A local equivariant index theorem for sub-signature operators

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Abstract

In this paper, we prove a local equivariant index theorem for sub-signature operators which generalizes Weiping Zhang's index theorem for sub-signature operators.

Keywords: Sub-signature operator; equivariant index.

1. Introduction

The Atiyah-Singer index Theorem ([1] [2]) gives a cohomological interpretation of the Fredholm index of an elliptic operator. The Atiyah-Bott-Segal-Singer index formula, which called the equivariant index theorem, is a generalization with group action of the Atiyah-Singer index theorem, was first direct proved by Patodi, Gilkey, Atiyah-Bott-Patodi partly by using invariant theory [3][4]. This theorem generalizes the Atiyah-Singer index theorem and the Atiyah-Bott fixed point formula for elliptic complexes, which is a generalization of the Lefschetz fixed point formula. In [5], Berline and Vergne gave a heat kernel proof of the Atiyah-Bott-Segal-Singer index formula. Moreover, Lafferty, Yu and Zhang [6] presented a simple and direct geometric proof of the equivariant index theorem for an orientation-preserving isometry on an even dimensional spin manifold by Clifford asymptotics of heat kernel. Furthermore, Ponge and H. Wang gave a different proof of the equivariant index formula by the Greiner's approach to the heat kernel asymptotics [7]. In [8], in order to prove family rigidity theorems, Liu and Ma proved the equivariant family index formula. In [9], Y. Wang gave another proof of the local equivarint index theorem for a family of Dirac operators by the Greiner's approach to the heat kernel asymptotics. In [10], using the Greiner's approach to the heat kernel asymptotics, Y. Wang proved the equivariant Gauss-Bonnet-Chern formula and gave the variation formulas for the equivariant Ray-Singer metric, which are originally due to J. M. Bismut and W. Zhang [11].

In parallel, Freed [12] considered the case of an orientation reversing involution acting on an odd dimensional spin manifold and gave the associated Lefschetz formulas by the K-theretical way. In [13], Wang constructed an even spectral triple by the Dirac operator and the orientation-reversing involution and computed the Connes-Chern character for this spectral triple. In [14], Liu and Wang proved an equivariant odd index theorem for Dirac operators with involution parity and the Atiyah-Hirzebruch vanishing theorems for odd dimensional spin manifolds. In [15] and [16], Zhang introduced the sub-signature operators and proved a local index formula for these operators. By computing the adiabatic limit of eta-invariants associated to the so-called sub-signature operators, a new proof of the Riemann-Roch-Grothendieck type formula of Bismut-Lott was given in [17] and [18]. The motivation of the present article is to prove a local equivariant index formula for sub-signature operators.

This paper is organized as follows: In Section 2, we recall some background on sub-signature operators. In Section 3.1, we prove a local equivariant index formula for sub-signature operators in even dimension. In

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Section 3.2, we prove a local equivariant odd dimensional index formula for sub-signature operators with an orientation-reversing involution.

2. The sub-signature operators

In this section, we give the standard setup (also see Section 1 in [15]). Let M be an oriented closed manifold of dimension n. Let E be an oriented sub-bundle of the tangent vector bundle TM. Let g^{TM} be a metric on TM. Let g^E be the induced metric on E. Let $E^\perp$ be the sub-bundle of TM orthogonal to E with respect to g^{TM}. Let $g^{E^\perp}$ be the metric on $E^\perp$ induced from g^{TM}. Then (TM, g^{TM}) has the following orthogonal splittings

$$TM = E \oplus E^\perp, \tag{2.1}$$
$$g^{TM} = g^E \oplus g^{E^\perp}. \tag{2.2}$$

Clearly, $E^\perp$ carries a canonically induced orientation. We identify the quotient bundle TM/E with $E^\perp$.

Let $\Omega(M) = \bigoplus_0^n \Omega^i(M) = \bigoplus_0^n \Gamma(\wedge^i(T^*M))$ be the set of smooth sections of $\wedge(T^*M)$. Let $*$ be the Hodge star operator of g^{TM}. Then $\Omega(M)$ inherits the following inner product

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge \overline{*\beta}, \quad \alpha, \beta \in \Omega(M). \tag{2.3}$$

We use g^{TM} to identify TM and T^*M. For any $e \in \Gamma(TM)$, let $e\wedge$ and i_e be the standard notation for exterior and interior multiplications on $\Omega(M)$. Let $c(e) = e\wedge - i_e$, $\hat{c}(e) = e\wedge + i_e$ be the Clifford actions on $\Omega(M)$ verifying that

$$c(e)c(e') + c(e')c(e) = -2\langle e, e' \rangle_{g^{TM}}, \tag{2.4}$$
$$\hat{c}(e)\hat{c}(e') + \hat{c}(e')\hat{c}(e) = 2\langle e, e' \rangle_{g^{TM}}, \tag{2.5}$$
$$c(e)\hat{c}(e') + \hat{c}(e')c(e) = 0. \tag{2.6}$$

Denote $k = \dim E$. Let $\{f_1, \cdots, f_k\}$ be an oriented (local) orthonormal basis of E. Set

$$\hat{c}(E, g^E) = \hat{c}(f_1) \cdots \hat{c}(f_k), \tag{2.7}$$

where $\hat{c}(E, g^E)$ does not depend on the choice of the orthonormal basis. Let

$$\epsilon = \mathrm{Id}_{\wedge^{even}(T^*M)} - \mathrm{Id}_{\wedge^{odd}(T^*M)}$$

be the Z_2-grading operator of

$$\wedge(T^*M) = \wedge^{even}(T^*M) \oplus \wedge^{odd}(T^*M).$$

Set

$$\tau(M, g^E) = \epsilon\hat{c}(E, g^E). \tag{2.8}$$

It is easy to check

$$\tau(M, g^E)^2 = (-1)^{\frac{k(k+1)}{2}}. \tag{2.9}$$

Let

$$\wedge_\pm(T^*M, g^E) = \left\{ \omega \in \wedge^*(T^*M), \tau(M, g^E)\omega = \pm\omega \right\}$$

the (even/odd) eigen-bundles of $\tau(M, g^E)$ and by $\Omega_\pm(M, g^E)$ the corresponding set of smooth sections. Let $\delta = \mathrm{d}^*$ be the formal adjoint operator of the exterior differential operator d on $\Omega(M)$ with respect to the inner product (2.3). Set

$$D_E = \frac{1}{2}\big(\hat{c}(E, g^E)(\mathrm{d} + \delta) + (-1)^k(\mathrm{d} + \delta)\hat{c}(E, g^E)\big). \tag{2.10}$$

Then we can check

$$D_E\tau(M,g^E) = -\tau(M,g^E)D_E, \tag{2.11}$$

$$D_E^* = (-1)^{\frac{k(k+1)}{2}}D_E, \tag{2.12}$$

where D_E^* is the formal adjoint operator of D_E with respect to the inner product (2.3). Set

$$\tilde{D}_E = (\sqrt{-1})^{\frac{k(k+1)}{2}}D_E.$$

From (2.11), $\tilde{D}_E$ is a formal self-adjoint first order elliptic differential operator on $\Omega(M)$ interchanging $\Omega_\pm(M,g^E)$.

