Complex mathematical statements useful as criterion for Landau-Siegel Zeros

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Abstract :

This is a mathematical article that deals with the domain of complex numbers in order to contribute in the worldwide collaboration of mathematicians who want to study and explain the Riemann's Zeta Function.

The aim of this work is to propose new mathematical statements of complex numbers and to give a new easy criterion that may help to identify or deny the possible Landau-Siegel Zeros. Furthermore, this work invites the mathematicians to study a new possible useful homomorphism.

Hence, this is a good article to be read by specialists of Analysis or Algebra and even by the beginners of mathematics who want to improve their skills.

Keywords : complex analysis; fixed-point; implication and negation; Landau-Siegel Zero; Riemann's Zeta function; Riemann's Hypothesis.

Introduction:

Many mathematical articles try to prove or disprove the Riemann's Hypothesis [1,2]. This is not only because of the "One Million Dollars Prize" but also because of the usefulness of Riemann's Zeta Function in many scientific fields.

After my thesis of mathematical physics that refuses the classical use of discrete mathematics in Newtonian Mechanics [3], I chose to show my respect to mathematics and especially Analysis by writing my personal articles that respect all the rules of classical mathematics [4,5]. This article is supposed to contribute in the collaboration of mathematicians who try to explain and simplify the study of Riemann's Zeta Function [6]. The purpose of this work is to propose new useful mathematical statements about complex numbers and to give a new easy criterion that may help to identify or deny the possible Landau-Siegel Zeros [7].

These are the considerations and notations of this article:

Let S be a complex number that respects that $S=\frac{1}{2}+ib$ where b is a strictly positive real number.

We can also write S as $S=\frac{1}{2}+i\frac{1}{\varepsilon}=\frac{1}{2}+i\frac{\tan(\theta)}{2}$ where $b=\frac{\tan(\theta)}{2}=\frac{1}{\varepsilon}$. (1)

In this case, we should only consider that $\frac{1}{\varepsilon}>0$ and $0<\theta<\frac{\pi}{2}$ and we have $S=\frac{e^{i\theta}}{2\cos(\theta)}$ (2)

We have also: $\theta=\arctan(\frac{2}{\varepsilon})=\frac{\pi}{2}-\arctan(\frac{\varepsilon}{2})$ (3)

and $\cos(\theta)=\cos(\arctan(\frac{2}{\varepsilon}))=\frac{\varepsilon}{\sqrt{\varepsilon^2+4}}$ (4)

and $\sin(\theta)=\sin(\arctan(\frac{2}{\varepsilon}))=\frac{2}{\sqrt{\varepsilon^2+4}}$ (5)

consequently: $\sin(\theta)\times\cos(\theta)=\frac{2\varepsilon}{\varepsilon^2+4}$ (6)

Let's consider also a function ***f*** of complex numbers defined from $\mathbb{C}$ in $\mathbb{C}$ as: $f(n)=\frac{n^{S+1}}{S}$ (7)

We will use the complex logarithm ***ln*** point by point during all this proof.

1. The considered complex function and its fixed-point:

Let's consider that the function ***f*** admits a fixed-point ***n*** .

We have: $f(n)=n\Leftrightarrow n^S=S\Leftrightarrow\frac{e^{i\theta}}{2\cos(\theta)}=n^{\frac{e^{i\theta}}{2\cos(\theta)}}$ (8)

And we have: $n^S=S\Leftrightarrow e^{S\times\ln(n)}=\frac{e^{i\theta}}{2\cos(\theta)}$ (9)

Since we have: $e^{S\times\ln(n)}=e^{S\times\ln(n)+i2k\pi}=e^{S\times(\ln(n)+i\frac{2k\pi}{S})}=\frac{e^{i\theta}}{2\cos(\theta)}$ $\forall k\in\mathbb{Z}$ (10)

Then we have ***k*** numbers that respect $n^S=S$ and these numbers are $(\ln(n)+i\frac{2k\pi}{S})$. (11)

However, the only real number among these ***k*** numbers is ***ln(n)*** which corresponds to k=0.

Hence we have only one unique real number ***ln(n)*** for each θ with $0<\theta<\frac{\pi}{2}$.

