Hopf structure for a Fock realization of the (p, q)-deformed white noise Heizenberg algebra

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Abstract. The Hopf algebraic structure problem for a two-parameter deformed white noise Heizenberg algebra based on the two-parameter deformation of canonical commutation relations is discussed. Firstly, in the basis of the (p, q)-Fock space we present the Fock realization of the (p, q)-deformed quantum Heizenberg algebra and we give its Hopf structure.

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1. Introduction

In recent years from the point of view of physical applicability in concrete physical problems and quantum algebras a lot of interest has been devoted to the study of the various quantum deformations of Heizenberg algebra. From mathematical point of view such popularity connected with numerous relations which exist between deformed oscillators and other quantum deformations (quantum groups, quantum algebras, quantum spaces etc). From the other side there are some hopes that in a physical studies of non-linear phenomena the deformed Heizenberg can play the role much the same as the usual boson oscillator in standard quantum mechanics. Such hopes are supported by several applications of the deformed oscillators in conformal field theory, lattice models [6, 9], nuclear spectroscopy [5, 7], in describing the systems with non-standard statistics and energy spectrum. For this reason, the interests in quantum deformations of Lie algebras [16, 14], Lie bialgebras

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[18, 21] and quantizations of Lie algebras [17] have been growing in the physical and mathematical literatures which are closely related to the Virasoro algebra. Among the quantum deformations of Lie algebras, the q-deformed Virasoro algebra has been most intensively considered [3, 13, 20], which can be viewed as a typical example of the physical application of quantum groups. Roughly speaking, the quantum Lie algebras in the context of these deformations are universal enveloping algebras deformed by one or more parameter(s) (q-deformation) and possess structures of Hopf algebras. More precisely, the q-deformed Heizenberg algebra $\mathrm{Heiz}_q(\mathcal{H})$ begin with Bożejko, Kümmerer, and Speicher in [8], [10] and [11] which introduce for $q \in (-1, 1)$ the q-analogues of Brownian motions and Gaussian processes. Their constructions were based on a suitably deformed Fock space $\mathcal{F}_q(\mathcal{H})$ on which the creation and annihilation operators satisfied the q-commutation relation:

$$A(\xi)A^*(\eta) - qA^*(\eta)A(\xi) = \langle \xi, \eta \rangle_{\mathcal{H}} \mathbf{1}, \qquad \forall \xi, \eta \in \mathcal{H}.$$

Furthermore, it has showed that $\mathrm{Heiz}_q(\mathcal{H})$ posses a structure of Hopf algebra. Although one-parameter deformations have been mostly studied, the multiparameter ones have aroused much interest because they become more flexible when we are dealing with applications to concrete physical models. This give our motivation to study the two-parametric deformations of the quantum algebras and its Hopf structure.

Our paper is organized as follows. Section 2 is devoted to the study of Fock representation of the generators of the quantum (p, q)-deformed white noise Heizenberg algebra $\mathrm{Heiz}_{p,q}(\mathcal{H})$ based on the so-called (p, q)-Fock space and the following (p, q)-deformed commutation relations :

$$\begin{array}{rcl} a(\xi)a^*(\eta) - pa^*(\eta)a(\xi) & = & q^N \langle \xi, \eta \rangle \mathbf{1}, \quad [N, a^*(\xi)] = a^*(\xi) \\ a(\xi)a^*(\eta) - qa^*(\eta)a(\xi) & = & p^N \langle \xi, \eta \rangle \mathbf{1}, \quad [N, a(\xi)] = -a(\xi). \end{array}$$

In Section 2, we discuss briefly the concept of a Hopf algebra and show that $\mathrm{Heiz}_{p,q}(\mathcal{H})$ is endowed with a non-trivial Hopf algebra structure.

2. Fock representation of (p, q)-deformed commutation relation

For p and q be two real numbers such that $0 < q < p \leq 1$, the natural number n has the following (p, q)-deformation:

$$[n]_{p,q} = \sum_{i=1}^{n} q^{i-1} p^{n-i} = \frac{p^n - q^n}{p - q}, \quad [0]_{p,q} = 0,$$

which is a natural generalization of the q-number that is we have

$$\lim_{p \to 1} [n]_{p,q} = [n]_q = \frac{1 - q^n}{1 - q}.$$

The (p, q)-factorial and (p, q)-binomial coefficients are defined as

$$[n]_{p,q}! = \prod_{k=1}^{n} [k]_{p,q} \text{ with } [0]_{p,q}! = 1; \quad \left(\begin{array}{c} n \\ k \end{array} \right)_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

The (p,q)-deformed Heizenberg algebra is defined by the three generators A, A^+, N satisfying the following (p,q)-deformed canonical commutations relations

$$\begin{array}{rcl} AA^+ - qA^+A &=& p^N, \quad [N,A] = -A \\ AA^+ - pA^+A &=& q^N, \quad [N,A^+] = A^+, \end{array}$$

where the commutator $[\cdot,\cdot]$ is defined by $[B,C] = BC - CB$. Now we will give a one-mode interacting Fock space representation of the previous algebra. In the same way as the usual Hermite polynomials are connected to the bosonic relation with $p = 1$ and $q \to 1$, the (p,q)-deformed relations are linked to (p,q)-analogues of the Hermite polynomials.

Definition 2.1. The polynomials $H_n^{(p,q)}$ with leading coefficient equals to 1 satisfying the recursion formula

$$\begin{cases} xH_n^{(p,q)}(x) = H_{n+1}^{(p,q)}(x) + [n]_{p,q}H_{n-1}^{(p,q)}(x), \quad n \in \mathbb{N} \\ H_{-1}^{(p,q)}(x) := 0, \quad H_0^{(p,q)}(x) := 1 \end{cases} \tag{2.1}$$

are called (p,q)-Hermite polynomials.