Definition 2.1. *The sub-signature operator $\tilde{D}_{E,+}$ with respect to (E,g^{TM}) is the restriction of $\tilde{D}_E$ on $\Omega_+(M,g^E)$.*

If we denote the restriction of $\tilde{D}_E$ on $\Omega_-(M,g^E)$ by $\tilde{D}_{E,-}$, then

$$\tilde{D}_{E,\pm}^* = \tilde{D}_{E,\mp}.$$

Recall that E is the subbundle of TM and that we have the orthogonal decomposition (2.1) of TM and the metric g^{TM}. Let P^E (resp. $P^{E^\perp}$) be the orthogonal projection from TM to E(resp. $E^\perp$). Let ∇^{TM} be the Levi-Civita connection of g^{TM}. We will use the same notation for its lift to$\Omega(M)$. Set

$$\nabla^E = P^E\nabla^{TM}P^E, \tag{2.13}$$

$$\nabla^{E^\perp} = P^{E^\perp}\nabla^{TM}P^{E^\perp}. \tag{2.14}$$

Then ∇^E(resp.$\nabla^{E^\perp}$) is a Euclidean connection on E(resp.$E^\perp$), and we will use the same notation for its lifting on $\Omega(E^*)$(resp. $\Omega(E^{\perp,*})$). Let S be the tensor defined by

$$\nabla^{TM} = \nabla^E + \nabla^{E^\perp} + S.$$

Then S takes values in skew-adjoint endomorphisms of TM, and interchanges E and $E^\perp$. Let $\{e_1,\cdots,e_n\}$ be an oriented(local) orthonormal base of TM. To specify the role of E, set $\{f_1,\cdots,f_k\}$ be an oriented (local) orthonormal basis of E. We will use the greek subscripts for the basis of E. Then by Proposition 1.4 in [15], we have

Proposition 2.2. *The following identity holds,*

$$\tilde{D}_E = (\sqrt{-1})^{\frac{k(k+1)}{2}}\big(\hat{c}(E,g^E)(\mathrm{d}+\delta) + \frac{1}{2}\sum_i c(e_i)(\nabla_{e_i}^{TM}\hat{c}(E,g^E))\big). \tag{2.15}$$

Similar to Lemma 1.1 in [15], we have

Lemma 2.3. *For any $X\in\Gamma(TM)$, the following identity holds,*

$$\nabla_X^{TM}\hat{c}(E,g^E) = -\hat{c}(E,g^E)\sum_\alpha \hat{c}(S(X)f_\alpha)\hat{c}(f_\alpha). \tag{2.16}$$

Let Δ^{TM}, Δ^E be the Bochner Laplacians

$$\Delta^{TM} = \sum_i^n(\nabla_{e_i}^{TM,2} - \nabla_{\nabla_{e_i}^{TM}e_i}^{TM}), \tag{2.17}$$

$$\Delta^E = \sum_i^k(\nabla_{e_i}^{E,2} - \nabla_{\nabla_{e_i}^{E}e_i}^{E}). \tag{2.18}$$

Let K be the scalar curvature of (M,g^{TM}). Let R^{TM} (*resp.*, R^E, $R^{E^\perp}$) be the curvature of ∇^{TM} (*resp.*, ∇^E, $\nabla^{E^\perp}$). Let $\{h_1,\cdots,h_{n-k}\}$ be an oriented(local) orthonormal base of $E^\perp$. Now we can state the following Lichnerowicz type formula for $\tilde{D}_E^2$. From Theorem 1.1 in [15], we have

Theorem 2.4. *[15] The following identity holds,*

$$\tilde{D}_E^2 = -\Delta^{TM} + \frac{K}{4} + \frac{1}{8}\sum_{1\leq i,j\leq n}\sum_{1\leq\alpha,\beta\leq k}\langle R^E(e_i,e_j)f_\beta, f_\alpha\rangle c(e_i)c(e_j)\hat{c}(f_\alpha)\hat{c}(f_\beta)$$
$$+\frac{1}{8}\sum_{1\leq i,j\leq n}\sum_{1\leq s,t\leq n-k}\langle R^{E^\perp}(e_i,e_j)h_t, h_s\rangle c(e_i)c(e_j)\hat{c}(h_s)\hat{c}(h_t) + \frac{1}{2}\sum_\alpha \hat{c}\big((\Delta^{TM}-\Delta^E)f_\alpha\big)\hat{c}(f_\alpha)$$
$$+\sum_{i,\alpha}\Big(\hat{c}(S(e_i)f_\alpha)\hat{c}(f_\alpha)\nabla_{e_i}^{TM} - \hat{c}(S(e_i)\nabla_{e_i}^E f_\alpha)\hat{c}(f_\alpha) + \frac{1}{2}\hat{c}\big(\nabla^E_{(\nabla_{e_i}^{TM}-\nabla_{e_i}^E)e_i}f_\alpha\big)\hat{c}(f_\alpha) + \frac{3}{4}\parallel S(e_i)f_\alpha)\parallel^2\Big)$$
$$+\frac{1}{4}\sum_{i,\alpha\neq\beta}\hat{c}(S(e_i)f_\alpha)\hat{c}(S(e_i)f_\beta)\hat{c}(f_\alpha)\hat{c}(f_\beta). \tag{2.19}$$

3. A local equivariant index Theorem for sub-signature operators

3.1. A local even dimensional equivariant index Theorem for sub-signature operators

Let M be a closed oriented Riemannian manifold of even dimension n and ϕ an orientation-preserving isometry on M preserving the orientation. Then the smooth map ϕ induces a map $\tilde{\phi} = \phi^{-1,*} : \wedge T_x^*M \to \wedge T_{\phi(x)}^*M$ on the exterior algebra bundle $\wedge T_x^*M$. Let $\tilde{D}_E$ be the sub-signature operator. We assume that $d\phi$ preserves E and $E^\perp$ and their orientations, then $\tilde{\phi}\hat{c}(E,g^E) = \hat{c}(E,g^E)\tilde{\phi}$. Then $\tilde{\phi}\tilde{D}_E = \tilde{D}_E\tilde{\phi}$. We will compute the equivariant index

$$\mathrm{Ind}_\phi(\tilde{D}_E^+) = \mathrm{Tr}(\tilde{\phi}|_{\ker\tilde{D}_E^+}) - \mathrm{Tr}(\tilde{\phi}|_{\ker\tilde{D}_E^-}). \tag{3.1}$$

We recall the Greiner's approach to the heat kernel asymptotics as in [7] and [19], [21]. Define the operator given by

$$(Q_0u)(x,s) = \int_0^\infty e^{-s\tilde{D}_E^2}[u(x,t-s)]dt, \quad u\in\Gamma_c(M\times\mathbb{R},\wedge T^*M), \tag{3.2}$$

maps continuously u to $D'(M\times\mathbb{R},\wedge T^*M))$ which is the dual space of $\Gamma_c(M\times\mathbb{R},\wedge T^*M))$. We have

$$(\tilde{D}_E^2+\frac{\partial}{\partial t})Q_0u = Q_0(\tilde{D}_E^2+\frac{\partial}{\partial t})u = u, \quad u\in\Gamma_c(M\times\mathbb{R},\wedge T^*M)). \tag{3.3}$$

Let $(\tilde{D}_E^2+\frac{\partial}{\partial t})^{-1}$ is the Volterra inverse of $\tilde{D}_E^2+\frac{\partial}{\partial t}$ as in [19]. Then

$$(\tilde{D}_E^2+\frac{\partial}{\partial t})Q_0 = I - R_1; \quad Q_0(\tilde{D}_E^2+\frac{\partial}{\partial t}) = 1 - R_2, \tag{3.4}$$

where R_1, R_2 are smoothing operators. Let

$$(Q_0u)(x,t) = \int_{M\times\mathbb{R}} K_{Q_0}(x,y,t-s)u(y,s)dyds, \tag{3.5}$$

and $k_t(x,y)$ is the heat kernel of $e^{-t\tilde{D}_E^2}$. We get

$$K_{Q_0}(x,y,t) = k_t(x,y) \text{ when } t>0, \quad \text{when } t<0,\ K_{Q_0}(x,y,t)=0. \tag{3.6}$$

Definition 3.1. *The operator Q_0 is called the Volterra ΨDO if*

(i) Q_0 has the Volterra property, i.e., it has a distribution kernel of the form $K_{Q_0}(x,y,t-s)$ where $K_{Q_0}(x,y,t)$ vanishes on the region $t<0$.