Consequently: $\ln(\frac{e^{i\theta}}{2\cos(\theta)})=\frac{e^{i\theta}}{2\cos(\theta)}\times\ln(n)$ (12)

Hence: $\ln(n)=\frac{2\cos(\theta)\times(\ln(e^{i\theta})-\ln(2\cos(\theta)))}{e^{i\theta}}$ (13)

Consequently: $\frac{\ln(n)}{2}=(\cos(\theta)-i\sin(\theta))\times\cos(\theta)\times(\ln(e^{i\theta})-\ln(2\cos(\theta)))$ (14)

Hence: $\frac{\ln(n)}{2}=(\cos(\theta)^2-i\frac{\sin(2\theta)}{2})\times(i\theta-\ln(\cos(\theta))-\ln(2))$ (15)

We considered that $\ln(e^{i\theta})=i\theta+i\times 2k\pi=i\theta$ with ***k=0*** (16)

because we have one unique complex number ***ln(n)*** for each $e^{i\theta}$ and for each $\cos(\theta)$ with $0<\theta<\frac{\pi}{2}$.

And thus: $\frac{\ln(n)}{2}=(-\cos(\theta)^2\times(\ln(\cos(\theta))+\ln(2))+\frac{\theta\times\sin(2\theta)}{2})$ +

$i(\theta\times\cos(\theta)^2+\frac{\sin(2\theta)\times\ln(\cos(\theta))}{2}+\frac{\sin(2\theta)\times\ln(2)}{2})$ (17)

And by using the formulas of the introduction, we get:

$\frac{\ln(n)}{2}=(\frac{-\varepsilon^2}{\varepsilon^2+4}\times(\ln(\frac{\varepsilon}{\sqrt{\varepsilon^2+4}})+\ln(2))+\arctan(\frac{2}{\varepsilon})\times\frac{2\varepsilon}{\varepsilon^2+4})$ +

$i(\arctan(\frac{2}{\varepsilon})\times\frac{\varepsilon^2}{\varepsilon^2+4}+\frac{2\varepsilon}{\varepsilon^2+4}\times\ln(\frac{\varepsilon}{\sqrt{\varepsilon^2+4}})+\frac{2\varepsilon}{\varepsilon^2+4}\times\ln(2))$ (18)

Consequently:

$\frac{\ln(n)}{2}=\frac{-\varepsilon^2}{\varepsilon^2+4}\times(\ln(\varepsilon)-\frac{\ln(\varepsilon^2+4)}{2}+\ln(2))+\arctan(\frac{2}{\varepsilon})\times\frac{2\varepsilon}{\varepsilon^2+4}$ +

$i(\arctan(\frac{2}{\varepsilon})\times\frac{\varepsilon^2}{\varepsilon^2+4}+\frac{\varepsilon}{\varepsilon^2+4}\times(2\ln(\varepsilon)-\ln(\varepsilon^2+4))+\frac{2\varepsilon}{\varepsilon^2+4}\times\ln(2))$ (19)

2. First investigation: if n is a real positive number:

The number ***n*** can't be equal to zero because zero is not a solution to the equation: $n^S=S$.

If n is a real strictly positive number then $\Im(\frac{\ln(n)}{2})=0$.

Hence we have: $\varepsilon\times\arctan(\frac{2}{\varepsilon})=-2\ln(\varepsilon)+\ln(\varepsilon^2+4)-2\ln(2)$ (20)

And thus, our equation (19) becomes:

$$\frac{\ln(n)}{2}=\frac{-\varepsilon^2}{\varepsilon^2+4}\times(\ln(\varepsilon)-\frac{\ln(\varepsilon^2+4)}{2}+\ln(2))+\arctan(\frac{2}{\varepsilon})\times\frac{2\varepsilon}{\varepsilon^2+4} \quad (21)$$

Hence:

$$\frac{\ln(n)}{2}=\frac{-\varepsilon^2}{\varepsilon^2+4}\times(\ln(\varepsilon)-\frac{\ln(\varepsilon^2+4)}{2}+\ln(2))+(-2\ln(\varepsilon)+\ln(\varepsilon^2+4)-2\ln(2))\times\frac{2\varepsilon}{\varepsilon^2+4} \quad (22)$$

Consequently:

$$\frac{\ln(n)}{2}=\frac{-\varepsilon}{\varepsilon^2+4}\times(\varepsilon\times(\ln(\varepsilon)-\frac{\ln(\varepsilon^2+4)}{2}+\ln(2))+\frac{4}{\varepsilon}\times(\ln(\varepsilon)-\ln\frac{(\varepsilon^2+4)}{2}+\ln(2))) \quad (23)$$

And thus: $\frac{\ln(n)}{2}=\frac{-\varepsilon}{\varepsilon^2+4}\times(\varepsilon+\frac{4}{\varepsilon})\times(\ln(\varepsilon)-\frac{\ln(\varepsilon^2+4)}{2}+\ln(2))$ (24)

