric immersed submanifold of dimension n in $\widetilde{M}^{2m+1}(\epsilon)$. Then M^n is referred to as a Legendrian submanifold if ζ is a normal vector field on M^n, i.e., M^n is a C- totally real submanifold, and $m = n$[1,27,32]. Legendrian submanifolds play a substantial role in contact geometry. From Riemannian geometric perspective, studying Legendrian submanifolds of Sasakian manifolds was initiated in 1970's. Many geometers have drawn significant attention to minimal Legendrian submanifolds in particular. The mean curvature vector H for an orthonormal frame $\{e_1, e_2, \cdots, e_n\}$ of tangent space TM on M^n is defined by

$$\mathcal{H} = \frac{1}{n} trace(h) = \frac{1}{n}\sum_{i=1}^{n} h\big(e_i, e_i\big), \tag{2.8}$$

where $n = dimM$. In addition, we set

$$(i) \quad ||h||^2 = \sum_{i,j=1}^{n} g(h(e_i,e_j), h(e_i,e_j)) \quad \text{and} \quad (ii) \quad \mathbf{S} = \sum_{\alpha=1}^{p}\sum_{\beta=p+1}^{n} \big(h^r_{\alpha\beta}\big)^2. \tag{2.9}$$

Now, we define an important Riemannian intrinsic invariant called the scalar curvature of M^n and it is denoted by $\widetilde{\tau}(T_x M^n)$, that is

$$2\widetilde{\tau}(T_x M^n) = \sum_{1\le\alpha<\beta\le n} K(e_\alpha \wedge e_\beta). \tag{2.10}$$

Let $K_{\alpha\beta}$ and $\widetilde{K}_{\alpha\beta}$ denotes the sectional curvature of the plane section spanned and e_α at x in the submanifold M^n and at the Riemannian space form $\widetilde{M}^m(c)$, respectively. Thus $K_{\alpha\beta}$ and $\widetilde{K}_{\alpha\beta}$ are the intrinsic and extrinsic sectional curvatures of the span $\{e_\alpha, e_\beta\}$ at x, thus from Gauss equation (2.7), we have

$$2\tau(T_xM^n) = K_{\alpha\beta} = 2\widetilde{\tau}(T_xM^n) + \sum_{r=n+1}^{2m+1}\Big(h^r_{\alpha\alpha}h^r_{\beta\beta} - (h^r_{\alpha\beta})^2\Big)$$
$$= \widetilde{K}_{\alpha\beta} + \sum_{r=n+1}^{2m+1}\Big(h^r_{\alpha\alpha}h^r_{\beta\beta} - (h^r_{\alpha\beta})^2\Big). \tag{2.11}$$

The following consequences are obtained from (2.7) and (2.11) as:

$$\tau(T_xN_1^p) = \sum_{r=n+1}^{2m+1}\sum_{1\le i<j\le p}\Big(h^r_{ii}h^r_{jj} - (h^r_{ij})^2\Big) + \widetilde{\tau}(T_xN_1^p). \tag{2.12}$$

Similarly, we have

$$\tau(T_xN_2^q) = \sum_{r=n+1}^{2m+1}\sum_{p+1\le a<b\le n}\Big(h^r_{aa}h^r_{bb} - (h^r_{ab})^2\Big) + \widetilde{\tau}(T_xN_2^q). \tag{2.13}$$

If φ preserves any tangent space of M^n, that is, $\varphi(T_xM) \subseteq T_xM$, for each $x \in M^n$, then M^n is called a invariant submanifold. Similarly, the totally real submanifold is defined as:- φ maps any tangent space of M^n into normal space, that is, $\varphi(T_xM) \subseteq T^\perp M$, for each $x \in M^n$.

Therefore, we shall prove interesting result which be very useful in geometry,

2.1 | Proof of Theorem 1.1

Proof. In view of Gauss equation (2.6), we get

$$n^2||\mathcal{H}||^2 = ||h||^2 + 2\tau(T_xM^) - 2\widetilde{\tau}(T_xM^n). \tag{2.14}$$

We assume that $\{e_1 \dots, e_p, e_{p+1}, \dots, e_n\}$ and $\{e_{n+1} \dots, e_{2m+1}\}$ be an orthonormal frames of $\mathfrak{X}(T_xM^n)$ and $\mathfrak{X}(T^\perp M^n)$, such that $\{e_1, \dots, e_p\}$ and
$\{e_{p+1}, \dots, e_n\}$ are the frames of $\mathfrak{X}(TN_1^p)$ and $\mathfrak{X}(TN_2^q)$. From (2.10), we have

$$\tau(T_xM^n) = \sum_{1\le\alpha<\beta\le n} K_{\alpha\beta} = \sum_{\alpha=1}^{p}\sum_{A=p+1}^{n} K_{\alpha A} + \sum_{1\le i<j\le p} K_{ij} + \sum_{p+1\le a<b\le n} K_{ab} \tag{2.15}$$

