Periodic solutions of a second-order iterative differential equation *

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Abstract. In this paper, we use Schauder and Banach fixed point theorem to study the existence, uniqueness and stability of periodic solutions of a class of iterative differential equation

$$\alpha x''(t) + \beta x'(t) + \gamma x(t) = \lambda_1(t)x(t) + \lambda_2(t)x(x(t)) + \cdots + \lambda_n(t)x^{[n]}(t) + f(t).$$

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1 Introduction

Delay differential equation of the form

$$x'(t) = f(t, x(t - \tau(t)))$$

has been discussed in [1] and [5]. In particularly, the delay function $\tau(t)$ depends not only on unknown function, but also state, $\tau(t, x(t))$ have been studied in many literatures in the last few years([6, 7], [13, 14]). In [2], Cooke pointed out that it is highly desirable to establish the existence and stability properties of periodic solutions for equations of the form

$$x'(t) + ax(t - h(t, x(t))) = F(t),$$

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in which the lag $h(t, x(t))$ implicitly involves $x(t)$. Eder [3] considered the iterative functional differential equation

$$x'(t) = x^{[2]}(t)$$

and obtains that every solution either vanishes identically or is strictly monotonic. Fečkan [4] studied the equation

$$x'(t) = f(x^{[2]}(t))$$

by obtaining an existence theorem for solutions satisfying $x(0) = 0$. In [8], Si and Cheng considered the analytic solutions of the form

$$x'(t) = x(at + bx(t)).$$

Further discussion is made in [9]-[11] for existence of analytic solutions of several iterative functional differential equations with state or state derivative dependent. In 2006, Liu and Li [6] considered the analytic solutions of the form

$$\alpha x''(t) + \beta x'(t) + \gamma x(t) = x(at + bx(t)) + h(t),$$

in a neighborhood of the origin. Recently, Si and Wang [12] studied the smooth solutions of

$$x'(t) = \lambda_1(t)x(t) + \lambda_2(t)x(x(t)) + \cdots + \lambda_n(t)x^{[n]}(t) + f(t). \tag{1.1}$$

Moreover, Zhao and Liu [16] considered the periodic solutions of (1.1).

In this note, we will study the existence of periodic solutions of

$$\alpha x''(t) + \beta x'(t) + \gamma x(t) = \lambda_1(t)x(t) + \lambda_2(t)x(x(t)) + \cdots + \lambda_n(t)x^{[n]}(t) + f(t). \tag{1.2}$$

For convenience, we will make use $C(\mathbb{R}, \mathbb{R})$ to denote the set of all real valued continuous functions map $\mathbb{R}$ into $\mathbb{R}$.

For $T > 0$, define

$$\mathcal{P}_T = \Big\{x \in C(\mathbb{R}, \mathbb{R}) : x(t + T) = x(t),\ \forall t \in \mathbb{R}\Big\}.$$

Then $\mathcal{P}_T$ is a Banach space with the norm

$$\|x\| = \max_{t\in[0,T]} |x(t)| = \max_{t\in\mathbb{R}} |x(t)|.$$

For $P > 0, L \geq 0$, define the sets

$$\mathcal{P}_T(P, L) = \Big\{x \in \mathcal{P}_T : \|x\| \leq P,\ |x(t_2) - x(t_1)| \leq L|t_2 - t_1|,\ \forall t_1, t_2 \in \mathbb{R}\Big\},$$

which is a closed convex and bounded subset of $\mathcal{P}_T$, and we wish to find T-periodic functions $x \in \mathcal{P}_T(P, L)$ satisfies (1.2).

2 Periodic solutions of (1.2)

In this section, the existence of periodic solutions of equation (1.2) will be proved. Let us state the Schauder fixed point theorem, which will be used to prove our main theorem.

Theorem 2.1 (Schauder) *Let Ω be a closed convex nonempty subset of a Banach space $(\mathbb{B}, \|\cdot\|)$. Suppose that A maps Ω into Ω and is compact and continuous. Then there exists $z \in \Omega$ with $z = Az$.*

Throughout this paper, we assume that all functions are continuous with respect to their arguments and following condition holds.

(H) $\lambda_i \in \mathcal{P}_T(P_i, L_i), i = 1, 2, \ldots, n$, and $f \in \mathcal{P}_T(P_f, L_f)$ are given.

We begin with the following lemma.

Lemma 2.1 ([15]) *It holds*

$$\mathcal{P}_T(P, L) = \Big\{x \in \mathcal{P}_T : \|x\| \leq P, |x(t_2) - x(t_1)| \leq L|t_2 - t_1|, \quad \forall t_1, t_2 \in [0, T]\Big\}. \tag{2.1}$$

Lemma 2.2 *For any $\varphi, \psi \in \mathcal{P}_T(P, L)$,*

$$||\varphi^{[n]} - \psi^{[n]}|| \leq \sum_{j=0}^{n-1} L^j \|\varphi - \psi\|, \ n = 1, 2, \ldots. \tag{2.2}$$

Proof. The result follows from the definition of $\mathcal{P}_T(P, L)$. $\square$

Now we rewrite (1.2) as a fixed point equation.

