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Existence and Uniqueness Results for Hilfer-Generalized Proportional Derivatives with Nonlocal Conditions †

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Abstract

In this paper, motivated by Hilfer and Hilfer-Katugampola fractional derivatives, we introduce new Hilfer-generalized proportional derivatives which interpolate the classical fractional derivatives of Hilfer, Riemann-Liouville, Caputo and generalized proportional fractional derivatives. We also present some important properties of the proposed derivatives. Furthermore, as an application, we show that this equation is equivalent to the Volterra integral equation and prove the existence, uniqueness of the solution to the Cauchy problem with the nonlocal initial condition. Finally, two examples were given to illustrate the results.

KEYWORDS:
Existence, Volterra integral equation, fixed point theorems, nonlocal condition, generalized proportional fractional derivative

1 | INTRODUCTION

For the last few decades, the fractional calculus, which is concerned in integrals and derivatives of arbitrary order of functions, have captured the interest of many scientists as an outcome of the good results they obtained when these scientists involved

† On Hilfer Proportional Derivatives.

the fractional integrals and derivatives in their reseaches for the sake of finding all the better of mathematical modelling of real world phenomena. The straight of it, the fractional calculus is as old as the classical calculus the tools of which are sometimes found uncapable for modeling some complex systems and it turned out the tools that the fractional calculus include are splendid when used modeling long-memory processes and many phenomena that appear in physics, chemistry, electricity, mechanics and many other disciplines [1,2,3,4,5,6].

Nonetheless, scientists have felt the exigence of other types of fractional operators that were limited to Riemann-Liouville fractional operators and Caputo fractional derivative till the turn of this century. Many scientists have proposed a variety of new fractional operators that contributed in the developement of the fractional calculus. Among these operators we mention the ones discussed in [7,8,9,10,11,12]. It is worthy mentioning that the fractional operators proposed in these works are specific cases of fractional integrals/derivatives with respect to another function that were mentioned in [2,5,13]. But all of these operators possess one of the most important peculiarity of the fractional operators: non-locality.

In 2015, the authors in [14] proposed a local derivative twith a non-integer order called it conformable derivative. The fractionalizion process of these local derivative will result in rediscovering the fractional non-local operators presented in [8,9]. In discussed the main concepts of the conformable derivatives and proposed the left and right conformable derivatives. Again it can be seen that the nonlocal fractional version of the proposed in [15] are presented in [12].

In all types of calculus fractional or calculus derivative of order 0 of a function should be equal to the function. This basic property of any derivative is lacked by the conformable derivative and in fact it is a deficit. To bypass this deficit, the authors in [16,17], redefined the conformal derivative so that it yields the function itself when the order of this local derivative is 0. This was followed by the work of Jarad et al. [18] where the authors proposed the fractional version of the redefined conformable derivative.

One of the most significant qualitative properties of differential equations is the existence and uniqueness of their solutions. The existence and uniqueness of solutions to differential equations that involve variety types of fractional derivatives and are governed by different types of intial/boundary conditions were tackled many scientists (see [20,21,22,24,25,26,27,28,29] and the references cited in them). But to the last of our knowledge no one has discussed the existence and uniqueness of solutions to impulsive differential equations including proportional fractional derivatives in the Hilfer setting. In this paper, we discuss the existence and uniqueness of solutions of a certain type of differential equations subject to non-local conditions and containing a Hilfer type proportional fractional derivative which will be defined. We also discuss the existence and uniqueness of solutions for the following initial value problem:

$$\begin{cases} \mathcal{D}_{a^+}^{p,q,\rho} x(t) = f(t, x(t)), & t \in J = [a, T], T > a \geq 0, \\ \mathcal{I}_{a^+}^{1-\gamma,\rho} x(a) = \sum_{i=1}^{m} c_i x(\tau_i), & p \leq \gamma = p + q - pq, \tau_i \in (a, T), \end{cases} \tag{1}$$

where $\mathcal{D}_{a^+}^{p,q,\rho}(\cdot)$ is the Hilfer generalized proportional derivative of order $(0 < p < 1)$, $\mathcal{I}_{a^+}^{1-\gamma,\rho}(\cdot)$ is the proportional fractional integral of order $1-\gamma > 0$, $c_i \in \mathbb{R}$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $\tau_i \in J$ satisfying $a < \tau_1 < \cdots < \tau_m < T$ for $i = 1, \cdots m$.

The remainder of the paper is composed as follows. In Section 2, we review some fundamental results that we need in the continuation. In Section 3, we define our proposed derivatives, the Hilfer-generalized proportional derivatives together with some it's preliminaries properties. In addition, we also examine the comparability between an initial value problem and a Volterra integral equation, from which we prove the existence and uniqueness of solution using Banach and Kransnoselkii's fixed point theorems. Examples clarify the obtained results are conferred in Section 4.

2 | PRELIMINARIES

In this section, we give some preliminaries facts, results and definitions of fractional calculus which are essential throughout this paper.

Let $-\infty < a < b < \infty$ be finite and infinite interval on $\mathbb{R}_+$. Denote $C[a, b]$, the spaces of continuous function f on $[a, b]$ with norm is defined by [5]

$$\|f\|_{C[a,b]} = \max_{t\in[a,b]} |f(t)|,$$

and $\mathcal{AC}^n[a, b]$, the space of n-times absolutely continuous, given by

$$\mathcal{AC}^n[a, b] = \{f : (a, b] \to \mathbb{R}; f^{n-1} \in \mathcal{AC}([a, b])\}.$$

The weighted space $C_\gamma[a,b]$ of a functions f on $(a,b]$ is defined by

$$C_\gamma[a,b] = \{f : (a,b] \to \mathbb{R}; (t-a)^\gamma f(t) \in C([a,b])\}, \quad 0 \le \gamma < 1,$$

with the norm

$$\|f\|_{C_\gamma[a,b]} = \|(t-a)^\gamma f(t)\|_{C[a,b]} = \max_{t\in[a,b]} |(t-a)^\gamma f(t)|.$$

The weighted space $C_\gamma^n[a,b]$ of a functions f on $(a,b]$ is defined by

$$C_\gamma^n[a,b] = \{f : (a,b] \to \mathbb{R}; f(t) \in C^{n-1}([a,b]); f^n(t) \in C_\gamma([a,b])\}, \quad 0 \le \gamma < 1,$$

with the norm

$$\|f\|_{C_\gamma^n[a,b]} = \sum_{k=0}^{n-1} \|f^k\|_{C[a,b]} + \|f^n\|_{C_\gamma[a,b]}.$$

Thus, for $n = 0$, we have $C_\gamma^0[a,b] = C_\gamma[a,b]$.

Definition 1. The fractional integral of order p with the lower limit a^+ for a function f is defined as

$$I_{a^+}^p f(t) = \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} f(\tau) d\tau, \quad p > 0, \quad n \in \mathbb{N},$$

provided the right side is point-wise defined on $[a,\infty)$, where $\Gamma(\cdot)$ denotes the gamma function.

Definition 2. The Reimann-Liouville derivative of order p with the lower limit a^+ for a function f is defined as

$${}^L D_{a+}^p f(t) = \frac{1}{\Gamma(n-p)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-p-1} f(\tau) d\tau, \quad p > 0, \quad n-1 < p < n, \quad n \in \mathbb{N},$$

provided the function f is absolutely continuous up to order $(n-1)$ derivatives,where $\Gamma(\cdot)$ denotes the gamma function.

Definition 3. [18] The left proportional fractional integral of order p and $\rho \in (0,1]$ of a function f is defined by

$$I_{a^+}^{p,\rho} f(t) = \frac{1}{\rho^p \Gamma(p)} \int_{a^+}^t e^{\frac{\rho-1}{\rho}(t-s)} (t-s)^{p-1} f(s) ds, \quad p \in \mathbb{C}, Re(p) > 0, \tag{2}$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 4. [18] The left proportional fractional derivative of order p and $\rho \in (0,1]$ of a function f is defined by

$$D_{a^+}^{p,\rho} f(t) = \frac{D^{n,\rho}}{\rho^{n-p}\Gamma(n-p)} \int_{a^+}^t e^{\frac{\rho-1}{\rho}(t-s)} (t-s)^{n-p-1} f(s) ds, \quad p \in \mathbb{C}, Re(p) > 0, \tag{3}$$

where $\Gamma(\cdot)$ is the Gamma-function and $n = [p] + 1$.