Using the virtues (1.3) and (2.10), we derive the following relation

$$\tau(T_xM^n) = \frac{q\Delta f}{f} + \tau(T_xN_1^p) + \tau(T_xN_2^q). \tag{2.16}$$

It follows from (3.6), (2.12) and (2.13), we have

$$\tau(T_xM^n) = \frac{q\Delta f}{f} + \widetilde{\tau}(T_xN_1^p) + \sum_{r=n+1}^{2n+1}\sum_{1\le\alpha<\beta\le p} h^r_{\alpha\alpha}h^r_{\beta\beta} - \sum_{r=n+1}^{2n+1}\sum_{1\le\alpha<\beta\le p}(h^r_{\alpha\beta})^2 + \widetilde{\tau}(T_xN_2^q)$$
$$+ \sum_{r=n+1}^{2n+1}\sum_{p+1\le a<b\le n} h^r_{aa}h^r_{bb} - \sum_{r=n+1}^{2n+1}\sum_{p+1\le a<b\le n}(h^r_{ab})^2. \tag{2.17}$$

Follows from (2.14) and (2.17), we get

$$\begin{aligned}\Big(\sum_{i=1}^{n} h_{ii}^{n+1}\Big)^2 &= \sum_{r=n+1}^{2n+1}\sum_{i=1}^{n}(h_{ii})^2 + \frac{2q\Delta f}{f} + 2\widetilde{\tau}(T_x N_1^p) + 2\widetilde{\tau}(T_x N_2^q)\\ &+ 2\sum_{r=n+1}^{2n+1}\sum_{1\le\alpha<\beta\le p} h^r_{\alpha\alpha}h^r_{\beta\beta} - 2\widetilde{\tau}(T_x M^n) - 2\sum_{r=n+1}^{2n+1}\sum_{1\le\alpha<\beta\le p}(h^r_{\alpha\beta})^2\\ &+ 2\sum_{r=n+1}^{2n+1}\sum_{p+1\le a<b\le n} h^r_{aa}h^r_{bb} - 2\sum_{r=n+1}^{2n+1}\sum_{p+1\le a<b\le n}(h^r_{ab})^2. \end{aligned} \tag{2.18}$$

After some computation, we derive

$$\begin{aligned}\Big(\sum_{i=1}^{n} h_{ii}^{n+1}\Big)^2 &= \sum_{r=n+1}^{2n+1}\sum_{i=1}^{p}(h_{ii})^2 + \sum_{r=n+1}^{2n+1}\sum_{j=p+1}^{n}(h_{jj})^2 + 2\sum_{r=n+1}^{2n+1}\sum_{\substack{i,j=1\\ i\neq j}}^{n}(h_{ij})^2\\ &+ \frac{2q\Delta f}{f} + 2\widetilde{\tau}(T_x N_1^p) + 2\sum_{r=n+1}^{2n+1}\sum_{1\le\alpha<\beta\le p} h^r_{\alpha\alpha}h^r_{\beta\beta}\\ &- 2\sum_{r=n+1}^{2n+1}\sum_{1\le\alpha<\beta\le p}(h^r_{\alpha\beta})^2 + 2\widetilde{\tau}(T_x N_2^q) - 2\widetilde{\tau}(T_x M^n)\\ &+ 2\sum_{r=n+1}^{2n+1}\sum_{p+1\le a<b\le n} h^r_{aa}h^r_{bb} - 2\sum_{r=n+1}^{2n+1}\sum_{p+1\le a<b\le n}(h^r_{ab})^2. \end{aligned} \tag{2.19}$$

As we assumed that M^n is a N_1-minimal Legendrian warped product submanifold, then

$$2\sum_{r=n+1}^{2n+1}\sum_{1\le\alpha<\beta\le p} h^r_{\alpha\alpha}h^r_{\beta\beta} + \sum_{r=n+1}^{2n+1}\sum_{i=1}^{n}(h_{ii})^2 = 0. \tag{2.20}$$

From the above, we find that

$$2\sum_{r=n+1}^{2n+1}\sum_{p+1\le a<b\le n} h^r_{aa}h^r_{bb} + \sum_{r=n+1}^{2n+1}\sum_{j=p+1}^{n}(h_{jj})^2 = \Big(\sum_{A=1}^{n} h_{AA}\Big)^2 \tag{2.21}$$

Substitute (2.20) and (2.21) in (2.19), we get

$$\begin{aligned}2\widetilde{\tau}(T_x M^n) &= \frac{2q\Delta f}{f} + 2\widetilde{\tau}(T_x N_1^p) + 2\widetilde{\tau}(T_x N_2^q)\\ &- 2\sum_{r=n+1}^{2n+1}\left\{\sum_{1\le\alpha<\beta\le p}(h^r_{\alpha\beta})^2 + \sum_{p+1\le a<b\le n}(h^r_{ab})^2 - \sum_{\substack{i,j=1\\ i\neq j}}^{n}(h_{ij})^2\right\} \end{aligned} \tag{2.22}$$

