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Neural field equations with neuron-dependent Heaviside-type activation function and spatial-dependent delay

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Summary

We introduce a neural field equation with a neuron-dependent Heaviside-type activation function and spatial-dependent delay. The basic object of the study is represented by a Volterra Hammerstein integral equation involving a discontinuous nonlinearity with respect to the state variable that is both time- and space-dependent. We replace the discontinuous nonlinearity by its multi-valued convexification and obtain the corresponding Volterra Hammerstein integral inclusion. We investigate the solvability of this inclusion using the properties of upper semi-continuous multi-valued mappings with convex closed values. Based on these results, we study the solvability of an initial-prehistory problem for the former neural field equation with the Heaviside-type activation function. The application of multi-valued analysis techniques allowed us to avoid some restrictive assumptions standardly used in the investigations of the solutions to neural field equations involving Heaviside-type activation functions.

KEYWORDS:
neural field equations, Volterra Hammerstein integral equations, Heaviside activation function, solvability

1 | INTRODUCTION

Neural field models are considered as a convenient approach to model electrical activity of large spatially structured ensembles of neurons.[1,2] The most well-known representative of such models is the so-called Amari neural field equation[3]

$$\partial_t u(t,x) = -\alpha u(t,x) + \int\limits_{\Omega} \omega(x,y) f\big(u(t,y)\big)\, dy, \quad t \geq 0,\ x \in \Omega, \tag{1}$$

where Ω represents a neural medium (neural field) that we assume here to be a compact subset of the m-dimensional real vector space, $u(t,x)$ denotes the activity of the neural field at the time t and spatial position x, $\alpha > 0$ is a relative time constant. The strength of connections between the neural elements is determined by the connectivity function ω, whereas the activation process of the neurons is described by the activation function f. A typical choice of the activation function is a non-decreasing continuous sigmoidal-shaped function having the values in the interval $[0,1]$.[2,4] Many works[5,6,7,8,9,10] also employ activation functions in the form of the Heaviside function $f(u) = \begin{cases} 0, & u \leq h, \\ 1, & u > h \end{cases}$ with some positive activation threshold h. The choice of a discontinuous activation in the neural field has its advantages and drawbacks. The main advantages concern the facilitation of numerical investigation of the corresponding models as well as the possibilities to derive closed form expressions for several specific types of solutions having physiological relevance (see the paper[11] and the references therein for more details). The weak points of the neural field models with the Heaviside-type activation arise from the fact that the initial-value problems are generally ill-posed even for the one of the simplest models (1).[12,13] The solvability of an initial value problem for the Amari equation with a Heaviside-type activation function was established[12] using the solvability results for smooth activation functions and compactness arguments techniques. In the present research we consider the solvability problem for a generalized version of (1) with discontinuous activation function, which takes into account some additional physiologically relevant assumptions. Namely, we introduce the neural field equation

$$\partial_t u(t,x) = -\alpha u(t,x) + \int_\Omega \omega(x,y) H\big(y, u(t-\tau(x,y),y)\big)dy, \quad t \geq 0,\ x \in \Omega, \tag{2}$$

where the non-negative function τ describes the effect of spatial-dependent delay in the neural field and the Heaviside-type activation function H is naturally assumed to be dependent on the taken neuron,[14,15] e.g. in the following way:

$$H(x,u) = \begin{cases} 0, & u \leq h(x), \\ 1, & u > h(x). \end{cases}$$

The presence of time delay in (2) yields the need in the initial-prehistory condition containing the states $u(t,x)$ for negative values of t:

$$u(t,x) = \varphi(t,x), \quad t \leq 0,\ x \in \Omega. \tag{3}$$

Taking into account (3), we represent (2) in terms of the following integral equation

$$u(t,x) = \int_0^t \exp(-\alpha(t-s)) \int_\Omega \omega(x,y) H(y, u(s-\tau(x,y),y))dyds + \varphi(0,x), \quad t \geq 0,\ x \in \Omega.$$

The time-integration kernel that is given here as $\exp(-\alpha(t-s))$ is usually called a memory function. We interchange the integration order, make the reassignment $s := s - \tau(x,y)$, and introduce a new spatiotemporal integration kernel

$$W(t,s,x,y) = \exp(-\alpha(t-s+\tau(x,y)))\omega(x,y)$$

that is responsible for the delay and the memory effects. We, thus, derive the following Volterra Hammerstein integral equation:

$$u(t,x) = \int_\Omega \int_{-\tau(x,y)}^{t-\tau(x,y)} W(t,s,x,y) H(y,u(s,y))dsdy + \varphi(0,x), \quad t \geq 0,\ x \in \Omega. \tag{4}$$