Remark 1. Observe that if $\rho = 1$ Definitions 3 and 4 coincides with the classical Definitions of Riemann-Liouville fractional integral and derivative (see, Definitions 1 and 2).

Next, we consider some significant properties of the proportional fractional derivative and integral operator as follow:

Proposition 1. [18] Let $p, \delta \in \mathbb{C}$ such that $Re(p) \ge 0$ and $Re(\delta) > 0$. Then, for any $\rho \in (0,1]$ we have

$$\begin{gathered}
\left(I_a^{p,\rho} e^{\frac{\rho-1}{\rho}s}(s-a)^{\delta-1}\right)(t) = \frac{\Gamma(\delta)}{\rho^p\Gamma(\delta+p)} e^{\frac{\rho-1}{\rho}t}(t-a)^{\delta+p-1}. \\
\left(D_a^{p,\rho} e^{\frac{\rho-1}{\rho}s}(s-a)^{\delta-1}\right)(t) = \frac{\rho^p\Gamma(\delta)}{\Gamma(\delta-p)} e^{\frac{\rho-1}{\rho}t}(t-a)^{\delta-p-1}. \\
\left(I_b^{p,\rho} e^{\frac{\rho-1}{\rho}(b-s)}(b-s)^{\delta-1}\right)(t) = \frac{\Gamma(\delta)}{\rho^p\Gamma(\delta+p)} e^{\frac{\rho-1}{\rho}(b-t)}(b-t)^{\delta+p-1}. \\
\left(D_b^{p,\rho} e^{\frac{\rho-1}{\rho}(b-s)}(b-s)^{q-1}\right)(t) = \frac{\rho^p\Gamma(\delta)}{\Gamma(\delta-p)} e^{\frac{\rho-1}{\rho}(b-t)}(b-t)^{\delta-p-1}.
\end{gathered} \tag{4}$$

Theorem 1. [18] If $\rho \in (0,1]$, $Re(p) > 0$ and $Re(q) > 0$. Then, if f is continuous and defined for $t \ge a$, we have

$$I_{a^+}^{p,\rho}(I_{a^+}^{q,\rho} f)(t) = I_{a^+}^{q,\rho}(I_{a^+}^{p,\rho} f)(t) = (I_{a^+}^{p+q,\rho} f)(t). \tag{5}$$

Theorem 2. [18] Let $\rho \in (0,1]$, $0 \leq m < [Re(p)] + 1$ and f be integrable function in each interval $[a,t]$, $t > a$. Then,

$$D^{m,\rho}_{a^+}(I^{p,\rho}_{a^+} f)(t) = (I^{p-m,\rho}_{a^+} f)(t). \tag{6}$$

Corollary 1. [18] Let $0 < Re(q) < Re(p)$ and $m-1 < Re(q) \leq m$. Then, we have

$$D^{q,\rho}_{a^+} I^{p,\rho}_{a^+} f(t) = I^{p-q,\rho}_{a^+} f(t).$$

Theorem 3. [18] Let f be integrable on $t \geq a$ and $Re(p) > 0$, $\rho \in (0,1]$, $n = [Re(p)] + 1$. Then, we have

$$D^{p,\rho}_{a^+} I^{p,\rho}_{a^+} f(t) = f(t).$$

Lemma 1. [18] For $p > 0$, $\rho \in (0,1]$ and m is a positive integer we have

$$(I^{p,\rho}_{a^+} D^{m,\rho}_{a^+} f)(t) = (D^{m,\rho}_{a^+} I^{p,\rho}_{a^+} f)(t) - \sum_{k=0}^{m-1} \frac{e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{p-m+k}}{\rho^{p-m+k}\Gamma(p+k-m+1)}(D^{k,\rho}_{a^+} f)(a). \tag{7}$$

In particular, if $m = 1$, we can obtain

$$(I^{p,\rho}_{a^+} D^{\rho}_{a^+} f)(t) = (D^{\rho}_{a^+} I^{p,\rho}_{a^+} f)(t) - \frac{e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{p-1}}{\rho^{p-1}\Gamma(p)} f(a). \tag{8}$$

Theorem 4. [18] Let $Re(p) > 0$, $n = -[-Re(p)]$, $f \in L_1(a,b)$ and $(I^{p,\rho}_{a^+} f)(t) \in AC^n[a,b]$. Then,

$$(I^{p,\rho}_{a^+} D^{p,\rho}_{a^+} f)(t) = f(t) - e^{\frac{\rho-1}{\rho}(t-a)} \sum_{j=1}^{n} \frac{(t-a)^{p-j}}{\rho^{p-j}\Gamma(p-j+1)}(I^{j-p,\rho}_{a} f)(a^+). \tag{9}$$

3 | MAIN RESULTS

In this section, we introduce the Hilfer proportional fractional derivatives and discuss some properties, equivalence with the Volterra integral equation. Furthermore, we study the existence and uniqueness of the Cauchy problem using fixed point theorems.

Definition 5. Let $n-1 < p < n$, $\rho \in (0,1]$ and $0 \leq q \leq 1$, with $n \in \mathbb{N}$. The left-sided/right-sided generalized proportional fractional derivative of order p and type q of a function f is defined by

$$\left(\mathcal{D}^{p,q,\rho}_{a\pm} f\right)(x) = \mathcal{I}^{q(n-p),\rho}_{a\pm}\left[\mathcal{D}^{\rho}\left(\mathcal{I}^{(1-q)(n-p),\rho}_{a\pm} f\right)\right](x), \tag{10}$$

where $\mathcal{D}^{\rho} f(x) = (1-\rho)f(x) + \rho f'(x)$ and $\mathcal{I}$ is the generalized proportional fractional integral defined in equation (2).

In particular, if $n = 1$, Definition 5 is equivalent with

$$\left(\mathcal{D}^{p,q,\rho}_{a\pm} f\right)(x) = \mathcal{I}^{q(1-p),\rho}_{a\pm}\left[\mathcal{D}^{\rho}\left(\mathcal{I}^{(1-q)(1-p),\rho}_{a\pm} f\right)\right](x). \tag{11}$$

Thus, throughout this paper, we discuss the case where $n = 1$, $0 < p < 1$, $0 \leq q \leq 1$ and $\gamma = p + q - pq$.

Remark 2. It is worth to specify that:

- If $\rho = 1$, Definition (5) coincides with the ones in [19] and if $q = 0$ coincide with the one in [18].
- The derivative is considered as an interpolator between the Riemann-Liouville fractional and Caputo fractional derivatives since

$$\mathcal{D}^{p,q,\rho}_{a^+} = \begin{cases} \mathcal{D}^{p}_{a^+}, & \rho \to 1, q = 0, \\ \mathcal{I}^{1-p}_{a^+}\mathcal{D}, & \rho \to 1, q = 1. \end{cases} \tag{12}$$

- The parameter γ satisfies

$$0 < \gamma \leq 1, \quad \gamma \geq p, \quad \gamma > q \quad 1-\gamma < 1 - q(1-p).$$

Property 1. The operator can be written as

$$\mathcal{D}^{p,q,\rho}_{a^+} = \mathcal{I}^{q(1-p),\rho}_{a^+}\mathcal{D}^{\rho}\mathcal{I}^{(1-\gamma),\rho}_{a^+} = \mathcal{I}^{q(1-p),\rho}_{a^+}\mathcal{D}^{\gamma,\rho}_{a^+}, \quad \gamma = p + q - pq.$$

