LIMIT CYCLES IN DISCONTINUOUS GENERALIZED LIÉNARD DIFFERENTIAL EQUATIONS

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ABSTRACT. The goal of this paper is to study the number of limit cycles that can bifurcate from the periodic orbits of a linear center perturbed by nonlinear functions inside the class of all generalized Liénard differential equations allowing discontinuities.

In particular our results show that for any $n \geq 1$ there are differential equations of the form $\ddot{x}+f(x,\dot{x})\dot{x}+x+\operatorname{sgn}(\dot{x})g(x)=0$, with f and g polynomials of degree n and 1 respectively, having $[n/2]+1$ limit cycles, where $[\cdot]$ denotes the integer part function.

1. INTRODUCTION

The study of Liénard differential equations has a long history and a lot of results were obtained, see [12] for example. In 1977 Lins, de Melo and Pugh studied the classical polynomial Liénard differential equations

$$\ddot{x} + f(x)\dot{x} + x = 0 \tag{1}$$

or equivalently a differential system

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -x \end{cases}$$

Here the dot denotes differentiation with respect to the time t, $f(x)$ is a polynomial of degree n, with $f(x) = F^{'}(x)$. They conjectured in [5] that the classical Liénard differential equation of degree $n \geqslant 1$ has at most $[n/2]$ limit cycles, where $[n/2]$ means the largest integer less than or equal to $n/2$. They also proved that the conjecture is true for $n = 2$. In [4] Chengzhi Li and Llibre proved the conjecture is is also true for $n = 3$. The conjecture for $n = 4$ is still open. Recently De Maesschalck and Dumortier proved in [8] that the classical Liénard equation of degree $n \geq 5$ can have $[n/2]+2$ limit cycles, where $[\cdot]$ denotes the integer part function. For $n \geq 6$ Dumortier, Panazzolo and Roussarie proved

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this conjecture is not true in [3].

In [9], the authors studied the number of limit cycles of the discontinuous classical Liénard differential equations $\ddot{x}+f(x)\dot{x}+x+\text{sgn}(\dot{x})g(x)=0$, with f and g polynomials of degree n and 1 respectively.

A large number of problems from mechanics and electrical engineering, theory of automatic control, economy, impact systems among others cannot be described with smooth dynamical systems (see for instance the book [2] and the references quoted therein). This is one of the reasons that the study of non-smooth dynamical systems has attracted many mathematicians. And of course in these problems the detection of limit cycles is of fundamental importance.

Thus we have been motivated by the Liénard equations and by importance of the non-smooth systems to study the limit cycles of the discontinuous generalized Liénard polynomial differential equations

$$\ddot{x}+f(x,\dot{x})\dot{x}+x+\text{sgn}(\dot{x})g(x)=0 \tag{2}$$

with f and g polynomials of degree n and 1, respectively. We study the number of limit cycles which can bifurcate from the periodic orbits of the linear center $\dot{x}=y, \dot{y}=-x$, perturbed inside the following class of discontinuous generalized Liénard polynomial differential systems

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x-\varepsilon\left(f(x,y)y+\text{sgn}(y)\left(k_1x+k_2\right)\right) \end{aligned} \tag{3}$$

where f is a polynomial of degree $n\in\mathbb{N}$ and $k_1,k_2\in\mathbb{R}$ In order to prove our main result we first study the piecewise linear generalized polynomial Liénard differential systems

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x-\varepsilon\left(f(x,y)y+\varphi_w(y)\left(k_1x+k_2\right)\right) \end{aligned} \tag{4}$$

where $\varphi_w:\mathbb{R}\to\mathbb{R}$ is the piecewise linear function

$$\varphi_w(y)=\begin{cases} -1, & \text{if } y<-w \\ \dfrac{y}{w}, & \text{if } -w<y<w \\ 1, & \text{if } y>w \end{cases} \tag{5}$$

Observe that taking $w\to 0$ in (4) we obtain discontinuous generalized Liénard polynomial differential systems. The classical results for studying the periodic orbits of differential systems require that the systems involved are of class at least $\mathcal{C}^2$. In 2004, Buica and Llibre [1] extended the averaging theory for studying periodic orbits to continuous differential systems using mainly the Brouwer degree theory.

Recently Llibre, Novaes and Teixeira [7], using the theory of regularization, developed the averaging theory of first order for studying periodic orbits to discontinuous piecewise differential systems with two systems. The displacement function that we construct here is the same as in [7], but adapted to the family of differential equations we consider. More precisely our main result are the following.