We conclude finally that: $\frac{\ln(n)}{2}=-\ln(\varepsilon)+\frac{\ln(\varepsilon^2+4)}{2}-\ln(2)$ (25)

Which is equivalent to: $\ln(n)=\ln(\frac{\varepsilon^2+4}{4\varepsilon^2})$ (26)

Also, we have: $\ln(n)=\ln(\frac{\varepsilon^2+4}{4\varepsilon^2})\Leftrightarrow n=\frac{\varepsilon^2+4}{4\varepsilon^2}\Leftrightarrow n=\frac{1}{4}+\frac{1}{\varepsilon^2}$ (27)

We proved also that: $\varepsilon\times\arctan(\frac{2}{\varepsilon})=-2\ln(\varepsilon)+\ln(\varepsilon^2+4)-2\ln(2)$ (28)

This means that: $\varepsilon\times\theta=\ln(\frac{\varepsilon^2+4}{4\varepsilon^2})$ (29)

And thus: $e^{\varepsilon\times\theta}=\frac{1}{4}+\frac{1}{\varepsilon^2}$ (30)

We conclude that: $n=e^{\varepsilon\times\theta}=e^{\varepsilon\times\arctan(\frac{2}{\varepsilon})}=e^{\frac{2\theta}{\tan(\theta)}}=e^{\frac{\arctan(2b)}{b}}$ (31)

We can also remark that we have: $e^{\varepsilon\times\theta}=\frac{1}{4}+\frac{1}{\varepsilon^2}\Leftrightarrow\frac{\varepsilon^2}{\varepsilon^2\times e^{\varepsilon\times\theta}-1}=4$ (32)

This means that: $\frac{\varepsilon^2}{(\varepsilon\times e^{\frac{\varepsilon}{2}\times\arctan(\frac{2}{\varepsilon})})^2-1}=4$ (33)

And we know that the maximum of $\frac{\arctan(x)}{x}$ is 1. (34)

Hence $\frac{\varepsilon}{2}\times\arctan(\frac{2}{\varepsilon})<1$ (35)

And thus: $e^{\frac{\varepsilon}{2}\times\arctan(\frac{2}{\varepsilon})}<e$ (36)

Finally, we conclude that we should have $\varepsilon\geq\frac{1}{e}$ (37)

which is equivalent to: $\tan(\theta)\leqslant 2e$ (38)

otherwise we will have a contradiction in the equation $\frac{\varepsilon^2}{(\varepsilon\times e^{\frac{\varepsilon}{2}\times\arctan(\frac{2}{\varepsilon})})^2-1}=4$ (39)

because ***4*** can't be equal to any negative value.

3. Second investigation: if n is a strictly negative real number:

Since we have: $e^{S\times\ln(n)}=e^{S\times\ln(n)+i2k\pi}=e^{S\times(\ln(n)+i\frac{2k\pi}{S})}=\frac{e^{i\theta}}{2\cos(\theta)}\qquad\forall\, k\in\mathbb{Z}$ (40)

Then we have ***k*** numbers that respect $n^S=S$ and these numbers are $(\ln(n)+i\frac{2k\pi}{S})$. (41)

If ***n*** is a strictly negative number then ***ln(n)*** is a complex number .

And we have $\exists k'\in\mathbb{Z}$ with $\ln(n)=\ln(-n)+i\pi+i2k'\pi$. (42)

Hence, we have $n^S=e^{S\times(\ln(n)+i\frac{2k\pi}{S})}=e^{S\times(\ln(-n)+i\pi+i2k'\pi+i\frac{2k\pi}{S})}$ (43)

And thus, we should have: $e^{i2k'\pi S+i2k\pi}=1$. (44)

We conclude that: $\exists k''\in\mathbb{Z}$ and $\exists k'''\in\mathbb{Z}$ with $k'\pi S+k\pi=k'''\pi$ (45)

Since b is a strictly positive number, then we should have obviously **k'=0**.