In order to handle the discontinuity in (4) we make use of the idea of A.F. Filippov (see Chapter 2, § 4[16]) and replace the Heaviside-type nonlinearity in the Hammerstein operator by its multi-valued convexification

$$\mathrm{H}(x,u) = \begin{cases} 0, & u < h(x), \\ [0,1], & u = h(x), \\ 1, & u > h(x). \end{cases}$$

We, thus, investigate the solvability of (3), (4) by studying the following Volterra Hammerstein integral inclusion with delay

$$u(t,x) \in \int_\Omega \int_{-\tau(x,y)}^{t-\tau(x,y)} W(t,s,x,y) \mathcal{H}(s,y,u(s,y))dsdy + I(t,x), \quad t \geq 0,\ x \in \Omega. \tag{5}$$

Here we take some further generalizations of (4) assuming the presence of the external input $I(t,x)$ and temporal dependence in the multi-valued nonlinearity $\mathcal{H}(t,x,u)$. We also relax the requirement of existence of a uniform bound for $\mathcal{H}(t,x,u)$ in the study of the problem (3), (5).

The formalisation of the equation (4) in terms of the inclusion (5) allowed us to avoid the following restriction on the set of solutions:

Assumption 1. The solutions to a neural field equation with a Heaviside-type activation function should have zero Lebesgue measure of the set of intersection points with the activation function threshold.

This assumption played a crucial role in the proof of solvability of an initial value problem for (1),[12] as well as in the investigations[11,17,18] of particular types of solutions to (1).

The present research, thus, opens a possibility to study the so-called sliding modes in the neural field equations with discontinuous activation functions. The existence of sliding modes was proved for gene regulatory networks with Heaviside-type gene activation,[19] which arouse the interest to similar investigations for the models of neural activity.

A number of works[20,21,22] contain the results on solvability of Hammerstein integral inclusions. The local solvability of a delayed Hammerstein integral inclusion was recently examined.[23] However, all these studies concern the case of time-dependent state variable. In the present paper we formulate statements on solvability of Hammerstein integral inclusions, where the state variable is both time- and space-dependent. Such setting allows to incorporate the effect of spatial-dependent delay in the modeling framework and, thus, increase the realism of the model.

The present paper is organized as follows. In Sect. 2 we introduce important notations, define the types of solutions to the problem (3), (5), and formulate and prove a statement on solvability of this problem. Based on this result, in Sect. 3 we investigate the solvability of the initial-prehistory problem (3) for the neural field equation (4) involving neuron-dependent Heaviside-type activation function and spatial-dependent delay. Sect. 4 provides concluding remarks and outlook.

2 | SOLVABILITY OF VOLTERRA HAMMERSTEIN INTEGRAL INCLUSION WITH DELAY

We denote R^l to be a l-dimensional real vector space with the norm $|\cdot|$. Let $\mathcal{B}$ be some Banach space with the norm $\|\cdot\|_{\mathcal{B}}$. We denote by $\mathbf{2}^{\mathcal{B}}$ the set of all nonempty subsets of $\mathcal{B}$; by $conv(\mathcal{B}) \subset \mathbf{2}^{\mathcal{B}}$ and $\overline{conv}(\mathcal{B}) \subset \mathbf{2}^{\mathcal{B}}$ – the sets of all convex and all convex closed subsets of $\mathcal{B}$, respectively. For any $M \in \mathbf{2}^{\mathcal{B}}$, $r > 0$, we define the set $B_{\mathcal{B}}(M, r) = \{\mathrm{b} \in \mathcal{B}, \|\mathrm{b} - \mathrm{m}\|_{\mathcal{B}} < r, \mathrm{m} \in M\}$ and understand the notation $B_{\mathcal{B}}(\mathrm{m}, r)$ as $B_{\mathcal{B}}(\{\mathrm{m}\}, r)$ for any $\mathrm{m} \in \mathcal{B}$.