Let $\Gamma(\mathbb{C}, \{\lambda_{n,p,q}\})$ be the weighted Fock space associated with

$$\lambda_{n,p,q} = \|H_n^{(p,q)}\|^2 = [n]_{p,q}! \ , i.e.,$$

$$\Gamma(\mathbb{C}, \{\lambda_{n,p,q}\}) = \Big\{(z_n)_{n\in\mathbb{N}}, z_n \in \mathbb{C};\ \sum_{n=0}^{+\infty} \lambda_{n,p,q}|z_n|^2 < \infty\Big\}$$

We define linear operators $A^\pm$ and N by

$$\begin{array}{rcl} A^+\Phi_n &=& \sqrt{[n+1]_{p,q}}\Phi_{n+1}, \quad n \geq 0, \\ A^-\Phi_n &=& \sqrt{[n]_{p,q}}\Phi_{n-1}, \quad n \geq 1, \quad A^-\Phi_0 = 0 \\ N\Phi_n &=& n\Phi_n, \quad n = 0,1,2,\cdots. \end{array}$$

Equipped with the natural domains, $A^\pm$ become closed operators which are mutually adjoint. Then $\Gamma := \Gamma(\mathbb{C}, \{\lambda_{n,p,q}\}, A^\pm)$ is called an interacting Fock space associated with $\lambda_{n,p,q}$. By a simple computation one can see that $A^\pm$ and N satisfy the relation of the (p,q)-deformed Heizenberg algebra.

Now our goal is to give the infinite dimensional analogue of the previous Fock representation. Let S_n denote the symmetric group of all permutations on $[\![1,n]\!] := \{1,\cdots,n\}$ and $I(\sigma)$ denote the number of inversions of the permutation $\sigma \in S_n$ defined by

$$I(\sigma) = \sharp(\mathrm{Inv}(\sigma)) := \sharp\big\{(i,j)\,|\,1 \leq i < j \leq n,\ \sigma(i) > \sigma(j)\big\},$$

where $\sharp(E)$ stands the cardinality of the set E. Analogously, the pair $(i,j) \in [\![1,n]\!]^2$ with $i < j$ is called a co-inversion in σ if $\sigma(i) < \sigma(j)$. The corresponding co-inversion is encoded by (i,j) and contained in the set

$$\mathrm{Cinv}(\sigma) := \big\{(i,j)\,|\,1 \leq i < j \leq n,\ \sigma(i) < \sigma(j)\big\}$$

with cardinality $C(\sigma) := \sharp(\mathrm{Cinv}(\sigma))$. Let $H = L^2(\mathbb{R}, dt)$ be the real Hilbert space with the norm $|\cdot|_0$ generated by the inner product $\langle\cdot,\cdot\rangle$ and denote

$\mathcal{H}$ its complexification. Denote $\mathcal{F}_0(\mathcal{H}) = \bigoplus_{n\geq 0} \mathcal{H}^{\otimes n}$ the full Fock space over $\mathcal{H}$ with the inner product $\langle \cdot, \cdot \rangle$ and $\mathcal{F}_0^{fin}(\mathcal{H})$ the linear span of vectors of the form $\xi_1 \otimes \cdots \otimes \xi_n \in \mathcal{H}^{\otimes n}, n \in \mathbb{N}$, where $\mathcal{H}^{\otimes 0} = \mathbb{C}\Omega$ for the vacuum vector $\Omega = (1, 0, 0, \cdots) \in \mathcal{F}_0(\mathcal{H})$. We equip $\mathcal{F}_0^{fin}(\mathcal{H})$ with the inner product

$$\langle \xi_1 \otimes \cdots \otimes \xi_n, \eta_1 \otimes \cdots \otimes \eta_m \rangle = \delta_{n,m} \prod_{k=1}^{n} \langle \xi_k, \eta_k \rangle.$$

Define the operator $\mathcal{T}_{p,q}$ on $\mathcal{F}_0^{fin}(\mathcal{H})$ by a linear extension of

$$\mathcal{T}_{p,q}\Omega = \Omega, \quad \mathcal{T}_{p,q}(\xi_1 \otimes \cdots \otimes \xi_n) = \sum_{\sigma \in S_n} q^{I(\sigma)} p^{C(\sigma)} \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(n)},$$

and put

$$\xi_1 \otimes_{p,q} \cdots \otimes_{p,q} \xi_n := \mathcal{T}_{p,q}(\xi_1 \otimes \cdots \otimes \xi_n), \quad \xi_i \in \mathcal{H},\ i \in [\![1, n]\!].$$

Definition 2.2. Define $\mathcal{F}_{p,q}^{(n)}(\mathcal{H})$ as the separable Hilbert space witch coincide with $\mathcal{H}^{\otimes n}$ as a set and has scalar product

$$\big\langle f^{(n)}, g^{(n)} \big\rangle_{p,q} := \big\langle f^{(n)}, g^{(n)} \big\rangle_{\mathcal{F}_{p,q}^{(n)}(\mathcal{H})} = \big\langle \mathcal{T}_{p,q} f^{(n)}, g^{(n)} \big\rangle. \tag{2.2}$$

Hence the (p, q)-Fock space denoted $\mathcal{F}_{p,q}(\mathcal{H})$ is defined by

$$\mathcal{F}_{p,q}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{F}_{p,q}^{(n)}(\mathcal{H}).$$

If we denote $\mathcal{F}_{p,q}^{fin}(\mathcal{H})$ the linear span of vectors of the form

$$\xi_1 \otimes_{p,q} \cdots \otimes_{p,q} \xi_n \in \mathcal{F}_{p,q}^{(n)}(\mathcal{H}), \quad n \in \mathbb{N},$$

one can see that $\langle \cdot, \cdot \rangle_{p,q}$ on $\mathcal{F}_{p,q}^{fin}(\mathcal{H})$ satisfy the following useful relation

$$\big\langle f^{(n)}, \xi_1 \otimes_{p,q} \cdots \otimes_{p,q} \xi_m \big\rangle_{p,q} = \delta_{n,m} [n]_{p,q}! \big\langle f^{(n)}, \xi_1 \otimes \cdots \otimes \xi_m \big\rangle. \tag{2.3}$$

Fore more details about the properties of the operator $\mathcal{T}_{p,q}$ and the construction of the (p, q)-Fock space we can see [4].