Thus from binomial properties, we arrive at

$$\sum_{1\le\alpha<\beta\le p}(h^r_{\alpha\beta})^2 + \sum_{p+1\le a<b\le n}(h^r_{ab})^2 - \sum_{\substack{i,j=1\\ i\neq j}}^{n}(h_{ij})^2 = \sum_{\alpha=1}^{p}\sum_{\beta=p+1}^{n}\big(h^r_{\alpha\beta}\big)^2. \tag{2.23}$$

For Legendrian warped product submanifolds, let us substituting $X = W = e_i$ and $Y = Z = e_j$ in (2.7), we get

$$\widetilde{R}(e_i, e_j, e_j, e_i) = \frac{c+3}{4}\left\{g(e_i, e_i)g(e_j, e_j) - g(e_i, e_j)g(e_j, e_i)\right\}.$$

Summing up over basis vectors of TM such that $1 \le i \neq j \le n$, it is easy to obtain that

$$2\widetilde{\tau}(TM) = \left(\frac{c+3}{4}\right)n(n-1). \tag{2.24}$$

Similarly, for TN_1^p, we derive

$$\widetilde{\tau}(TN_1^p) = \left(\frac{c+3}{8}\right)p(p-1). \tag{2.25}$$

Now, one derives for TN_2^q, we have

$$\widetilde{\tau}(TN_2^q) = \left(\frac{c+3}{8}\right)q(q-1). \tag{2.26}$$

Therefore, combining the Eqs. (2.22), (2.23), (2.25) and (2.26), we get the required result (1.4). Thus, the proof is completed. □

3 | CLASSIFICATIONS OF THE RICCI CURVATURE AND DIVERGENCE OF THE HESSIAN TENSOR

In this section, we studied some applications of the derived inequality with equality cases. Let identify any $(0,2)$-tensor T on M with a $(1,1)$-tensor by equation

$$g(T(Z),Y) = T(Z,Y).$$

for all $Y, Z \in \Gamma(TM)$. Thus , we get

$$div(\phi T) = \varphi div T + T(\nabla\phi, \bullet) \ \text{ and } \ \nabla(\phi T) = \phi\nabla T + d\phi \otimes T,$$

for all $\phi \in C^\infty(M)$. In particular, we have $div(\phi g) = d\phi$. Moreover, the following general facts are well known in the literature

$$(1) \ div\nabla^2\phi = Ric(\nabla\phi, \bullet) + d\Delta\phi \ \text{ and } (ii) \ \frac{1}{2}d\|\nabla\varphi\|^2 = \nabla^2\phi(\nabla, \bullet). \tag{3.1}$$

3.1 | Proof of Theorem 1.2

Proof. We use the Ricci identity (3.1). Applying these Ricci identity on the warping function $\phi = \mu = \ln f$, which implies that

$$div\nabla^2\mu = d\big(\Delta\mu\big) + \mathcal{R}ic(\nabla\mu,). \tag{3.2}$$

From the hypothesis, M^n is a compact warped product submanifold with free boundary, and then taking integration along the volume element dV, we get

$$\Delta\mu = \int_{M^n} \big(div\nabla^2\mu\big)\mathrm{dV} - \int_{M^n} \mathcal{R}ic(\nabla\mu,)\mathrm{dV}. \tag{3.3}$$

Using the Green theorem on a compact manifold M^n, one gets $\int_{M^n} \Delta f dV = 0$. Using the results of Yano and Kon from (see[35]), it follows $\Delta f = -div(\nabla f)$ and from the Green lemma $\int_{M^n} div(X)dV = 0$ for an arbitrary vector field X on M^n. Thus, we get $\int_{M^n} div\nabla^2\mu dV = 0$. Therefore, (3.3) implies that

$$\Delta\mu = -\int_{M^n} \mathcal{R}ic(\nabla\mu,)\mathrm{dV}. \tag{3.4}$$

On the other hand, from (1.4) we have

$$\|h(u,v)\|^2 = q\|\nabla\mu\|^2 + \left(\frac{c+3}{4}\right)pq - q\Delta\mu. \tag{3.5}$$

From (3.4) and (3.5), we obtaine

$$\|h(u,v)\|^2 = q\|\nabla\mu\|^2 + \left(\frac{c+3}{4}\right)pq + q\int_{M^n} \mathcal{R}ic(\nabla\mu,)\mathrm{dV}. \tag{3.6}$$

If the equality (1.6) is satisfied, then from (3.6) we get the following condition

$$q\|\nabla\mu\|^2 = 0,$$

which is equivalent to $||\nabla\mu||^2 = 0 \Rightarrow \nabla\mu = 0$. This means that $grad \ln f = 0$. it shows that f is a constant function on M^n. Hence, from Remark 6.1, we conclude that M^n is a trivial Legendrian warped product submanifold of a Sasakian Space form $\tilde{M}^{2m+1}(c)$. This is proof of Theorem 1.2. □

3.2 | Proof of Corollary 1.2

As we have considered that M^n is a Ricci flat. This means that Ricci curvature of M^n is vanished everywhere, that is

$$\mathcal{R}ic(\nabla\mu,) = 0.$$

Substituting the above equation in (1.6) , we get required proof of the corollary.