Lemma 2.3 *Suppose $\alpha, \beta, \gamma \neq 0$, then $x \in \mathcal{P}_T$ is a solution of equation* (1.2) *if and only if*

$$x(t) = \frac{1}{\alpha} E(\alpha, \beta_1) E(\alpha, \beta_2) \int_t^{t+T} \int_u^{u+T} \Big(\Phi_x(s) + f(s)\Big) e^{\frac{\beta_1}{\alpha}(s-u)} e^{\frac{\beta_2}{\alpha}(u-t)} ds du \tag{2.3}$$

where

$$E(\alpha, \beta_1) = \frac{1}{e^{\frac{\beta_1}{\alpha}T} - 1}, \ E(\alpha, \beta_2) = \frac{1}{e^{\frac{\beta_2}{\alpha}T} - 1} \tag{2.4}$$

and

$$\Phi_x(t) = \Sigma_{i=1}^n \lambda_i(t) x^{[i]}(t), \tag{2.5}$$

$\beta_1 = \frac{\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2}, \beta_2 = \frac{\beta - \sqrt{\beta^2 - 4\alpha\gamma}}{2}$ *or* $\beta_1 = \frac{\beta - \sqrt{\beta^2 - 4\alpha\gamma}}{2}, \beta_2 = \frac{\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2}$ *and we see* $\beta_1 + \beta_2 = \beta, \beta_1\beta_2 = \alpha\gamma$.

Proof. By direct calculation, we can see that (2.3) is a T-periodic solution of (1.2).

Suppose $x(t)$ is a T-periodic solution of (1.2), then it is easy to find Eq (1.2) can be written in the form of

$$x''(t)e^{\frac{\beta_1}{\alpha}t}+\frac{\beta_1}{\alpha}x'(t)e^{\frac{\beta_1}{\alpha}t}+\frac{\beta_2}{\alpha}x'(t)e^{\frac{\beta_1}{\alpha}t}+\frac{\gamma}{\alpha}x(t)e^{\frac{\beta_1}{\alpha}t}=\frac{1}{\alpha}\Big(\Sigma_{i=1}^{n}\lambda_i(t)x^{[i]}(t)+f(t)\Big)e^{\frac{\beta_1}{\alpha}t},$$

or

$$\Big(x'(t)e^{\frac{\beta_1}{\alpha}t}\Big)'+\frac{\beta_2}{\alpha}\Big(x(t)e^{\frac{\beta_1}{\alpha}t}\Big)'=\frac{1}{\alpha}\Big(\Sigma_{i=1}^{n}\lambda_i(t)x^{[i]}(t)+f(t)\Big)e^{\frac{\beta_1}{\alpha}t}. \tag{2.6}$$

Integrating (2.6) from t to $t+T$ and using the fact $x(t+T)=x(t)$ obtain

$$x'(t)+\frac{\beta_2}{\alpha}x(t)=\frac{1}{\alpha}\int_t^{t+T}\Big(\Sigma_{i=1}^{n}\lambda_i(s)x^{[i]}(s)+f(s)\Big)\frac{e^{\frac{\beta_1}{\alpha}(s-t)}}{e^{\frac{\beta_1}{\alpha}T}-1}ds,$$

Therefore,

$$x(t)=\frac{1}{\alpha}\int_t^{t+T}\int_u^{u+T}\Big(\Sigma_{i=1}^{n}\lambda_i(s)x^{[i]}(s)+f(s)\Big)\frac{e^{\frac{\beta_1}{\alpha}(s-u)}}{e^{\frac{\beta_1}{\alpha}T}-1}\frac{e^{\frac{\beta_2}{\alpha}(u-t)}}{e^{\frac{\beta_2}{\alpha}T}-1}dsdu.$$

This completes the proof. □

Now we will need to construct a mapping that satisfy the hypotheses of Theorem 2.1. To this aim, consider the map $A:\mathcal{P}_T(P,L)\to\mathcal{P}_T$ defined as follows:

$$(Ax)(t)=\frac{1}{\alpha}E(\alpha,\beta_1)E(\alpha,\beta_2)\int_t^{t+T}\int_u^{u+T}\Big(\Phi_x(s)+f(s)\Big)e^{\frac{\beta_1}{\alpha}(s-u)+\frac{\beta_2}{\alpha}(u-t)}dsdu, \tag{2.7}$$

where $E(\alpha,\beta_1),E(\alpha,\beta_2)$ and $\Phi_x(t)$ are defined as in Lemma 2.3.

Lemma 2.4 *Suppose* (H) *holds and $\alpha,\beta,\gamma\neq 0$, then operator A is continuous and compact on $\mathcal{P}_T(P,L)$.*

Proof. Take $\varphi,\psi\in\mathcal{P}_T(P,L), t\in\mathbb{R}$, then by (2.2),

$$\begin{aligned}&|(A\varphi)(t)-(A\psi)(t)|\\ \le\ &\frac{1}{|\alpha|}|E(\alpha,\beta_1)||E(\alpha,\beta_2)|\left|\int_t^{t+T}\int_u^{u+T}|\Phi_\varphi(s)-\Phi_\psi(s)|e^{\frac{\beta_1}{\alpha}(s-u)+\frac{\beta_2}{\alpha}(u-t)}dsdu\right|\\ \le\ &\frac{1}{|\gamma|}\sum_{i=1}^{n}\sum_{j=0}^{i-1}L^jP_i\|\varphi-\psi\|,\end{aligned} \tag{2.8}$$

thus A is continuous.