Proof. From the above definition, we have

$$\begin{aligned}\left(\mathcal{D}_{a^+}^{p,q,\rho} f\right)(x) &= \mathcal{I}_{a^+}^{q(1-p),\rho}\left[\mathcal{D}^{\rho}\left(\mathcal{I}_{a^+}^{(1-q)(1-p),\rho} f\right)\right](x)\\ &= \mathcal{I}_{a^+}^{q(1-p),\rho}\left\{\frac{\mathcal{D}^{\rho}}{\rho^{(1-\gamma)}\Gamma((1-\gamma))}\int_a^t e^{\frac{\rho-1}{\rho}(t-\tau)}(t-\tau)^{(1-\gamma)-1} f(\tau)d\tau\right\}\\ &= \left(\mathcal{I}_{a^+}^{q(1-p),\rho}\mathcal{D}^{\gamma,\rho} f\right)(x).\end{aligned}$$

□

We introduce the following spaces

$$C_{1-\gamma}^{p,q}[a,b] = \{f \in C_{1-\gamma}[a,b], \mathcal{D}_{a^+}^{p,q,\rho} f \in C_{1-\gamma}[a,b]\}$$

and

$$C_{1-\gamma}^{\gamma}[a,b] = \{f \in C_{1-\gamma}[a,b], \mathcal{D}_{a^+}^{\gamma,\rho} f \in C_{1-\gamma}[a,b]\}.$$

Since $\mathcal{D}_{a^+}^{p,q,\rho} = \mathcal{I}_{a^+}^{q(1-p),\rho}\mathcal{D}_{a^+}^{\gamma,\rho}$, it then becomes

$$C_{1-\gamma}^{\gamma}[a,b] \subset C_{1-\gamma}^{p,q}[a,b].$$

Lemma 2. Suppose $0 < p < 1$, $\rho \in (0,1]$ and $0 \le \gamma < 1$. If $f \in C_{\gamma}[a,b]$ then

$$\mathcal{I}_{a^+}^{p,\rho} f(a) = \lim_{x\to a^+}\mathcal{I}_{a^+}^{p,\rho} f(x) = 0, \quad 0 \le \gamma < p.$$

Proof. Considering $f \in C[a,b]$, it implies that $f \in C_{\gamma}[a,b]$ and $(x-a)^{\gamma} \in C[a,b]$. Therefore, there exist $M > 0$ such that

$$(x-a)^{\gamma} f(x) < M, \quad x \in [a,b],$$

and

$$|\mathcal{I}_{a^+}^{p,\rho} e^{\frac{\rho-1}{\rho}t} f(x)| < M\left[\mathcal{I}_{a^+}^{p,\rho} e^{\frac{\rho-1}{\rho}t}(t-a)^{-\gamma}\right](x).$$

It follows from (1), that

$$|\mathcal{I}_{a^+}^{p,\rho} e^{\frac{\rho-1}{\rho}t} f(x)| < M\left[\frac{\Gamma(1-\gamma)}{\Gamma(p+1-\gamma)} e^{\frac{\rho-1}{\rho}x}(x-a)^{p-\gamma}\right],$$

which implies that, the right-hand side $\to 0$ as $x \to a^+$. □

Lemma 3. Let $0 < p < 1$, $\rho \in (0,1]$, $0 \le q \le 1$ and $\gamma = p + q - pq$. If $f \in C_{1-\gamma}^{\gamma}[a,b]$ then

$$\mathcal{I}_{a^+}^{\gamma,\rho}\mathcal{D}_{a^+}^{\gamma,\rho} f = \mathcal{I}_{a^+}^{p,\rho}\mathcal{D}_{a^+}^{p,q,\rho} f$$

and

$$\mathcal{D}_{a^+}^{\gamma,\rho}\mathcal{I}_{a^+}^{p,\rho} f = \mathcal{D}_{a^+}^{q(1-p),\rho} f.$$

Proof. Using Theorem 1 and property (1), yields

$$\begin{aligned}\mathcal{I}_{a^+}^{\gamma,\rho}\mathcal{D}_{a^+}^{\gamma,\rho} f &= \mathcal{I}_{a^+}^{\gamma,\rho}\left(\mathcal{I}_{a^+}^{-q(1-p),\rho}\mathcal{D}_{a^+}^{p,q,\rho} f\right)\\ &= \mathcal{I}_{a^+}^{p+q-pq,\rho}\mathcal{I}_{a^+}^{-q(1-p),\rho}\mathcal{D}_{a^+}^{p,q,\rho} f\\ &= \mathcal{I}_{a^+}^{p,\rho}\mathcal{D}_{a^+}^{p,q,\rho} f.\end{aligned}$$

Furthermore, in view of Theorem 1 and Definition (5), we can see that

$$\begin{aligned}\mathcal{D}_{a^+}^{\gamma,\rho}\mathcal{I}_{a^+}^{p,\rho} f &= \mathcal{D}_{a^+}^{\rho}\mathcal{I}_{a^+}^{1-\gamma,\rho}\mathcal{I}_{a^+}^{p,\rho} f\\ &= \mathcal{D}_{a^+}^{\rho}\mathcal{I}_{a^+}^{1-q+pq,\rho} f\\ &= \mathcal{D}_{a^+}^{q(1-p),\rho} f.\end{aligned}$$

□

Lemma 4. Suppose $f \in L^1(a,b)$ such that $\mathcal{D}_{a^+}^{q(1-p),\rho} f$ exists in $L^1(a,b)$. Then

$$\mathcal{D}_{a^+}^{p,q,\rho} \mathcal{I}_{a^+}^{p,\rho} f = \mathcal{I}_{a^+}^{q(1-p),\rho} \mathcal{D}_{a^+}^{q(1-p),\rho} f.$$

Proof. It follows from Definitions 4 and 10, that

$$\begin{aligned} \mathcal{D}_{a^+}^{p,q,\rho} \mathcal{I}_{a^+}^{p,\rho} f &= \mathcal{I}_{a^+}^{q(1-p),\rho} \mathcal{D}_{a^+}^{\rho} \mathcal{I}_{a^+}^{(1-q)(1-p),\rho} \\ &= \mathcal{I}_{a^+}^{q(1-p),\rho} \mathcal{D}_{a^+}^{\rho} \mathcal{I}_{a^+}^{1-q(1-p),\rho} \\ &= \mathcal{I}_{a^+}^{q(1-p),\rho} \mathcal{D}_{a^+}^{q(1-p),\rho} f. \end{aligned}$$

□

Lemma 5. Let $0 < p < 1$, $\rho \in (0,1]$, and $0 \leq \gamma < 1$. If $f \in C_\gamma[a,b]$ and $\mathcal{I}_{a^+}^{1-p,\rho} f \in C_\gamma^1[a,b]$, then

$$\mathcal{I}_{a^+}^{p,\rho} \mathcal{D}_{a^+}^{p,\rho} f(x) = f(x) - e^{\frac{\rho-1}{\rho}(x-a)} \frac{(x-a)^{p-1}}{\rho^{p-1}\Gamma(p)} (I_a^{1-p,\rho} f)(a^+),$$

for all $x \in (a,b]$.