4 | APPLICATION TO THE ORDINARY DIFFERENTIAL EQUATIONS

4.1 | Proof of Theorem 1.3

Proof. Let we define the following equation as

$$\left\|Hess(\mu) + t\mu I\right\|^2 = \|Hess(\mu)\|^2 + t^2(\mu)^2\|I\|^2 + 2t\mu g(Hess(\mu), I).$$

But we know that $\|I\|^2 = trace(II^*) = p$ and $g(Hess(\mu), I^*) = tr(Hess(\mu)I^*) = trHess(\mu)$. Then the proceeding equation takes the form taking clue from definition of the Laplacian, we have

$$\left\|Hess(\mu) + t\mu I\right\|^2 = \|Hess(\mu)\|^2 + pt^2(\mu)^2 - 2t\mu\Delta\mu. \tag{4.1}$$

Let λ_1 is an eigenvalue of the eigenfunction $\mu = \ln f$, then $\Delta\mu = \lambda_1\mu$. Thus we get

$$\left\|Hess(\mu) + t\mu I\right\|^2 = \|Hess(\mu)\|^2 + \left(pt^2 - 2t\lambda_1\right)(\mu)^2. \tag{4.2}$$

On the other hand, we obtain

$$\Delta\left(\frac{\phi^2}{2}\right) = -div\left(\nabla\left(\frac{\phi^2}{2}\right)\right) = -div\left(\phi\nabla\phi\right) = -\phi\Delta\phi + \|\nabla\phi\|^2.$$

Then setting $\phi = \mu = \ln f$, we have

$$\Delta\left(\frac{\mu^2}{2}\right) = -\mu\Delta\mu + \|\nabla\mu\|^2.$$

For eigenvalue $\lambda_1 > 0$, and $\Delta\mu = \lambda_1\mu$, we have

$$\frac{\lambda_1\mu^2}{2} = -\lambda_1\mu^2 + \|\nabla\mu\|^2.$$

which implies that

$$\mu^2 = \frac{2}{3\lambda_1}\|\nabla\mu\|^2. \tag{4.3}$$

It follows from (4.2) and (4.3), we find that

$$\|Hess(\mu) + t\mu I\|^2 = \|Hess(\mu)\|^2 + \frac{2}{3}\left(\frac{pt^2}{\lambda_1} - 2t\right)\|\nabla\mu\|^2. \tag{4.4}$$

In particular, $t = \frac{\lambda_1}{p}$ on (4.4) by taking integration, we get

$$\int\limits_{N_1^p\times\{q\}} \left\|Hess(\mu) + \frac{\lambda_1}{p}\mu I\right\|^2 dV = \int\limits_{N_1^p\times\{q\}} \|Hess(\mu)\|^2 dV - \frac{2\lambda_1}{3p}\int\limits_{N_1^p\times\{q\}} \|\nabla\mu\|^2 dV. \tag{4.5}$$

Again taking integration on (1.4) and involving the Green lemma, we have

$$\int\limits_{N_1^p\times\{q\}} \|\nabla\mu\|^2 dV = \frac{1}{q}\int\limits_{N_1^p\times\{q\}} \left\{\left\|h(u,v)\right\|^2 - \left(\frac{c+3}{4}\right)pq\right\} dV. \tag{4.6}$$

From (4.5) and (4.6), we derive

$$\int_{N_1^p\times\{q\}} \Big\|Hess(\mu)+\frac{\lambda_1}{p}\mu I\Big\|^2 dV = \int_{N_1^p\times\{q\}} \|Hess(\mu)\|^2 dV - \frac{2\lambda_1}{3p}\int_{N_1^p\times\{q\}} \left\{\frac{1}{q}\Big\|h(u,v)\Big\|^2 - \Big(\frac{c+3}{4}\Big)p\right\} dV. \tag{4.7}$$

If the Eq. (1.8) is satisfied, then from (4.7), we get

$$\Big\|Hess(\mu)+\frac{\lambda_1}{p}\mu I\Big\|^2 = 0 \;\Rightarrow\; Hess(\mu) = -\frac{\lambda_1}{p}\mu I. \tag{4.8}$$

Since, the warping function $\mu = \ln f$ is non-constant because of warped product manifold M^n is non-trivial, Eq (4.8), gives Obata's[30] differential equation with constant $c = \frac{\lambda_1}{p} > 0$ as $\lambda_1 > 0$, and therefore N_1^p is isometric to the sphere $\mathbb{S}^n(\frac{\lambda_1}{p})$ with a constant sectional curvature is $\frac{\lambda_1}{p}$. The converse part is straightforward from (4.8). This complete the proof of the theorem. □

5 | APPLICATION OF BOCHNER FORMULA AS PROOF OF THEOREM ??