Definition 2.3. For each $\xi \in \mathcal{H}$, we define the (p, q)-creation operator $a^*(\xi)$ and the (p, q)-annihilation operator $a(\xi)$ on the dense subspace $\mathcal{F}_{p,q}^{fin}(\mathcal{H})$ as follows:

$$a^*(\xi)\Omega = \xi,$$

$$a^*(\xi) f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n = \xi \otimes_{p,q} f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n \tag{2.4}$$

and

$$a(\xi)\Omega = 0,$$

$$a(\xi) f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n = \xi \otimes_{p,q}^{1} (f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n),$$

where $f \otimes_{p,q}^1 g$ is the left 1-contraction of $f \in \mathcal{H}$ and $g \in \mathcal{H}^{\otimes n}$. More precisely, we have

$$\xi \otimes_{p,q}^1 (f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n) = \sum_{i=1}^{n} q^{i-1} p^{n-i} \langle \xi, f_i \rangle f_1 \otimes_{p,q} \cdots \otimes_{p,q} \breve{f}_i \otimes_{p,q} \cdots \otimes_{p,q} f_n, \tag{2.5}$$

where $\langle \cdot, \cdot \rangle$ denote the inner product on $\mathcal{H}$ and the symbol $\breve{f}_i$ means that f_i has to be deleted in the tensor product.

Lemma 2.4. *The (p,q)-creation and (p,q)-annihilation operators fulfill the (p,q)-commutations relations of the (p,q)-deformed quantum oscillator algebra, i.e.,*

$$a(\xi)a^*(\eta) - pa^*(\eta)a(\xi) = q^N \langle \xi, \eta \rangle \mathbf{1}, \tag{2.6}$$

$$a(\xi)a^*(\eta) - qa^*(\eta)a(\xi) = p^N \langle \xi, \eta \rangle \mathbf{1}, \quad \forall \xi, \eta \in \mathcal{H} \tag{2.7}$$

$$[N, a^*(\xi)] = a^*(\xi), \quad [N, a(\xi)] = -a(\xi), \tag{2.8}$$

where N is the standard number operator defined by

$$N f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n = n f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n. \tag{2.9}$$

Proof. For any $n \in \mathbb{N}$ and $\xi, \eta, f_1, \cdots, f_n \in \mathcal{H}$ we have

$$\begin{aligned} & a(\xi)a^*(\eta)(f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n) \\ = \;& a(\xi)(\eta \otimes_{p,q} f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n) \\ = \;& p^n \langle \xi, \eta \rangle f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n \\ + \;& \sum_{i=2}^{n+1} q^{i-1} p^{n+1-i} \langle \xi, f_{k-1} \rangle \eta \otimes_{p,q} f_1 \otimes_{p,q} \cdots \breve{f}_{k-1} \otimes_{p,q} \cdots \otimes_{p,q} f_n. \end{aligned} \tag{2.10}$$

On the other hand, one can see that

$$\begin{aligned} & \sum_{i=2}^{n+1} q^{i-1} p^{n+1-i} \langle \xi, f_{i-1} \rangle \eta \otimes_{p,q} f_1 \otimes_{p,q} \cdots \breve{f}_{i-1} \otimes_{p,q} \cdots \otimes_{p,q} f_n \\ = \;& \eta \otimes_{p,q} \Big(\sum_{i=1}^{n} q^i p^{n-i} \langle \xi, f_i \rangle \otimes_{p,q} f_1 \otimes_{p,q} \cdots \breve{f}_{i-1} \otimes_{p,q} \cdots \otimes_{p,q} f_n \Big) \\ = \;& qa^* \Big(\sum_{i=1}^{n} q^i p^{n-i} \langle \xi, f_i \rangle \otimes_{p,q} f_1 \otimes_{p,q} \cdots \breve{f}_{i-1} \otimes_{p,q} \cdots \otimes_{p,q} f_n \Big) \\ = \;& qa^*(\eta)a(\xi)(f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n). \end{aligned} \tag{2.11}$$

Therefore, by (2.10) and (2.11) we obtain

$$\begin{aligned} & a(\xi)a^*(\eta)(f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n) \\ & \quad = p^n \langle \xi, \eta \rangle f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n + qa^*(\eta)a(\xi)(f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n) \end{aligned}$$

which proves (2.6). Note the symmetry of this relation under the exchange of p and q gives (2.7) and by a same calculus we obtain (2.8). □

Proposition 2.5. *Let $\xi \in \mathcal{H}$.*

1. *The operators $a^*(\xi)$ and $a(\xi)$ are adjoints of each other on $\mathcal{F}_{p,q}^{fin}(\mathcal{H})$ with respect to $\langle\cdot,\cdot\rangle_{p,q}$.*
2. *The operators $a^*(\xi)$ and $a(\xi)$ are bounded on $\mathcal{F}_{p,q}(\mathcal{H})$.*