Proof. The proof is similar to the ones in [18]. □

Lemma 6. Let $0 < p < 1$, $\rho \in (0,1]$, $0 \leq q \leq 1$ and $\gamma = p + q - pq$. If $f \in C_{1-\gamma}[a,b]$ and $\mathcal{D}_{a^+}^{p,q,\rho} f$ then $\mathcal{D}_{a^+}^{p,q,\rho} \mathcal{I}_{a^+}^{p,\rho} f$ exists in (a,b) and

$$\mathcal{D}_{a^+}^{p,q,\rho} \mathcal{I}_{a^+}^{p,\rho} f(x) = f(x), \quad x \in (a,b].$$

Proof. Now, using Lemmas 2, 4 and 5, we have

$$\begin{aligned} \left(\mathcal{D}_{a^+}^{p,q,\rho} \mathcal{I}_{a^+}^{p,\rho} f\right)(x) &= \left(\mathcal{I}_{a^+}^{q(1-p),\rho} \mathcal{D}_{a^+}^{q(1-p),\rho} f\right)(x) \\ &= f(x) - e^{\frac{\rho-1}{\rho}(x-a)} \frac{(x-a)^{q(1-p)-1}}{\rho^{q(1-p)-1}\Gamma(q(1-p))} (I_a^{1-q(1-p),\rho} f)(a^+) \\ &= f(x). \end{aligned}$$

□

Lemma 7. Let $0 < p < 1$, $\rho \in (0,1]$, $0 \leq q \leq 1$ and $0 < \gamma < 1$. If $f \in C_{1-\gamma}[a,b]$ and $\mathcal{I}_{a^+}^{1-\gamma,\rho} f$, then

$$\mathcal{I}_{a^+}^{p,\rho} \mathcal{D}_{a^+}^{p,q,\rho} f(x) = f(x) - e^{\frac{\rho-1}{\rho}(x-a)} \frac{(x-a)^{\gamma-1}}{\rho^{\gamma-1}\Gamma(\gamma)} (I_a^{1-\gamma,\rho} f)(a^+), \quad x \in (a,b].$$

Proof. It follows from Definition 5 and Lemma 5, we have

$$\begin{aligned} \left(\mathcal{I}_{a^+}^{p,\rho} \mathcal{D}_{a^+}^{p,q,\rho} f\right)(x) &= \mathcal{I}_{a^+}^{p,\rho} \left(\mathcal{I}_{a^+}^{\gamma-p,\rho} \mathcal{D}_{a^+}^{\gamma,\rho} f\right)(x) \\ &= \mathcal{I}_{a^+}^{\gamma,\rho} \mathcal{D}_{a^+}^{\gamma,\rho} f(x) \\ &= f(x) - e^{\frac{\rho-1}{\rho}(x-a)} \frac{(x-a)^{\gamma-1}}{\rho^{\gamma-1}\Gamma(\gamma)} (I_a^{1-\gamma,\rho} f)(a^+). \end{aligned}$$

□

3.1 | Equivalent mixed type Volterra integral equation

We consider the following Hilfer proportional fractional differential equation with nonlocal condition described by:

$$\begin{cases} \mathcal{D}_{a^+}^{p,q,\rho} x(t) = f(t, x(t)), & t \in J = [a,T], \\ \mathcal{I}_{a^+}^{1-\gamma,\rho} x(a) = \sum_{i=1}^m c_i x(\tau_i), & p \leq \gamma = p + q - pq, \tau_i \in [a,T]. \end{cases} \tag{13}$$

Lemma 8. Let $0 < p < 1$, $0 \leq q \leq 1$ and $\gamma = p + q - pq$ and let $f : J \times \mathbb{R} \to \mathbb{R}$ be a function such that $f \in C_{1-\gamma}[J, \mathbb{R}]$ for any $x \in C_{1-\gamma}[J, \mathbb{R}]$. If $x \in C^{\gamma}_{1-\gamma}[J, \mathbb{R}]$ then x satisfies problems (13) if and only if x satisfies the mixed-type integral equation:

$$\begin{aligned} x(t) =& \frac{\Lambda}{\rho^p \Gamma(p)} e^{\frac{(\rho-1)}{\rho}(t-a)} (t-a)^{\gamma-1} \sum_{i=1}^{m} c_i \int_{a^+}^{\tau_i} e^{\frac{(\rho-1)}{\rho}(\tau_i - s)} (\tau_i - s)^{p-1} f(s, x(s)) ds \\ &+ \frac{1}{\rho^p \Gamma(p)} \int_{a^+}^{t} e^{\frac{(\rho-1)}{\rho}(t-s)} (t-s)^{p-1} f(s, x(s)) ds, \end{aligned} \tag{14}$$

where

$$\Lambda = \frac{1}{\rho^{\gamma-1}\Gamma(\gamma) - \sum_{i=1}^{m} c_i e^{\frac{(\rho-1)}{\rho}(\tau_i - a)} (\tau_i - a)^{\gamma-1}}. \tag{15}$$

Proof. Suppose, $x \in C^{\gamma}_{1-\gamma}[J, \mathbb{R}]$ be a solution of (13). We show that x is also a solution of (14). In view of Lemma 7, we have

$$x(t) = \frac{(t-a)^{\gamma-1}}{\rho^{\gamma-1}\Gamma(\gamma)} e^{\frac{(\rho-1)}{\rho}(t-a)} \mathcal{I}^{1-\gamma,\rho}_{a^+} x(a^+) + \frac{1}{\rho^p \Gamma(p)} \int_{a^+}^{t} e^{\frac{(\rho-1)}{\rho}(t-s)} (t-s)^{p-1} f(s, x(s)) ds. \tag{16}$$

Now, substituting $t = \tau_i$ and multiplying c_i both sides of equation (16), yields

$$c_i x(\tau_i) = \frac{(\tau_i - a)^{\gamma-1}}{\rho^{\gamma-1}\Gamma(\gamma)} e^{\frac{(\rho-1)}{\rho}(\tau_i - a)} c_i \mathcal{I}^{1-\gamma,\rho}_{a^+} x(a^+) + c_i \mathcal{I}^{p,\rho}_{a^+} f(\tau_i), \tag{17}$$

this implies that

$$\begin{aligned} \sum_{i=1}^{m} c_i x(\tau_i) =& \frac{1}{\rho^{\gamma-1}\Gamma(\gamma)} \sum_{i=1}^{m} c_i e^{\frac{(\rho-1)}{\rho}(\tau_i - a)} (\tau_i - a)^{\gamma-1} \mathcal{I}^{1-\gamma,\rho}_{a^+} x(a^+) \\ &+ \frac{1}{\rho^p \Gamma(p)} \sum_{i=1}^{m} c_i \int_{a^+}^{\tau_i} e^{\frac{(\rho-1)}{\rho}(\tau_i - s)} (\tau_i - s)^{p-1} f(s, x(s)) ds. \end{aligned} \tag{18}$$

Therefore, using the initial condition $\mathcal{I}^{1-\gamma,\rho}_{a^+} x(a) = \sum_{i=1}^{m} c_i x(\tau_i)$, we get

$$\mathcal{I}^{1-\gamma;\varphi}_{a^+} x(a^+) = \frac{\rho^{\gamma-1}\Gamma(\gamma)}{\rho^p \Gamma(p)} \Lambda \sum_{i=1}^{m} c_i \int_{a^+}^{\tau_i} e^{\frac{(\rho-1)}{\rho}(\tau_i - s)} (\tau_i - s)^{p-1} f(s, x(s)) ds. \tag{19}$$

Hence, the result follows by substituting (19) in (16). This implies that $x(t)$ satisfies (14).
Conversely, suppose that $x \in C^{\gamma}_{1-\gamma}$ satisfies equation (14), then, we show that x is also satisfies equation (13). Applying $\mathcal{D}^{\gamma,\rho}_{a^+}$ to both sides of (14) and in view of proposition 1, Lemma 2 and Definition 5, we have

$$\begin{aligned} \mathcal{D}^{\gamma,\rho}_{a^+} x(t) =& \mathcal{D}^{\gamma,\rho}_{a^+} \left(\frac{\Lambda}{\rho^p \Gamma(p)} e^{\frac{(\rho-1)}{\rho}(t-a)} (t-a)^{\gamma-1} \sum_{i=1}^{m} c_i \int_{a^+}^{\tau_i} e^{\frac{(\rho-1)}{\rho}(\tau_i - s)} (\tau_i - s)^{p-1} f(s, x(s)) ds \right) \\ &+ \mathcal{D}^{\gamma,\rho}_{a^+} \left(\frac{1}{\rho^p \Gamma(p)} \int_{a^+}^{t} e^{\frac{(\rho-1)}{\rho}(t-s)} (t-s)^{p-1} f(s, x(s)) ds \right) \\ =& \left(\mathcal{D}^{q(1-p),\rho}_{a^+} f(t, x(t)) \right)(x). \end{aligned} \tag{20}$$