Finally, we conclude that we should investigate about the unique complex number ***ln(n)*** that corresponds to each ***θ*** with $0<\theta<\frac{\pi}{2}$ and that respects: $\ln(n)=\ln(-n)+i\pi$. (46)

In this case, we use $\Im(\frac{\ln(n)}{2})=\frac{\pi}{2}$ (47)

and we get:

$$\arctan(\frac{2}{\varepsilon})\times\frac{\varepsilon^2}{\varepsilon^2+4}+\frac{\varepsilon}{\varepsilon^2+4}\times(2\ln(\varepsilon)-\ln(\varepsilon^2+4))+\frac{2\varepsilon}{\varepsilon^2+4}\times\ln(2)=\frac{\pi}{2} \quad (48)$$

Hence: $$\arctan(\frac{2}{\varepsilon})\times\frac{\varepsilon^2}{\varepsilon^2+4}=\frac{\pi}{2}-\frac{\varepsilon}{\varepsilon^2+4}\times(2\ln(\varepsilon)-\ln(\varepsilon^2+4))-\frac{2\varepsilon}{\varepsilon^2+4}\times\ln(2) \quad (49)$$

Consequently: $$\arctan(\frac{2}{\varepsilon})\times\frac{\varepsilon^2}{\varepsilon^2+4}=\frac{\pi}{2}-\frac{\varepsilon}{\varepsilon^2+4}\times(2\ln(\varepsilon)-\ln(\varepsilon^2+4))-\frac{2\varepsilon}{\varepsilon^2+4}\times\ln(2) \quad (50)$$

And thus: $$\arctan(\frac{2}{\varepsilon})\times\frac{2\varepsilon}{\varepsilon^2+4}=\frac{\pi}{\varepsilon}-\frac{2}{(\varepsilon^2+4)}\times(2\ln(\varepsilon)-\ln(\varepsilon^2+4))-\frac{4}{\varepsilon^2+4}\times\ln(2) \quad (51)$$

We conclude that: $$\theta\times\frac{2\varepsilon}{\varepsilon^2+4}=\frac{\pi}{\varepsilon}-\frac{2}{\varepsilon^2+4}\times\ln(\frac{4\varepsilon^2}{\varepsilon^2+4}) \quad (52)$$

which is equivalent to: $$\ln(\frac{4\varepsilon^2}{\varepsilon^2+4})=(-\theta\times\frac{2\varepsilon}{\varepsilon^2+4}+\frac{\pi}{\varepsilon})\times\frac{\varepsilon^2+4}{2}=-\varepsilon\theta+\frac{\varepsilon\pi}{2}+\frac{2\pi}{\varepsilon} \quad (53)$$

Consequently, we get: $$\frac{4\varepsilon^2}{\varepsilon^2+4}=e^{-\varepsilon\theta}\times e^{\frac{\varepsilon\pi}{2}}\times e^{\frac{2\pi}{\varepsilon}} \quad (54)$$

Hence: $$\frac{4\varepsilon^2}{e^{-\varepsilon\theta}\times e^{\frac{\varepsilon\pi}{2}}\times e^{\frac{2\pi}{\varepsilon}}}=\varepsilon^2+4 \quad (55)$$

And thus: $$4=\varepsilon^2\times(\frac{4}{e^{-\varepsilon\theta}\times e^{\frac{\varepsilon\pi}{2}}\times e^{\frac{2\pi}{\varepsilon}}}-1) \quad (56)$$

Hence, in order to avoid the contradiction, we should have: $$\frac{4}{e^{-\varepsilon\theta}\times e^{\frac{\varepsilon\pi}{2}}\times e^{\frac{2\pi}{\varepsilon}}}>1 \quad . \quad (57)$$

And this is equivalent to: $$\frac{(2e^{\frac{\varepsilon}{2}\theta})^2}{e^{\frac{\varepsilon\pi}{2}}\times e^{\frac{2\pi}{\varepsilon}}}>1\Leftrightarrow\frac{(2e^{\frac{\varepsilon}{2}\arctan(\frac{2}{\varepsilon})})^2}{e^{\frac{\varepsilon\pi}{2}}\times e^{\frac{2\pi}{\varepsilon}}}>1 \quad (58)$$

And we proved that: $$e^{\frac{\varepsilon}{2}\times\arctan(\frac{2}{\varepsilon})}<e \quad (59)$$

And thus we should have: $$e^{\frac{\varepsilon\pi}{2}}\times e^{\frac{2\pi}{\varepsilon}}<(2e)^2 \quad (60)$$

in order to avoid the contradiction. Now let's check if this inequality is possible.