(ii) The parabolic homogeneity of the heat operator $Q_0+\frac{\partial}{\partial t}$, i.e. the homogeneity with respect to the dilations of $\mathbb{R}^n\times\mathbb{R}^1$ given by

$$\lambda\cdot(\xi,\tau) = (\lambda\xi,\lambda^2\tau), \qquad (\xi,\tau)\in\mathbb{R}^n\times\mathbb{R}^1, \quad \lambda\neq 0. \tag{3.7}$$

In the following, for $g \in S(\mathbb{R}^{n+1})$ and $\lambda \neq 0$, we let g_λ be the tempered distribution defined by

$$\langle g_\lambda(\xi,\tau), u(\xi,\tau)\rangle = |\lambda|^{-(n+2)} \left\langle g_\lambda(\xi,\tau), u(\lambda^{-1}\xi, \lambda^{-2}\tau)\right\rangle, \quad u \in S(\mathbb{R}^{n+1}). \tag{3.8}$$

Definition 3.2. *A distribution $g \in S(\mathbb{R}^{n+1})$ is parabolic homogeneous of degree m, $m \in Z$, if for any $\lambda \neq 0$, we have $g_\lambda = \lambda^m g$.*

Let $\mathbb{C}_-$ denote the complex halfplane $\{\mathrm{Im}\tau < 0\}$ with closure $\overline{\mathbb{C}_-}$. Then:

Lemma 3.3. *[19] Let $q(\xi,\tau) \in C^\infty((\mathbb{R}^n \times \mathbb{R})/0)$ be a parabolic homogeneous symbol of degree m such that:*

(i) q extends to a continuous function on $(\mathbb{R}^n \times \overline{\mathbb{C}_-})\backslash 0$ in such way to be holomorphic in the last variable when the latter is restricted to $\mathbb{C}_-$.

Then there is a unique $g \in S(\mathbb{R}^{n+1})$ agreeing with q on $\mathbb{R}^{n+1}\backslash 0$ so that:

(ii) g is homogeneous of degree m;

(iii) The inverse Fourier transform $\breve{g}(x,t)$ vanishes for $t < 0$.

Let U be an open subset of $\mathbb{R}^n$. We define Volterra symbols and Volterra ΨDOs on $U \times \mathbb{R}^{n+1}\backslash 0$ as follows.

Definition 3.4. *$S_V^m(U \times \mathbb{R}^{n+1})$, $m \in \mathbb{Z}$, consists in smooth functions $q(x,\xi,\tau)$ on $U \times \mathbb{R}^n \times \mathbb{R}$ with an asymptotic expansion $q \sim \sum_{j\geq 0} q_{m-j}$, where:*

(i) $q_l \in C^\infty(U \times [(\mathbb{R}^n \times \mathbb{R})/0]$ is a homogeneous Volterra symbol of degree l, i.e. q_l is parabolic homogeneous of degree l and satisfies the property (i) in Lemma 2.3 with respect to the last $n+1$ variables;

(ii) The sign $\sim$ means that, for any integer N and any compact K, U, there is a constant $C_{NK\alpha\beta k} > 0$ such that for $x \in K$ and for $|\xi| + |\tau|^{\frac{1}{2}} > 1$ we have

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\tau^k (q - \sum_{j<N} q_{m-j})(x,\xi,\tau)| \leq C_{NK\alpha\beta k}(|\xi| + |\tau|^{\frac{1}{2}})^{m-N-|\beta|-2k}. \tag{3.9}$$

Definition 3.5. *$\Psi_V^m(U \times \mathbb{R})$, $m \in \mathbb{Z}$, consists in continuous operators Q_0 from $C_c^\infty(U_x \times \mathbb{R}_t)$ to $C^\infty(U_x \times \mathbb{R}_t)$ such that:*

(i) Q_0 has the Volterra property;

(ii) $Q_0 = q(x, D_x, D_t) + R$ for some symbol q in $S_V^m(U \times \mathbb{R})$ and some smoothing operator R.

In what follows, if Q_0 is a Volterra ΨDO, we let $K_{Q_0}(x,y,t-s)$ denote its distribution kernel, so that the distribution $K_{Q_0}(x,y,t)$ vanishes for $t < 0$.

Definition 3.6. *Let $q_m(x,\xi,\tau) \in C^\infty(U \times (\mathbb{R}^{n+1}/0))$ be a homogeneous Volterra symbol of order m and let $g_m \in C^\infty(U) \otimes \mathbb{S}'(\mathbb{R}^{n+1})$ denote its unique homogeneous extension given by Lemma 2.3. Then:*

(i) $\breve{q}_m(x,y,t)$ is the inverse Fourier transform of $g_m(x,\xi,\tau)$ in the last $n+1$ variables;

(ii) $q_m(x, D_x, D_t)$ is the operator with kernel $\breve{q}_m(x, y-x, t)$.

Proposition 3.7. *The following properties hold.*

1) Composition. Let $Q_j \in \Psi_V^{m_j}(U \times \mathbb{R})$, $j = 1,2$ have symbol q_j and suppose that Q_1 or Q_2 is properly supported. Then $Q_1 Q_2$ is a Volterra ΨDO of order $m_1 + m_2$ with symbol $q_1 \circ q_2 \sim \sum \frac{1}{\alpha!}\partial_\xi^\alpha q_1 D_x^\alpha q_2$. 2) Parametrices. An operator Q is the order m Volterra ΨDO with the paramatrix P then

$$QP = 1 - R_1, \quad PQ = 1 - R_2 \tag{2.10}$$

where R_1, R_2 are smoothing operators.

Proposition 3.8. *The differential operator $\tilde{D}_E^2 + \partial_t$ is invertible and its inverse $(\tilde{D}_E^2 + \partial_t)^{-1}$ is a Volterra ΨDO of order -2.*

We denote by M^ϕ the fixed-point set of ϕ, and for $a=0,\cdots,n$, we let $M^\phi=\bigcup_{0\leq a\leq n}M^\phi_a$, where M^ϕ_a is an a-dimensional submanifold. Given a fixed-point x_0 in a component M^ϕ_a, consider some local coordinates $x=(x^1,\cdots,x^a)$ around x_0. Setting $b=n-a$, we may further assume that over the range of the domain of the local coordinates there is an orthonormal frame $e_1(x),\cdots,e_b(x)$ of N^ϕ_z. This defines fiber coordinates $v=(v_1,\cdots,v_b)$. Composing with the map $(x,v)\in N^\phi(\varepsilon_0)\to \exp_x(v)$ we then get local coordinates $x^1,\cdots,x^a,v^1,\cdots,v^b$ for M near the fixed point x_0. We shall refer to this type of coordinates as *tubular coordinates.* Then $N^\phi(\varepsilon_0)$ is homeomorphic with a tubular neighborhood of M^ϕ. Set $i_{M^\phi}:M^\phi\hookrightarrow M$ be an inclusion map. Since $\mathrm{d}\phi$ preserves E and $E^\perp$, considering the oriented (local) orthonormal basis $\{f_1,\cdots,f_k,h_1,\cdots,h_{n-k}\}$, set

$$\mathrm{d}\phi_{x_0}=\begin{pmatrix}\exp(L_1) & 0\\ 0 & \exp(L_2)\end{pmatrix},\tag{3.10}$$

where $L_1\in\mathfrak{so}(k)$ and $L_2\in\mathfrak{so}(n-k)$

Let

$$\widehat{A}(R^{M^\phi})=\det{}^{\frac{1}{2}}\left(\frac{R^{M^\phi}/4\pi}{\sinh(R^{M^\phi}/4\pi)}\right);\quad \nu_\phi(R^{N^\phi}):=\det{}^{-\frac{1}{2}}(1-\phi^N e^{-\frac{R^{N^\phi}}{2\pi}}).\tag{3.11}$$

The aim of this section is to prove the following result.