Let K be some compact subset of R^l. Denote by $C(K, \mathcal{B})$ the space of continuous functions $u : K \to \mathcal{B}$ with the norm $\|u\|_{C(K,\mathcal{B})} = \max\limits_{z\in K} \|u(z)\|_{\mathcal{B}}$. We denote by $L(K, \mathcal{B})$ the space of Bochner integrable functions $w : K \to \mathcal{B}$ with the norm $\|w\|_{L(K,\mathcal{B})} = \int\limits_K \|w(z)\|_{\mathcal{B}} dz$. Denote by $L_\infty(K, \mathcal{B})$ the space of essentially bounded functions $\chi : K \to \mathcal{B}$ with the norm $\|\chi\|_{L_\infty(K,\mathcal{B})} = \operatorname*{vrai\,sup}\limits_{z\in K} \|\chi(z)\|_{\mathcal{B}}$.

We consider here the inclusion (5) with the initial-prehistory condition (3), where $\tau : \Omega\times\Omega \to [0,\infty)$, $\sup\limits_{(x,y)\in\Omega\times\Omega} \tau(x, y) = d$, $\mathcal{H} : [-d,\infty)\times\Omega\times R^n \to \overline{conv}(R^k)$, $W : [0,\infty)\times[-d,\infty)\times\Omega\times\Omega \to R^{n\times k}$, $\varphi : [-d, 0]\times\Omega \to R^n$, $I : [0,\infty)\times\Omega \to R^n$.

For any $t > 0$, we denote $\Omega_t = [0, t]\times\Omega$ and assume that for any $T > 0$, the above mappings satisfy the following conditions:

$(A1)$ For any $(t, x) \in \Omega_T$, the integration kernel $W(t,\cdot,x,\cdot) \in L_\infty([-d, T]\times\Omega, R^{n\times k})$ and the function $(t, x) \mapsto \|W(t,\cdot,x,\cdot)\|_{L_\infty([-d,T]\times\Omega,R^{n\times k})}$ is bounded.

$(A2)$ For any measurable set $\mathbb{I} \subset [-d, T]\times\Omega$ and any $(t_0, x_0) \in [-d, T]\times\Omega$, it holds true that

$$\lim_{(t,x)\to(t_0,x_0)} \iint\limits_{\mathbb{I}\cap\Omega_t} W(t, s, x, y)dyds = \iint\limits_{\mathbb{I}\cap\Omega_{t_0}} W(t_0, s, x_0, y)dyds.$$

$(A3)$ For almost all $(t, x) \in [-d, T]\times\Omega$, the function $\mathcal{H}(t, x,\cdot) : R^n \to \overline{conv}(R^k)$ is upper semi-continuous.

$(A4)$ The mapping $\mathcal{H}(\cdot,\cdot,u)$ is integrally bounded, i.e. for any $\rho > 0$, there exists some non-negative function $\eta_\rho \in L([-d, T]\times\Omega, R)$ such that for all $u \in B_{R^n}(0, \rho)$, it holds true that

$$\max_{w\in\mathcal{H}(t,x,u)} |w| \le \eta_\rho(t, x); \tag{6}$$

and for any $\rho, \varepsilon > 0$, there exists $\delta > 0$ such that for all $t, t_0 \in [0, T]$ and $x, x_0 \in \Omega$, the inequalities $0 \le t - t_0 < \delta$ and $|x - x_0| < \delta$ imply

$$\int\limits_0^t \int\limits_\Omega |W(t, s, x, y) - W(t_0, s, x_0, y)|\eta_\rho(s, y)dyds < \varepsilon. \tag{7}$$

$(A5)$ The delay $\tau : \Omega\times\Omega \to [0,\infty)$ is continuous.

$(A6)$ $\varphi \in C([-d, 0]\times\Omega, R^n)$.

$(A7)$ $I \in C([0, T]\times\Omega, R^n)$, $I(0,\cdot) = \varphi(0,\cdot)$.

Definition 1. We define a *T-local solution* to the problem (3), (5) to be a function $\mathbf{u}_T \in C(\Omega_T, R^n)$ satisfying the inclusion (5) on Ω_T and the condition (3). A mapping $\mathbf{u} : [0,\infty)\times\Omega \to R^n$ such that for any $T \in (0,\infty)$, the restriction $\mathbf{u}_T \in C(\Omega_T, R^n)$ of $\mathbf{u}$ on Ω_T is a T-local solution to (3), (5) is called a *global solution* to the problem (3), (5) . We define a *$\mathcal{T}$-maximally extended solution* ($\mathcal{T} > 0$) to the problem (3), (5) to be a mapping $\widetilde{\mathbf{u}}_{\mathcal{T}} : [0,\mathcal{T})\times\Omega \to R^n$ such that

- for any $T \in (0, \mathcal{T})$, the restriction $\mathbf{u}_T \in C(\Omega_T, R^n)$ of $\widetilde{\mathbf{u}}_{\mathcal{T}}$ on Ω_T is a local solution to (3), (5);
- $\lim\limits_{T \to \mathcal{T}-0} \|\mathbf{u}_T\|_{C(\Omega_T, R^n)} = \infty$.