Since $\mathcal{D}^{p,q,\rho}_{a^+} x \in C_{1-\gamma}[J, \mathbb{R}]$, then by definition of $C^{\gamma}_{1-\gamma}[J, \mathbb{R}]$ equation (20) implies that

$$\mathcal{D}^{q(1-p),\rho}_{a^+} f = \mathcal{D} \mathcal{I}^{1-q(1-\rho),\rho}_{a^+} f \in C_{1-\gamma,\rho}[J, \mathbb{R}].$$

For $f \in C_{1-\gamma}[J, \mathbb{R}]$ and from Lemma 3, we can see that $\mathcal{I}^{1-q(1-p),\rho}_{a^+} f \in C_{1-\gamma,\rho}[J, \mathbb{R}]$, this implies that $\mathcal{I}^{1-q(1-p),\rho}_{a^+} f \in C^{1}_{1-\gamma}[J, \mathbb{R}]$ from the definition of $C^{n}_{\gamma}[J, \mathbb{R}]$.

Apply $\mathcal{I}_{a^+}^{q(1-p),\rho}$ on both sides of (20) and in view of proposition 1, Lemma 5 and Definition 5, yields

$$\begin{aligned}\mathcal{I}_{a^+}^{q(1-p),\rho}\mathcal{D}_{a^+}^{\gamma,\rho}x(t) &= \mathcal{I}_{a^+}^{q(1-p),\rho}\mathcal{D}_{a^+}^{q(1-p),\rho}f(t,x(t)). \\ &= f(t,x(t)) - \frac{\left(\mathcal{I}_{a^+}^{1-q(1-p),\rho}f\right)(a)}{\Gamma(q(1-p))}(t-a)^{q(p-1)-1} \\ &= f(t,x(t)).\end{aligned} \tag{21}$$

Finally, we show that if $x \in C_{1-\gamma}^{\gamma}[J,\mathbb{R}]$ satisfying (14), it also satisfies the initial condition. So, by applying $\mathcal{I}_{0^+}^{1-\gamma,\rho}$ to both sides of (14) and using proposition 1 and Theorem 1, we obtain

$$\begin{aligned}\mathcal{I}_{a^+}^{1-\gamma,\rho}x(t) = \mathcal{I}_{a^+}^{1-\gamma,\rho}&\left(\frac{\Lambda}{\rho^p\Gamma(p)}e^{\frac{(\rho-1)}{\rho}(t-a)}(t-a)^{\gamma-1}\sum_{i=1}^m c_i\int_{a^+}^{\tau_i}e^{\frac{(\rho-1)}{\rho}(\tau_i-s)}(\tau_i-s)^{p-1}f(s)ds\right) \\ &+ \mathcal{I}_{a^+}^{1-\gamma;\varphi}\left(\frac{1}{\rho^p\Gamma(p)}\int_{a^+}^{t}e^{\frac{(\rho-1)}{\rho}(t-s)}(t-s)^{p-1}f(s)ds\right), \\ &= \frac{\rho^{\gamma-1}\Gamma(\gamma)}{\rho^p\Gamma(p)}\Lambda e^{\frac{(\rho-1)}{\rho}(t-a)}\sum_{i=1}^m c_i\int_{a^+}^{\tau_i}e^{\frac{(\rho-1)}{\rho}(\tau_i-s)}(\tau_i-s)^{p-1}f(s)ds \\ &+ \mathcal{I}_{a^+}^{1-q(1-p),\rho}f(t).\end{aligned} \tag{22}$$

Taking limit as $t \to a^+$ in equation (22) and the condition $1-q < 1-p(1-r)$, we get

$$\mathcal{I}_{a^+}^{1-\gamma,\rho}x(a^+) = \frac{\rho^{\gamma-1}\Gamma(\gamma)}{\rho^p\Gamma(p)}\Lambda\sum_{i=1}^m c_i\int_{a^+}^{\tau_i}e^{\frac{(\rho-1)}{\rho}(\tau_i-s)}(\tau_i-s)^{p-1}f(s,x(s))ds. \tag{23}$$

Now, substituting $t=\tau_i$ and multiply through by c_i in (14), we get

$$\begin{aligned}c_ix(\tau_i) =& \frac{\Lambda}{\rho^p\Gamma(p)}e^{\frac{(\rho-1)}{\rho}(\tau_i-a)}(\tau_i-a)^{\gamma-1}\sum_{i=1}^m c_i\int_{a^+}^{\tau_i}e^{\frac{(\rho-1)}{\rho}(\tau_i-s)}(\tau_i-s)^{p-1}f(s)ds \\ &+ \frac{c_i}{\rho^p\Gamma(p)}\int_{a^+}^{\tau_i}e^{\frac{(\rho-1)}{\rho}(\tau_i-s)}(\tau_i-s)^{p-1}f(s,x(s))ds,\end{aligned} \tag{24}$$

which implies that

$$\begin{aligned}\sum_{i=1}^m c_ix(\tau_i) &= \Lambda\sum_{i=1}^m c_i\mathcal{I}_{a^+}^{p,\rho}f(\tau_i)\sum_{i=1}^m c_ie^{\frac{(\rho-1)}{\rho}(\tau_i-a)}(\tau_i-a)^{\gamma-1} + \sum_{i=1}^m c_i\mathcal{I}_{a^+}^{p,\rho}f(\tau_i) \\ &= \sum_{i=1}^m c_i\mathcal{I}_{a^+}^{p,\rho}f(\tau_i)\left(1+\Lambda\sum_{i=1}^m c_ie^{\frac{(\rho-1)}{\rho}(\tau_i-a)}(\tau_i-a)^{\gamma-1}\right),\end{aligned} \tag{25}$$

hence

$$\sum_{i=1}^m c_ix(\tau_i) = \frac{\rho^{\gamma-1}\Gamma(\gamma)}{\rho^p\Gamma(p)}\Lambda\sum_{i=1}^m c_i\int_{a^+}^{\tau_i}e^{\frac{(\rho-1)}{\rho}(\tau_i-s)}(\tau_i-s)^{p-1}f(s,x(s))ds. \tag{26}$$

Therefore, in view of (23) and (26), we obtain

$$\mathcal{I}_{a^+}^{1-\gamma,\rho}x(a^+) = \sum_{i=1}^m c_ix(\tau_i). \tag{27}$$

□

Remark 3. We present the numerical solution of equation (14) and present these solutions in figure $1-3$, as shown below to show the proposed Hilfer-proportional fractional derivative (see, Definition 10) covers the existing ones of Riemann-Liouville, generalized proportional and Hilfer fractional derivatives.

Our first existence and uniqueness results were based on Banach contraction principle.

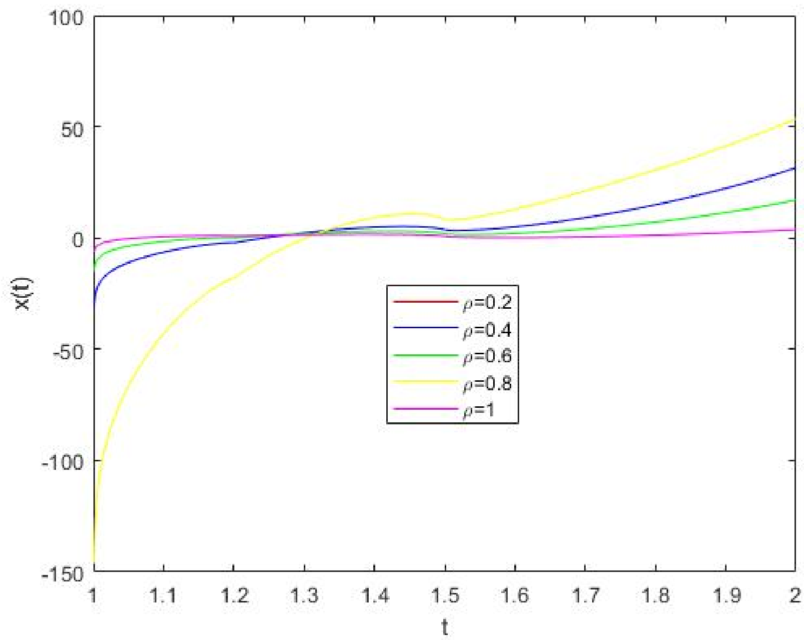

FIGURE 1 Graph of $x(t)$, for the Hilfer-fractional derivatives ($\rho = 1$) and Hilfer-generalized proportional fractional derivatives ($\rho \in (0, 1)$).