Proof. 1. By using (2.3), (2.4) and (2.5), then for any $f_i, g_j \in \mathcal{H}, i \in \{1,\cdots,n-1\}, j \in \{1,\cdots n\}$, we have

$$\begin{aligned}
&\Big\langle a^*(\xi) f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_{n-1}, g_1 \otimes_{p,q} \cdots \otimes_{p,q} g_n \Big\rangle_{p,q} \\
&= [n]_{p,q}! \sum_{\sigma \in S_n} q^{I(\sigma)} p^{C(\sigma)} \langle \eta_{\sigma(1)}, g_1\rangle \cdots \langle \eta_{\sigma(n)}, g_n\rangle \\
&= [n]_{p,q}! \sum_{\sigma \in S_n} q^{I(\sigma^{-1})} p^{C(\sigma^{-1})} \langle \eta_1, g_{\sigma^{-1}(1)}\rangle \cdots \langle \eta_n, g_{\sigma^{-1}(n)}\rangle \\
&= [n]_{p,q}! \sum_{\rho \in S_n} q^{I(\rho)} p^{C(\rho)} \langle \xi, g_{\rho(1)}\rangle \cdots \langle f_{n-1}, g_{\rho(n)}\rangle \\
&= [n]_{p,q}! \Big\langle f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_{n-1}, \xi \otimes_{p,q}^1 g_1 \otimes_{p,q} \cdots \otimes_{p,q} g_n \Big\rangle,
\end{aligned}$$

where $\eta_1 = \xi$ and $\eta_i = f_{i-1}, i = 2, \cdots, n$. For convenience, we put

$$\overrightarrow{f} = f_1 \otimes \cdots \otimes f_{n-1}, \quad \overrightarrow{f}_{p,q} = f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_{n-1}.$$

Then by (2.5) we have

$$\begin{aligned}
&\Big\langle \overrightarrow{f}, \xi \otimes_{p,q}^1 g_1 \otimes_{p,q} \cdots \otimes_{p,q} g_n \Big\rangle \\
&= \Big\langle \overrightarrow{f}, \mathcal{T}_{p,q}\Big(\sum_{i=1}^n q^{i-1} p^{n-i} \langle \xi, g_i\rangle g_1 \otimes \cdots \otimes \breve{g}_i \otimes \cdots \otimes g_n \Big) \Big\rangle \\
&= \Big\langle \overrightarrow{f}_{p,q}, \sum_{i=1}^n q^{i-1} p^{n-i} \langle \xi, g_i\rangle g_1 \otimes \cdots \otimes \breve{g}_i \otimes \cdots \otimes g_n \Big\rangle.
\end{aligned}$$

Hence we deduce that

$$\begin{aligned}
&\Big\langle a^*(\xi) f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_{n-1}, g_1 \otimes_{p,q} \cdots \otimes_{p,q} g_n \Big\rangle_{p,q} \\
&= [n]_{p,q}! \Big\langle \overrightarrow{f}_{p,q}, \sum_{i=1}^n q^{i-1} p^{n-i} \langle \xi, g_i\rangle g_1 \otimes \cdots \otimes \breve{g}_i \otimes \cdots \otimes g_n \Big\rangle \\
&= \Big\langle f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_{n-1}, a(\xi) g_1 \otimes_{p,q} \cdots \otimes_{p,q} g_n \Big\rangle_{p,q}
\end{aligned}$$

which follows the proof.

2. For any $\xi \in \mathcal{H}$ and $f_i \in \mathcal{H}, i \in \{1, 2, \cdots, n\}$, we have

$$\begin{aligned}
&\|a(\xi) f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n\|_{p,q}^2 \\
&= \Big\langle a(\xi) f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n, a(\xi) f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n \Big\rangle_{p,q} \\
&= \Big\langle f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n, a^*(\xi) a(\xi) f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n \Big\rangle_{p,q}
\end{aligned}$$

On the other hand by using (2.6) and (2.7) we get

$$a^*(\xi)a(\eta) = \frac{p^N - q^N}{p-q} = [N]_{p,q}\langle \xi, \eta\rangle, \tag{2.12}$$

where the operator $[N]_{p,q}$ is defined by

$$[N]_{p,q}(f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n) = [n]_{p,q} f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n. \tag{2.13}$$

Then Eq. (2.6) yields

$$\begin{array}{rcl} a^*(\xi)a(\xi) f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n & = & \langle \xi, \xi\rangle [N]_{p,q}\big(f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n\big) \\ & = & [n]_{p,q} |\xi|_0^2\, f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n \end{array}$$

and we obtain

$$\begin{array}{rcl} \|a(\xi) f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n\|_{p,q}^2 & = & [n]_{p,q} |\xi|_0^2 \|f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n\|_{p,q}^2 \\ & \leq & \dfrac{|\xi|_0^2}{p-q} \|f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n\|_{p,q}^2. \end{array}$$

Hence we prove that $\|a^*(\xi)\|_{OP} \leq \dfrac{|\xi|_0}{\sqrt{p-q}}$ which proves (2).

□

Definition 2.6. The (p,q)-deformed white noise Heizenberg algebra $\mathrm{Heiz}_{p,q}(\mathcal{H})$ is defined by generators

$$\Big\{ a(\xi),\ a^*(\eta),\ N;\quad \xi, \eta \in \mathcal{H} \Big\}$$

satisfying

$$(a^*(\xi))^* = a(\xi), \qquad N^* = N$$

and the commutation relations (2.6)-(2.8).

3. Hopf structure of the (p,q)-deformed white noise Heizenberg algebra

In this section, we give a direct construction of the Hopf algebraic structures of the Fock realization of the (p,q)-deformed white noise Heizenberg algebra $\mathrm{Heiz}_{p,q}(\mathcal{H})$. This structures will be done by several lemma bellow.