We recall now Bochner formula (see e.g.[6]) which states that for a differential function $\mu = \ln f$ defined on a Riemannian manifold, the following relation holds:

$$\frac{1}{2}\Delta\|\nabla\mu\|^2 = \|Hess(\mu)\|^2 + Ric(\nabla\mu,\nabla\mu) + g\big(\nabla\mu,\nabla(\Delta\mu)\big).$$

Integrating above equation with the aid of Stokes theorem, we get

$$\int_{N_1^p\times\{q\}} \|Hess(\mu)\|^2 dV + \int_{N_1^p\times\{q\}} Ric(\nabla\mu,\nabla\mu) dV + \int_{N_1^p\times\{q\}} g\big(\nabla\mu,\nabla(\Delta\mu)\big) dV = 0. \tag{5.1}$$

Now by using $\Delta\mu = \lambda_1\mu$ and some rearrangement in (5.1), we derive

$$\int_{N_1^p\times\{q\}} \|Hess(\mu)\|^2 dV = -\lambda_1 \int_{N_1^{n_1}\times\{q\}} \|\nabla\mu\|^2 dV - \int_{N_1^p\times\{q\}} Ric(\nabla\mu,\nabla\mu) dV. \tag{5.2}$$

Inserting (5.2) into (4.5), we get

$$\int_{N_1^p\times\{q\}} \Big\|Hess(\mu)+\frac{\lambda_1}{p}\mu I\Big\|^2 dV = -\lambda_1 \int_{N_1^p\times\{q\}} \|\nabla\mu\|^2 dV - \frac{2\lambda_1}{3p}\int_{N_1^p\times\{q\}} \|\nabla\mu\|^2 dV - \int_{N_1^p\times\{q\}} Ric(\nabla\mu,\nabla\mu) dV,$$

which implies that

$$\int_{N_1^p\times\{q\}} \Big\|Hess(\mu)+\frac{\lambda_1}{p}\mu I\Big\|^2 dV = -\lambda_1\Big(\frac{3p+2}{3p}\Big) \int_{N_1^p\times\{q\}} \|\nabla\mu\|^2 dV - \int_{N_1^p\times\{q\}} Ric(\nabla\mu,\nabla\mu) dV. \tag{5.3}$$

It follows from (4.6) and (5.3), we find that

$$\int_{N_1^p\times\{q\}} \Big\|Hess(\mu)+\frac{\lambda_1}{p}\mu I\Big\|^2 dV = -\lambda_1\Big(\frac{3p+2}{3pq}\Big) \int_{N_1^p\times\{q\}} \Big\|h(u,v)\Big\|^2 dV - \int_{N_1^p\times\{q\}} Ric(\nabla\mu,\nabla\mu) dV + \lambda_1\Big(\frac{3p+2}{3pq}\Big)\int_{N_1^p\times\{q\}} \Big(\frac{c+3}{4}\Big)pq\, dV.$$

It is equivalent to the following

$$\int_{N_1^p\times\{q\}} \Big\|Hess(\mu)+\frac{\lambda_1}{p}\mu I\Big\|^2 dV + \int_{N_1^p\times\{q\}} Ric(\nabla\mu,\nabla\mu) dV = \lambda_1\Big(\frac{3p+2}{3pq}\Big)\int_{N_1^p\times\{q\}} \left\{\Big(\frac{c+3}{4}\Big)pq - \Big\|h(u,v)\Big\|^2\right\} dV. \tag{5.4}$$

The equality in (1.9) is satisfied if and only if the following equality holds in (5.4), that is

$$\int_{N_T^p\times\{q\}} \left\|Hess(\mu)+\frac{\lambda_1}{p}\mu I\right\|^2 dV = 0,$$

which means that

$$\nabla^2\mu = -\frac{\lambda_1}{p}\mu I. \tag{5.5}$$

Therefore, we again invoke the result of Obata's[30] differential equation with constant $c = \frac{\lambda_1}{p} > 0$ as $\lambda_1 > 0$, and therefore N_1^p is isometric to the sphere $\mathbb{S}^n(\frac{\lambda_1}{p})$. The converse part immediately follows from (5.5). This complete proof of theorem.