Theorem 1. Let the conditions $(A1)$ – $(A7)$ be satisfied. Then for some $T > 0$, the problem (3), (5) has a T-local solution. Any local solution can be extended to a global solution or to a maximally extended solution.

Proof. For each fixed $T > 0$, we introduce the following mappings. A one-to-one mapping $\mathcal{G}_T$ defined by the relation $(\mathcal{G}_T u)(t,x) = \begin{cases} u(t,x), & (t,x) \in \Omega_T, \\ \varphi(t,x), & (t,x) \in [-d,0] \times \Omega; \end{cases}$ a multi-valued mapping $\mathcal{N}_T$ that puts into correspondence to any function $v : [-d,T] \times \Omega \to R^n$ the set of all measurable selections of the mapping $(t,x) \mapsto \mathcal{H}\big(t,x,v(t,x)\big)$; a linear integral operator $\mathcal{I}_T$ given as $(\mathcal{I}_T w)(t,x) = \int\limits_{\Omega} \int\limits_{-\tau(x,y)}^{t-\tau(x,y)} W(t,s,x,y) w(s,y) ds$.

Choose any $T > 0$, $r > \max\left\{ \max\limits_{(t,x) \in [-d,0] \times \Omega} |\varphi(t,x)|, \max\limits_{(t,x) \in \Omega_T} |I(t,x)| \right\}$, $\xi \in C(\Omega_T, R^n)$, denote $\mathbb{M}_r^{\xi} = B_{C(\Omega_T, R^n)}(\xi, r)$, and define $\overline{\varphi}_0(t,x) \equiv \varphi(0,x)$, $(t,x) \in \Omega_T$. Show that the composition $\mathcal{I}_T \mathcal{N}_T \mathcal{G}_T : \mathbb{M}_r^{\overline{\varphi}_0} \to \overline{conv}(C(\Omega_T, R^n))$ is upper semi-continuous and the set $\mathcal{I}_T \mathcal{N}_T \mathcal{G}_T(\mathbb{M}_r^{\overline{\varphi}_0})$ is relatively compact in $C(\Omega_T, R^n)$.

The conditions $(A1)$ and $(A2)$ guarantee that the mapping $\mathcal{I}_T$ is a continuous operator acting from $L([-d,T] \times \Omega, R^k)$ to $C(\Omega_T, R^n)$ (see chapter III, § 5.5[24]). The condition $(A1)$ imply that for any $\rho > 0$, the set

$$\mathcal{I}_T\big(\{w \in L([-d,T] \times \Omega, R^k), |w(t,x)| \leq \eta_\rho(t,x) \text{ for almost all } (t,x) \in [-d,T] \times \Omega\}\big),$$

is relatively compact in $C(\Omega_T, R^n)$. For any $u \in \mathbb{M}_r^{\overline{\varphi}_0}$, there exists a measurable selection $w : [-d,T] \times \Omega \to R^k$ of the mapping $(t,x) \mapsto \mathcal{H}\big(t,x,(\mathcal{G}_T u)(t,x)\big)$ (see Theorem 1.3.5[25]). Thus, the set $\mathcal{I}_T \mathcal{N}_T \mathcal{G}_T(\mathbb{M}_r^{\overline{\varphi}_0})$ is relatively compact in $C(\Omega_T, R^n)$. The assumption $(A3)$ and the linearity of the operator $\mathcal{I}_T$ imply the fact that $\mathcal{I}_T \mathcal{N}_T \mathcal{G}_T : \mathbb{M}_r^{\overline{\varphi}_0} \to conv(C(\Omega_T, R^n))$.