Now we will show that A is a compact map. It is easy to see that $\mathcal{P}_T(P,L)$ is a uniformly bounded and equicontinuous on $\mathbb{R}$, then using Arzela-Ascoli theorem we know $\mathcal{P}_T(P,L)$ is a compact set. Since A is continuous, it maps compact sets into compact sets, therefore A is compact. This completes the proof. $\square$

Theorem 2.2 *Suppose* (H) *holds and* $\alpha, \beta, \gamma \neq 0$. *Furthermore, the following inequalities hold*

$$P_f \leq (|\gamma| - \sum_{i=1}^{n} P_i)P, \tag{2.9}$$

$$|E(\alpha,\beta_2)|(P\sum_{i=1}^{n} P_i + P_f)e^{\frac{|\beta_2|}{|\alpha|}T}\Big(1 + \frac{1}{|E(\alpha,\beta_2)|} + e^{\frac{|\beta_2|}{|\alpha|}T}\Big) < |\beta_1|L, \tag{2.10}$$

then Eq. (1.2) *has a periodic solution in* $\mathcal{P}_T(P,L)$.

Proof. For any $\varphi \in \mathcal{P}_T(P,L)$. By (2.9), we have

$$\begin{aligned}
& |(A\varphi)(t)| \\
\leq\ & \frac{1}{|\alpha|}|E(\alpha,\beta_1)||E(\alpha,\beta_2)|\left|\int_t^{t+T}\int_u^{u+T}\Big(\Phi_\varphi(s)+f(s)\Big)e^{\frac{\beta_1}{\alpha}(s-u)+\frac{\beta_2}{\alpha}(u-t)}dsdu\right| \\
\leq\ & \frac{1}{|\gamma|}(P\sum_{i=1}^{n} P_i + P_f) \\
\leq\ & P.
\end{aligned}$$

Without loss of generality, assume $t_1, t_2 \in [0,T]$, we obtain

$$\begin{aligned}
& \left|\int_{t_2}^{t_2+T}\int_u^{u+T}\Big(\Phi_\varphi(s)+f(s)\Big)e^{\frac{\beta_1}{\alpha}(s-u)+\frac{\beta_2}{\alpha}(u-t_2)}dsdu\right. \\
& \left. -\int_{t_1}^{t_1+T}\int_u^{u+T}\Big(\Phi_\varphi(s)+f(s)\Big)e^{\frac{\beta_1}{\alpha}(s-u)+\frac{\beta_2}{\alpha}(u-t_1)}dsdu\right| \\
\leq\ & \left|\int_{t_2}^{t_2+T}\int_u^{u+T}\Big(\Phi_\varphi(s)+f(s)\Big)e^{\frac{\beta_1}{\alpha}(s-u)+\frac{\beta_2}{\alpha}u}dsdu\right|\left|e^{-\frac{\beta_2}{\alpha}t_2} - e^{-\frac{\beta_2}{\alpha}t_1}\right| \\
& +e^{-\frac{\beta_2}{\alpha}t_1}\left|\int_{t_2}^{t_2+T}\int_u^{u+T}\Big(\Phi_\varphi(s)+f(s)\Big)e^{\frac{\beta_1}{\alpha}(s-u)+\frac{\beta_2}{\alpha}u}dsdu - \right. \\
& \left.\int_{t_1}^{t_1+T}\int_u^{u+T}\Big(\Phi_\varphi(s)+f(s)\Big)e^{\frac{\beta_1}{\alpha}(s-u)+\frac{\beta_2}{\alpha}u}dsdu\right|
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \frac{|\beta_2|}{|\gamma|}\frac{(P\sum_{i=1}^n P_i+P_f)}{|E(\alpha,\beta_1)||E(\alpha,\beta_2)|}e^{\frac{|\beta_2|}{|\alpha|}T}|t_2-t_1| \\
& +e^{-\frac{\beta_2}{\alpha}t_1}\left|\int_{t_1+T}^{t_2+T}\int_u^{u+T}\Big(\Phi_\varphi(s)+f(s)\Big)e^{\frac{\beta_1}{\alpha}(s-u)+\frac{\beta_2}{\alpha}u}dsdu\right| \\
& +e^{-\frac{\beta_2}{\alpha}t_1}\left|\int_{t_2}^{t_1}\int_u^{u+T}\Big(\Phi_\varphi(s)+f(s)\Big)e^{\frac{\beta_1}{\alpha}(s-u)+\frac{\beta_2}{\alpha}u}dsdu\right| \\
\leq\ & \frac{|\beta_2|}{|\gamma|}\frac{(P\sum_{i=1}^n P_i+P_f)}{|E(\alpha,\beta_1)||E(\alpha,\beta_2)|}e^{\frac{|\beta_2|}{|\alpha|}T}|t_2-t_1|+\frac{|\beta_2|}{|\gamma|}\frac{(P\sum_{i=1}^n P_i+P_f)}{|E(\alpha,\beta_1)|}e^{\frac{2|\beta_2|}{|\alpha|}T}|t_2-t_1| \\
& +\frac{|\beta_2|}{|\gamma|}\frac{(P\sum_{i=1}^n P_i+P_f)}{|E(\alpha,\beta_1)|}e^{\frac{|\beta_2|}{|\alpha|}T}|t_2-t_1| \\
=\ & \frac{|\beta_2|}{|\gamma|}\frac{(P\sum_{i=1}^n P_i+P_f)}{|E(\alpha,\beta_1)|}e^{\frac{|\beta_2|}{|\alpha|}T}\Big(1+\frac{1}{|E(\alpha,\beta_2)|}+e^{\frac{|\beta_2|}{|\alpha|}T}\Big)|t_2-t_1|,
\end{aligned}
\tag{2.11}
$$

