ON THE ZERO-HOPF BIFURCATION OF THE GENERALIZED A CHEN–WANG SYSTEM

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Abstract. In this work, we show that a zero–Hopf bifurcation takes place in the differential system as parameters vary. Using averaging theory, we prove the existence of two periodic orbits bifurcating from the zero–Hopf equilibrium for the generalized a Chen–Wang system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = a + by + cz - x^2 - xz + 3y^2 \end{cases}$$

where a, b and c are real arbitrary parameters. The prime denotes derivative with respect to the independent variable t.

1. Introduction and statement of the main results

The study of the periodic solutions of a differential equation is one of the main objectives of the qualitative theory of differential equations. In general, the periodic solutions are studied numerically because, usually, their analytical study is very difficult. In this work we perform an analytic analysis on the existence of periodic solutions of the differential system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = a + by + cz - x^2 - xz + 3y^2 \end{cases} \tag{1}$$

by applying averaging theory of first order. More precisely, we will prove that a zero–Hopf bifurcation occurs in system (1) bifurcating two periodic solutions from the zero–Hopf equilibria as parameters vary.

There are several works studying zero–Hopf bifurcation see for instance Guckenheimer [6], Guckenheimer and Holmes [5], Han [7], Kuznetsov [8], Llibre [9], Marsden, Scheurle [11]. It has been shown that, under specific conditions, some elaborated invariant sets of the unfolding could be bifurcated from a zero–Hopf equilibrium.

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In the next result, we characterize when the equilibrium points of system (1) are zero-Hopf equilibria.

Proposition 1.The differential system (1) *has a unique zero-Hopf equilibrium localized at the origin of coordinates when* $a = 0, b = -w^2, c = 0$.

Theorem 2. Assume that in the generalized a Chen–Wang system (1) *we have*

$$\begin{aligned} a &= \varepsilon^2 \left(a_2 + w^4/4\right), \quad a_2 > 0, \quad b = -w^2 + \varepsilon b_1, \quad \omega > 0. \\ c &= \pm\sqrt{a} + \varepsilon. \end{aligned} \tag{2}$$

Then for $\varepsilon \neq 0$ *sufficiently small the differential system* (1) *has two periodic solutions* $(x_i(t,\varepsilon), y_i(t,\varepsilon), z_i(t,\varepsilon))$ *bifurcating from the zero–Hopf equilibrium of Proposition* 1*, namely*

$$\begin{aligned} &(\varepsilon \frac{2V_i^* - 2wR^* \cos(wt) - w^2\sqrt{4a_2 + w^4}}{2w^2} + O\left(\varepsilon^2\right), \varepsilon R^* \sin(wt) + O\left(\varepsilon^2\right) \\ &, \varepsilon w R^* \cos(wt) + O\left(\varepsilon^2\right)) \end{aligned} \tag{3}$$

where

$$\begin{aligned} R^* &= \frac{\sqrt{-\left(8\,w^2 - 2\right)\left(w^4\sqrt{4\,a_2 + w^4} - 4\,w^2 a_2 + 4\,a_2 - w^6\right)w}}{\left(4\,w^2 - 1\right)\left(w^2 - 2\right)}, \\ V^* &= \frac{w^2\left(w^2 + (-1)^i\sqrt{4\,a_2 + w^4}\right)}{w^2 - 2}, \text{ for } i = 1, 2. \end{aligned}$$

2. Averaging theory of first order

The averaging theory is a classical and matured tool for studying the behavior of the dynamics of nonlinear smooth dynamical systems, and in particular of their periodic orbits. The method of averaging has a long history that starts with the classical works of Lagrange and Laplace who provided an intuitive justification of the process. The first formalization of this procedure is due to Fatou [4] in 1928. Important practical and theoretical contributions in this theory were made by Krylov and Bogoliubov [3] in the 1930s and Bogoliubov [2] in 1945. The averaging theory of first order for studying periodic orbits can be found in [12], see also [5]. It can be summarized as follows.

We also present a result from the averaging theory that we shall need for proving Theorem 2, for a general introduction to the averaging theory see the book of Sanders, Verhulst and Murdock [10].