Theorem 3.9. *(Local Equivariant Sub-Signature Index Theorem. Even Dimension)*
Let $x_0\in M^\phi$, then

$$\begin{aligned}\lim_{t\to 0}\mathrm{Str}\left[\tilde{\phi}(x_0)K_t(x_0,\phi(x_0))\right] &= (\frac{1}{\sqrt{-1}})^{\frac{k}{2}}2^{\frac{n}{2}}\Big\{\widehat{A}(R^{M^\phi})\nu_\phi(R^{N^\phi})i^*_{M^\phi}\Big[\det{}^{\frac{1}{2}}\Big(\cosh(\frac{R^E}{4\pi}-\frac{L_1}{2})\Big)\\ &\quad\times\det{}^{\frac{1}{2}}\Big(\frac{\sinh(\frac{R^{E^\perp}}{4\pi}-\frac{L_2}{2})}{\frac{R^{E^\perp}}{4\pi}-\frac{L_2}{2}}\Big)\mathrm{Pf}\Big(\frac{R^{E^\perp}}{4\pi}-\frac{L_2}{2}\Big)\Big]\Big\}^{(a,0)}(x_0),\end{aligned}\tag{3.12}$$

where $L_1\in so(k),L_2\in so(n-k)$ and $\mathrm{Pf}\Big(\frac{R^{E^\perp}}{4\pi}-\frac{L_2}{2}\Big)$ denotes the Pfaffian of $\Big(\frac{R^{E^\perp}}{4\pi}-\frac{L_2}{2}\Big)$.

Next we give a detailed proof of Theorem 3.9. Let $Q=(\tilde{D}^2_E+\partial_t)^{-1}$. For $x\in M^\phi$ and $t>0$ set

$$I_Q(x,t):=\widetilde{\phi}(x)^{-1}\int_{N^\phi_x(\varepsilon)}\phi(\exp_x v)K_Q(\exp_x v,\exp_x(\phi'(x)v),t)dv.\tag{3.13}$$

Here we use a trivialization over $\wedge(T^*M)$ about the tubular coordinates. Using the tubular coordinates, we have

$$I_Q(x,t)=\int_{|v|<\varepsilon}\widetilde{\phi}(x,0)^{-1}\widetilde{\phi}(x,v)K_Q(x,v;x,\phi'(x)v;t)dv.\tag{3.14}$$

Let

$$q^{\wedge(T^*M)}_{m-j}(x,v;\xi,\nu;\tau):=\widetilde{\phi}(x,0)^{-1}\widetilde{\phi}(x,v)q_{m-j}(x,v;\xi,\nu;\tau).\tag{3.15}$$

We mention the following result

Proposition 3.10. *[7] Let $Q\in\Psi^m_V(M\times\mathbb{R},\wedge(T^*M))$, $m\in\mathbb{Z}$. Uniformly on each component M^ϕ_a*

$$I_Q(x,t)\sim\sum_{j\geq 0}t^{-(\frac{a}{2}+[\frac{m}{2}]+1)}I^j_Q(x)\qquad\text{as}\quad t\to 0^+,\tag{3.16}$$

where $I^j_Q(x)$ is defined by

$$I^{(j)}_Q(x):=\sum_{|\alpha|\leq m-[\frac{m}{2}]+2j}\int\frac{v^\alpha}{\alpha!}\left(\partial^\alpha_v q^{\wedge(T^*M)}_{2[\frac{m}{2}]-2j+|\alpha|}\right)^\vee(x,0;0,(1-\phi'(x))v;1)dv.\tag{3.17}$$

Similar to Theorem 1.2 in [8] and Section 2 (d) in [24], we have

$$\begin{aligned}\mathrm{Str}_{\tau}[\tilde{\phi}\exp(-t\tilde{D}_E^2)] &= (\sqrt{-1})^{\frac{k}{2}}\int_M \mathrm{Str}_{\epsilon}\Big[\hat{c}(E,g^E)k_t(x,\phi(x))\Big]\mathrm{d}x\\ &= (\sqrt{-1})^{\frac{k}{2}}\int_M \mathrm{Str}_{\epsilon}[\hat{c}(E,g^E)K_{(\tilde{D}_E^2+\partial_t)^{-1}}(x,\phi(x),t)]\mathrm{d}x. \qquad (3.18)\end{aligned}$$

We will compute the local index in this trivialization. Let (V,q) be a finite dimensional real vector space equipped with a quadratic form. Let $C(V,q)$ be the associated Clifford algebra, i.e., the associative algebra generated by V with the relations $v\cdot w+w\cdot v=-2q(v,w)$ for $v,w\in V$. Let $\{e_1,\cdots,e_n\}$ be an orthomormal basis of (V,q), let $C(V,q)\hat{\otimes}C(V,-q)$ be the grading tensor product of $C(V,q)$ and $C(V,-q)$, and $\wedge^*V\hat{\otimes}\wedge^*V$ be the grading tensor product of $\wedge^*V$ and $\wedge^*V$. Define the symbol map:

$$\sigma: C(V,q)\hat{\otimes}C(V,-q)\to\wedge^*V\hat{\otimes}\wedge^*V; \qquad (3.19)$$

where $\sigma(c(e_{j_1})\cdots c(e_{j_l})\otimes 1)=e^{j_1}\wedge\cdots\wedge e^{j_1}\otimes 1$, $\sigma(1\otimes\hat{c}(e_{j_1})\cdots\hat{c}(e_{j_l}))=1\otimes\hat{e}^{j_1}\wedge\cdots\wedge\hat{e}^{j_1}$. Using the interior multiplication $\iota(e_j):\wedge^*V\to\wedge^{*-1}V$ and the exterior multiplication $\varepsilon(e_j):\wedge^*V\to\wedge^{*+1}V$, we define representations of $C(V,q)$ and $C(V,-q)$ on the exterior algebra:

$$c: C(V,q)\to\mathrm{End}\wedge V,\ e_j\mapsto c(e_j):\ \varepsilon(e_j)-\iota(e_j); \qquad (3.20)$$

$$\hat{c}: C(V,-q)\to\mathrm{End}\wedge V,\ e_j\mapsto \hat{c}(e_j):\ \varepsilon(e_j)+\iota(e_j). \qquad (3.21)$$

The tensor product of these representations yields an isomorphism of superalgebras

$$c\otimes\hat{c}: C(V,q)\hat{\otimes}C(V,-q)\to\mathrm{End}\wedge V \qquad (3.22)$$

which we will also denote by c. We obtain a supertrace (i.e., a linear functional vanishing on supercommutators) on $C(V,q)\hat{\otimes}C(V,-q)$ by setting $\mathrm{Str}(a)=\mathrm{Str}_{\mathrm{End}\wedge V}[c(a)]$ for $a\in C(V,q)\hat{\otimes}(V,-q)$, where $\mathrm{Str}_{\mathrm{End}\wedge V}$ is the canonical supertrace on End$V$.