By (2.11) and (2.10), we have

$$
\begin{aligned}
& \Big|(A\varphi)(t_2)-(A\varphi)(t_1)\Big| \\
\leq\ & \frac{1}{|\alpha|}|E(\alpha,\beta_1)||E(\alpha,\beta_2)|\left|\int_{t_2}^{t_2+T}\int_u^{u+T}\Big(\Phi_\varphi(s)+f(s)\Big)e^{\frac{\beta_1}{\alpha}(s-u)+\frac{\beta_2}{\alpha}(u-t_2)}dsdu\right. \\
& \left.-\int_{t_1}^{t_1+T}\int_u^{u+T}\Big(\Phi_\varphi(s)+f(s)\Big)e^{\frac{\beta_1}{\alpha}(s-u)+\frac{\beta_2}{\alpha}(u-t_1)}dsdu\right| \\
\leq\ & \frac{1}{|\beta_1|}|E(\alpha,\beta_2)|(P\sum_{i=1}^n P_i+P_f)e^{\frac{|\beta_2|}{|\alpha|}T}\Big(1+\frac{1}{|E(\alpha,\beta_2)|}+e^{\frac{|\beta_2|}{|\alpha|}T}\Big)|t_2-t_1| \\
\leq\ & L|t_2-t_1|
\end{aligned}
$$

Therefore $(A\varphi)(t)\in\mathcal{P}_T(P,L)$. So by Lemma 2.4, we see that all the conditions of Schauder's theorem are satisfied on $\mathcal{P}_T(P,L)$. Thus there exists a fixed point x in $\mathcal{P}_T(P,L)$ such that $x=Ax$, from Lemma 2.3, x is a T-periodic solution of equation (1.2). This completes the proof. □

Remark 2.1 If $\alpha=0$, Eq. (1.2) change to

$$
\beta x'(t)+\gamma x(t)=\lambda_1(t)x(t)+\lambda_2(t)x(x(t))+\cdots+\lambda_n(t)x^{[n]}(t)+f(t), \tag{2.12}
$$

and we have the following results which similarly as [16].

Proposition 2.1 *Suppose $\beta\neq 0$, $\lambda_1(t)\neq\gamma$ and $\lambda_1\in\mathcal{P}_T$, then $x\in\mathcal{P}_T$ is a solution of (2.12) if and only if*

$$
x(t)=\frac{1}{\beta}\int_t^{t+T}\Big(\Psi_x(u)+f(u)\Big)G(\beta,\gamma,\lambda_1)du \tag{2.13}
$$

where

$$G(\beta,\gamma,\lambda_1)=\frac{e^{\frac{1}{\beta}\int_t^u(\gamma-\lambda_1(s))ds}}{e^{\frac{1}{\beta}\int_0^T(\gamma-\lambda_1(s))ds}-1}, \tag{2.14}$$

and

$$\Psi_x(t)=\Sigma_{i=2}^n\lambda_i(t)x^{[i]}(t). \tag{2.15}$$

The proof as Lemma 2.2 in [16], here we omit it.

Remark 2.2 It is easy to see there exists constants m and M such that

$$m\le G(\beta,\gamma,\lambda_1)\le M,$$

here $m=\frac{e^{\frac{1}{\beta}(\gamma-P_1)T}}{e^{\frac{1}{\beta}(\gamma+P_1)T}-1}, M=\frac{e^{\frac{1}{\beta}(\gamma+P_1)T}}{e^{\frac{1}{\beta}(\gamma-P_1)T}-1}$ for $\beta>0$ and $m=\frac{e^{\frac{1}{\beta}(\gamma+P_1)T}}{e^{\frac{1}{\beta}(\gamma-P_1)T}-1}, M=\frac{e^{\frac{1}{\beta}(\gamma-P_1)T}}{e^{\frac{1}{\beta}(\gamma+P_1)T}-1}$ for $\beta<0$. Furthermore, we see

$$\|G(\beta,\gamma,\lambda_1)\|\le\Gamma(\beta,\gamma,\lambda_1)e^{\frac{1}{|\beta|}(\gamma+P_1)T}=\overline{M}, \tag{2.16}$$

here $\Gamma(\beta,\gamma,\lambda_1)=\max\left\{\frac{1}{|e^{\frac{1}{\beta}(\gamma-P_1)T}-1|},\frac{1}{|e^{\frac{1}{\beta}(\gamma+P_1)T}-1|}\right\}$.