Show that the composition $\mathcal{I}_T \mathcal{N}_T \mathcal{G}_T : \mathbb{M}_r^{\overline{\varphi}_0} \to conv(C(\Omega_T, R^n))$ is a closed mapping. Choose sequences $\{u^i\} \subset C(\Omega_T, R^n)$, $\{\vartheta^i\} \subset C(\Omega_T, R^n)$ such that $\vartheta^i \in \mathcal{I}_T \mathcal{N}_T \mathcal{G}_T u^i$ and

$$\|u^i - u^0\|_{C(\Omega_T, R^n)} \to 0, \ \|\vartheta^i - \vartheta^0\|_{C(\Omega_T, R^n)} \to 0 \ (i \to \infty)$$

for some $u^0 \in C(\Omega_T, R^n)$ and $\vartheta^0 \in C(\Omega_T, R^n)$. By the definition of $\mathcal{G}_T : C(\Omega_T, R^n) \to C([-d,T] \times \Omega, R^n)$, the sequence $\{v^i = \mathcal{G}_T u^i\} \subset C([-d,T] \times \Omega, R^n)$ converges to $v^0 = \mathcal{G}_T u^0 \in C([-d,T] \times \Omega, R^n)$, as $i \to \infty$. Choose the sequence $\{w^i\}$, $w^i \in \mathcal{N}_T v^i$, such that $\vartheta^i = \mathcal{I}_T w^i$.

We now consider the sequence $\{w^i\} \subset L([-d,T] \times \Omega, R^n)$ as the sequence of Bochner integrable mappings $w^i : [-d,T] \to L(\Omega, R^n)$. The condition $(A4)$ and the fact that $v^i \to v^0$ allow to apply Kolmogorov–Riesz compactness theorem[26] and, subsequently, Proposition 4.2.1,[25] thus, obtaining a subsequence of the sequence $w^i \subset L([-d,T], L(\Omega, R^k))$, which weakly converges to some $w^0 \in L([-d,T], L(\Omega, R^k))$ (without loss of generality we will keep the notation $\{w^i\}$ for this subsequence). Now, using Mazur's lemma (see e.g. Section 5.1, Theorem 2[27]), we find a subsequence $\widehat{w}^i = \sum\limits_{j=i}^{\infty} \beta_{ij} w^j$ such that

$$\|\widehat{w}^i - w^0\|_{L([-d,T], L(\Omega, R^k))} \to 0 \ (i \to \infty), \tag{8}$$

where the coefficients β_{ij} satisfy the conditions

(i) for all $i = 1, 2, \dots$ it holds true that $\sum\limits_{j=i}^{\infty} \beta_{ij} = 1$,

(ii) for each $i = 1, 2, \dots$ one can find a number j_0 such that $\beta_{ij} = 0$ for all $j > j_0$.

The relation (8) is equivalent to $\|\widehat{w}^i - w^0\|_{L([-d,T] \times \Omega, R^k)} \to 0$, as $i \to \infty$. The latter fact implies (see Section 41, Theorem 4[28]) the existence of a subsequence of the sequence $\{\widehat{w}^i\}$ converging to w^0 almost everywhere on $[-d,T] \times \Omega$ (without loss of generality we will use the notation $\{\widehat{w}^i\}$ for this subsequence).

For almost all $(t,x) \in [-d,T] \times \Omega$, due to upper semi-continuity of $\mathcal{H}(t,x,\cdot)$, for any $\varepsilon > 0$, there exists a number $i_0 = i_0(\varepsilon, t, x)$ such that for all $i > i_0$, it holds true that

$$\mathcal{H}\big(t,x,(\mathcal{G}_T u^i)(t,x)\big) \subset B_{R^k}\big(\mathcal{H}\big(t,x,(\mathcal{G}_T u^0)(t,x)\big), \varepsilon\big).$$

Consequently, we have $w^i(t,x) \in B_{R^k}\big(\mathcal{H}\big(t,x,(\mathcal{G}_T u^0)(t,x)\big), \varepsilon\big)$ for $i > i_0$. Due to the fact that an ε-neighborhood of a convex set is convex, we obtain $\widehat{w}^i(t,x) \in B_{R^k}\big(\mathcal{H}\big(t,x,(\mathcal{G}_T u^0)(t,x)\big), \varepsilon\big)$ for $i > i_0$. By the virtue of the closedness of the set

$\mathcal{H}\big(t,x,(\mathcal{G}_T u^0)(t,x)\big)$, the latter relation implies that $w^0(t,x) \in \mathcal{H}\big(t,x,(\mathcal{G}_T u^0)(t,x)\big)$ for almost all $(t,x) \in [-d,T]\times\Omega$, so that $w^0 \in \mathcal{N}_T\mathcal{G}_T u^0$. Putting

$$\widehat{\vartheta}^i = \mathcal{I}_T\widehat{w}^i = \mathcal{I}_T\sum_{j=i}^{\infty}\beta_{ij}w^j = \sum_{j=i}^{\infty}\beta_{ij}\mathcal{I}_T w^j = \sum_{j=i}^{\infty}\beta_{ij}\vartheta^j$$

we obtain $\lim\limits_{i\to\infty}\|\widehat{\vartheta}^i - \vartheta^0\|_{C(\Omega_T,R^n)} = 0$, which due to the continuity of $\mathcal{I}_T$ implies that

$$\vartheta^0 = \mathcal{I}w^0 \in \mathcal{I}_T\mathcal{N}_T\mathcal{G}_T u^0.$$