Lemma 3.11. *For $1\le i_1<\cdots<i_p\le n$,$1\le j_1<\cdots<j_q\le n$, when $p=q=n$,*

$$\mathrm{Str}[c(e_{i_1})\cdots c(e_{i_n})\hat{c}(e_{i_1})\cdots\hat{c}(e_{i_n})]=(-1)^{\frac{n(n+1)}{2}}2^n \qquad (3.23)$$

and otherwise equals zero.

We will also denote the volume element in $\wedge V\hat{\otimes}\wedge V$ by $\omega=e^1\wedge\cdots\wedge e^n\wedge\hat{e}^1\wedge\cdots\wedge\hat{e}^n$. For $a\in\wedge V\hat{\otimes}\wedge V$, let $\mathrm{T}a$ be the coefficient of ω. The linear functional $\mathrm{T}:\wedge V\hat{\otimes}\wedge V\to\mathrm{R}$ is called the Berezin trace. Then for a $a\in C(V,q)\hat{\otimes}(V,.q)$, we have $\mathrm{Str}_s(a)=(-1)^{\frac{n(n+1)}{2}}2^n(\mathrm{T}\sigma)(a)$. We define the Getzler order as follows:

$$\mathrm{deg}\partial_j=\frac{1}{2}\mathrm{deg}\partial_t=-\mathrm{deg}x^j=1,\quad \mathrm{deg}c(e_j)=1,\quad \mathrm{deg}\hat{c}(e_j)=0. \qquad (3.24)$$

Let $Q\in\Psi_V^*(\mathbb{R}^n\times\mathbb{R},\wedge^*T^*M)$ have symbol

$$q(x,\xi,\tau)\sim\sum_{k\le m'}q_k(x,\xi,\tau), \qquad (3.25)$$

where $q_k(x,\xi,\tau)$ is an order k symbol. Then taking components in each subspace $\wedge^jT^*M\otimes\wedge^lT^*M$ of $\wedge T^*M\otimes\wedge T^*M$ and using Taylor expansions at $x=0$ give formal expansions

$$\sigma[q(x,\xi,\tau)]\sim\sum_{j,k}\sigma[q_k(x,\xi,\tau)]^{(j,l)}\sim\sum_{j,k,\alpha}\frac{x^\alpha}{\alpha!}\sigma[\partial_x^\alpha q_k(0,\xi,\tau)]^{(j,l)}. \qquad (3.26)$$

The symbol $\frac{x^\alpha}{\alpha!}\sigma[\partial_x^\alpha q_k(0,\xi,\tau)]^{(j,l)}$ is the Getzler homogeneous of $k+j-|\alpha|$. Therefore, we can expand $\sigma[q(x,\xi,\tau)]$ as

$$\sigma[q(x,\xi,\tau)]\sim\sum_{j\ge0}q_{(m-j)}(x,\xi,\tau),\qquad q_{(m)}\neq0, \qquad (3.27)$$

where $q_{(m-j)}$ is a Getzler homogeneous symbol of degree $m-j$.

Definition 3.12. *The integer m is called as the Getzler order of Q. The symbol $q_{(m)}$ is the principal Getzler homogeneous symbol of Q. The operator $Q_{(m)} = q_{(m)}(x, D_x, D_t)$ is called the model operator of Q.*

Let $e_1, \ldots, e_n$ be an oriented orthonormal basis of $T_{x_0}M$ such that $e_1, \cdots, e_a$ span $T_{x_0}M^\phi$ and $e_{a+1}, \cdots, e_n$ span $N^\phi_{x_0}$. This provides us with normal coordinates $(x_1, \cdots, x_n) \to \exp_{x_0}(x^1e_1 + \cdots + x^ne_n)$. Moreover using parallel translation enables us to construct a synchronous local oriented tangent frame $e_1(x), ..., e_n(x)$ such that $e_1(x), \cdots, e_a(x)$ form an oriented frame of TM^ϕ_a and $e_{a+1}(x), \cdots, e_n(x)$ form an (oriented) frame N^τ (when both frames are restricted to M^ϕ). This gives rise to trivializations of the tangent and exterior algebra bundles. Write

$$\phi'(0) = \begin{pmatrix} 1 & 0 \\ 0 & \phi^N \end{pmatrix} = \exp(A_{ij}), \tag{3.28}$$

where $A_{ij} \in \mathfrak{so}(n)$.

Let $\wedge(n) = \wedge^*\mathbb{R}^n$ be the exterior algebra of $\mathbb{R}^n$. We shall use the following gradings on $\wedge(n)\hat{\otimes} \wedge (n)$,

$$\wedge(n)\hat{\otimes} \wedge (n) = \bigoplus_{\substack{1 \le k_1, k_2 \le a \\ 1 \le \bar{l}_1, \bar{l}_2 \le b}} \wedge^{k_1,\bar{l}_1}(n)\hat{\otimes} \wedge^{k_2,\bar{l}_2}(n), \tag{3.29}$$

where $\wedge^{k,\bar{l}}(n)$ is the space of forms $dx^{i_1} \wedge \cdots \wedge dx^{i_{k+\bar{l}}}$ with $1 \le i_1 < \cdots < i_k \le a$ and $a+1 \le i_{k+1} < \cdots < i_{k+\bar{l}} \le n$. Given a form $\omega \in \wedge(n)\hat{\otimes}\wedge(n)$, denote by $\omega^{(k_1,\bar{l}_1),(k_2,\bar{l}_2)}$ its component in $\wedge(n)^{(k_1,\bar{l}_1)}\hat{\otimes}\wedge^{(k_2,\bar{l}_2)}(n)$. We denote by $|\omega|^{(a,0),(a,0)}$ the Berezin integral $|\omega^{(*,0),(*,0)}|^{(a,0),(a,0)}$ of its component $\omega^{(*,0),(*,0)}$ in $\wedge^{(*,0),(*,0)}(n)$.

Let $A \in Cl(V,q)\hat{\otimes}Cl(V,-q)$, then

$$\begin{aligned} \mathrm{Str}[\widetilde{\phi}A] &= (-1)^{\frac{n}{2}}2^n(-\frac{1}{4})^{\frac{b}{2}}det(1-\phi^N)|\sigma(A)|^{((a,0),(a,0))} \\ &+(-1)^{\frac{n}{2}}2^n\sum_{0\le l_1<b, 0\le l_2\le b}|\sigma(\widetilde{\phi})^{((0,l_1),(0,l_2))}\sigma(A)^{((a,b-l_1),(a,b-l_2))}|^{(n,n)}. \end{aligned} \tag{3.30}$$

In order to calculate $\mathrm{Str}[\widetilde{\phi}A]$, we need to consider the representation of $|\sigma(\widetilde{\phi})^{((0,b),(0,l_2))}\sigma(A)^{((a,0),(a,b-l_2))}|^{(n,n)}$. Let the matrix ϕ^N equal

$$\phi^N = \begin{pmatrix} A_{\frac{a}{2}+1} & & & \\ & \ddots & 0 & \\ & & \ddots & \\ & 0 & & \ddots & \\ & & & & A_{\frac{n}{2}} \end{pmatrix}, \ A_{\frac{a}{2}+1} = \begin{pmatrix} \cos\theta_{\frac{a}{2}+1} & \sin\theta_{\frac{a}{2}+1} \\ -\sin\theta_{\frac{a}{2}+1} & \cos\theta_{\frac{a}{2}+1} \end{pmatrix}, \ A_{\frac{n}{2}} = \begin{pmatrix} \cos\theta_{\frac{n}{2}} & \sin\theta_{\frac{n}{2}} \\ -\sin\theta_{\frac{n}{2}} & \cos\theta_{\frac{n}{2}} \end{pmatrix}. \tag{3.31}$$