We can notice that: $$\frac{\varepsilon\pi}{2}+\frac{2\pi}{\varepsilon}-4=\frac{\varepsilon^2\pi-8\varepsilon+4\pi}{2\varepsilon} \quad (61)$$

and that the discriminant of the quadratic equation $\varepsilon^2\pi-8\varepsilon+4\pi$ is negative, Hence $\frac{\varepsilon^2\pi-8\varepsilon+4\pi}{2\varepsilon}>0$ (62)

And thus: $\frac{\varepsilon\pi}{2}+\frac{2\pi}{\varepsilon}>4$ (63)

Consequently: $e^{\frac{\varepsilon\pi}{2}}\times e^{\frac{2\pi}{\varepsilon}}>e^4$ (64)

And we should have: $e^{\frac{\varepsilon\pi}{2}}\times e^{\frac{2\pi}{\varepsilon}}<4e^2$ (65)

However $e^4>4e^2$ (66)

so we conclude that we can never have $e^{\frac{\varepsilon\pi}{2}}\times e^{\frac{2\pi}{\varepsilon}}<(2e)^2$ (67)

and this leads to the contradiction because we conclude that: $\varepsilon^2\times(\frac{4}{e^{-\varepsilon\theta}\times e^{\frac{\varepsilon\pi}{2}}\times e^{\frac{2\pi}{\varepsilon}}}-1)$ can't be equal to ***4***.

And thus we have always: $\arctan(\frac{2}{\varepsilon})\times\frac{\varepsilon^2}{\varepsilon^2+4}+\frac{\varepsilon}{\varepsilon^2+4}\times(2\ln(\varepsilon)-\ln(\varepsilon^2+4))+\frac{2\varepsilon}{\varepsilon^2+4}\times\ln(2)\neq\frac{\pi}{2}$ (68)

Finally we conclude that if ***n<0*** then we have always $n^S\neq S$.

4. First conclusion:

If we have: $n^S=S$ and $S=\frac{1}{2}+ib$ and $n=A+iC$ with ***A*** and ***C*** are real numbers, then we have the following implications:

First of all: $n\in\mathbb{R}$ and $S=\frac{1}{2}+ib\Rightarrow n>0$ (69)

And also: $n\in\mathbb{R}$ and $S=\frac{1}{2}+ib\Rightarrow n=e^{\varepsilon\times\theta}=e^{\varepsilon\times\arctan(\frac{2}{\varepsilon})}=e^{\frac{2\theta}{\tan(\theta)}}=e^{\frac{\arctan(2b)}{b}}$ (70)

Which is equivalent to: $n\neq e^{\frac{\arctan(2b)}{b}}\Rightarrow n\notin\mathbb{R}$ or $(S\neq\frac{1}{2}+ib)$ (71)

And we have also: $n\in\mathbb{R}$ and $S=\frac{1}{2}+ib\Rightarrow b\leqslant e$ (72)

And also: $b>e\Rightarrow n\notin\mathbb{R}$ or $(S\neq\frac{1}{2}+ib)$ (73)

5. Third investigation:: if $n=e^{iB}$ ***with B a real number and*** $\forall k'\in\mathbb{Z} \quad B\neq k'\pi$

The statement $\forall k'\in\mathbb{Z} \quad B\neq k'\pi$ means that $n\notin\mathbb{R}$. (74)

We have $n=e^{iB}\Rightarrow\exists k\in\mathbb{Z}$ with $\ln(n)=i\times(B+2k\pi)$ with $n^S=S$. (75)

We proved that: $\frac{\ln(n)}{2}=\frac{-\varepsilon^2}{\varepsilon^2+4}\times(\ln(\varepsilon)-\frac{\ln(\varepsilon^2+4)}{2}+\ln(2))+\arctan(\frac{2}{\varepsilon})\times\frac{2\varepsilon}{\varepsilon^2+4}$ +

$i(\arctan(\frac{2}{\varepsilon})\times\frac{\varepsilon^2}{\varepsilon^2+4}+\frac{\varepsilon}{\varepsilon^2+4}\times(2\ln(\varepsilon)-\ln(\varepsilon^2+4))+\frac{2\varepsilon}{\varepsilon^2+4}\times\ln(2))$ (76)

We know that if $n=e^{iB}$ then the real part of ***ln(n)*** is null.