Thus, the closedness and, consequently, the upper semi-continuity of the composition $\mathcal{I}_T\mathcal{N}_T\mathcal{G}_T : \mathbb{M}_r^{\overline{\varphi}_0} \to \overline{conv}(C(\Omega_T,R^n))$.

Now we choose some sufficiently large $T > 0$, put $r_1 = 1 + \max\left\{\max\limits_{(t,x)\in[-d,0]\times\Omega}|\varphi(t,x)|, \max\limits_{(t,x)\in\Omega_T}|I(t,x)|\right\}$, and using the conditions $(A2)$ and $(A4)$ find the maximal $T_1 \in (0,T]$ such that

$$\max_{(t,x)\in\Omega_{2T_1}}\left|\int\limits_{\Omega}\int\limits_{-\tau(x,y)}^{t-\tau(x,y)} W(t,s,x,y)\eta_{r_1}(s,y)ds\,dy + I(t,x) - \varphi(0,x)\right| \le r.$$

Thus, for the set $\mathbb{M}_{r_1}^{\overline{\varphi}_0}$, we have $\mathcal{I}_{T_1}\mathcal{N}_{T_1}\mathcal{G}_{T_1}(\mathbb{M}_{r_1}^{\overline{\varphi}_0})+I \subset \mathbb{M}_{r_1}^{\overline{\varphi}_0}$. Taking into account the obtained above properties of the composition $\mathcal{I}_T\mathcal{N}_T\mathcal{G}_T$ $(\forall T > 0)$, we apply Bohnenblust–Karlin theorem[29] and prove the existence of a fixed point $\mathbf{u}_{T_1} \in C(\Omega_{T_1},R^n)$ that is a T_1-local solution to the problem (3), (5).

Choose any T_1-local solution $\mathbf{u}_{T_1} \in C(\Omega_{T_1},R^n)$ to the problem (3), (5) and find a measurable selection $\mathbf{w}_{T_1} \in \mathcal{H}(\cdot,\cdot,\mathbf{u}_{T_1}(\cdot,\cdot))$ $(\mathbf{w}_{T_1} \in L([-d,T_1]\times\Omega,R^k))$ such that

$$u(t,x) = \int\limits_{\Omega}\int\limits_{-\tau(x,y)}^{t-\tau(x,y)} W(t,s,x,y)\mathbf{w}_{T_1}(s,y)ds\,dy + I(t,x),\ \ (t,x)\in\Omega_{T_1}.$$

We introduce a new time-variable $t' = t - T_1$ and the functions

$$\varphi' : [-d-T_1,0]\times\Omega \to R^n,\ \varphi'(t',x) = \begin{cases}\varphi(t'+T_1,x), & (t',x)\in[-d-T_1,-T_1]\times\Omega,\\ \mathbf{u}_{T_1}(t'+T_1,x), & (t',x)\in[-T_1,0]\times\Omega,\end{cases}$$

and

$$I' : [0,\infty)\times\Omega \to R^n,\ I'(t',x) = I(t'+T_1,x) + \int\limits_{\Omega}\int\limits_{-\tau(x,y)}^{T_1-\tau(x,y)} W(T_1,s,x,y)\mathbf{w}_{T_1}(s,y)ds\,dy.$$

Now, we consider the problem (3), (5) with $t = t'$, $I = I'$, and $\varphi = \varphi'$ (we will refer to this new problem as $(3'),(5')$). As the new functions φ' and I' satisfy the conditions $(A6)$ and $(A7)$, we apply the procedure described above and prove the existence of a T_1'-local solution $\mathbf{u}'_{T_1'}$ to $(3'),(5')$ for some $T_1' > 0$. We, thus, obtain a T_2-local solution $\mathbf{u}_{T_2} : \Omega_{T_2} \to R^n$ to (3), (5), where $T_2 = T_1 + T_1'$, $\mathbf{u}_{T_2}(t,\cdot) = \begin{cases}\mathbf{u}_{T_1}(t,\cdot), & t\in[0,T_1],\\ \mathbf{u}'_{T_1'}(t,\cdot), & t\in[T_1,T_2],\end{cases}$ We finally note that, due to the definition of the functions $\varphi' : [-d-T_1,0]\times\Omega \to R^n$, $I' : [0,\infty)\times\Omega \to R^n$ and the formulation $(3'),(5')$ of the problem, the T_2-local solution $\mathbf{u}_{T_2}$ is continuous at (T_1,x) for any $x\in\Omega$, i.e. $\mathbf{u}_{T_2} \in C(\Omega_{T_2},R^n)$.