We consider the initial value problems

$$\dot{x} = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x,\varepsilon), \quad x\,(0) = x_0, \tag{4}$$

and the averaged differential equation

$$\dot{y} = \varepsilon f(y), \quad y(0) = x_0. \tag{5}$$

with x, y, and x_0 in some open subset Ω of $\mathbb{R}^n$, $t \in [0,\infty)$, $\varepsilon \in (0,\varepsilon_0]$, We assume that F_1 and F_2 are periodic of period T in the variable t, and we set

$$f(y) = \frac{1}{T}\int_0^T F_1\,(t,y)\,dt. \tag{6}$$

We will also use the notation $D_x f$ for all the first derivatives of f, and $D_{xx} f$ for all the second derivatives of f.
For a proof of the next result, see [12].
Theorem 3. *Assume that F_1, $D_x F_1$, $D_{xx} F_1$ and $D_x F_2$ are continuous and bounded by a constant independent of ε in $[0,\infty)\times\Omega\times(0,\varepsilon_0]$, and that $y(t)\in\Omega$ for $t\in[0,1/\varepsilon]$. Then, the following statements hold :*

1. *For $t \in [0, 1/\varepsilon]$, we have $x(t) - y(t) = O(\varepsilon)$ as $\varepsilon \to 0$.*
2. *If p is a singular point of system* (6) *such that*

$$\det D_y f(p) \neq 0, \tag{7}$$

   then there exists a periodic solution $x(t,\varepsilon)$ of period T for system (4) *which is close to p and such that $x(0,\varepsilon) - p = O(\varepsilon)$ as $\varepsilon \to 0$.*
3. *The stability of the periodic solution $x(t,\varepsilon)$ is given by the stability of the singular point.*

3. Zero-Hopf bifurcation

In this section, we prove *Proposition 1* and *Theorem 2*.
Proof of Proposition 1 The differential system (1) has two equilibrium points $e_\pm = (\pm\sqrt{a}, 0, 0)$ when $a > 0$, which collide at the origin when $a = 0$. The proof is made computing directly the eigenvalues at each equilibrium point.Note that the characteristic polynomial of the linear part of system (1) at the equilibrium point $e_\pm$ is

$$P\,(\lambda) = \lambda^3 + (\pm\sqrt{a} - c)\lambda^2 - b\lambda \pm 2\sqrt{a} \tag{8}$$

As $p(\lambda)$ is a polynomial of degree 3, it has either one, two (then one has multiplicity 2), or three real zeros. Using the discriminant of $P\,(\lambda)$, it follows that $P\,(\lambda)$ has a unique real root. for more details see [1].

In order to study the zero-Hopf bifurcation we imposing that

$$P\,(\lambda) = (\lambda - \varepsilon)\left(\lambda^2 + w^2\right) \tag{9}$$

This occurs if and only if the coefficients of this equation are $\pm 2\sqrt{a} + \varepsilon w^2 = 0, b + w^2 = 0, -c \pm \sqrt{a} + \varepsilon = 0$.

We obtain $a = \varepsilon^2 \dfrac{w^4}{4}, b = -w^2, c = \pm\sqrt{a} + \varepsilon$.This completes the proof of the proposition.

Proof of Theorem 2 The differential system (1) satisfying (2) has two equilibria, namely $p_\pm = (\pm \dfrac{\varepsilon\sqrt{(4a_2 + w^2)}}{2}, 0, 0)$, First we study the periodic solutions bifurcating from the zero–Hopf equilibrium near the equilibrium p_-.

For applying the averaging theory of first order to system (1) satisfying (2) we translate the equilibrium point p_- to the origin by doing the change of variables

$$(x, y, z) = \left(x_1 - \frac{\varepsilon\sqrt{(4a_2 + w^4)}}{2}, y_1, z_1 \right) \tag{10}$$

The differential system in the new variables (x_1, y_1, z_1) is

$$\begin{cases} \dot{x}_1 = y_1, \\ \dot{y}_1 = z_1, \\ \dot{z}_1 = -w^2 y_1 - x_1^2 - x_1 z_1 + 3y_1^2 + \varepsilon \left(b_1 y_1 + z_1 + \sqrt{(4a_2 + w^4)} x_1 \right) \end{cases} \tag{11}$$

We need to write the linear part of system (11) at the equilibrium point $(0, 0, 0)$ in its real Jordan normal form, i.e. into the form

$$\begin{pmatrix} 0 & -w & 0 \\ w & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in order to facilitate the application of the averaging theory, given by Theorem 3, for computing the zero–Hopf bifurcation. Then, doing the change of variables $(x_1, y_1, z_1) \to (X, Y, Z)$ given by