Hence we get: $\frac{\varepsilon^2}{\varepsilon^2+4}\times(\ln(\varepsilon)-\frac{\ln(\varepsilon^2+4)}{2}+\ln(2))=\arctan(\frac{2}{\varepsilon})\times\frac{2\varepsilon}{\varepsilon^2+4}$ (77)

Consequently: $\frac{\varepsilon}{2}\times(\ln(\varepsilon)-\frac{\ln(\varepsilon^2+4)}{2}+\ln(2))=\arctan(\frac{2}{\varepsilon})$ (78)

We can use this result in the imaginary part and we get:

$$\frac{\ln(n)}{2}=i(\arctan(\frac{2}{\varepsilon})\times\frac{\varepsilon^2}{\varepsilon^2+4}+\frac{\varepsilon}{\varepsilon^2+4}\times(2\ln(\varepsilon)-\ln(\varepsilon^2+4))+\frac{2\varepsilon}{\varepsilon^2+4}\times\ln(2)) \quad (79)$$

Hence:

$$\frac{\ln(n)}{2}=i(\frac{\varepsilon}{2}\times(\ln(\varepsilon)-\frac{\ln(\varepsilon^2+4)}{2}+\ln(2))\times\frac{\varepsilon^2}{\varepsilon^2+4}+\frac{\varepsilon}{\varepsilon^2+4}\times(2\ln(\varepsilon)-\ln(\varepsilon^2+4))+\frac{2\varepsilon}{\varepsilon^2+4}\times\ln(2)) \quad (80)$$

Consequently:

$$\frac{\ln(n)}{2}=i(\frac{\varepsilon}{4}\times(2\ln(\varepsilon)-\ln(\varepsilon^2+4)+2\ln(2))\times\frac{\varepsilon^2}{\varepsilon^2+4}+\frac{\varepsilon}{\varepsilon^2+4}\times(2\ln(\varepsilon)-\ln(\varepsilon^2+4)+2\ln(2))) \quad (81)$$

And thus:

$$\frac{\ln(n)}{2}=i(2\ln(\varepsilon)-\ln(\varepsilon^2+4)+2\ln(2))\times\frac{\varepsilon}{\varepsilon^2+4}\times(1+\frac{\varepsilon^2}{4}) \quad (82)$$

We conclude that: $\frac{\ln(n)}{2}=i\frac{\varepsilon}{4}\times(2\ln(\varepsilon)-\ln(\varepsilon^2+4)+2\ln(2))$ (83)

which means that: $\frac{\ln(n)}{2}=i\frac{\varepsilon}{4}\times\ln(\frac{4\varepsilon^2}{\varepsilon^2+4})$ (84)

Also, we proved that: $\frac{\varepsilon}{2}\times(\ln(\varepsilon)-\frac{\ln(\varepsilon^2+4)}{2}+\ln(2))=\arctan(\frac{2}{\varepsilon})=\theta$ (85)

Hence: $\frac{\varepsilon}{4}\times\ln(\frac{4\varepsilon^2}{\varepsilon^2+4})=\theta$ (86)

And thus: $\frac{\ln(n)}{2}=i\theta$ (87)

We conclude that: $n=e^{i2\theta}$ (88)

And from: $\frac{\varepsilon}{4}\times\ln(\frac{4\varepsilon^2}{\varepsilon^2+4})=\theta$ we conclude that: $e^{4\theta}=(\frac{4\varepsilon^2}{\varepsilon^2+4})^{\varepsilon}$ (89)

And we have also: $\ln(\frac{4\varepsilon^2}{\varepsilon^2+4})=\frac{\theta\times\varepsilon}{4}$ (90)

We know that: $\cos(\theta)^2=\frac{\varepsilon^2}{\varepsilon^2+4}$ Hence we have: $\cos(\theta)^2=e^{\frac{\theta\times 4}{\varepsilon}}$ (91)

We remark that: $0<\theta<\frac{\pi}{2}\Rightarrow 1<e^{4\frac{\theta}{\varepsilon}}<e^{2\frac{\pi}{\varepsilon}}$ (92)

Consequently we have: $1<4\times\cos(\theta)^2<e^{2\frac{\pi}{\varepsilon}}$ (93)

And since: $0<\theta<\frac{\pi}{2}$ then we have: $\cos(\theta)^2>\frac{1}{4}\Rightarrow\cos(\theta)>\frac{1}{2}$ (94)

We conclude that: $\theta<\frac{\pi}{3}$ and since the function ***tangent*** is a monotonic increasing function in

$0<\theta<\frac{\pi}{2}$, then we get: $b=\frac{\tan(\theta)}{2}<\frac{\tan(\frac{\pi}{3})}{2}$ (95)

We can also remark that $n^S=S\Rightarrow S=e^{(S\times i2\theta)}$ (96)

Hence: $\frac{1}{2}+ib=e^{(-b\times 2\theta+i\theta)}$ (97)