Lemma 3.1. *There is a unique algebraic homomorphism*

$$\Delta : \mathcal{H}_{p,q} \longrightarrow Heiz_{p,q}(\mathcal{H}) \times Heiz_{p,q}(\mathcal{H})$$

with

$$\Delta(N) = N \otimes 1 + 1 \otimes N, \tag{3.1}$$

$$\Delta(a^*(\eta)) = a^*(\eta) \otimes q^{N/2} + q^{N/2} \otimes a^*(\eta), \tag{3.2}$$

$$\Delta(a(\xi)) = a(\xi) \otimes q^{-N/2} + q^{-N/2} \otimes a(\xi). \tag{3.3}$$

Proof. We need to show that $\Delta(N), \Delta(a^*(\eta))$ and $\Delta(a(\xi))$ satisfy the (p,q)-commutation relation. Firstly, one can see that

$$\begin{aligned}
&\Delta(a^*(\eta))\Delta(a(\xi)) \\
&= \big(a^*(\eta)\otimes q^{N/2} + q^{N/2}\otimes a^*(\eta)\big)\big(a(\xi)\otimes q^{-N/2} + q^{-N/2}\otimes a(\xi)\big) \\
&= a^*(\eta)a(\xi)\otimes 1 + a^*(\eta)q^{-N/2}\otimes q^{N/2}a(\xi) \\
&+ q^{N/2}a(\xi)\otimes a^*(\eta)q^{-N/2} + 1\otimes a^*(\eta)a(\xi)
\end{aligned}$$

and

$$\begin{aligned}
&\Delta(a(\xi))\Delta(a^*(\eta)) \\
&= \big(a(\xi)\otimes q^{-N/2} + q^{-N/2}\otimes a(\xi)\big)\big(a^*(\eta)\otimes q^{N/2} + q^{N/2}\otimes a^*(\eta)\big) \\
&= a(\xi)a^*(\eta)\otimes 1 + a(\xi)q^{N/2}\otimes q^{-N/2}a^*(\eta) \\
&+ q^{-N/2}a^*(\eta)\otimes a(\xi)q^{N/2} + 1\otimes a(\xi)a^*(\eta).
\end{aligned}$$

Then by using (3.2) and (3.3) we obtain

$$\begin{aligned}
&\Delta(a(\xi))\Delta(a^*(\eta)) - q\Delta(a^*(\eta))\Delta(a(\xi)) \\
&= p^N\langle\xi,\eta\rangle\mathbf{1}\otimes 1 + (a^*(\eta)q^{-N/2}\otimes q^{N/2}a(\xi) - qa(\xi)q^{N/2}\otimes q^{-N/2}a^*(\eta)) \\
&+ (q^{N/2}a(\xi)\otimes a^*(\eta)q^{-N/2} - qq^{-N/2}a^*(\eta)\otimes a(\xi)q^{N/2}) + 1\otimes p^N\langle\xi,\eta\rangle\mathbf{1}.
\end{aligned}$$

On the other hand by using a basis $(\zeta_k)_k$ of the Hilbert space $\mathcal{H}$, we get

$$\begin{aligned}
&\Big(a^*(\eta)q^{-N/2}\otimes q^{N/2}a(\xi) - qa(\xi)q^{N/2}\otimes q^{-N/2}a^*(\eta)\Big)\zeta_k^{\otimes n}\otimes\zeta_k^{\otimes n} \\
&= q^{-n/2}a^*(\eta)\zeta_k^{\otimes n}\otimes q^{n-1/2}a(\xi)\zeta_k^{\otimes n} \\
&- q^{n+1/2}a(\xi)\zeta_k^{\otimes n}\otimes q^{-(n+1)/2}a^*(\eta)\zeta_k^{\otimes n} \\
&= q^{-1/2}[n]_{p,q}\langle\xi,\zeta_k^{\otimes n}\rangle\Big(\eta\otimes\zeta_k^{\otimes n}\otimes\zeta_k^{\otimes(n-1)} - \zeta_k^{\otimes n}\otimes\eta\otimes\zeta_k^{\otimes n}\Big)
\end{aligned}$$

and

$$\begin{aligned}
&\Big(q^{N/2}a(\xi)\otimes a^*(\eta)q^{-N/2} - qq^{-N/2}a^*(\eta)\otimes a(\xi)q^{N/2}\Big)\zeta_k^{\otimes n}\otimes\zeta_k^{\otimes n} \\
&= q^{n-1/2}a(\xi)\zeta_k^{\otimes n}\otimes q^{-n/2}a^*(\eta)\zeta_k^{\otimes n} \\
&- q^{-(n+1)/2}a^*(\eta)\zeta_k^{\otimes n}\otimes q^{n/2}a(\xi)\zeta_k^{\otimes n} \\
&= q^{-1/2}[n]_{p,q}\langle\xi,\zeta_k^{\otimes n}\rangle\Big(\zeta_k^{\otimes n}\otimes\eta\otimes\zeta_k^{\otimes n} - \eta\otimes\zeta_k^{\otimes n}\otimes\zeta_k^{\otimes(n-1)}\Big).
\end{aligned}$$

Hence we get

$$\begin{aligned}
\Delta(a(\xi))\Delta(a^*(\eta)) - q\Delta(a^*(\eta))\Delta(a(\xi)) &= p^N\langle\xi,\eta\rangle\mathbf{1}\otimes 1 + 1\otimes p^N\langle\xi,\eta\rangle\mathbf{1} \\
&= \Delta(p^N\langle\xi,\eta\rangle\mathbf{1}),
\end{aligned}$$

and by a similar calculus we obtain

$$[\Delta(N),\Delta(a^*(\eta))] = \Delta(a^*(\eta)), \quad [\Delta(N),\Delta(a(\xi))] = -\Delta(a(\xi)).$$