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{w} \\ 0 & 1 & 0 \\ w & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

the differential system (11) having its linear part in its real Jordan form is

$$\left\{\begin{array}{l}\dot{X}=-wY-\frac{1}{w}\left(-\frac{X}{w}+\frac{Z}{w^2}\right)^2-\left(-\frac{X}{w}+\frac{Z}{w^2}\right)X+\frac{3}{w}Y^2\\ \qquad+\varepsilon\frac{1}{w}\left(b_1Y+wX+\left(-\frac{X}{w}+\frac{Z}{w^2}\right)\sqrt{4\,a_2+w^4}\right)\\ \dot{Y}=wX\\ \dot{Z}=-\left(-\frac{X}{w}+\frac{Z}{w^2}\right)^2-\left(-\frac{X}{w}+\frac{Z}{w^2}\right)wX+3\,Y^2\\ \qquad+\varepsilon\left(b_1Y+wX+\left(-\frac{X}{w}+\frac{Z}{w^2}\right)\sqrt{4\,a_2+w^4}\right)\end{array}\right. \tag{12}$$

Consider the cylindrical coordinates (r,θ,Z) defined by $X=r\cos\theta, Y=r\sin\theta, Z=Z$ then the differential system (12) becomes

$$\left\{\begin{array}{l}\dot{r}=-\frac{\cos(\theta)\left(-w^4b_1r\sin(\theta)+w^3\sqrt{4\,a_2+w^4}r\cos(\theta)-w^5r\cos(\theta)-w^2\sqrt{4\,a_2+w^4}Z\right)\varepsilon}{w^5}\\ \dot{\theta}=\frac{2\,\sin(\theta)\,w^4r^2\left(\cos(\theta)\right)^2+\sin(\theta)\,w^2r^2\left(\cos(\theta)\right)^2+\sin(\theta)\,w^3r\cos(\theta)\,Z}{w^5r}\\ +\frac{-2\,\sin(\theta)\,wr\cos(\theta)\,Z+\sin(\theta)\,Z^2-3\,\sin(\theta)\,r^2w^4+w^6r}{w^5r}\\ +\frac{\left(-\sin(\theta)\,w^5r\cos(\theta)+\sin(\theta)\,w^3\sqrt{4\,a_2+w^4}r\cos(\theta)-\sin(\theta)\,w^2\sqrt{4\,a_2+w^4}Z\right)\varepsilon}{w^5r}\\ +\frac{\left(w^4b_1r\left(\cos(\theta)\right)^2-w^4b_1r\right)\varepsilon}{w^5r}\\ \dot{Z}=-\left(-\frac{r\cos(\theta)}{w}+\frac{Z}{w^2}\right)^2-\left(-\frac{r\cos(\theta)}{w}+\frac{Z}{w^2}\right)wr\cos(\theta)+3\,r^2\left(\sin(\theta)\right)^2\\ +\varepsilon\left(b_1r\sin(\theta)+wr\cos(\theta)+\left(-\frac{r\cos(\theta)}{w}+\frac{Z}{w^2}\right)\sqrt{4\,a_2+w^4}\right)\end{array}\right. \tag{13}$$

Doing the rescaling $(r,Z)=(\varepsilon R,\varepsilon V)$ we obtain

$$\left\{\begin{array}{l}
\dot{R}=(-\dfrac{\cos(\theta)}{w^5}(-w^4b_1R\sin(\theta)+w^3\sqrt{4\,a_2+w^4}R\cos(\theta)-w^5R\cos(\theta)-w^2\sqrt{4\,a_2+w^4}V\\
-2\,w^4R^2\left(\cos(\theta)\right)^2-w^3R\cos(\theta)\,V-V^2-w^2R^2\left(\cos(\theta)\right)^2+3\,R^2w^4+2\,wR\cos(\theta)\,V))\varepsilon\\
\dot{\theta}=\dfrac{1}{Rw^5}(-\sin(\theta)\,w^5R\cos(\theta)+\sin(\theta)\,w^3\sqrt{4\,a_2+w^4}R\cos(\theta)-\sin(\theta)\,w^2\sqrt{4\,a_2+w^4}V\\
+w^4b_1R\left(\cos(\theta)\right)^2-w^4b_1R+2\,\sin(\theta)\,w^4R^2\left(\cos(\theta)\right)^2+\sin(\theta)\,w^2R^2\left(\cos(\theta)\right)^2\\
-2\,\sin(\theta)\,wR\cos(\theta)\,V+\sin(\theta)\,V^2-3\,\sin(\theta)\,R^2w^4+\sin(\theta)\,w^3R\cos(\theta)\,V)\varepsilon+w\\
\dot{V}=-\dfrac{\cos(\theta)}{w^5}(\,-w^4b_1R\sin(\theta)+w^3\sqrt{4\,a_2+w^4}R\cos(\theta)-w^5R\cos(\theta)\\
-w^2\sqrt{4\,a_2+w^4}V-2\,w^4R^2\left(\cos(\theta)\right)^2-w^3R\cos(\theta)\,V-V^2-w^2R^2\left(\cos(\theta)\right)^2\\
+3\,R^2w^4+2\,wR\cos(\theta)\,V)\varepsilon
\end{array}\right. \tag{14}$$