Consider the map $B:\mathcal{P}_T(P,L)\to\mathcal{P}_T$ defined as follows:

$$(Bx)(t)=\frac{1}{\beta}\int_t^{t+T}\Big(\Psi_x(u)+f(u)\Big)G(\beta,\gamma,\lambda_1)du, \tag{2.17}$$

where $G(\beta,\gamma,\lambda_1)$ and Ψ_x are defined as in Proposition 2.1.

Proposition 2.2 *Suppose* (H) *holds and* $\beta\neq0,\lambda_1(t)\neq\gamma$, *then operator* B *is continuous and compact on* $\mathcal{P}_T(P,L)$.

The proof as Lemma 2.4, here we omit it.

In the following theorem, we obtain the similarly result as in [16], but the condition $\lambda_1(t)<\gamma$ changes to $\lambda_1(t)\neq\gamma$ in this paper.

Theorem 2.3 *Suppose* (H) *holds and* $\beta\neq0,\lambda_1(t)\neq\gamma$. *Furthermore, the following inequalities hold*

$$\overline{M}T(P\sum_{i=2}^nP_i+P_f)\le|\beta|P, \tag{2.18}$$

$$\overline{M}(P\sum_{i=2}^{n}P_i+P_f)e^{\frac{T}{|\beta|}(|\gamma|+P_1)}\Big(2+\frac{T}{|\beta|}(|\gamma|+P_1)\Big)<|\beta|L, \tag{2.19}$$

then Eq. (2.12) *has a periodic solution in* $\mathcal{P}_T(P,L)$.

The proof as Theorem 2.2, here we omit it.

3 Uniqueness and stability

In this section, uniqueness and stability of (1.2) and (2.12) will be proved.

Theorem 3.1 *In addition to the assumption of Theorem* 2.2, *suppose that*

$$\sum_{i=1}^{n}\sum_{j=0}^{i-1}L^jP_i<|\gamma|, \tag{3.1}$$

then (1.2) *has a unique solution in* $\mathcal{P}_T(P,L)$.

Proof. We know from the proof of Theorem 2.2 that $A:\mathcal{P}_T(P,L)\to\mathcal{P}_T(P,L)$, Moreover, by (2.8), we get

$$\|A\varphi-A\psi\|\le\frac{1}{|\gamma|}\sum_{i=1}^{n}\sum_{j=0}^{i-1}L^jP_i\|\varphi-\psi\|,\quad \varphi,\psi\in\mathcal{P}_T(P,L),$$

(3.1) means $\frac{1}{|\gamma|}\sum_{i=1}^{n}\sum_{j=0}^{i-1}L^jP_i<1$, so the fixed point must be unique by the Banach fixed point theorem. □

Theorem 3.2 *The unique solution obtained in Theorem* 3.1 *depends continuously on the given functions* $\lambda_i(t)$ *and* $f(t)$ *for* $i=1,2,\dots,n$.

Proof. Let functions $\lambda_i(t)$, $f(t)$ and $\mu_i(t)$, $\widetilde{f}(t)$ in $\mathcal{P}_T(P_i,L_i)$ and $\mathcal{P}_T(P_f,L_f)$ be given. Then we consider the corresponding operators $A,\widetilde{A}$ defined by (2.7). Assuming corresponding conditions (2.9), (2.10) and (3.1), there are two unique corresponding functions $x(t)$ and $\widetilde{x}(t)$ in $\mathcal{P}_T(P,L)$ such that

$$x=Ax,\quad \widetilde{x}=\widetilde{A}\widetilde{x}.$$

Then we have

$$\|x-\widetilde{x}\| \le \|Ax - A\widetilde{x}\| + \|A\widetilde{x} - \widetilde{A}\widetilde{x}\| \le \frac{1}{|\gamma|}\sum_{i=1}^{n}\sum_{j=0}^{i-1} L^j P_i \|x-\widetilde{x}\| + \|A\widetilde{x} - \widetilde{A}\widetilde{x}\|,$$

which implies

$$\|x-\widetilde{x}\| \le \frac{|\gamma|}{|\gamma| - \sum_{i=1}^{n}\sum_{j=0}^{i-1} L^j P_i}\|A\widetilde{x} - \widetilde{A}\widetilde{x}\|.$$

Now, for $t \in [0, T]$, we note

$$\begin{aligned}
& \Big|\int_t^{t+T}\int_u^{u+T}\Big(\sum_{i=1}^{n}(\lambda_i(s)-\mu_i(s))\widetilde{x}^{[i]}(s)\Big)e^{\frac{\beta_1}{\alpha}(s-u)+\frac{\beta_2}{\alpha}(u-t)}dsdu\Big| \\
\le\ & P\sum_{i=1}^{n}\|\lambda_i - \mu_i\|\Big|\int_t^{t+T}\int_u^{u+T} e^{\frac{\beta_1}{\alpha}(s-u)+\frac{\beta_2}{\alpha}(u-t)}dsdu\Big| \\
\le\ & \frac{|\alpha|}{|\gamma|}\frac{P}{|E(\alpha,\beta_1)||E(\alpha,\beta_2)|}\sum_{i=1}^{n}\|\lambda_i - \mu_i\|, \qquad (3.2)
\end{aligned}$$

$$\begin{aligned}
& \Big|\int_t^{t+T}\int_u^{u+T}(f(s)-\widetilde{f}(s))e^{\frac{\beta_1}{\alpha}(s-u)+\frac{\beta_2}{\alpha}(u-t)}dsdu\Big| \\
\le\ & \frac{|\alpha|}{|\gamma|}\frac{1}{|E(\alpha,\beta_1)||E(\alpha,\beta_2)|}\|f-\widetilde{f}\| \qquad (3.3)
\end{aligned}$$