Theorem 1. *For every $n \geq 1$ and $|\varepsilon|$ sufficiently small the maximum number of limit cycles of piecewise generalized Liénard polynomial differential systems bifurcating from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$ is $[n/2] + 1$. In order to guarantee that the limit cycles dont vanishes then $w \to 0$, that is, to obtain the number of limit cycles of discontinuous generalized Liénard polynomial systems (3) we have the following result.*

Corollary 2. *For every $n \geq 1$ and $|\varepsilon|$ sufficiently small the maximum number of limit cycles of discontinuous generalized Liénard polynomial differential systems (3) bifurcating from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$ is $[n/2] + 1$.*

Comparing the mentioned result from [10], that smooth generalized Liénard polynomial differential systems have at least $[n/2]$ limit cycles with **corollary 2** we can say that the non-smooth generalized Liénard polynomial differential systems can have at least one more limit cycle than the smooth ones. The proof of **theorem 1** is based on the first-order averaging method. In **section 2** we will present this method in the form obtained in [1] where differentiability of the vector field is not needed. **Theorem 1** and **corollary 2** are proved in sections **3** and **4** respectively.

2. The first-order averaging theory

Consider the differential system

$$\dot{x}(t) = \varepsilon F_1(t, x) + \varepsilon^2 R(t, x, \varepsilon), \tag{6}$$

where $F_1 : \mathbb{R} \times D \to \mathbb{R}^n, R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ are continuous functions, T -periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. We define $F_{10} : D \to \mathbb{R}^n$ as

$$F_{10}(z) = \frac{1}{T} \int_0^T F_1(s, z) ds,$$

and we assume that the following hypotheses (i) and (ii) hold.
(i) F_1 and R are locally Lipschitz with respect to x.
(ii) $F_{10}(0) = 0$ and there exists a neighborhood V of 0 such that $F_{10}(z) \neq 0$ for all $z \in \bar{V} \backslash \{0\}$ and $d_B(F_{10}, V, 0) \neq 0$.

So Then for $|\varepsilon| > 0$ sufficiently small there exists a T -periodic solution $\psi(\cdot,\varepsilon)$ of system (6) such that $\psi(0,\varepsilon) \to 0$ as $\varepsilon \to 0$ The expression $d_B\left(F_{10}, V, 0\right) \neq 0$ means that the Brouwer degree of the function $F_{10} : V \to \mathbb{R}^n$ at the fixed point 0 is not zero.

3. Proof of Theorem 1

We shall need the rst-order averaging theory to prove Theorem 1. In order to apply the first-order averaging method we write system (4) in polar coordinates (r,θ) where $x = r\cos(\theta)$, $y = r\sin(\theta)$, $r > 0$. In this way system (4) is written in the standard form for applying the averaging theory. If we write $f(x,y) = \sum\limits_{i+j=0}^{n} a_{ij}x^i y^j$ aixi then system (4) becomes

$$\begin{cases} \dot{r} = -\varepsilon\left(\sum\limits_{i+j=0}^{n} a_{ij} r^{i+j+1}\cos^i\theta\sin^{j+2}\theta + \varphi_w(r\sin\theta)\left(k_1 r\cos\theta + k_2\right)\sin\theta\right) \\ \dot{\theta} = -1 - \dfrac{\varepsilon}{r}\left(\sum\limits_{i+j=0}^{n} a_{ij} r^{i+j+1}\cos^{i+1}\theta\sin^{j+1}\theta + \varphi_w(r\sin\theta)\left(k_1 r\cos\theta + k_2\right)\cos\theta\right) \end{cases} \tag{7}$$

Taking θ as the new independent variable system (7) becomes

$$\frac{dr}{d\theta} = \varepsilon\left(\left(\sum_{i+j=0}^{n} a_i r^{i+j+1}\cos^i\theta\sin^{j+2}\theta\right) + \varphi_w(r\sin\theta)\left(k_1 r\cos\theta\sin\theta + k_2\sin\theta\right)\right) + O\left(\varepsilon^2\right)$$

where

$$\varphi_w(r\sin\theta) = \begin{cases} -1, & \text{if } \sin\theta < -\dfrac{w}{r} \\ \dfrac{r\sin\theta}{w}, & \text{if } -\dfrac{w}{r} < \sin\theta < \dfrac{w}{r} \\ 1, & \text{if } \sin\theta > \dfrac{w}{r} \end{cases} \tag{8}$$

and

$$\begin{aligned} F_{10}(r) &= \frac{1}{2\pi}\int_0^{2\pi}\left(\left(\sum_{i+j=0}^{n} a_i r^{i+j+1}\cos^i\theta\sin^{j+2}\theta\right)\right. \\ &\qquad \left. + \varphi_w(r\sin\theta)\left(k_1 r\cos\theta\sin\theta + k_2\sin\theta\right)\right) d\theta \end{aligned}$$