And thus: $\frac{1}{2}=e^{(-2\times b\theta)}\times\cos(\theta)$ (98)

with $b=e^{(-2\times b\theta)}\times\sin(\theta)$ (99)

Consequently: $\cos(\theta)=\frac{e^{(2\times b\theta)}}{2}$ (100)

and we have $\cos(\theta)<1$ Hence we have: $e^{2\times b\theta}<2$ (101)

Finally we conclude that: $e^{2b\times\arctan(2b)}<2$ (102)

6. Second conclusion:

If we have: $n^S=S$ and $S=\frac{1}{2}+ib$ and $n=e^{iB}$ with ***B*** is a real number, then we have the following implications:

First of all:

$$\exists B\in\mathbb{R} \quad \text{with} \quad n=e^{iB} \quad \text{and} \quad \forall k'\in\mathbb{Z} \quad B\neq k'\pi \quad \text{and} \quad S=\frac{1}{2}+ib\Rightarrow n=e^{i2\theta}\Rightarrow\exists k\in\mathbb{Z} \quad \text{with} \quad B=2\theta+2k\pi$$

(103)

Which is equivalent to:

$$\exists B\in\mathbb{R} \quad \text{with} \quad n=e^{iB} \quad \text{and} \quad \forall k'\in\mathbb{Z} \quad B\neq k'\pi \quad \text{and} \quad S=\frac{1}{2}+ib\Rightarrow\exists k\in\mathbb{Z} \quad \text{with} \quad B=2\arctan(2b)+2k\pi$$

(104)

And also to:

$$\forall k\in\mathbb{Z} \quad B\neq 2\arctan(2b)+2k\pi\Rightarrow\forall B\in\mathbb{R} \quad \text{we have} \quad n\neq e^{iB} \quad \text{or} \quad \exists k'\in\mathbb{Z} \quad B=k'\pi \quad \text{or} \quad S\neq\frac{1}{2}+ib$$

(105)

And we have also:

$$\exists B\in\mathbb{R} \quad \text{with} \quad n=e^{iB} \quad \text{and} \quad \forall k'\in\mathbb{Z} \quad B\neq k'\pi \quad \text{and} \quad S=\frac{1}{2}+ib\Rightarrow b<\frac{\tan\left(\frac{\pi}{3}\right)}{2}=\frac{\sqrt{3}}{2} \qquad (106)$$

And also: $b\geq\frac{\sqrt{3}}{2}\Rightarrow\forall B\in\mathbb{R}$ we have $n\neq e^{iB}$ or $\exists k'\in\mathbb{Z}$ $B=k'\pi$ or $S\neq\frac{1}{2}+ib$ (107)

And also: $\exists B\in\mathbb{R}$ with $n=e^{iB}$ and $\forall k'\in\mathbb{Z}$ $B\neq k'\pi$ and $S=\frac{1}{2}+ib\Rightarrow e^{2b\times\arctan(2b)}<2$

(108)

And also: $e^{2b\times\arctan(2b)}\geq 2\Rightarrow\forall B\in\mathbb{R}$ we have $n\neq e^{iB}$ or $\exists k'\in\mathbb{Z}$ $B=k'\pi$ or $S\neq\frac{1}{2}+ib$

(109)

7. Third conclusion:

If we have: $n^S = S$ and $S = \frac{1}{2} + ib$ and $n = e^{iB}$ with ***B*** is a real number, then we have the following conclusions:

We know that: $b > e \Rightarrow b \geq \frac{\sqrt{3}}{2}$ (110)

And we know that the function $2x \times \arctan(2x)$ is an increasing monotonic function when x is a real strictly positive number. Hence: $b > e \Rightarrow 2b \times \arctan(2b) > 2e \times \arctan(2e)$ (111)

Consequently: $b > e \Rightarrow e^{2b \times \arctan(2b)} > e^{2e \times \arctan(2e)} \simeq 1902.2422648$ (112)

And thus: $b > e \Rightarrow e^{2b \times \arctan(2b)} > 2$ (113)

We conclude finally that :

$$b > e \Rightarrow (\forall B \in \mathbb{R} \quad \text{we have} \quad n \neq e^{iB} \quad \text{and} \quad n \notin \mathbb{R}) \quad \text{or} \quad (S \neq \frac{1}{2} + ib) \tag{114}$$

And we know that: $n \notin \mathbb{R}$ causes that $\forall k' \in \mathbb{Z} \quad B \neq k' \pi$.