On the next step, we choose any T_2-local solution $\mathbf{u}_{T_2} \in C(\Omega_{T_2},R^n)$ to the problem (3), (5). We introduce $t'' = t - T_2$,

$$\varphi'' : [-d-T_2,0]\times\Omega \to R^n,\ \varphi''(t'',x) = \begin{cases}\varphi(t''+T_2',x), & (t'',x)\in[-d-T_2,-T_2]\times\Omega,\\ \mathbf{u}_{T_2}(t''+T_2,x), & (t'',x)\in[-T_2,0]\times\Omega,\end{cases}$$

and

$$I'' : [0,\infty)\times\Omega \to R^n,\ I''(t'',x) = I(t''+T_2,x) + \int\limits_{\Omega}\int\limits_{-\tau(x,y)}^{T_2-\tau(x,y)} W(T_2,s,x,y)\mathbf{w}_{T_2}(s,y)ds\,dy,$$

where $\mathbf{w}_{T_2} \in L([-d,T_2]\times\Omega,R^k)$ such that

$$u(t,x) = \int\limits_{\Omega}\int\limits_{-\tau(x,y)}^{t-\tau(x,y)} W(t,s,x,y)\mathbf{w}_{T_2}(s,y)ds\,dy + I(t,x),\ \ (t,x)\in\Omega_{T_2},$$

and repeat the procedure.

We, thus, obtain a sequence $\{\mathbf{u}_{T_i}\}$, $i = 1, 2, \ldots$, of local solutions to (3), (5) such that for any $i_1 < i_2$, $\mathbf{u}_{T_{i_1}}(t,x) = \mathbf{u}_{T_{i_2}}(t,x)$ on $\Omega_{T_{i_1}}$. We find $\lim_{i\to\infty} T_i = \widehat{T}$. Take any $t \in (0, \widehat{T})$. For some number i, we have $t \in (T_{i-1}, T_i]$, and, therefore, $\mathbf{u}_t \in C(\Omega_t, R^n)$ is a t-local solution to the problem (3), (5) . We, thus, constructed the mapping

$$(0, \widehat{T}) \ni t \mapsto \mathbf{u}_t \in C(\Omega_t, R^n). \tag{9}$$

If $\widehat{T} = \infty$, then this mapping is obviously a global solution to the problem (3), (5). Prove that for the case $\widehat{T} < \infty$, the mapping (9) is a maximally extended solution to (3), (5). There are the following two possibilities:

(a) $\lim\limits_{t\to\widehat{T}-0} \|\mathbf{u}_t\|_{C(\Omega_t,R^n)} = \infty$,

(b) $\lim\limits_{t\to\widehat{T}-0} \|\mathbf{u}_t\|_{C(\Omega_t,R^n)} = \mathfrak{N} < \infty$.

The case (a) means that the mapping (9) is a maximally extended solution to (3), (5) . In the case (b), we find a uniform bound $r > 1 + \max\left\{ \mathfrak{N}, \max\limits_{(t,x)\in[-d,0]} |\varphi(t,x)|, \max\limits_{(t,x)\in[0,T]} |I(t,x)| \right\}$ which implies that in the process of the local solutions construction we have $T_i = iT_1$, $i = 1, 2, \ldots$. The latter relation and, hence, the case (b) contradicts with the assumption that $\lim\limits_{i\to\infty} T_i = \widehat{T} < \infty$. Thus, in the case $\widehat{T} < \infty$ we always obtain a maximally extended solution. □

Corollary 1. Let the conditions $(A1)$ – $(A7)$ be fulfilled and the function $\eta_\rho \in L([-d,T]\times\Omega, [0,\infty))$ in the estimates (6), (7) does not depend on the value of $\rho > 0$. Then for some $T > 0$, the problem (5), (3) has a T-local solution and each local solution can be extended to a global solution.

The validity of this statement follows from the proof of Theorem 1.