That means that Δ is an algebraic homomorphism. Consequently, $\mathrm{Heiz}_{p,q}(\mathcal{H})$ is a bialgebra. □

Lemma 3.2. *The comultiplication* Δ *on* $Heiz_{p,q}(\mathcal{H})$ *is coassociative, i.e.,*

$$(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta.$$

Proof. We have to check that all the generators of $\mathrm{Heiz}_{p,q}(\mathcal{H})$ are mapped both ways by $(1\otimes\Delta)\Delta$ and $(\Delta\otimes 1)\Delta$ to the same image, which simply involves straightforward calculations. We shall take $a^*(\eta)$ as an example (others can be done similarly). So we have

$$\begin{array}{rcl} (1 \otimes \Delta)\Delta(a^*(\eta)) & = & (1 \otimes \Delta)(a^*(\eta) \otimes q^{N/2} + q^{N/2} \otimes a^*(\eta)) \\ & = & a^*(\eta) \otimes \Delta(q^{N/2}) + q^{N/2} \otimes \Delta(a^*(\eta)) \\ & = & a^*(\eta) \otimes (q^{N/2} \otimes 1 + 1 \otimes q^{N/2}) \\ & + & q^{N/2} \otimes (a^*(\eta) \otimes q^{N/2} + q^{N/2} \otimes a^*(\eta)) \\ & = & a^*(\eta) \otimes q^{N/2} \otimes 1 + a^*(\eta) \otimes 1 \otimes q^{N/2} \\ & + & q^{N/2} \otimes a^*(\eta) \otimes q^{N/2} + q^{N/2} \otimes q^{N/2} \otimes a^*(\eta) \end{array}$$

and

$$\begin{array}{rl} & (\Delta \otimes 1)\Delta(a^*(\eta)) \\ = & (\Delta \otimes 1)(a^*(\eta) \otimes q^{N/2} + q^{N/2} \otimes a^*(\eta)) \\ = & \Delta(a^*(\eta)) \otimes q^{N/2} + \Delta(q^{N/2}) \otimes a^*(\eta) \\ = & (a^*(\eta) \otimes q^{N/2} + q^{N/2} \otimes a^*(\eta)) + (q^{N/2} \otimes 1 + 1 \otimes q^{N/2}) \otimes a^*(\eta) \\ = & a^*(\eta) \otimes q^{N/2} \otimes q^{N/2} + q^{N/2} \otimes a^*(\eta) \otimes q^{N/2} + q^{N/2} \otimes 1 \otimes a^*(\eta) \\ + & q^{N/2} \otimes q^{N/2} \otimes a^*(\eta). \end{array}$$

This gives the statement. □

Now for simplicity of notation we will denote $\mathrm{Heiz}_{p,q}(\mathcal{H}) \otimes \mathrm{Heiz}_{p,q}(\mathcal{H})$ by $\mathrm{Heiz}^{\otimes 2}_{p,q}(\mathcal{H})$.

Lemma 3.3. *There is a unique homomorphism of* $\mathbb{C}$*-algebras*

$$\varepsilon : Heiz_{p,q}(\mathcal{H}) \longrightarrow \mathbb{C}$$

with

$$\varepsilon(1) = 1 \quad \text{and} \quad \varepsilon(N) = \varepsilon(a(\xi)) = \varepsilon(a^*(\xi)) = 0.$$

Moreover, the following diagrams are commutative

$$\begin{tikzcd} Heiz_{p,q}(\mathcal{H}) \arrow[r, "\Delta"] \arrow[d, "id"] & Heiz^{\otimes 2}_{p,q}(\mathcal{H}) \arrow[d, "1\otimes\varepsilon"] \\ Heiz_{p,q}(\mathcal{H}) \arrow[r, "\pi_1"] & Heiz^{\otimes 2}_{p,q}(\mathcal{H}) \end{tikzcd} \tag{3.4}$$

$$\begin{tikzcd} Heiz_{p,q}(\mathcal{H}) \arrow[r, "\Delta"] \arrow[d, "id"] & Heiz^{\otimes 2}_{p,q}(\mathcal{H}) \arrow[d, "\varepsilon\otimes 1"] \\ Heiz_{p,q}(\mathcal{H}) \arrow[r, "\pi_2"] & Heiz^{\otimes 2}_{p,q}(\mathcal{H}) \end{tikzcd} \tag{3.5}$$

namely, $(1\otimes\varepsilon)\Delta = \pi_1 \circ id$ *and* $(\varepsilon\otimes 1)\Delta = \pi_2 \circ id$*, where* π_1 *(resp.* π_2*) denotes the isomorphism* $u \mapsto u \otimes 1$ *(resp.* $u \mapsto 1 \otimes u$*) for any* $u \in Heiz_{p,q}(\mathcal{H})$*.*

Proof. It is straightforward to see that $(\varepsilon(1), \varepsilon(N), \varepsilon(a(\xi)), \varepsilon(a^*(\eta))) = (1, 0, 0, 0)$ satisfy the (p, q)-commutation relations (2.6)–(2.8). So we have the algebraic homomorphism ε. For the commutativity of the two diagrams, it can be easily checked on the generators. The homomorphism ε is called the counit of $\mathrm{Heiz}_{p,q}(\mathcal{H})$. $\square$

Lemma 3.4. *There is a unique linear map S of $Heiz_{p,q}(\mathcal{H})$ with*

$$\begin{cases} S(a^*(\eta)) = -q^{-N/2}a^*(\eta)q^{N/2} \\ S(a(\xi)) = -q^{-N/2}a(\xi)q^{N/2} \\ S(N) = N. \end{cases} \tag{3.6}$$