We denote

$$F_{10}(r) = \frac{1}{2\pi}\left(F_{10a}(r) + F_{10b}(r)\right),$$

where

$$F_{10a}(r) = \int_0^{2\pi}\left(\sum_{i+j=0}^{n} a_{i,j} r^{i+j+1}\cos^i\theta\sin^{j+2}\theta\right) d\theta$$

and

$$F_{10b}(r)=\int_0^{2\pi}\left(\varphi_w(r\sin\theta)\left(k_1 r\cos\theta\sin\theta+k_2\sin\theta\right)\right)d\theta$$

In order to calculate the exact expression of F_{10a} we use the following formulas

$$\int_0^{2\pi}\cos^i\theta\sin^{j+2}\theta d\theta=\begin{cases}0 \quad \text{if } i \text{ is odd, or } j \text{ is odd}\\ \pi\alpha_{ij} \text{ if } i \text{ is even and } j \text{ is even,}\end{cases}$$

$$\int_0^{2\pi}\cos^i\theta\sin\theta d\theta \quad =0, \quad \text{for} \quad i=0,1,\ldots$$

Hence

$$F_{10a}(r)=\sum_{\substack{i+j=0\\ i \text{ even}, \ j \text{ even}}}^{n}\pi\alpha_{ij}a_{ij}r^{i+j+1} \tag{9}$$

In order to calculate the expression of F_{10b} we define for each $r_1>0$ the function

$$\begin{aligned}I_1\left(r_1,w\right)&=\textstyle\int_0^{2\pi}\left(\varphi_w\left(r_1\sin\theta\right)\left(k_1 r\cos\theta\sin\theta+k_2\sin\theta\right)\right)d\theta\\ &=\begin{cases}\pi k_2\frac{r_1}{w} & 0<r_1\le w\\ 2k_2\left(\dfrac{r_1}{w}\operatorname{arccsc}\left(\dfrac{r_1}{w}\right)+\dfrac{\sqrt{r_1^2-w^2}}{r_1}\right) & r_1\ge w\end{cases}\end{aligned} \tag{10}$$

Thas the averaged function F_{10} is given by

$$F_{10}\left(r_1\right)=\left(\sum_{\substack{i+j=0\\ i \text{ even}, \ j \text{ even}}}^{n}\pi\alpha_{ij}a_{ij}r_1^{i+j+1}\right)+I_1\left(r_1,w\right),$$

We have to find the zeroes of equation $F_{10}\left(r_1\right)=0$. We shall divide our study in two cases. At really, we are interested just in the zeroes for $r_1>w$, but we shall also consider the case $0<r_1\le w$ for completion. If we write

$$\begin{aligned}
F_{10}^{I}(r_1) &= \frac{1}{2\pi}\left(\left(\sum_{\substack{i+j=0\\ i \text{ even}, \ j \text{ even}}}^{n} \pi\alpha_{ij}a_{ij}r_1^{i+j+1}\right)+\pi k_2\frac{r_1}{w}\right)\\
&= \frac{r_1}{2}\left(\left(\sum_{\substack{i+j=0\\ i \text{ even}, \ j \text{ even}}}^{n} \alpha_{ij}a_{ij}r_1^{i+j}\right)+\frac{k_2}{w}\right)\\
&= \frac{r_1}{2}\left(\left(\sum_{\substack{i+j=2\\ i \text{ even}, \ j \text{ even}}}^{n} \alpha_{ij}a_{ij}r_1^{i+j}\right)+\left(\alpha_{00}a_{00}+\frac{k_2}{w}\right)\right)
\end{aligned}$$

and

$$F_{10}^{II}(r_1) = \frac{1}{2\pi}\left(\sum_{\substack{i+j=0\\ i \text{ even}, \ j \text{ even}}}^{n} \pi\alpha_{ij}a_{ij}r_1^{i+j+1}+2k_2\left(\frac{r_1}{w}\operatorname{arccsc}\left(\frac{r_1}{w}\right)+\frac{\sqrt{r_1^2-w^2}}{r_1}\right)\right)$$

then

$$F_{10}(r_1) = \begin{cases} F_{10}^{I}(r_1) & , \quad r_1 < w \\ F_{10}^{II}(r_1) & , \quad r_1 \geq w \end{cases}$$