8. Example of usefulness for Riemann's Zeta Function:

Let's consider that: $\exists \gamma \in \mathbb{R}$ with $n = e^{i\gamma}$ and $f(n) = n \Leftrightarrow n^S = S$. (115)

We consider also that: ***S=a+ib*** with ***b****>****e***.

In this case we already have $n = e^{i\gamma} = e^{iB}$ and $n^S = S$ and we can also have $n \in \mathbb{R}$.

However, since b>e then $a \neq \frac{1}{2}$.

If we need to have $b \leq e$, we can also find a set of pairs ***a*** and ***b*** linked by the same relationship and respecting that: $a \neq \frac{1}{2}$. However, in this case, we should be sure that $\forall k \in \mathbb{Z} \quad B \neq 2\arctan(2b) + 2k\pi$ while $n = e^{i\gamma} = e^{iB}$ and $n^S = S$. In this case, if we know **B** and it is a small number then we can fix ***b*** with $\frac{B}{2} < \arctan(2b) < \frac{B}{2} + \pi$ (116)

or with $\frac{B}{2}-\pi<\arctan(2b)<\frac{B}{2}$ (117)

and if ***B*** is a big number we should find a positive integer ***k*** respecting that

$$\frac{B}{2}+(k-1)\pi<\arctan(2b)<\frac{B}{2}+k\pi \quad (118)$$

or $\frac{B}{2}-k\pi<\arctan(2b)<\frac{B}{2}-(k-1)\pi$. (119)

We can find a set of different real numbers ***γ (Gamma)*** and each one of these numbers respects that $e^{(S\times i\gamma)}=S$ and thus each number ***γ*** creates a set of pairs of real numbers ***(a,b)*** linked by a special relationship that respects that $a\neq\frac{1}{2}$.

We can define this relationship since we have $n^S=S\Rightarrow S=e^{(S\times i\gamma)}$

Hence: $a+ib=e^{(-b\gamma+ia\gamma)}$ (120)

And thus we have: $a=e^{(-b\gamma)}\times\cos(a\gamma)$ (121)

And we have: $b=e^{(-b\gamma)}\times\sin(a\gamma)$ (122)

Consequently: $|S|^2=a^2+b^2=e^{(-2b\gamma)}$ (123)

This result can be useful in many mathematical cases as a criterion since we can find one useful representative of all the ***γ (Gamma)*** numbers possibilities easily from formula (123) and we have:

$$\gamma=\frac{-\ln(a^2+b^2)}{2b} \quad (124)$$

Furthermore the number ***γ (Gamma)*** always exists since we use b>0.

<u>The example of usefulness:</u>

For example, if ***S=a+ib*** is a non-trivial zero of Riemann's Zeta Function and we use $n=e^{i\gamma}$ with $\gamma=\frac{-\ln(a^2+b^2)}{2b}$ as demonstrated above, then we have not only:

$$a=e^{(-b\gamma)}\times\cos(a\gamma) \quad \text{and} \quad b=e^{(-b\gamma)}\times\sin(a\gamma) \quad \text{with} \quad S=e^{(S\times i\gamma)} \quad (125)$$

But we have also: $a\neq\frac{1}{2}$ (126)

This means that ***S*** is a Landau-Siegel Zero. And in this case, we have of course ***b>e*** since the study of non-trivial zeros ***S*** of Riemann's Zeta Function is only interesting when ***b*** gets big. Furthermore, we won't need in this case a criterion for ***b*** negative because Riemann's Xi Function causes that if ***S*** is a non-trivial Zero of Riemann's Zeta Function then ***(1-S)*** is also a non-trivial Zero of Riemann's Zeta Function.

9. Related Algebraic mathematical projects:

*The work that remains about the functions **f** of complex numbers defined from* $\mathbb{C}$ *in* $\mathbb{C}$ *as* $f(n)=\frac{n^{S+1}}{S}$ *is when **S=a+ib** with **a** and **b** are real numbers, and* $f(n)=n \Leftrightarrow n^{S}=S$ *:*

1) *How can we describe in this case the homomorphism that may link the set **V** containing all the pairs of real numbers* (a,b) *excepting the pairs* $(\frac{1}{2},b)$ *with the set of fixed-points of the functions **f*** that depend on each complex number S *? This is interesting because the set **V** contains obviously any possible Landau-Siegel Zero.*

2) *How can we describe in this case the homomorphism that may link the set W containing all the pairs of real numbers* $(\frac{1}{2},b)$ *with the set of fixed-points of the functions **f*** that depend on each complex number S *? This is interesting because the set **W** contains the zeros that respect Riemann's Hypothesis.*

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