5.1 | Proof of Theorem 1.5

Proof. Now consider the Eq. (5.4), we have

$$\begin{aligned}\int_{N_1^p\times\{q\}} \left\|Hess(\mu)+\frac{\lambda_1}{p}\mu I\right\|^2 dV =& -\lambda_1\Big(\frac{3p+2}{3pq}\Big)\int_{N_1^p\times\{q\}} \left\|h(u,v)\right\|^2 dV - \int_{N_T^p\times\{q\}} Ric(\nabla\mu,\nabla\mu)dV\\ &+\lambda_1\Big(\frac{3p+2}{3pq}\Big)\int_{N_1^p\times\{q\}}\Big(\frac{c+3}{4}\Big)pq dV.\end{aligned} \tag{5.6}$$

As we considered the Ricci curvature of N_1^p is non-negative $Ric \geq 0$, then

$$\int_{N_1^p\times\{q\}} Ric(\nabla\mu,\nabla\mu)dV \geq 0 \;\Rightarrow\; -\int_{N_1^p\times\{q\}} Ric(\nabla\mu,\nabla\mu)dV \leq 0.$$

Using the proceeding in (5.6), we get

$$\int_{N_1^q\times\{n_2\}} \left\|Hess(\mu)+\frac{\lambda_1}{n}\mu I\right\|^2 dV + \lambda_1\Big(\frac{3p+2}{3pq}\Big)\int_{N_1^p\times\{q\}}\Big(\frac{c+3}{4}\Big)pq dV - \lambda_1\Big(\frac{3p+2}{3pq}\Big)\int_{N_1^p\times\{q\}}\left\|h(u,v)\right\|^2 dV. \tag{5.7}$$

The following inequality holds from (5.7) if and only if the condition (1.10) is satisfied of Theorem 1.5, as $p \neq -\frac{2}{3}$, we get

$$\left\|Hess(\mu)+\frac{\lambda_1}{p}\mu I\right\|^2 \leq 0. \tag{5.8}$$

On the other hand, we known that

$$\left\|Hess(\mu)+\frac{\lambda_1}{p}\mu I\right\|^2 \geq 0. \tag{5.9}$$

Combining (5.8) and (5.9), we arrive at

$$Hess(\mu) = -\frac{\lambda_1}{p}\mu I. \tag{5.10}$$

From[30], N_1^p is isometric to the sphere $\mathbb{S}^p(\frac{\lambda_1}{p})$. This complete the proof the theorem. □

5.2 | Proof of Theorem 1.6

In hypothesis of the theorem, we assumed that the Ricci curvature of base manifold N_1^p is positive and hence from Myers's theorem which stat that a complete Riemannian manifold with positive Ricci curvature is compact. Therefore, N_1^p is connected and compact base manifold. Then from (5.6), we get

$$\int_{N_1^p\times\{q\}} \left\|Hess(\mu)+\frac{\lambda_1}{n}\mu I\right\|^2 dV < \lambda_1\Big(\frac{3p+2}{3pq}\Big)\int_{N_1^p\times\{q\}}\left\{\Big(\frac{c+3}{4}\Big)pq - \left\|h(u,v)\right\|^2\right\}dV. \tag{5.11}$$

Follows the statement of the theorem and the equation (1.11) is satisfied, then

$$\left\|Hess(\mu)+\frac{\lambda_1}{p}\mu I\right\|^2<0.$$

which implies that

$$Hess(\mu)=-\frac{\lambda_1}{p}\mu I. \tag{5.12}$$

As we assumed that $\lambda_1<0$ in the hypothesis of theorem, therefore we invoke the result from[21]. Then, N_1^p is isometric to a warped product of the Euclidean line and a complete Riemannian manifold, where warping function satisfies the equation $\frac{d^2\sigma}{dt^2}+\lambda_1\sigma=0$, where σ is a warping function. This complete the proof of theorem.

6 | PROOF OF THEOREM ??

Let the equation can be expressed as

$$\left\|Hess(\mu)-tI\right\|^2=\|Hess(\mu)\|^2+t^2\|I\|^2-2tg(Hess(\mu),I). \tag{6.1}$$

which implies that the fact of Hessian $Hess(\mu)$ and identity operator I, that is

$$\left\|Hess(\mu)-tI\right\|^2=\|Hess(\mu)\|^2+t^2p-2t\Delta\mu.$$

In particular, $t=\frac{\lambda_1}{p}$ putting in the above equation with integrated along volume element dV, we derive

$$\int\limits_{N_1^p\times\{q\}}\left\|Hess(\mu)-\frac{\lambda_1}{p}I\right\|^2dV=\int\limits_{N_1^p\times\{q\}}\left(\|Hess(\mu)\|^2+\frac{\lambda_1^2}{p}\right)dV \tag{6.2}$$