Proof. We need to show that $(S(N), S(a^*(\eta)), S(a(\xi)))$ satisfies the (p, q)-commutation relation in $\mathrm{Heiz}_{p,q}(\mathcal{H})$. By using (2.6), we have

$$\begin{aligned} & S(a^*(\eta))S(a(\xi)) - qS(a(\xi))S(a^*(\eta)) \\ &\quad = (-q^{-N/2}a^*(\eta)q^{N/2})(-q^{N/2}a(\xi)q^{N/2}) \\ &\quad - q(-q^{N/2}a(\xi)q^{N/2})(-q^{-N/2}a^*(\eta)q^{N/2}) \\ &\quad = q^{-N/2}a^*(\eta)a(\xi)q^{N/2} - qq^{-N/2}a(\xi)a^*(\eta)q^{N/2} \\ &\quad = q^{-N/2}(a^*(\eta)a(\xi) - qa(\xi)a^*(\eta))q^{N/2} \\ &\quad = q^{-N/2}p^N q^{N/2}\langle \xi, \eta\rangle \mathbf{1}. \end{aligned}$$

On the other hand one can see that the action of $q^{-N/2}p^N q^{N/2}\langle \xi, \eta\rangle \mathbf{1}$ and $p^N\langle \xi, \eta\rangle \mathbf{1}$ coincide on the basis $(\zeta_k^{\otimes n})_k$ of the Hilbert space $\mathcal{H}^{\otimes n}$. Thus we deduce that

$$S(a^*(\eta))S(a(\xi)) - qS(a(\xi))S(a^*(\eta)) = S(p^N\langle \xi, \eta\rangle \mathbf{1}),$$

namely, the map S preserves (2.6). One can similarly check that (2.7) and (2.8) are also preserved by S. So there is a homomorphism $S : \mathrm{Heiz}_{p,q}(\mathcal{H}) \longrightarrow \mathrm{Heiz}_{p,q}(\mathcal{H})$ satisfying (3.6). Now S^2 is an ordinary homomorphism from $\mathrm{Heiz}_{p,q}(\mathcal{H})$ to $\mathrm{Heiz}_{p,q}(\mathcal{H})$. Moreover by using (3.6) one can see that the action of S^2 on the generators is given by

$$\begin{aligned} S^2(a^*(\eta)) &= S(S(a^*(\eta))) = -q^{-N/2}(S(a^*(\eta)))q^{n/2} \\ &= -q^{-N/2}(-q^{-N/2}a^*(\eta)q^{N/2}) = q^{-N}a^*(\eta)q^N, \\ S^2(a(\xi)) &= S(S(a(\xi))) = q^{-N}a(\xi)q^N, \\ S^2(N) &= S(S(N)) = S(N) = N. \end{aligned}$$

By a same argument we can easily verify that $q^{-N}a^*(\eta)q^N$ and $a^*(\eta)$ coincide on the basis $(\zeta_k^{\otimes n})_k$ and we deduce that $S^2 = id$, which implies that S is bijective. $\square$

The map S from Lemma 3.4 is called the antipode of $\mathrm{Heiz}_{p,q}(\mathcal{H})$. It is clear that the inverse S^{-1} of S is also an antiautomorphism, which is given by

$$\begin{aligned} S^{-1}(a^*(\eta)) &= -q^{-N/2}a^+(\eta)q^{N/2}, \\ S^{-1}(a(\xi)) &= -q^{-N/2}a(\xi)q^{N/2}, \\ S^{-1}(N) &= N. \end{aligned}$$

Lemma 3.5. *The following diagrams are commutative*

$$\begin{tikzcd} Heiz_{p,q}(\mathcal{H}) \arrow[r, "\Delta"] \arrow[d, "\sigma\circ\varepsilon"] & Heiz_{p,q}^{\otimes 2}(\mathcal{H}) \arrow[d, "1\otimes S"] \\ \mathcal{H}_{p,q} \arrow[r, "m"] & Heiz_{p,q}^{\otimes 2}(\mathcal{H}) \end{tikzcd} \tag{3.7}$$

$$\begin{tikzcd} Heiz_{p,q}(\mathcal{H}) \arrow[r, "\Delta"] \arrow[d, "\sigma\circ\varepsilon"] & Heiz_{p,q}^{\otimes 2}(\mathcal{H}) \arrow[d, "S\otimes 1"] \\ Heiz_{p,q}(\mathcal{H}) \arrow[r, "m"] & Heiz_{p,q}^{\otimes 2}(\mathcal{H}) \end{tikzcd} \tag{3.8}$$

where

$$m : Heiz_{p,q}^{\otimes 2}(\mathcal{H}) \longrightarrow Heiz_{p,q}(\mathcal{H})$$

is the multiplication map, namely, $m(u\otimes v) = uv$ *for all* $u, v \in Heiz_{p,q}(\mathcal{H})$, *and where* $\sigma : \mathbb{C} \longrightarrow Heiz_{p,q}(\mathcal{H})$ *is the embedding* $\sigma(a) = a1$ *for all* $a \in \mathbb{C}$.