3 | SOLVABILITY OF NEURAL FIELD EQUATION WITH NEURON-DEPENDENT HEAVISIDE-TYPE ACTIVATION FUNCTION AND SPATIAL-DEPENDENT DELAY

We apply here the results of the previous section to the problem (3), (4), where

$(A8)$ $H : \Omega \times R^n \to R^k$, $H = (H_1, \ldots, H_k)$, $H_i(x,u) = \begin{cases} 0, & u \le h_i(x), \\ 1, & u > h_i(x), \end{cases}$ $h_i : \Omega \to R$ are Lebesgue measurable for all $i = 1, \ldots, k$,

and the rest of the functions involved are defined in Sect. 2 and satisfy the corresponding assumptions $(A1)$, $(A2)$, $(A5)$, and $(A6)$.

Definition 2. We define a *generalized T-local solution* to the problem 3, (4) to be a function $\mathbf{u}_T \in C(\Omega_T, R^n)$ satisfying the inclusion

$$u(t,x) \in \int\limits_{\Omega} \int\limits_{-\tau(x,y)}^{t-\tau(x,y)} W(t,s,x,y)\mathrm{H}(y, u(s,y))\,ds\,dy + \varphi(0,x), \tag{10}$$

$$\mathrm{H} : \Omega \times R^n \to \mathbf{2}^{R^k}, \mathrm{H} = (\mathrm{H}_1, \ldots, \mathrm{H}_k),\ \mathrm{H}_i(x,u) = \begin{cases} 0, & u < h_i(x), \\ [0,1], & u = h_i(x), \quad i = 1, \ldots, k, \\ 1, & u > h_i(x), \end{cases}$$

on the set Ω_T and the prehistory condition (3). A mapping $\mathbf{u} : [0,\infty) \times \Omega \to R^n$ such that for any $T \in (0,\infty)$, the restriction $\mathbf{u}_T \in C(\Omega_T, R^n)$ of $\mathbf{u}$ on Ω_T is a T-local solution to 3, (4), is called a *generalized global solution* to the problem 3, (4).

We now note that for any $x \in \Omega$, the multi-valued mapping $\mathrm{H}(x,\cdot) : R^n \to \mathbf{2}^{R^k}$ is upper semi-continuous and has convex closed values. Moreover, the estimates (6) and (7) are satisfied for $\mathcal{H}(\cdot,\cdot,\cdot) = \mathrm{H}(\cdot,\cdot)$ with the function $\eta_\rho \equiv 1$ that is independent of the choice of $\rho > 0$.

The application of Corollary 1 to the problem (3), (10) implies the following result.

Theorem 2. Let the conditions $(A1)$, $(A2)$, $(A5)$, $(A6)$, and $(A8)$ be satisfied. Then the problem (3), (4) has a generalized global solution.

4 | CONCLUSIONS

In this paper the solvability of an initial-prehistory problem for a neural field equation with neuron-dependent Heaviside-type activation function and spatial-dependent delay was established. To the best of our knowledge, neural field models with neuron-dependent activation functions, being physiologically relevant, have not yet been considered. The main investigation framework in our research was a Volterra Hammerstein integral inclusion obtained from the former neural field equation by convexification of the discontinuous activation function. The generality of this approach allowed us to omit the standard Assumption 1 used in the investigations [11,12,17,18] of neural field models involving Heaviside-type activation functions. The latter fact, in turn, allows to study the so-called sliding modes in the neural field equations. The investigation of the sliding modes and of their stability/instability properties can, thus, be considered as a direction for future extensions of the present work. Other direction can be the study of the parametric dependence properties of solutions to neural field equations with discontinuous activation. The parametric dependence results can be exploited in the proofs of connection between the solutions to the neural field equations involving Heaviside-type activation functions and the corresponding equations with sufficiently "steep" sigmoidal activation functions. Another possible application of the parametric dependence results is the investigation of control problems for models of neural activity that gained the attention of the mathematical neuroscientists community in the recent years. [30,31,32,33,34,35] Based on the results on continuous dependence of solutions to neural field equations with continuous activation, the corresponding control problems for such neural field equations were investigated. [36] The planned study of the parametric dependence properties of the solutions to (4) could be exploited to obtain similar results for a wide class of neural field models with Heaviside-type activation functions.

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Author contributions

The idea of this research and the derivation of the neural field equation with neuron-dependent Heaviside-type activation function and spatial-dependent delay belong to VV. EZ and EB have made equal contributions to the investigation of the mathematical model.

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Conflict of interest

The authors declare no potential conflict of interests.

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