From Lemma 3.2 in [20], then

Lemma 3.13. *We have*

$$\begin{aligned} \tilde{\phi} &= (\frac{1}{2})^{\frac{n-a}{2}}\prod_{j=\frac{a}{2}+1}^{n}\Big[(1+\cos\theta_j) - (1-\cos\theta_j)c(e_{2j-1})c(e_{2j})\hat{c}(e_{2j-1})\hat{c}(e_{2j}) \\ &+\sin\theta_j\big(c(e_{2j-1})c(e_{2j}) - \hat{c}(e_{2j-1})\hat{c}(e_{2j})\big)\Big]. \end{aligned} \tag{3.32}$$

Then we obtain

$$\begin{aligned}
\sigma(\tilde{\phi})^{((0,b),(0,l_2))} &= (\frac{1}{2})^{\frac{n-a}{2}}\sigma\Big\{\prod_{j=\frac{a}{2}+1}^{n}\Big[-(1-\cos\theta_j)c(e_{2j-1})c(e_{2j})\hat{c}(e_{2j-1})\hat{c}(e_{2j}) \\
&\quad +\sin\theta_j\big(c(e_{2j-1})c(e_{2j})\big)\Big]\Big\}^{((0,b),(0,l_2))} \\
&= (\frac{1}{2})^{\frac{n-a}{2}}e^{a+1}\wedge\cdots\wedge e^n\sigma\Big\{\prod_{j=\frac{a}{2}+1}^{n}\Big[-(1-\cos\theta_j)\hat{c}(e_{2j-1})\hat{c}(e_{2j})+\sin\theta_j\Big]\Big\}^{(0,l_2)} \\
&= (\frac{1}{2})^{\frac{n-a}{2}}e^{a+1}\wedge\cdots\wedge e^n\sigma\Big\{\prod_{j=\frac{a}{2}+1}^{n}2\sin\frac{\theta_j}{2}\Big[\cos\frac{\theta_j}{2}-\sin\frac{\theta_j}{2}\hat{c}(e_{2j-1})\hat{c}(e_{2j})\Big]\Big\}^{(0,l_2)} \\
&= (\frac{1}{2})^{\frac{n-a}{2}}e^{a+1}\wedge\cdots\wedge e^n\det{}^{\frac{1}{2}}(1-\phi^N)\sigma\Big[\exp\big(-\frac{1}{4}\sum_{1\leq i,j\leq n}A_{ij}\hat{c}(e_i)\hat{c}(e_j)\big)\Big]^{(0,l_2)} \\
&= (\frac{1}{2})^{\frac{n-a}{2}}e^{a+1}\wedge\cdots\wedge e^n\det{}^{\frac{1}{2}}(1-\phi^N)\sigma\Big[\exp\big(-\frac{1}{4}\sum_{1\leq i,j\leq k}(L_1)_{ij}\hat{c}(f_i)\hat{c}(f_j) \\
&\quad -\frac{1}{4}\sum_{1\leq i,j\leq n-k}(L_2)_{k+i,k+j}\hat{c}(h_i)\hat{c}(h_j)\big)\Big]^{(0,l_2)}. \qquad (3.33)
\end{aligned}$$

Next we calculate $|\sigma(A)|^{((a,0),(a,b-l_2))}$. In the following, we shall use the following "curvature forms": $R' := (R_{i,j})_{1\leq i,j\leq a}$, $R'' := (R_{a+i,a+j})_{1\leq i,j\leq b}$. Let

$$\dot{R} = \frac{1}{4}\sum_{1\leq\alpha,\beta\leq k}\langle R^E f_\alpha, f_\beta\rangle\hat{c}(f_\alpha)\hat{c}(f_\beta),$$
$$\ddot{R} = \frac{1}{4}\sum_{1\leq s,t\leq n-k}\langle R^{E^\perp} h_s, h_t\rangle\hat{c}(h_s)\hat{c}(h_t);$$

and

$$\tilde{\dot{R}} = \frac{1}{4}\sum_{1\leq\alpha,\beta\leq k}\langle(R^E-L_1)f_\alpha, f_\beta\rangle\hat{c}(f_\alpha)\hat{c}(f_\beta),$$
$$\tilde{\ddot{R}} = \frac{1}{4}\sum_{1\leq s,t\leq n-k}\langle(R^{E^\perp}-L_2)h_s, h_t\rangle\hat{c}(h_s)\hat{c}(h_t).$$

By (2.19), let $F=\tilde{D}_E^2$, we get

Proposition 3.14. *The model operator of F is*

$$\begin{aligned}
F_{(2)} &= -\sum_{r=1}^{n}\big(\partial_r+\frac{1}{8}\sum_{1\leq i,j,l\leq n}\langle R^{TM}(e_i,e_j)e_l,e_r\rangle y_l e^i\wedge e^j\big)^2 \\
&\quad +\frac{1}{8}\sum_{1\leq i,j\leq n}\sum_{1\leq\alpha,\beta\leq k}\langle R^E(e_i,e_j)f_\beta,f_\alpha\rangle e^i\wedge e^j\hat{c}(f_\alpha)\hat{c}(f_\beta) \\
&\quad +\frac{1}{8}\sum_{1\leq i,j\leq n}\sum_{1\leq s,t\leq n-k}\langle R^{E^\perp}(e_i,e_j)h_t,h_s\rangle e^i\wedge e^j\hat{c}(h_s)\hat{c}(h_t). \qquad (3.34)
\end{aligned}$$

From the representation of $F_{(2)}$, we get the model operator of $\frac{\partial}{\partial t}+\tilde{D}_E^2$ is $\frac{\partial}{\partial t}+F_{(2)}$. And we have

$$(\frac{\partial}{\partial t}+F_{(2)})K_{Q_{(-2)}}(x,y,t)=0. \qquad (3.35)$$

Similar to Lemma 2.9 in [7], we get

Lemma 3.15. *Let $Q\in\Psi^{(-2)}(\mathbb{R}^n\times\mathbb{R},\wedge(T^*M))$ be a parametrix for $(F_{(2)}+\partial_t)^{-1}$. Then*
(1) Q has Getzler order -2 and its model operator is $(F_{(2)}+\partial_t)^{-1}$.
(2) For all $t>0$,

$$\begin{aligned}&(\sqrt{-1})^{\frac{k}{2}}\hat{c}(E,g^E)I_{(F_{(2)}+\partial_t)^{-1}}(0,t)\\ =\ &(\sqrt{-1})^{\frac{k}{2}}\hat{c}(E,g^E)\frac{(4\pi t)^{-\frac{a}{2}}}{\det^{\frac12}(1-\phi^N)}\det{}^{\frac12}\Big(\frac{\frac{tR'}{2}}{\sinh(\frac{tR'}{2})}\Big)\det{}^{-\frac12}(1-\phi^Ne^{-tR''})\exp\big(t(\tilde{\dot{R}}+\tilde{\ddot{R}})\big).\end{aligned}\tag{3.36}$$

Similar to Lemma 3.6 in [9]. we have

Lemma 3.16. *$Q\in\Psi^*_V(\mathbb{R}^n\times\mathbb{R},\wedge(T^*M))$ has the Getzler order m and model operator $Q_{(m)}$. Then as $t\to0^+$*
(1) $\sigma[I_Q(0,t)]^{(j,l)}=O(t^{\frac{j-m-a-1}{2}})$, if $m-j$ is odd.
(2) $\sigma[I_Q(0,t)]^{(j,l)}=O(t^{\frac{j-m-a-2}{2}})I_{Q(m)}(0,1)^{(j,l)}+O(t^{\frac{j-m-a}{2}})$, if $m-j$ is even.
In particular, for $m=-2$ and $j=a$ and a is even we get

$$\sigma[I_Q(0,t)]^{((a,0),(a,b-l_2))}=I_{Q(-2)}(0,1)^{((a,0),(a,b-l_2))}+O(t^{\frac12}).\tag{3.37}$$