3.2 | Existence and uniqueness result

Now, we prove the uniqueness of solution to (13) by means of Banach contraction principle. Therefore, we need the following assumptions.

(H_1) Let $f : J \times \mathbb{R} \to \mathbb{R}$ be a function such that $f \in C_{1-\gamma}[J, \mathbb{R}]$ for any $x \in C_{1-\gamma}[J, \mathbb{R}]$.

(H_2) There exists a constant $K > 0$ such that

$$|f(t, u) - f(t, \bar{u})| \leq K|u - \bar{u}|$$

for any $u, \bar{u} \in \mathbb{R}$ and $t \in J$.

(H_3) Suppose that

$$K\psi < 1,$$

where

$$\psi = \frac{\mathcal{B}(\gamma, p)}{\rho^p \Gamma(p)} \left(|\Lambda| \sum_{i=1}^{m} c_i (\tau_i - a)^{p+\gamma-1} + (T - a)^p \right). \tag{28}$$

Theorem 5. Let $0 < p < 1$, $0 \leq q \leq 1$ and $\gamma = p + q - pq$. Suppose that the assumptions (H_1), (H_2) and (H_3) are satisfied. Then, problem (13) has a unique solution in the space $C_{1-\gamma}[J, \mathbb{R}]$.

Proof. Define the operator $T : C_{1-\gamma}[J, \mathbb{R}] \to C_{1-\gamma}[J, \mathbb{R}]$ by

$$\begin{aligned}(Tx)(t) =& \frac{\Lambda}{\rho^p \Gamma(p)} e^{\frac{(\rho-1)}{\rho}(t-a)} (t-a)^{\gamma-1} \sum_{i=1}^{m} c_i \int_{a^+}^{\tau_i} e^{\frac{(\rho-1)}{\rho}(\tau_i - s)} (\tau_i - s)^{p-1} f(s, x(s)) ds \\ &+ \frac{1}{\rho^p \Gamma(p)} \int_{a^+}^{t} e^{\frac{(\rho-1)}{\rho}(t-s)} (t-s)^{p-1} f(s, x(s)) ds,\end{aligned} \tag{29}$$

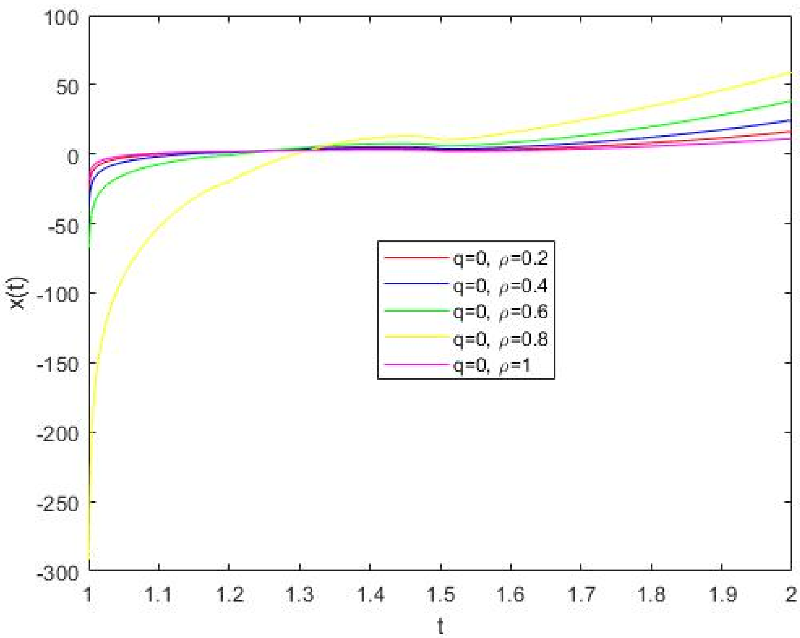

FIGURE 2 Graph of $x(t)$, for the Riemann-Lioville fractional derivatives ($q = 0,\ \rho = 1$), and generalized proportional fractional derivatives ($q = 0,\ \rho \in (0, 1)$).

it follows that, the operator T is well defined. Now for any $x_1, x_2 \in C^{p,q}_{1-\gamma}[J, \mathbb{R}]$ and $t \in J$, we have

$$\begin{aligned}
&|((Tx_1)(t) - (Tx_2)(t))(t-a)^{1-\gamma}| \\
&\le \frac{|\Lambda|}{\rho^p\Gamma(p)}\left|e^{\frac{(\rho-1)}{\rho}(t-a)}\right|\sum_{i=1}^{m} b_i\int_{a^+}^{\tau_i}\left|e^{\frac{(\rho-1)}{\rho}(\tau_i-s)}\right|(\tau_i-s)^{p-1}|f(s,x_1(s)) - f(s,x_2(s))|ds \\
&+ \frac{1}{\rho^p\Gamma(p)}\int_{a^+}^{t}\left|e^{\frac{(\rho-1)}{\rho}(t-s)}\right|(t-s)^{p-1}|f(s,x_1(s)) - f(s,x_2(s))|ds.
\end{aligned} \tag{30}$$

Since $|e^{\frac{(\rho-1)}{\rho}(t-s)}| < 1$, for all $\rho \in (0, 1]$, this implies that

$$\begin{aligned}
&|((Tx_1)(t) - (Tx_2)(t))(t-a)^{1-\gamma}| \\
&\le \frac{K|\Lambda|}{\rho^p\Gamma(p)}\left(\sum_{i=1}^{m} b_i\int_{a^+}^{\tau_i}(\tau_i-s)^{p-1}(s-a)^{\gamma-1}ds\right)\|x_1 - x_2\|_{C_{1-\gamma}[J,\mathbb{R}]} \\
&+ \frac{K}{\rho^p\Gamma(p)}(t-a)^{1-\gamma}\left(\int_{a^+}^{t}(t-s)^{p-1}(s-a)^{\gamma-1}ds\right)\|x_1 - x_2\|_{C_{1-\gamma}[J,\mathbb{R}]} \\
&\le \frac{K|\Lambda|}{\rho^p\Gamma(p)}\mathcal{B}(\gamma,p)\sum_{i=1}^{m} c_i(\tau_i-a)^{p+\gamma-1}\|x_1 - x_2\|_{C_{1-\gamma}[J,\mathbb{R}]} \\
&+ \frac{K}{\rho^p\Gamma(p)}(T-a)^p\mathcal{B}(\gamma,p)\|x_1 - x_2\|_{C_{1-\gamma}[J,\mathbb{R}]}.
\end{aligned} \tag{31}$$