In system (14) we take θ as the new independent variable, and we get

$$\left\{\begin{array}{l}
\dfrac{dR}{d\theta}=-\dfrac{\cos(\theta)}{w^6}(-w^4b_1R\sin(\theta)+w^3\sqrt{4\,a_2+w^4}R\cos(\theta)-w^5R\cos(\theta)\\
\qquad -w^2\sqrt{4\,a_2+w^4}V+2\,w^4R^2\left(\cos(\theta)\right)^2+w^3R\cos(\theta)\,V\\
\qquad +V^2+w^2R^2\left(\cos(\theta)\right)^2-3\,w^4R^2-2\,wR\cos(\theta)\,V)\varepsilon+O\left(\varepsilon^2\right)\\
\qquad =\varepsilon F_1\left(\theta,R,V\right)+O\left(\varepsilon^2\right)\\
\dfrac{dV}{d\theta}=\dfrac{-1}{w^5}(-w^4b_1R\sin(\theta)+w^3\sqrt{4\,a_2+w^4}R\cos(\theta)-w^5R\cos(\theta)\\
\qquad -w^2\sqrt{4\,a_2+w^4}V+2\,w^4R^2\left(\cos(\theta)\right)^2+w^3R\cos(\theta)\,V+V^2\\
\qquad +w^2R^2\left(\cos(\theta)\right)^2-3\,w^4R^2-2\,wR\cos(\theta)\,V)\varepsilon+O\left(\varepsilon^2\right)\\
\qquad =\varepsilon F_2\left(\theta,R,V\right)+O\left(\varepsilon^2\right)
\end{array}\right. \tag{15}$$

Using the notation of the averaging theory described in section 2 we have that if we take $t=\theta, T=2\pi, x=\left(R,V\right)^T$ and

$$F_1\left(\theta,R,V\right)=\begin{pmatrix} F_{11}\left(\theta,R,V\right)\\ F_{12}\left(\theta,R,V\right)\end{pmatrix} \quad \text{and } f\left(R,V\right)=\begin{pmatrix} f_1\left(R,V\right)\\ f_2\left(R,V\right)\end{pmatrix}$$

It is immediate to check that system (15) satisfies all the assumptions of Theorem 3.

Now we compute the integrals (6). We obtain that

$$\left\{\begin{array}{l}
f_1\left(R,V\right)=\dfrac{1}{2\pi}\int_0^{2\pi}F_{11}\left(\theta,R,V\right)d\theta\\
\qquad\qquad =-\dfrac{R\left(w^2V-2\,V-w^4+\sqrt{4\,a_2+w^4}w^2\right)}{2w^5}\\
f_2\left(R,V\right)=\dfrac{1}{2\pi}\int_0^{2\pi}F_{12}\left(\theta,R,V\right)d\theta\\
\qquad\qquad =\dfrac{-w^2R^2-2\,V^2+4\,w^4R^2+2\,w^2\sqrt{4\,a_2+w^4}V}{2w^5}
\end{array}\right.$$

The system $f_1(R,V) = f_2(R,V) = 0$ has a unique solution (R^*,V^*) with $R^*>0$, namely

$$\begin{aligned} R^* &= \frac{\sqrt{-\left(8\,w^2-2\right)\left(w^4\sqrt{4\,a_2+w^4}-4\,w^2a_2+4\,a_2-w^6\right)w}}{\left(4\,w^2-1\right)\left(w^2-2\right)}, \\ V^* &= \frac{w^2\left(w^2+\left(-1\right)^i\sqrt{4\,a_2+w^4}\right)}{w^2-2}, \end{aligned}$$

if $w \neq \pm\sqrt{2}, w \neq \pm\dfrac{1}{2}$ and $\left(8\,w^2-2\right)\left(w^4\sqrt{4\,a_2+w^4}-4\,w^2a_2+4\,a_2-w^6\right) < 0$.