Taking the help from virtue (5.2), we obtain

$$\int\limits_{N_1^p\times\{q\}}\left\|Hess(\mu)-\frac{\lambda_1}{p}I\right\|^2dV=-\lambda_1\int\limits_{N_1^p\times\{q\}}\|\nabla\mu\|^2dV-\int\limits_{N_1^p\times\{q\}}Ric(\nabla\mu,\nabla\mu)dV+\int\limits_{N_1^p\times\{q\}}\frac{\lambda_1^2}{p}dV. \tag{6.3}$$

Taking account of (4.6) and from the above equation, we get

$$\begin{aligned}\int\limits_{N_1^p\times\{q\}}\left\|Hess(\mu)-\frac{\lambda_1}{p}I\right\|^2dV=&-\frac{\lambda_1}{q}\int\limits_{N_T^p\times\{q\}}\left\|h(u,v)\right\|^2dV-\int\limits_{N_1^p\times\{q\}}Ric(\nabla\mu,\nabla\mu)dV\\&+\lambda_1\int\limits_{N_1^p\times\{q\}}\left\{\frac{\lambda_1}{p}+\left(\frac{c+3}{4}\right)pq\right\}dV\end{aligned} \tag{6.4}$$

In view our assumption that the Ricci curvature is non-negative, then the above equation implies that

$$\int\limits_{N_1^p\times\{q\}}\left\|Hess(\mu)-\frac{\lambda_1}{p}I\right\|^2dV\leq\lambda_1\int\limits_{N_1^p\times\{q\}}\left\{\frac{\lambda_1}{p}+\left(\frac{c+3}{4}\right)pq\right\}dV-\frac{\lambda_1}{q}\int\limits_{N_1^p\times\{q\}}\left\|h(u,v)\right\|^2dV. \tag{6.5}$$

If Eq. (6.13) is satisfied then from (6.5), we arrive at

$$\int\limits_{N_T^p\times\{q\}}\left\|Hess(\mu)-\frac{\lambda_1}{p}I\right\|^2dV\leq 0.$$

It follows from the definition of the norm

$$Hess(\mu)=\frac{\lambda_1}{p}I,$$

which implies that

$$Hess_{\mu}(X,X)=\frac{\lambda_1}{p}g(X,X), \tag{6.6}$$

for any $X \in \mathfrak{X}(N_1)$. Note that if the potential function $\mu = \ln f$ is a constant then M^n is a trivial warped product submanifold that leads to a contradiction as M^n is non-trivial. Hence equation (6.6) is Tashiro[34] differential equation and therefore N_1^p is isometric to the Euclidean space $\mathbb{R}^p$ [33,34] with positive constant $c = \frac{\lambda_1}{p} > 0$, as $\lambda_1 > 0$. This complete the proof of the theorem.

Remark 6.1. Assume that $\mathbb{S}^{2m+1}$ be a $(2m+1)$-sphere, we put $Jz = \xi$, for any point $z \in \mathbb{S}^{2m+1}$ and J is an almost complex structure of complex $n+1$-space $\mathbb{C}^{m+1}$. Let us consider the orthogonal projection map $\pi : T_z\mathbb{C}^{m+1} \to T_z\mathbb{S}^{2m+1}$ such that $\varphi = \pi \circ J$, it can be easily seen that (φ, η, ξ, g) is a Saskian structure in $\mathbb{S}^{2m+1}$, where η is a one-form dual to ξ, and g is standard metric tensor field on $\mathbb{S}^{2m+1}$. Therefore, the odd-dimension sphere $\mathbb{S}^{2m+1}$ can be considered a Sasakian manifold of constant φ-sectional curvature one. Let M^n be a submanifold in $\mathbb{S}^{2n+1}$, then M^n is a Legendrian submanifold in $\mathbb{S}^{2n+1}(1)$ if and only if it is a Legendrian submanifold in $\mathbb{S}^{2n+1}(\epsilon)$.

Motivated by above Remark and Theorems 1.1, 1.2,1.3, 1.4, 1.5, 1.6 and 1.7, we have following corollaries.

Corollary 6.1. Let $\Psi : M^n = N_1^p \times_f N_2^q \longrightarrow \widetilde{M}^{2+1}(c)$ be an isometric immersion from a Legendrian warped product submanifold $N_1^p \times_f N_2^q$ into a Sasakian space form $\mathbb{S}^{2n+1}$. If N_1^p is minimal in $\mathbb{S}^{2n+1}$, then the following equality is satisfied

$$\|h(u,v)\|^2 = q\Big\{ ||\nabla\mu||^2 + p - \Delta\mu \Big\}, \tag{6.7}$$

where ∇ and Δ are gradient and the Laplacian of the warping function $\mu = \ln f$ on N_1^p. Moreover, $u = \{e_i\}_{1\leq i\leq p}$ and $v = \{e_j^*\}_{1\leq j\leq q}$ are vector fields on N_1^p and N_2^q, respectively.