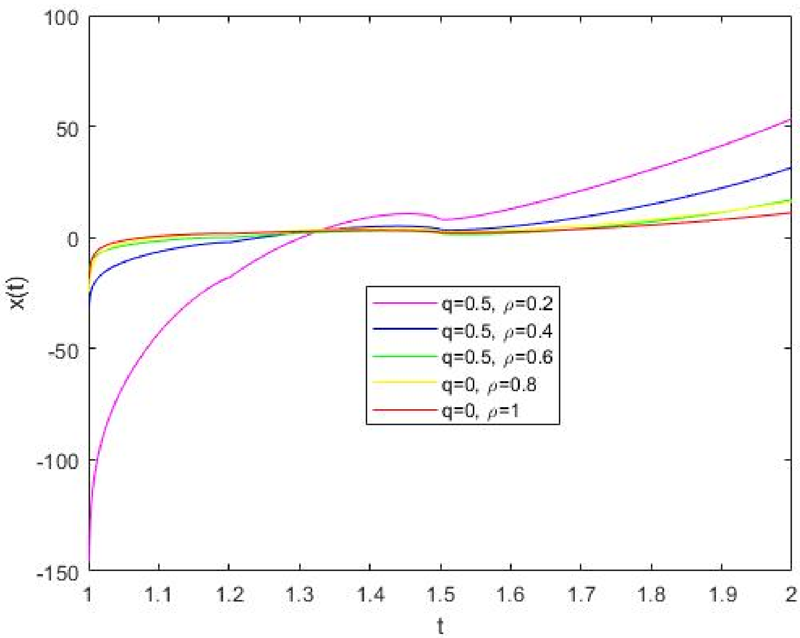

FIGURE 3 Graph of $x(t)$, for the Riemann-Lioville fractional derivatives ($q = 0,\ \rho = 1$), generalized proportional fractional derivatives ($q = 0,\ \rho = 0.8$) and Hilfer-generalized proportional fractional derivatives ($q \in (0, 1),\ \rho \in (0, 1)$).

Therefore,

$$\begin{aligned} &\|(Nx_1) - (Nx_2)\|_{C_{1-\gamma}} \\ &\leq \frac{K}{\rho^p \Gamma(p)} \mathcal{B}(\gamma, p) \left(|\Lambda| \sum_{i=1}^{m} c_i (\tau_i - a)^{p+\gamma-1} + (T - a)^p \right) \|x_1 - x_2\|_{C_{1-\gamma}[J,\mathbb{R}]} \\ &\leq K\psi \|x_1 - x_2\|_{C_{1-\gamma}[J,\mathbb{R}]}. \end{aligned} \tag{32}$$

Hence, it follows from (28) that N is a contraction map. Thus, as consequences of Banach contraction principle, problem (13) has a unique solution. □

Next, we prove the existence results using the concepts of Krasnoselskii's fixed point theorem.

Theorem 6. (see[30]) Let P be a closed, convex and nonempty subset of a Banach space X, and let F_1, F_2 be operators such that:

1. $F_1 x + F_2 y \in P$ whenever $x, y \in P$.
2. F_1 is compact and continuous.
3. F_2 is a contraction mapping.

Then there exist $z \in P$ such that $z = F_1 z + F_2 z$.

3.2.1 | Existence result via Krasnoselskii's fixed point theorem

In this subsection, we prove the existence of solutions for problem (13) using the concepts of Krasnoselskii's fixed point theorem[30]. Therefore, we set-up the following hypotheses.
(A_1) Let $f : J \times \mathbb{R} \to \mathbb{R}$ be a function such that $f \in C_{1-\gamma}[J, \mathbb{R}]$ for any $x \in C_{1-\gamma}[J, \mathbb{R}]$.
(A_2) There exist a constants $M > 0$ such that

$$|f(t, z) - f(t, \bar{z})| \leq M|z - \bar{z}|$$

for any $z, \bar{z} \in \mathbb{R}$ and $t \in J$.
(A_3) Suppose that

$$\Delta M < 1,$$

where

$$\Delta = \frac{\mathcal{B}(\gamma, p)}{\rho^p \Gamma(p)} |\Lambda| \sum_{i=1}^{m} c_i (\tau_i - a)^{p+\gamma-1}. \tag{33}$$

Theorem 7. Let $0 < p < 1$, $0 \le q \le 1$ and $\gamma = p + q - pq$. Suppose that the hypotheses (A_1), (A_2) and (A_3) are satisfied. Then, problem (13) has at least one solution in the space $\mathcal{C}_{1-\gamma}[J, \mathbb{R}]$.

Proof. As well as $\|\eta\|_{\mathcal{C}_{1-\gamma}[J,\mathbb{R}]} = \sup_{t \in J} |(t-a)^{1-\gamma} \eta(t)|$ and choose $\kappa \ge M \|\eta\|_{\mathcal{C}_{1-\gamma}[J,\mathbb{R}]}$, where

$$M = \frac{\mathcal{B}(\gamma, p)}{\rho^p \Gamma(p)} \left(|\Lambda| \sum_{i=1}^{m} c_i (\tau_i - a)^{p+\gamma-1} + (T-a)^p \right), \tag{34}$$

we consider $\mathcal{B}_\kappa = \{x \in \mathbb{C}[J, \mathbb{R}] : \|x\|_{\mathcal{C}_{1-\gamma}[J,\mathbb{R}]} \le \kappa\}$. Define the operators $\mathbb{T}_1$ and $\mathbb{T}_2$ on $\mathcal{B}_\kappa$ by

$$\mathbb{T}_1 x(t) = \frac{1}{\rho^p \Gamma(p)} \int_{a^+}^{t} e^{\frac{(\rho-1)}{\rho}(t-s)} (t-s)^{p-1} f(s, x(s)) ds,$$

$$\mathbb{T}_2 x(t) = \frac{\Lambda}{\rho^p \Gamma(p)} e^{\frac{(\rho-1)}{\rho}(t-a)} (t-a)^{\gamma-1} \sum_{i=1}^{m} c_i \int_{a^+}^{\tau_i} e^{\frac{(\rho-1)}{\rho}(\tau_i-s)} (\tau_i - s)^{p-1} f(s, x(s)) ds,$$

for all $t \in [a, T]$. Now, for for every $x, y \in \mathcal{B}_\kappa$, yields

$$\begin{aligned}
&|(\mathbb{T}_1 x(t) + \mathbb{T}_2 y(t))(t-a)^{1-\gamma}| \\
&\le \frac{(t-a)^{1-\gamma}}{\rho^p \Gamma(p)} \int_{a^+}^{t} (t-s)^{p-1} (s-a)^{\gamma-1} |f(s, x(s))(s-a)^{1-\gamma}| ds \\
&+ \frac{|\Lambda|}{\rho^p \Gamma(p)} \sum_{i=1}^{m} c_i \int_{a^+}^{\tau_i} (\tau_i - s)^{p-1} (\tau_i - a)^{\gamma-1} |f(s, y(s))(\tau_i - a)^{1-\gamma}| ds \\
&\le \|\eta\| \left[\frac{\mathcal{B}(\gamma, p)}{\rho^p \Gamma(p)} |\Lambda| \sum_{i=1}^{m} c_i (\tau_i - a)^{p+\gamma-1} + \frac{\mathcal{B}(\gamma, p)}{\rho^p \Gamma(p)} (T-a)^p \right] \\
&\le \|\eta\| M \\
&\le \kappa.
\end{aligned} \tag{35}$$

This implies that, $\mathbb{T}_1 x + \mathbb{T}_2 y \in \mathcal{B}_\kappa$.
Step 2. We show that $\mathbb{T}_2$ is a contraction.
Now, let $x, y \in \mathcal{C}_{1-\gamma}[J, \mathbb{R}]$ and $t \in J$, then

$$\begin{aligned}
&\left| (\mathbb{T}_2 x(t) - \mathbb{T}_2 y(t))(t-a)^{1-\gamma} \right| \\
&= \left| \Lambda e^{\frac{(\rho-1)}{\rho}(t-a)} \sum_{i=1}^{m} c_i \mathcal{I}_{a^+}^{p,\rho} (f(s, x(s)) - f(s, y(s)))(\tau_i) \right| \\
&\le \frac{L|\Lambda|}{\rho^p \Gamma(p)} \sum_{i=1}^{m} c_i \int_{a^+}^{\tau_i} (\tau_i - s)^{p-1} (\tau_i - s)^{\gamma-1} |x(s) - y(s)| ds \\
&\le \left[\frac{L|\Lambda|}{\rho^p \Gamma(p)} \sum_{i=1}^{m} c_i (\tau_i - a)^{p+\gamma-1} \mathcal{B}(\gamma, p) \right] \|x - y\|_{\mathcal{C}_{1-\gamma}[J,\mathbb{R}]} \\
&\le L\Delta \|x - y\|_{\mathcal{C}_{1-\gamma}[J,\mathbb{R}]}.
\end{aligned} \tag{36}$$

Hence, it follows from (33), that $\mathbb{T}_2$ is a contraction.
Step 3. We show that the operator $\mathbb{T}_1$ is continuous and compact.