Proof. By using (3.1), (3.2) and (3.3) we get

$$\begin{aligned} m\circ(S\otimes 1)\circ\Delta(N) &= m\circ(S\otimes 1)(N\otimes 1 + 1\otimes N) \\ &= m\circ(S(N)\otimes 1 + S(1)\otimes N) \\ &= m\circ(-N\otimes 1 + 1\otimes N) = 0 = \eta\circ\varepsilon(N), \end{aligned}$$

$$\begin{aligned} m\circ(S\otimes 1)\circ\Delta(a^*(\eta)) &= m\circ(S\otimes 1)(a^*(\eta)\otimes q^{N/2} + q^{N/2}\otimes a^*(\eta)) \\ &= m\circ(S(a^*(\eta))\otimes q^{N/2} + S(q^{N/2})\otimes a^*(\eta)) \\ &= m\circ(-q^{N/2}a^*(\eta)q^{-N/2}\otimes q^{N/2} + q^{N/2}\otimes a^*(\eta)) \\ &= -q^{N/2}a^*(\eta) + q^{N/2}a^*(\eta) \\ &= 0 = \eta\circ\varepsilon(a^*(\eta)), \end{aligned}$$

$$\begin{aligned} m\circ(S\otimes 1)\circ\Delta(a(\xi)) &= m\circ(S\otimes 1)(a(\xi)\otimes q^{-N/2} + q^{-N/2}\otimes a(\xi)) \\ &= m\circ(S(a(\xi))\otimes q^{-N/2} + S(q^{-N/2})\otimes a(\xi)) \\ &= m\circ(-q^{-N/2}Aq^{N/2}\otimes q^{-N/2} + q^{-N/2}\otimes a(\xi)) \\ &= -q^{-N/2}a(\xi) + q^{-N/2}a(\xi) \\ &= 0 = \eta\circ\varepsilon(a(\xi)). \end{aligned}$$

Similarly, we have

$$m\circ(1\otimes S)\circ\Delta(X) = \eta\circ\varepsilon(X), \qquad X = N, a^*(\eta), a(\xi),$$

and (3.7) is established. □

Definition 3.6. An algebra $\mathcal{A}$ together with algebraic homomorphisms $\Delta : \mathcal{A} \longrightarrow \mathcal{A}\otimes\mathcal{A}$, $\varepsilon : \mathcal{A} \longrightarrow \mathbb{C}$ and a linear map $S : \mathcal{A} \longrightarrow \mathcal{A}$ is called a Hopf algebra, if Δ is coassociative and if the diagrams (3.4), (3.5), (3.7) and (3.8) (with $\mathrm{Heiz}_{p,q}(\mathcal{H})$ is replaced by $\mathcal{A}$) commute i.e.,

$$\begin{aligned} (\Delta\otimes 1)\circ\Delta &= (1\otimes\Delta)\circ\Delta \quad \text{(coassociativity)} \\ (\varepsilon\otimes 1)\circ\Delta &= (1\otimes\varepsilon)\circ\Delta = id \quad \text{(counitary)} \\ m\circ(S\otimes 1)\circ\Delta &= m\circ(1\otimes S)\circ\Delta = \eta\circ\varepsilon. \end{aligned}$$

We call Δ, ε and S the coproduct, the counit and the antipode of the Hopf algebra, respectively.

So by collecting Lemmas 3.1-3.5 we obtain the following main result:

Theorem 3.7. *$\big(Heiz_{p,q}(\mathcal{H}), \Delta, \varepsilon, S\big)$ defined by (2.6)–(2.8) and (3.1)-(3.6) is a Hopf algebra.*

Our next aim is to discuss the pointwise structure of the Hopf algebra associated with creation and annihilation operators at points of the space $\mathbb{R}$. At least informally, for each $t \in \mathbb{R}$ we may consider a delta function at t, denoted by δ_t and let b_t and b_t^* the standard pointwise annihilation and creation operators on $\mathcal{F}_{p,q}(\mathcal{H})$ defined by

$$\begin{aligned} b_t^* f^{(n)} &= \delta_t \otimes_{p,q} f^{(n)}, \\ (b_t f^{(n)})(t_1, \cdots, t_{n-1}) &= \sum_{i=1}^{n} q^{i-1} p^{n-i} f^{(n)}(t_1, \cdots, t_{i-1}, t, t_i, \cdots t_{n-1}). \end{aligned}$$

Hence one can see that the (p, q)-creation and (p, q)-annihilation operators are given as the smeared operators in terms of b_t and b_t^*, i.e.,

$$a(\xi) = \int_{\mathbb{R}} \xi(t) b_t dt, \quad a^*(\xi) = \int_{\mathbb{R}} \xi(t) b_t^* dt. \tag{3.9}$$

Let $\widetilde{b}_t$ the operator defined by

$$(\widetilde{b}_t f^{(n)})(t_1, \cdots, t_{n-1}) = n f^{(n)}(t_1, \cdots, t_{i-1}, t, t_i, \cdots t_{n-1}),$$

so we can see that $n_t := b_t^* \widetilde{b}_t$ is the standard number operator. By using the relation of the $\mathrm{Heiz}_{p,q}(\mathcal{H})$ satisfied by the operators $a(\xi), a^*(\xi)$ and the number operator N on $\mathcal{F}_{p,q}(\mathcal{H})$, it's easily to verify that the pointwise creation and annihilation operators satisfy the following (p, q)-commutation relations

$$\begin{cases} b_t b_s^* - q b_s^* b_t = p^{b_s^* \widetilde{b}_t} \delta(s,t), \\ b_t b_s^* - p b_s^* b_t = q^{b_s^* \widetilde{b}_t} \delta(s,t), \\ n_s b_t^* - b_t^* n_s = b_t^* \delta(s,t), \\ n_s b_t - b_t n_s = -b_t \delta(s,t), \end{cases} \tag{3.10}$$

where $\delta(s, t)$ is understood as:

$$\int_{\mathbb{R}^2} f^{(2)}(s,t) \delta(s,t) dt ds = \int_{\mathbb{R}} f^{(2)}(t,t) dt.$$

Corollary 3.8. *$(Heiz_{p,q}(\mathcal{H}), \Delta, \varepsilon, S)$ defined by (3.10) is a Hopf algebra.*

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