With all these preparations, we are going to prove the local even dimensional equivariant index theorem for sub-signature operators. Substituting (3.33), (3.36) into (3.30), we obtain

$$\begin{aligned}&\lim_{t\to0}\mathrm{Str}_\varepsilon\Big[\tilde{\phi}(x_0)(\sqrt{-1})^{\frac{k}{2}}\hat{c}(E,g^E)I_{(F+\partial_t)^{-1}}(x_0,t)\Big]\\ =\ &(-1)^{\frac{n}{2}}2^n(\frac12)^{\frac{n-a}{2}}(4\pi)^{-\frac{a}{2}}(\sqrt{-1})^{\frac{k}{2}}\big|\widehat{A}(R^{M^\phi})\nu_\phi(R^{N^\phi})\sigma\big[\hat{c}(f_1)\cdots\hat{c}(f_k)\exp(\tilde{\dot{R}}+\tilde{\ddot{R}})\big]\big|^{((a,0),n)}\\ =\ &(\frac{1}{\sqrt{-1}})^{\frac{k}{2}}2^{\frac{n}{2}}\Big\{\widehat{A}(R^{M^\phi})\nu_\phi(R^{N^\phi})i^*_{M^\phi}\Big[\det{}^{\frac12}\Big(\cosh(\frac{R^E}{4\pi}-\frac{L_1}{2})\Big)\\ &\times\det{}^{\frac12}\Big(\frac{\sinh(\frac{R^{E^\perp}}{4\pi}-\frac{L_2}{2})}{\frac{R^{E^\perp}}{4\pi}-\frac{L_2}{2}}\Big)\mathrm{Pf}\Big(\frac{R^{E^\perp}}{4\pi}-\frac{L_2}{2}\Big)\Big]\Big\}^{(a,0)}(x_0).\end{aligned}\tag{3.38}$$

Where we have used the algebraic result of Proposition 3.13 in [22] , and the Berezin integral in the right hand side of (3.38) is the application of the following lemma.

Lemma 3.17. *Let $L_1\in so(k),L_2\in so(n-k)$, we have*

$$\begin{aligned}&|\sigma\big[\hat{c}(f_1)\cdots\hat{c}(f_k)\exp(\tilde{\dot{R}}+\tilde{\ddot{R}})\big]|^{(n)}\\ =\ &(-1)^{\frac{n-k}{2}}\det{}^{\frac12}\Big(\cosh(\frac{R^E-L_1}{2})\Big)\det{}^{\frac12}\Big(\frac{\sinh(\frac{R^{E^\perp}-L_2}{2})}{(R^{E^\perp}-L_2)/2}\Big)\mathrm{Pf}\Big(\frac{R^{E^\perp}-L_2}{2}\Big).\end{aligned}\tag{3.39}$$

Proof. In order to compute this differential form, we make use of the Chern root algorithm (see [9]). Assume that $n=\dim M$ and $k=\dim E$ are both even integers. As in [5], let $L_1\in so(k),L_2\in so(n-k)$, we write

$$R^E-L_1=\begin{pmatrix}\begin{pmatrix}0&-\theta_1\\ \theta_1&0\end{pmatrix}&&0\\ &\ddots&\\ 0&&\begin{pmatrix}0&-\theta_{-\frac{k}{2}}\\ \theta_{-\frac{k}{2}}&0\end{pmatrix}\end{pmatrix},R^{E^\perp}-L_2=\begin{pmatrix}\begin{pmatrix}0&-\hat{\theta}_1\\ \hat{\theta}_1&0\end{pmatrix}&&0\\ &\ddots&\\ 0&&\begin{pmatrix}0&-\hat{\theta}_{\frac{n-k}{2}}\\ \hat{\theta}_{\frac{n-k}{2}}&0\end{pmatrix}\end{pmatrix}.\tag{3.40}$$

Then we obtain

$$\begin{aligned}\frac{1}{4}\sum_{1\le\alpha,\beta\le k}\langle(R^E-L_1)f_\alpha,f_\beta\rangle\hat{c}(f_\alpha)\hat{c}(f_\beta) &= \frac{1}{2}\sum_{1\le\alpha<\beta\le k}\langle(R^E-L_1)f_\alpha,f_\beta\rangle\hat{c}(f_\alpha)\hat{c}(f_\beta)\\ &= \frac{1}{2}\sum_{1\le j\le\frac{k}{2}}\theta_j\hat{c}(f_{2j-1})\hat{c}(f_{2j});\end{aligned}\tag{3.41}$$

$$\begin{aligned}\frac{1}{4}\sum_{1\le s,t\le n-k}\langle(R^{E^\perp}-L_2)h_s,h_t\rangle\hat{c}(h_s)\hat{c}(h_t) &= \frac{1}{2}\sum_{1\le s<t\le n-k}\langle(R^{E^\perp}-L_2)h_s,h_t\rangle\hat{c}(h_s)\hat{c}(h_t)\\ &= \frac{1}{2}\sum_{1\le l\le\frac{n-k}{2}}\hat{\theta}_l\hat{c}(h_{2l-1})\hat{c}(h_{2l}).\end{aligned}\tag{3.42}$$

Then the left hand side of (3.39) is

$$\begin{aligned}&\Big|\sigma\Big(\hat{c}(f_1)\cdots\hat{c}(f_k)\exp(\tilde{\dot{R}}+\tilde{\ddot{R}})\Big)\Big|^{(n)}\\ =\ &\Big|\sigma\Big(\hat{c}(f_1)\cdots\hat{c}(f_k)\prod_{1\le j\le\frac{k}{2}}\exp(\frac{1}{2}\theta_j\hat{c}(f_{2j-1})\hat{c}(f_{2j}))\prod_{1\le l\le\frac{n-k}{2}}\exp(\frac{1}{2}\hat{\theta}_l\hat{c}(h_{2l-1})\hat{c}(h_{2l}))\Big)\Big|^{(n)}\\ =\ &\Big|\sigma\Big(\hat{c}(f_1)\cdots\hat{c}(f_k)\prod_{1\le j\le\frac{k}{2}}\Big[\cos\frac{\theta_j}{2}-\sin\frac{\theta_j}{2}\hat{c}(f_{2j-1})\hat{c}(f_{2j})\Big]\prod_{1\le l\le\frac{n-k}{2}}\Big[\cos\frac{\hat{\theta}_l}{2}-\sin\frac{\hat{\theta}_l}{2}\hat{c}(h_{2l-1})\hat{c}(h_{2l})\Big]\Big)\Big|^{(n)}\\ =\ &(-1)^{\frac{n-k}{2}}\prod_{1\le j\le\frac{k}{2}}\cos\frac{\theta_j}{2}\prod_{1\le l\le\frac{n-k}{2}}\sin\frac{\hat{\theta}_l}{2}.\end{aligned}\tag{3.43}$$

Now we consider the right hand side of (3.39),

$$\big(R^E-L_1\big)^{2p}=(-1)^p\begin{pmatrix}\begin{pmatrix}\theta_1^{2p}&0\\0&\theta_1^{2p}\end{pmatrix}&&0\\&\ddots&\\0&&\begin{pmatrix}\theta_{\frac{k}{2}}^{2p}&0\\0&\theta_{\frac{k}{2}}^{2p}\end{pmatrix}\end{pmatrix}.\tag{3.44}$$