Clearly, the operator $\mathbb{T}_1$ is also continuous, due the fact that the function f is continuous. Thus, for any $x \in C_{1-\gamma}[J,\mathbb{R}]$, we have

$$\|\mathbb{T}_1 x\| \leq \|\eta\| \frac{\mathcal{B}(p,\gamma)}{\rho^p \Gamma(p)} (T-a)^p.$$

This shows that the operator $\mathbb{T}_1$ is uniformly bounded on $\mathcal{B}_\kappa$. Thus, it remains to shows that $\mathbb{T}_1$ is compact. Denoting $\sup\limits_{(t,x)\in J\times\mathcal{B}_\kappa} |f(t,x(t))| = \delta < \infty$ and for any $a < \tau_1 < \tau_2 < T$, we obtain

$$\begin{aligned}
&|(\tau_2-a)^{1-\gamma}(\mathbb{T}_1 x(\tau_2)) + (\tau_1-a)^{1-\gamma}(\mathbb{T}_1 x(\tau_1))| \\
&= \Bigg| \frac{(\tau_2-a)^{1-\gamma}}{\rho^p\Gamma(p)} \int_{a^+}^{\tau_2} e^{\frac{(\rho-1)}{\rho}(\tau_2-s)}(\tau_2-s)^{p-1} f(s,x(s))ds \\
&- \frac{(\tau_1-a)^{1-\gamma}}{\rho^p\Gamma(p)} \int_{a^+}^{\tau_1} e^{\frac{(\rho-1)}{\rho}(\tau_1-s)}(\tau_1-s)^{p-1} f(s,x(s))ds \Bigg| \\
&\leq \frac{1}{\rho^p\Gamma(p)} \int_{a^+}^{\tau_2} [(\tau_2-a)^{1-\gamma}(\tau_2-s)^{p-1} - (\tau_1-a)^{1-\gamma}(\tau_1-s)^{p-1}]|f(s,x(s))|ds \\
&+ \frac{1}{\rho^p\Gamma(p)} \int_{\tau_1}^{\tau_2} (\tau_2-a)^{1-\gamma}(\tau_2-s)^{p-1}|f(s,x(s))|ds \\
&\leq \frac{\delta}{\rho^p\Gamma(p+1)} \left((\tau_2-a)^{p+1-\gamma} - (\tau_1-a)^{p+1-\gamma} \right) \\
&\to 0, \ \text{ as } \ \tau_2 \to \tau_1.
\end{aligned} \tag{37}$$

As a consequences of Arzela-Ascoli theorem, the operator $\mathbb{T}_1$ is compact on $\mathcal{B}_\kappa$. Thus, by Theorem 6, problem (13) has at least one solution on $[a,T]$. □

4 | EXAMPLES

Example 1. Consider the fractional differential equation which involves Hilfer generalized proportional derivatives of the form:

$$\begin{cases} \mathcal{D}_{0^+}^{\frac{2}{3},\frac{1}{2},1} x(t) = \frac{1}{25e^{2t}}\left(\frac{\cos 2t}{1+|x(t)|}\right) + \frac{3}{2}, & t \in J = [0,2], \\ \mathcal{I}_{0^+}^{1-\gamma,1} x(0) = 2x(\frac{2}{5}). \end{cases} \tag{38}$$

By comparing (13) with (38), we get:
$p = \frac{2}{3}$, $q = \frac{1}{2}$, $\rho = 1$, $\gamma = \frac{5}{6}$, $a = 0$, $T = 2$, $c_1 = 2$ since $m = 1$, $\tau_1 = \frac{2}{5} \in J$ and $f : J \times \mathbb{R} \to \mathbb{R}$ is a function defined by

$$f(t,u) = \frac{1}{25e^{2t}}\left(\frac{\cos 2t}{1+|u|}\right) + \frac{3}{2}, \quad t \in J, \quad u \in \mathbb{R}_+.$$

Thus, f is continuous and for all $u, v \in \mathbb{R}_+$ and $t \in J$, we have
$|f(t,u) - f(t,v)| \leq \frac{1}{25}|u-v|$. Thus, it follows that conditions (A_1) and (A_3) are true with $K = L = \frac{1}{25}$. Therefore, by simple calculation, we can see that $|\Lambda| \approx 0.8325$ and $\psi \approx 3.3633$, which implies that

$$K\psi \approx 0.1345 < 1.$$

Thus, all the assumptions of Theorem 5 are satisfied. Hence, problem (13) has a unique solution on J.
Similarly, we can also find out that $\Delta \approx 1.3413 > 0$ and $L\Delta \approx 0.0536 < 1$. Thus, we can conclude that, by Theorem 7, problem (13) has at least one solution on J.

Example 2. Consider the fractional differential equation which involves Hilfer generalized proportional derivatives of the:

$$\begin{cases} \mathcal{D}_{0^+}^{\frac{2}{3},\frac{1}{2},\frac{1}{5}} x(t) = \frac{1}{25e^{2t}} \left(\frac{\cos 2t}{1+|x(t)|} \right) + \frac{3}{2}, & t \in \boldsymbol{J} = [0,2], \\ \mathcal{I}_{0^+}^{1-\gamma,\frac{1}{5}} x(0) = 2x(\frac{2}{5}). \end{cases} \tag{39}$$

Repeating application of the same procedure as Example 1 above, we get the following values, $|\Lambda| \approx 0.9943$, $\psi \approx 13.2055$ and $\Delta \approx 5.8376$. Thus

$$K\psi \approx 0.4902 < 1,$$

which implies that, by Theorem 5, problem (13) has a unique solution on $\boldsymbol{J}$. Furthermore,

$$L\Delta \approx 0.2335 < 1,$$

thus, by Theorem 7, problem (13) has at least one solution on $\boldsymbol{J}$.

It is worth mentioning here that the proposed Hilfer-generalized proportional derivative covers the classical ones, that is:

- If $\rho \to 1$ and $q \in [0,1]$, the formulation for this problem, reduce to Hilfer fractional derivatives[5,31], (see figure 1).
- If $\rho \in (0,1)$ and $q \in [0,1]$, we obtain the proposed Hilfer-generalized proportional fractional derivatives, which we can see that it covers the classical Hilfer fractional derivative, as shown in figure 1 above.
- If $\rho \to 1$ and $q = 0$, the formulation for this problem, reduce to Riemann-Liouville fractional derivative[5], (see figure 2).
- If $\rho \in (0,1)$ and $q = 0$, we obtain the generalized proportional fractional derivative[18], which we can see from figure 2 it cover the classical Riemann-Liouville fractional derivative.
- If $q, \rho \in (0,1)$, it easily to figure-out from figure 3, that the new proposed derivatives covers the classical ones of Hilfer, Riemann-Liouville and generalized proportional fractional derivative.

5 | CONCLUSIONS

In this paper, we defined the proportional fractional derivatives in the Hilfer setting. We used some known theorems from the fixed point theory that enabled us to prove the existence and uniqueness of solutions to a specific type of fractional initial value problem involving the defined Hilfer proportional fractional derivative. In additions, to show the effectiveness of our results, we presented some examples. In fact, the Hilfer proportional derivative contains three parameters. The existence of more parameters is useful especially when one considers the stability and other qualitative aspects of differential equations involving fractional derivatives.

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Author contributions

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Conflict of interest

The authors declare no potential conflict of interests.

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