From (3.2)-(3.3), we arrive at

$$\begin{aligned}
& \|x-\widetilde{x}\| \\
\le\ & \frac{|\gamma|}{|\gamma| - \sum_{i=1}^{n}\sum_{j=0}^{i-1} L^j P_i}\|A\widetilde{x} - \widetilde{A}\widetilde{x}\| \\
\le\ & \frac{|\gamma|}{|\alpha|}\frac{|E(\alpha,\beta_1)||E(\alpha,\beta_2)|}{|\gamma| - \sum_{i=1}^{n}\sum_{j=0}^{i-1} L^j P_i}\times \\
& \times\Big(\Big|\int_t^{t+T}\int_u^{u+T}\Big(\sum_{i=1}^{n}(\lambda_i(s)-\mu_i(s))\widetilde{x}^{[i]}(s)\Big)e^{\frac{\beta_1}{\alpha}(s-u)+\frac{\beta_2}{\alpha}(u-t)}dsdu\Big| \\
& +\Big|\int_t^{t+T}\int_u^{u+T}(f(s)-\widetilde{f}(s))e^{\frac{\beta_1}{\alpha}(s-u)+\frac{\beta_2}{\alpha}(u-t)}dsdu\Big|\Big) \\
\le\ & \frac{1}{|\gamma| - \sum_{i=1}^{n}\sum_{j=0}^{i-1} L^j P_i}\Big(P\sum_{i=1}^{n}\|\lambda_i - \mu_i\| + \|f-\widetilde{f}\|\Big).
\end{aligned}$$

This completes the proof. $\square$

Remark 3.1 For Eq. (2.12), we have the following results which similarly as [16].

Theorem 3.3 *Assume the conditions in Theorem* 2.3 *hold and*

$$T\sum_{i=2}^{n}\sum_{j=0}^{i-1}L^jP_i<|\beta|. \tag{3.4}$$

Then (2.12) *has a unique periodic solution in* $\mathcal{P}_T(P,L)$.

Theorem 3.4 *The unique solution obtained in Theorem* 3.3 *depends continuously on the given functions* $\lambda_i(t)$ *and* $f(t)$, $i=1,2,\ldots,n$.

4 Examples

Example 4.1 Let us show that the conditions in Theorem 2.2 do not self-contradict. Consider the following equation:

$$x''(t)+2x'(t)-3x(t)=\frac{1}{10^3}\sin(t)x(t)+\frac{1}{10^3}\cos(t)x(x(t))+\frac{1}{10}\sin(t), \tag{4.1}$$

where $\alpha=1,\beta=2,\gamma=-3,\lambda_1(t)=\frac{1}{10^3}\sin(t),\lambda_2(t)=\frac{1}{10^3}\cos(t),f(t)=\frac{1}{10}\sin(t)$. $\beta_1=\frac{\beta+\sqrt{\beta^2-4\alpha\gamma}}{2}=3,\beta_2=\frac{\beta-\sqrt{\beta^2-4\alpha\gamma}}{2}=-1,\ P_1=P_2=\frac{1}{10^3},L_1=L_2=\frac{1}{10^3},P_f=L_f=\frac{1}{10}$,

$$E(\alpha,\beta_1)=\frac{1}{e^{\frac{\beta_1}{\alpha}T}-1}=\frac{1}{e^{6\pi}-1},\ E(\alpha,\beta_2)=\frac{1}{e^{\frac{\beta_2}{\alpha}T}-1}=\frac{e^{2\pi}}{1-e^{2\pi}},$$

here $T=2\pi$. Take $P=1,L=3269$, then

$$P_f=\frac{1}{10}\le 3-\frac{2}{10^3}=(|\gamma|-P_1-P_2)P,$$

and

$$\begin{aligned}
&|E(\alpha,\beta_2)|(P(P_1+P_2)+P_f)e^{\frac{|\beta_2|}{|\alpha|}T}\Big(1+\frac{1}{|E(\alpha,\beta_2)|}+e^{\frac{|\beta_2|}{|\alpha|}T}\Big)\\
=&\ \frac{e^{2\pi}}{e^{2\pi}-1}\frac{51}{500}e^{2\pi}(2+\frac{1}{e^{2\pi}}+e^{2\pi})=\frac{51e^{4\pi}}{500(e^{2\pi}-1)}(2+\frac{1}{e^{2\pi}}+e^{2\pi})\\
<&\ 9804.31\\
<&\ 9807=|\beta_1|L,
\end{aligned}$$

By Theorem 2.2, Eq. (4.1) has a 2π-periodic solution x such that $|x(t)|\le 1$, and $|x(t_2)-x(t_1)|\le 3269|t_2-t_1|,\ \forall t_1,t_2\in\mathbb{R}$.