Corollary 6.2. Let $\Psi : M^n = N_1^p \times_f N_2^q$ be an isometric immersion of a compact Legendrian warped product submanifold $N_1^p \times_f N_2^q$ into a Sasakian space form $\mathbb{S}^{2n+1}$ such that N_1^p is minimal in $\mathbb{S}^{2n+1}$ If the following equality is satisfied for the warped product submanifold M^n

$$\|h(u,v)\|^2 = q\left\{ p + \int\limits_M \mathcal{R}ic(\nabla\mu,)\mathrm{dV} \right\}. \tag{6.8}$$

Then, Legendrian warped product submanifold $N_1^p \times_f N_2^q$ into a Sasakian space form, is simply a Riemannian product of N_1^p and N_2^q.

Corollary 6.3. Let $\Psi : M^n = N_1^p \times_f N_2^q$ be an isometric immersion of a Legendrian warped product submanifold $N_1^p \times_f N_2^q$ into a Sasakian space form $\mathbb{S}^{2n+1}$ with base N_1^p is minimal in $\mathbb{S}^{2n+1}$, connected and compact. Then N_1^p is isometric to the sphere $\mathbb{S}^p(\frac{\lambda_1}{p})$ with constant sectional curvature is equal to $\frac{\lambda_1}{p}$ if and only if the following equality is satisfied

$$\|Hess(\mu)\|^2 = \frac{2\lambda_1}{3pq}\left\{ \|h(u,v)\|^2 - pq \right\}, \tag{6.9}$$

Theorem 1.4 implies that

Corollary 6.4. Let $\Psi : M^n = N_1^p \times_f N_2^q$ be an isometric immersion of a Legendrian warped product $N_1^p \times_f N_2^q$ into a Sasakian space form $\mathbb{S}^{2n+1}$ with base N_1^p is minimal in $\mathbb{S}^{2n+1}$, connected and compact. Then N_1^p is isometric to the sphere $\mathbb{S}^p(\frac{\lambda_1}{p})$ with constant sectional curvature is equal to $\frac{\lambda_1}{p}$ if and only if the following equality is satisfied

$$Ric(\nabla\mu, \nabla\mu) = \lambda_1\Big(\frac{3p+2}{3pq}\Big)\left\{ pq - \Big\|h(u,v)\Big\|^2 \right\}. \tag{6.10}$$

where $\lambda_1 > 0$ is a positive eigenvalue associated to the eigenfunction $\mu = \ln f$.

Theorem 1.5 gives the following

Corollary 6.5. Let $\Psi : M^n = N_1^p \times_f N_2^q$ be an isometric immersion of a Legendrian warped product submanifold $N_1^p \times_f N_2^q$ into a Sasakian space form $\mathbb{S}^{2n+1}$ with with base N_1^p is minimal in $\mathbb{S}^{2n+1}$, connected and complete. If N_1^p has non-negative Ricci curvature then its is isometric to the sphere $\mathbb{S}^p(\frac{\lambda_1}{p})$ if and only if the following equality is satisfied

$$\|h(u,v)\|^2 = pq, \tag{6.11}$$

where $\lambda_1 > 0$.

Similarly, from Theorem 1.6, we have

Corollary 6.6. Let $\Psi : M^n = N_1^p \times_f N_2^q$ be an isometric immersion of a Legendrian warped product submanifold $N_1^p \times_f N_2^q$ into a Sasakian space form $\mathbb{S}^{2n+1}$ with base N_1^p is minimal in $\mathbb{S}^{2n+1}$, complete and has positive Ricci curvature. Then N_1^p is isometric to a warped product of the Euclidean line and a complete Riemannian manifold whose warping function ψ satisfies the equation that $\frac{d^2\psi}{dt^2} + \lambda_1\psi = 0$ if and only if the following equality is satisfied

$$\|h(u, v)\|^2 = pq, \tag{6.12}$$

where $\lambda_1 < 0$ is a negative eigenvalue associated to the eigenfunction $\mu = \ln f$.

At the last result follows from Theorem 1.7, we get

Corollary 6.7. Let $\Psi : M^n = N_1^p \times_f N_2^q$ be an isometric immersion of a Legendrian warped product submanifold $N_1^p \times_f N_2^q$ into a Sasakian space form $\mathbb{S}^{2n+1}$ with base N_1^p is minimal in $\mathbb{S}^{2n+1}$, complete and has positive Ricci curvature. Then N_1^p is isometric to the Euclidean space $\mathbb{R}^p$ if and only if the following equality is satisfied

$$\Big\|h(u, v)\Big\|^2 = q^2\Big\{\frac{\lambda_1}{p} + p\Big\}, \tag{6.13}$$

with $\lambda_1 > 0$ is positive eigenvalue of the non-constant warping function $\mu = \ln f$.

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