Then

$$\det{}^{\frac{1}{2}}\Big(\cosh(\frac{R^E-L_1}{2})\Big)=\prod_{j=1}^{\frac{k}{2}}\Big(\sum_{p=0}^{\infty}(\frac{\theta_j}{2})^{2p}\frac{(-1)^p}{(2p)!}\Big)=\prod_{j=1}^{\frac{k}{2}}\cosh\frac{\sqrt{-1}\theta_j}{2}=\prod_{j=1}^{\frac{k}{2}}\frac{e^{\frac{\sqrt{-1}\theta_j}{2}}+e^{-\frac{\sqrt{-1}\theta_j}{2}}}{2}=\prod_{j=1}^{\frac{k}{2}}\cos\frac{\theta_j}{2}.\tag{3.45}$$

Similarly, we have

$$\det{}^{\frac{1}{2}}\Big(\frac{\sinh(\frac{R^{E^\perp}-L_2}{2})}{(R^{E^\perp}-L_2)/2}\Big)=\prod_{j=1}^{\frac{n-k}{2}}\frac{\sin\frac{\hat{\theta}_j}{2}}{\frac{\hat{\theta}_j}{2}}.\tag{3.46}$$

On the other hand,

$$\mathrm{Pf}(\frac{R^{E^\perp}-L_2}{2})=T\Big(\exp(\sum_{s<t}\langle\frac{R^{E^\perp}-L_2}{2}h_s,h_t\rangle h^s\wedge h^t)\Big)=T\Big(\exp(\sum_{1\le j\le\frac{n-k}{2}}\frac{\hat{\theta}_j}{2}h^{2j-1}\wedge h^{2j})\Big)=\prod_{j=1}^{\frac{n-k}{2}}\frac{\hat{\theta}_j}{2}.\tag{3.47}$$

Combining these equations, the proof of lemma 3.17 is complete. □

To summarize, we have proved Theorem 3.9.

3.2. The local odd dimensional equivariant index Theorem for sub-signature operators

In this section, we give a proof of a local odd dimensional equivariant index theorem for sub-signature operators. Let M be an odd dimensional oriented closed Riemannian manifold. Using (2.19) in Section 2, we may define the sub-signature operators $\tilde{D}_E$. Let γ be an orientation reversing involution isometric acting on M. Set $d\gamma$ preserves E and $E^{\perp}$ and preserves the orientation of E , then $\tilde{\gamma}\hat{\tau}(E,g^E)=\hat{\tau}(E,g^E)\tilde{\gamma}$, where $\tilde{\gamma}$ is the lift on the exterior algebra bundle $\wedge T^*M$ of γ. There exists a self-adjoint lift $\widetilde{\gamma}:\ \Gamma(M;\wedge(T^*M))\to\Gamma(M;\wedge(T^*M))$ of γ satisfying

$$\widetilde{\gamma}^2=1;\quad \tilde{D}_E\widetilde{\gamma}=-\widetilde{\gamma}\tilde{D}_E. \tag{3.48}$$

Now the $+1$ and -1 eigenspaces of $\widetilde{\gamma}$ give a splitting

$$\Gamma(M;\wedge(T^*M))\cong\Gamma^+(M;\wedge(T^*M))\oplus\Gamma^-(M;\wedge(T^*M))) \tag{3.49}$$

then the sub-signature operator interchanges $\Gamma^+(M;\wedge(T^*M))$ and $\Gamma^-(M;\wedge(T^*M))$, and $\hat{c}(E,g^E)$ preserves $\Gamma^+(M;\wedge(T^*M))$ and $\Gamma^-(M;\wedge(T^*M))$.

Denotes by $\tilde{D}_E^+$ the restriction of $\tilde{D}_E$ to $\Gamma^+(M,\wedge(T^*M))$. We assume $\dim E=k$ is even, then $(\tilde{D}_E)\hat{c}(E,g^E)=\hat{c}(E,g^E)(\tilde{D}_E)$ and $\hat{c}(E,g^E)$ is a linear map from $\ker\tilde{D}_E^{\pm}$ to $\ker\tilde{D}_E^{\pm}$.

The purpose of this section is to compute

$$\mathrm{ind}_{\hat{c}(E,g^E)}[(\tilde{D}_E^+)]=\mathrm{Tr}(\hat{c}(E,g^E)|_{\ker\tilde{D}_E^+})-\mathrm{Tr}(\hat{c}(E,g^E)|_{\ker\tilde{D}_E^+}). \tag{3.50}$$

By the Mckean-Singer formular, we have

$$\begin{aligned}\mathrm{ind}_{\hat{c}(E,g^E)}(\tilde{D}_E^+)&=\int_M(\sqrt{-1})^{\frac{k}{2}}\mathrm{Tr}[\tilde{\gamma}\hat{c}(E,g^E)k_t(x,\gamma(x))]dx\\&=\int_M(\sqrt{-1})^{\frac{k}{2}}\mathrm{Tr}[\tilde{\gamma}\hat{c}(E,g^E)K_{(F+\partial_t)^{-1}}(x,\gamma(x),t)]dx.\end{aligned} \tag{3.51}$$

Let

$$R^E-L_1=\begin{pmatrix}\begin{pmatrix}0&-\theta_1\\\theta_1&0\end{pmatrix}&&0\\&\ddots&\\0&&\begin{pmatrix}0&-\theta_{-\frac{k}{2}}\\\theta_{-\frac{k}{2}}&0\end{pmatrix}\end{pmatrix},\ R^{E^{\perp}}-L_2=\begin{pmatrix}\begin{pmatrix}0&-\hat{\theta}_1\\\hat{\theta}_1&0\end{pmatrix}&&&0\\&\ddots&&\\0&&\begin{pmatrix}0&-\hat{\theta}_{\frac{n-k-1}{2}}\\\hat{\theta}_{\frac{n-k-1}{2}}&0\end{pmatrix}&\\&&&0\end{pmatrix}; \tag{3.52}$$

and

$$\mathrm{Pf}(\frac{R^{E^{\perp}}-L_2}{2})=\prod_{j=1}^{\frac{n-k-1}{2}}\frac{\hat{\theta}_j}{2}. \tag{3.53}$$

Similar to Theorem 3.9, we get the main Theorem in this section.

Theorem 3.18. *(Local odd dimensional equivariant index Theorem for sub-signature operators)*
Let $x_0\in M^{\gamma}$, then

$$\begin{aligned}&\lim_{t\to0}\mathrm{Tr}\left[\tilde{\gamma}(x_0)\hat{c}(E,g^E)I_{(F+\partial_t)^{-1}}(x_0,t)\right]\\=&-(\frac{1}{\sqrt{-1}})^{\frac{k}{2}-1}2^{\frac{n}{2}}\Big\{\widehat{A}(R^{M^{\gamma}})\nu_{\phi}(R^{N^{\gamma}})i^*_{M^{\gamma}}\Big[\det{}^{\frac12}\Big(\cosh(\frac{R^E}{4\pi}-\frac{L_1}{2})\Big)\\&\times\det{}^{\frac12}\Big(\frac{\sinh(\frac{R^{E^{\perp}}}{4\pi}-\frac{L_2}{2})}{\frac{R^{E^{\perp}}}{4\pi}-\frac{L_2}{2}}\Big)\mathrm{Pf}\Big(\frac{R^{E^{\perp}}}{4\pi}-\frac{L_2}{2}\Big)\Big]\Big\}^{(a,0)}(x_0).\end{aligned} \tag{3.54}$$

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