Example 4.2 Let us show that the conditions in Theorem 3.1 do not self-contradict. Consider the following equation:

$$x''(t) + 2x'(t) - 3x(t) = \frac{1}{10^3}\sin(t)x(t) + \frac{1}{10^3}\cos(t)x(x(t)) + \frac{1}{100}\sin(t), \qquad (4.2)$$

where $\alpha = 1, \beta = 2, \gamma = -3, \lambda_1(t) = \frac{1}{10^3}\sin(t), \lambda_2(t) = \frac{1}{10^3}\cos(t), f(t) = \frac{1}{100}\sin(t)$. $\beta_1 = \frac{\beta+\sqrt{\beta^2-4\alpha\gamma}}{2} = 3, \beta_2 = \frac{\beta-\sqrt{\beta^2-4\alpha\gamma}}{2} = -1,\ P_1 = P_2 = \frac{1}{10^3}, L_1 = L_2 = \frac{1}{10^3}, P_f = L_f = \frac{1}{100}$,

$$E(\alpha,\beta_1) = \frac{1}{e^{\frac{\beta_1}{\alpha}T} - 1} = \frac{1}{e^{6\pi}-1},\ E(\alpha,\beta_2) = \frac{1}{e^{\frac{\beta_2}{\alpha}T} - 1} = \frac{e^{2\pi}}{1-e^{2\pi}},$$

here $T = 2\pi$. Take $P = 1, L = 385$, then

$$P_f = \frac{1}{100} \le 3 - \frac{2}{10^3} = (|\gamma| - P_1 - P_2)P,$$

and

$$\begin{aligned} & |E(\alpha,\beta_2)|(P(P_1+P_2)+P_f)e^{\frac{|\beta_2|}{|\alpha|}T}\Big(1 + \frac{1}{|E(\alpha,\beta_2)|} + e^{\frac{|\beta_2|}{|\alpha|}T}\Big) \\ = \ & \frac{e^{2\pi}}{e^{2\pi}-1}\frac{3}{250}e^{2\pi}(2 + \frac{1}{e^{2\pi}} + e^{2\pi}) = \frac{3e^{4\pi}}{250(e^{2\pi}-1)}(2 + \frac{1}{e^{2\pi}} + e^{2\pi}) \\ < \ & 1153.45 \\ < \ & 1155 = |\beta_1|L, \end{aligned}$$

By Theorem 2.2, Eq. (4.2) has a 2π-periodic solution x such that $|x(t)| \le 1$, and $|x(t_2) - x(t_1)| \le 385|t_2 - t_1|,\ \forall t_1, t_2 \in \mathbb{R}$. Furthermore, we see

$$P_1 + P_2(1 + L) = 0.387 < 3 = |\gamma|,$$

then by Theorem 3.1, Eq. (4.2) has a unique 2π-periodic solution.

We see that the solution of Example 4.2 satisfies the properties of a solution of Example 4.1 and we do not know if a solution of Example 4.1 is different from the one of Example 4.2. Thus, we discuss this in the next case.

Example 4.3 Consider

$$x''(t) + 2x'(t) - 3x(t) = \frac{1}{10^3}\sin(t)x(t) + \frac{1}{10^3}\cos(t)x(x(t)) + \delta\sin(t), \qquad (4.3)$$

where $\delta > 0$ is a parameter. Noting $\alpha = 1, \beta = 2, \gamma = -3,\ \beta_1 = \frac{\beta+\sqrt{\beta^2-4\alpha\gamma}}{2} = 3, \beta_2 = \frac{\beta-\sqrt{\beta^2-4\alpha\gamma}}{2} = -1$ and $\lambda_1(t) = \frac{1}{10^3}\sin(t), \lambda_2(t) = \frac{1}{10^3}\cos(t)$ as in Example 4.1. Here $f(t) = \delta\sin(t)$, taking $P_f = L_f = \delta,\ P_1 = P_2 = L_1 = L_2 = \frac{1}{10^3}$.

$$E(\alpha,\beta_1) = \frac{1}{e^{\frac{\beta_1}{\alpha}T} - 1} = \frac{1}{e^{6\pi}-1},\ E(\alpha,\beta_2) = \frac{1}{e^{\frac{\beta_2}{\alpha}T} - 1} = \frac{e^{2\pi}}{1-e^{2\pi}},$$

here $T = 2\pi$. Next, we consider $P(\delta)$ and $L(\delta)$ as variables to be defined by δ. Then (2.9), (2.10) and (3.1) have the forms

$$\delta \leq \frac{1499}{500} P(\delta), \tag{4.4}$$

$$\frac{e^{4\pi}}{e^{2\pi}-1}(\frac{P(\delta)}{500} + \delta)(2 + \frac{1}{e^{2\pi}} + e^{2\pi}) < 3L(\delta), \tag{4.5}$$

$$\frac{1}{10^3} + \frac{1}{10^3}(1 + L(\delta)) < 3, \tag{4.6}$$

By (4.4) and (4.5), we have

$$P(\delta) \geq \frac{500}{1499}\delta \tag{4.7}$$

and

$$L(\delta) > \frac{\delta e^{4\pi}}{e^{2\pi}-1}\frac{500}{1499}(2 + \frac{1}{e^{2\pi}} + e^{2\pi}) \doteq \delta \times 3377.856428315845. \tag{4.8}$$