From now we take w small so that there are no zeros of F_{10} in the interval $(0,w)$ as F_{10}^{I} is a polynomial and $r_1=0$ is a root, we can assure that there is such interval.
Now we study the existence of zeros for $r_1 > w$. Denote

$$I(r_1,w) = 2\left(\frac{r_1}{w}\operatorname{arccsc}\left(\frac{r_1}{w}\right)+\frac{\sqrt{r_1^2-w^2}}{r_1}\right),$$

We have:
(i) For each w fixed $\dfrac{\partial^2 I(r_1,w)}{\partial r_1^2} = -\dfrac{4w^2}{r^3\sqrt{r^2-w^2}} < 0$ so the graph of $I(.,w)$ is concave.
(ii) For each w fixed $\dfrac{\partial I(r_1,w)}{\partial r_1} = \dfrac{2}{w}\operatorname{arccsc}\left(\dfrac{r_1}{w}\right) - \dfrac{2\sqrt{r_1^2-w^2}}{r_1^2}$ and,

(iia) $\displaystyle\lim_{r_1\to w}\frac{\partial I}{\partial r_1}(r_1,w) = \frac{\pi}{w}$, and, (iib) $\lim_{r_1\to\infty} \dfrac{\partial I}{\partial r_1}(r_1, w) = 0.$

By (i) we have that $\dfrac{\partial I}{\partial r_1}$ is decreasing. Then by (i), (iia) and (iib) we obtain that $\dfrac{\partial I}{\partial r_1}(r_1, w) > 0$, so the graph of $I(., w)$ is strictly increasing. Moreover as $\dfrac{\partial I}{\partial r_1}(r_1, w) < \frac{\pi}{w}$ it follows that the graph of $I(., w)$ is below of the straight line $\dfrac{\pi}{w}r_1$.

(iii) $\lim_{r_1\to\infty} I(r_1, w) = 4, I(., w) : (0, \infty) \to (0, 4)$ and I is a C^1-diffeomorphism.

Thus the averaging function F_{10} is C^1 Now we need solve

$$\sum_{\substack{i+j=0\\ i \text{ even},\ \ j \text{ even}}}^{n} \pi\alpha_{ij}a_{ij}r_1^{i+j+1} + k_2 I(r_1, w) = 0, \tag{11}$$

For simplification, we denote $k_2 = \pi$.

Note that, if n is odd, then (11) writes as

$$-r_1\left(\sum_{\substack{i+j=0\\ i \text{ even},\ \ j \text{ even}}}^{n-1} \pi\alpha_{ij}a_{ij}r_1^{i+j}\right) = I(r_1, w), \tag{12}$$

while if n is even, then (11) writes as

$$-r_1\left(\sum_{\substack{i+j=0\\ i \text{ even},\ \ j \text{ even}}}^{n} \pi\alpha_{ij}a_{ij}r_1^{i+j}\right) = I(r_1, w), \tag{13}$$

The left hand sides of equations (12) and (13) are polynomials of odd degree, with zero as a root. Both systems have the same number of solutions. From now we consider n even, so we will prove the existence of $\dfrac{n}{2} + 1$ limit cycles. The greatest integer function $[n/2] + 1$ is needed just to deal with the case n odd, and the adaptation of the proof is straightforward. Denote

$$\mathcal{P}(r_1) = -r_1\mathcal{P}_0(r_1) \tag{14}$$

Note that:

i) $\lim_{r_1\to w^+} I(r_1, w) = \pi$ and $\lim_{r_1\to\infty} I(r_1, w) = 4$,

ii) $\mathcal{P}(0) = 0$ and the polynomial $\mathcal{P}$ has at most $\dfrac{n}{2}$ positive roots (the nonzero roots are symmetric)

iii) $\mathcal{P}'(r_1)$ is a polynomial of degree n, and its zeros are also symmetric, so there are at most $n/2$ positive critical points (maxima or minima). So if $\mathcal{P}'(0) > 0$ then equation (13) has at most $\frac{n}{2}+1$ solutions, and if $\mathcal{P}'(0) < 0$, then equation (13) has at most $\frac{n}{2}$ solutions. We note that condition (iii) shows that there are no more than two solutions for equation (13) between two zeroes of $\mathcal{P}$.