The Jacobian (7) at (R^*,V^*) takes the value

$$-\frac{4\,a_2-4\,w^2a_2-\;w^6+w^4\sqrt{4\,a_2+w^4}}{w^6\left(w^2-2\right)} \neq 0$$

Then, by Theorem 3, it follows that for $\varepsilon \neq 0$ sufficiently small system (15) has a periodic solution $\left(R\left(\theta,\varepsilon\right), V\left(\theta,\varepsilon\right)\right)$ such that $\left(R\left(0,\varepsilon\right), V\left(0,\varepsilon\right)\right) \to (R^*,V^*)$ when $\varepsilon \to 0$. writes an approximation of this periodic solution in the form

$$\left(R\left(\theta,\varepsilon\right), V\left(\theta,\varepsilon\right)\right) = \left(R^*+O\left(\varepsilon\right), V^*+O\left(\varepsilon\right)\right)$$

This periodic solution becomes for the differential system (14) in the form

$$\left(R\left(t,\varepsilon\right), \theta\left(t,\varepsilon\right), V\left(t,\varepsilon\right)\right) = \left(R^*+O\left(\varepsilon\right), wt+O\left(\varepsilon\right), V^*+O\left(\varepsilon\right)\right)$$

In the same way, in the differential system (13) it becomes

$$\left(r\left(t,\varepsilon\right), \theta\left(t,\varepsilon\right), Z\left(t,\varepsilon\right)\right) = \left(\varepsilon R^*+O\left(\varepsilon^2\right), wt+O\left(\varepsilon\right), \varepsilon V^*+O\left(\varepsilon^2\right)\right)$$

Passing to a periodic solution to the differential system (12) we get

$$\left(X\left(t,\varepsilon\right), Y\left(t,\varepsilon\right), Z\left(t,\varepsilon\right)\right) = \left(\varepsilon R^*\cos\left(wt\right)+O\left(\varepsilon^2\right), \varepsilon R^*\sin\left(wt\right)+O\left(\varepsilon^2\right), \varepsilon V^*+O\left(\varepsilon^2\right)\right)$$

In the differential system (11) the periodic solution writes

$$\begin{aligned} (x_1(t,\varepsilon), y_2(t,\varepsilon), z_3(t,\varepsilon)) \;=\; & (\varepsilon\frac{V_i^*-wR^*\cos\left(wt\right)}{w^2}+O\left(\varepsilon^2\right), \\ & \varepsilon R^*\sin\left(wt\right)+O\left(\varepsilon^2\right), \varepsilon wR^*\cos\left(wt\right)+O\left(\varepsilon^2\right)) \end{aligned}$$

Finally, for the differential system (1) the periodic solution becomes the solution (3) for $i=1$ of the statement of Theorem 2. Computing the eigenvalues of the jacobian matrix

$$\left.\frac{\partial\left(f_1,f_2\right)}{\partial\left(R,V\right)}\right|_{(R,V)=(R^*,V^*)}$$

we obtain

$$\frac{-2\,w^2+\sqrt{4\,a_2+w^4}w^2\pm\sqrt{24\,w^4-12\,w^4\sqrt{4\,a_2+w^4}-12\,a_2w^4-3\,w^8+48\,w^2a_2+4\,w^6\sqrt{4\,a_2+w^4}-32\,a_2+8\,w^6}}{2(w^2-2)w^3}$$

So, by statement 3 of Theorem 3 the sability of the periodic solution associated to the zero (R^*, V^*) cannot be studied due to the difficulty of studying the sign of the real part of the two eigenvalues. If instead of translating the equilibrium point p_- at the origin doing the change of variables (10), we translate at the origin the equilibrium point p_+ and repeat all the previous computations we shall obtain for the differential system (1) the periodic solution (3) for $i = 2$ of the statement of Theorem 2. This concludes the proof of Theorem 2.

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