From (4.6), we get

$$L(\delta) < 2998. \tag{4.9}$$

For (4.5), by (4.7) and (4.9), we have

$$\frac{500}{1499}\delta \leq P(\delta) < 500\left(\frac{8994(e^{2\pi}-1)}{e^{2\pi}(e^{4\pi}+2e^{2\pi}+1)} - \delta\right). \tag{4.10}$$

and

$$0 < \delta < \frac{1499}{1500}\left(\frac{8994(e^{2\pi}-1)}{e^{2\pi}(e^{4\pi}+2e^{2\pi}+1)}\right) \doteq 0.010376817040576716. \tag{4.11}$$

Thus, if we taking $P(\delta)$ and $L(\delta)$ satisfy (4.7) and (4.8), Theorem 2.2 is satisfied and Eq. (4.3) has a 2π-periodic solution such that $|x(t)| \leq P(\delta)$ and $|x(t_2)-x(t_1)| \leq L(\delta)|t_2-t_1|$, $\forall t_1, t_2 \in \mathbb{R}$. Furthermore, if we taking $P(\delta)$ and $L(\delta)$ satisfy (4.8)-(4.11), by Theorem 2.3, we know the 2π-periodic solution of (4.3) is a unique one.

Remark 4.1 Obviously, in Example 4.1, $P = 1, L = 3269, \delta = \frac{1}{10}$ satisfy (4.7) and (4.8), so (4.1) has periodic solutions. In Example 4.2, $P = 1, L = 385, \delta = \frac{1}{100}$ satisfies (4.8)-(4.11), then (4.2) has a unique periodic solution.

Example 4.4 Now, let us consider the first order differential equation

$$2x'(t) - 3x(t) = \frac{1}{10^3}\sin(t)x(t) + \frac{1}{10^3}\cos(t)x(x(t)) + \delta\sin(t), \tag{4.12}$$

where $\delta > 0$ is a parameter. As Example 4.3, $\beta = 2, \gamma = -3$, $\lambda_1(t) = \frac{1}{10^3}\sin(t)$, $\lambda_2(t) = \frac{1}{10^3}\cos(t)$, $f(t) = \delta\sin(t)$, taking $P_f = L_f = \delta$, $P_1 = P_2 = L_1 = L_2 = \frac{1}{10^3}$, and $T = 2\pi$. We see $\overline{M} = \frac{1}{e^{\frac{2999}{1000}\pi}-1}$, Then (2.18), (2.19) and (3.4) have the forms

$$\frac{2\pi}{e^{\frac{2999}{1000}\pi} - 1}(\frac{1}{10^3}P(\delta) + \delta) \leq 2P(\delta), \tag{4.13}$$

$$\frac{1}{e^{\frac{2999}{1000}\pi} - 1}(\frac{1}{1000}P(\delta) + \delta)e^{\frac{3001}{1000}\pi}(2 + \frac{3001}{1000}\pi) < 2L(\delta), \tag{4.14}$$

$$2\pi(1 + L(\delta))\frac{1}{1000} < 2. \tag{4.15}$$

By (4.13) and (4.14) we have

$$P(\delta) \geq \frac{\pi\delta}{e^{\frac{2999}{1000}\pi} - 1 - \frac{\pi}{1000}} \tag{4.16}$$

and

$$L(\delta) > \frac{P(\delta) + 1000\delta}{2000(e^{\frac{2999}{1000}\pi} - 1)}(2 + \frac{3001}{1000}\pi)e^{\frac{3001}{1000}\pi}. \tag{4.17}$$

Next, (4.15) shows us

$$L(\delta) < \frac{1000}{\pi} - 1. \tag{4.18}$$

For (4.16), by (4.17) and (4.18), we have

$$\frac{\pi\delta}{e^{\frac{2999}{1000}\pi} - 1 - \frac{\pi}{1000}} \leq P(\delta) < \frac{2 \times 10^6(1000 - \pi)(e^{\frac{2999}{1000}\pi} - 1)}{\pi(2000 + 3001\pi)e^{\frac{3001}{1000}\pi}} - 1000\delta. \tag{4.19}$$

and

$$0 < \delta < \frac{2 \times 10^6(1000 - \pi)(e^{\frac{2999}{1000}\pi} - 1)}{\pi(2000 + 3001\pi)e^{\frac{3001}{1000}\pi}\left(\frac{\pi}{e^{\frac{2999}{1000}\pi}-1-\frac{\pi}{1000}} + 1000\right)} \doteq 55.18009. \tag{4.20}$$

Thus, if we taking $P(\delta)$ and $L(\delta)$ satisfy (4.16) and (4.17), Theorem 2.3 is satisfied and Eq. (4.12) has a 2π-periodic solution such that $|x(t)| \leq P(\delta)$ and $|x(t_2) - x(t_1)| \leq L(\delta)|t_2 - t_1|$, $\forall t_1, t_2 \in \mathbb{R}$. Furthermore, if we taking $P(\delta)$ and $L(\delta)$ satisfy (4.16)-(4.20), by Theorem 3.3, we know the 2π-periodic solution of (4.12) is a unique one.

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