4. **Proof of Corollary 2**

Let $\Sigma = \{y = 0\}$ be a section for the flow of (3) and (4). Define $P_0 : \Sigma \to \Sigma$ $P_w : \Sigma \to \Sigma$ the first map associated to (3) and (4) respectively. Note that both maps P_0 and P_w are analytic for $w > 0$, because: i) P_0 is a composition of two analytic functions: the Poincaré maps of (3) for $y > 0$ and $y < 0$, considering the cross section $y = 0$, and ii) P_w is a compositions of four analytic functions, the Poincaré maps of (4) with respect to the cross sections $y = \pm\omega$ Moreover $\lim_{w\to 0} P_w = P_0$ (see [11]).

Now, from Theorem 1 we have that:

Case 1) If $r < w$, then we can discard P_w, as $r \to 0$ when $w \to 0$, and all limit cycles for $r < w$ disappears.

Case 2) If $r > w$ for each $w > 0, P_w$ has at most $[n/2] + 1$ fixed point $\bar{y}_w^i \in \Sigma, i = 1, \ldots, [n/2]+1$, with $\bar{y}_w^i \neq 0$. For each $\bar{y}_w^i$ there exists a $\bar{r}_{1,w}^i > w$ that satisfy (11). These points are the points of intersection between the graph of the function $I_1(,w)$ and the curve $h(r_1)$. We will show that these points are stable. Now note that for each $r_1 > w$ we have (i) $\dfrac{\partial I_1}{\partial w} = \dfrac{2}{w^2}\left(\dfrac{w\sqrt{r^2-w^2}}{r} - r\operatorname{arccsc}\left(\dfrac{r}{w}\right)\right)$

(ii) $\lim\limits_{r\to w} \dfrac{\partial I_1}{\partial w}(r_1, w) = -\dfrac{\pi}{w}$

(iii) $\lim\limits_{r\to\infty} \dfrac{\partial I_1}{\partial w}(r_1, w) = 0$

(iv) $\dfrac{\partial^2 I_1}{\partial r\partial w} = \dfrac{2}{w^2}\left(\dfrac{w\left(w^2+r^2\right)}{r^2\sqrt{r^2-w^2}} - \operatorname{arccsc}\left(\dfrac{r}{w}\right)\right)$

(v) If $f_1(r,w) = \dfrac{w\left(w^2+r^2\right)}{r^2\sqrt{r^2-w^2}} - \operatorname{arccsc}\left(\dfrac{r}{w}\right)$ then $\lim\limits_{r\to w} f_1(r,w) = +\infty$ and $\lim\limits_{r\to\infty} f_1(r,w) = 0$. Moreover

$$\frac{\partial f_1}{\partial r}(r,w) = -\frac{2w^3\sqrt{r^2-w^2}\left(2r^2-w^2\right)}{r\left(r^3-rw^2\right)^2} < 0$$

then we can conclude

$$\frac{\partial^2 I_1}{\partial r \partial w} = \frac{2}{w^2}\left(\frac{w\left(w^2+r^2\right)}{r^2\sqrt{r^2-w^2}} - \operatorname{arccsc}\left(\frac{r}{w}\right)\right) > 0.$$

By (i) to (v) we obtain $\frac{\partial I_1}{\partial w}(r, w) < 0$. So for r_1 fixed with $r_1 > w$ we have $I_1\left(r_1, .\right)$ decreasing with w. This implies that if $w_1 > w_2$ then $\bar{r}^i_{1,w_1} < \bar{r}^i_{1,w_2}$ Moreover an upper bounded of $\bar{r}^i_{1,w}$ for $i = 1, \ldots, [n/2]+1$ is 4. This implies that $\lim_{w\to 0} \bar{r}^i_{1,w} = \bar{r}^i_1$ exists and $0 < w < \bar{r}^i_1 \leq 4$, for each $i = 1, \ldots, [n/2]+1$ From those arguments it follows that the fixed point $\bar{r}^i_{1,w}, i = 1, \ldots, [n/2]+1$ of the averaging equation associated to the fixed point $\bar{y}^i_w$ of P_w have a non- zero limit when $w \to 0$. This implies that $\bar{y}^i_w \to \bar{y}^i$ and $\bar{y}^i \neq 0$, for each $i = 1, \ldots, [n/2]+1$. Now as $\lim_{w\to 0} P_w\left(\bar{y}^i_w\right) = P_0\left(\bar{y}^i\right)$ we obtain $P_0\left(\bar{y}^i\right) = \bar{y}^i$. This concludes the proof of **corollary 2**.

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