The study of cases of fixed points of a complex function depending on the inputs of this function

Akram Louiz
Independent researcher, Settat, Morocco.
louiz.akram@gmail.com

Abstract :

This is a mathematical article that deals with the domain of complex numbers in order to present new results by using a complex function and studying its fixed points.

The aim of this work is to propose new mathematical statements of complex numbers as a probable criterion that can be linked to Landau-Siegel Zeros. Hence, this is a good article to be read by specialists of Calculus and Analysis and even by the beginners of mathematics who want to improve their skills in the field of complex numbers.

Keywords : complex analysis; fixed-point; implication and negation; Landau-Siegel Zero; Riemann's Zeta function; Riemann's Hypothesis.

Introduction:

After my thesis of mathematical physics that refuses the classical use of discrete mathematics in Newtonian Mechanics [1], I chose to focus on subjects of pure mathematics and especially calculus and analysis [2,3] or the number theory [4,5] by writing my personal articles that respect all the rules of classical mathematics. The purpose of this work is to study two cases of fixed points of a complex function by using a complex number ***S*** that respects that $S=\frac{1}{2}+ib$ where ***b*** is a strictly positive real number. The choice of the number ***S*** in its form as an example in this proof is because we are trying to make the example of this article more useful and more attractive since the zeros of Riemann Hypothesis are supposed to have the same form which is $S=\frac{1}{2}+ib$. We wish that this article proposes new useful mathematical statements about complex numbers and gives a probable criterion that may help to identify or deny the possible Landau-Siegel Zeros [6].

These are the considerations and notations of this article:

Let ***S*** be a complex number that respects that $S=\frac{1}{2}+ib$ where ***b*** is a strictly positive real number.

We can also write ***S*** as $S=\frac{1}{2}+i\frac{1}{\varepsilon}=\frac{1}{2}+i\frac{\tan(\theta)}{2}$ where $b=\frac{\tan(\theta)}{2}=\frac{1}{\varepsilon}$. (1)

In this case, we should only consider that $\frac{1}{\varepsilon}>0$ and $0<\theta<\frac{\pi}{2}$ and we have $S=\frac{e^{i\theta}}{2\cos(\theta)}$ (2)

We have also: $\theta=\arctan(\frac{2}{\varepsilon})=\frac{\pi}{2}-\arctan(\frac{\varepsilon}{2})$ (3)

and $\cos(\theta)=\cos(\arctan(\frac{2}{\varepsilon}))=\frac{\varepsilon}{\sqrt{\varepsilon^2+4}}$ (4)

and $\sin(\theta)=\sin(\arctan(\frac{2}{\varepsilon}))=\frac{2}{\sqrt{\varepsilon^2+4}}$ (5)

consequently: $\sin(\theta)\times\cos(\theta)=\frac{2\varepsilon}{\varepsilon^2+4}$ (6)

Let's consider also a function ***f*** of complex numbers ***z*** defined from $\mathbb{C}$ in $\mathbb{C}$ as:

$$f(z)=\frac{z^{S+1}}{S} \quad (7)$$

We will use the complex logarithm ***ln(z)*** point by point during all this proof.

1. The considered complex function and its fixed-point:

We have: $f(z)=z\Leftrightarrow z^S=S\Leftrightarrow\frac{e^{i\theta}}{2\cos(\theta)}=z^{\frac{e^{i\theta}}{2\cos(\theta)}}$ (8)

And we have: $z^S=S\Leftrightarrow e^{S\times\ln(z)}=\frac{e^{i\theta}}{2\cos(\theta)}$ (9)

Consequently: $\ln(S)=\ln(z^S)\Leftrightarrow\ln(\frac{e^{i\theta}}{2\cos(\theta)})=\frac{e^{i\theta}}{2\cos(\theta)}\times\ln(z)$ (10)

Hence: $\ln(z)=\frac{2\cos(\theta)\times(\ln(e^{i\theta})-\ln(2\cos(\theta)))}{e^{i\theta}}$ (11)

Consequently: $\frac{\ln(z)}{2}=(\cos(\theta)-i\sin(\theta))\times\cos(\theta)\times(\ln(e^{i\theta})-\ln(2\cos(\theta)))$ (12)

Hence: $\frac{\ln(z)}{2}=(\cos(\theta)^2-i\frac{\sin(2\theta)}{2})\times(i\theta-\ln(\cos(\theta))-\ln(2))$ (13)

We considered that $\ln(e^{i\theta})=i\theta+i\times 2k\pi=i\theta$ with ***k=0*** (14)

because we have one unique complex number ***ln(z)*** for each $e^{i\theta}$ and for each $\cos(\theta)$ with $0<\theta<\frac{\pi}{2}$.

And thus: $\frac{\ln(z)}{2}=(-\cos(\theta)^2\times(\ln(\cos(\theta))+\ln(2))+\frac{\theta\times\sin(2\theta)}{2})$ +

$$i(\theta\times\cos(\theta)^2+\frac{\sin(2\theta)\times\ln(\cos(\theta))}{2}+\frac{\sin(2\theta)\times\ln(2)}{2}) \quad (15)$$

And by using the formulas of the introduction, we get:

$$\frac{\ln(z)}{2}=(\frac{-\varepsilon^2}{\varepsilon^2+4}\times(\ln(\frac{\varepsilon}{\sqrt{\varepsilon^2+4}})+\ln(2))+\arctan(\frac{2}{\varepsilon})\times\frac{2\varepsilon}{\varepsilon^2+4}) \quad +$$

$$i(\arctan(\frac{2}{\varepsilon})\times\frac{\varepsilon^2}{\varepsilon^2+4}+\frac{2\varepsilon}{\varepsilon^2+4}\times\ln(\frac{\varepsilon}{\sqrt{\varepsilon^2+4}})+\frac{2\varepsilon}{\varepsilon^2+4}\times\ln(2)) \quad (16)$$

Consequently:

$$\frac{\ln(z)}{2}=\frac{-\varepsilon^2}{\varepsilon^2+4}\times(\ln(\varepsilon)-\frac{\ln(\varepsilon^2+4)}{2}+\ln(2))+\arctan(\frac{2}{\varepsilon})\times\frac{2\varepsilon}{\varepsilon^2+4} \quad +$$

$$i(\arctan(\frac{2}{\varepsilon})\times\frac{\varepsilon^2}{\varepsilon^2+4}+\frac{\varepsilon}{\varepsilon^2+4}\times(2\ln(\varepsilon)-\ln(\varepsilon^2+4))+\frac{2\varepsilon}{\varepsilon^2+4}\times\ln(2)) \quad (17)$$

2. First investigation: if z is a real positive number:

The number ***z*** can't be equal to zero because zero is not a solution to the equation: $z^S=S$.

Since we have: $e^{S\times\ln(z)}=e^{S\times\ln(z)+i2k\pi}=e^{S\times(\ln(z)+i\frac{2k\pi}{S})}=\frac{e^{i\theta}}{2\cos(\theta)}$ $\forall\, k\in\mathbb{Z}$ (18)

Then we have ***k*** numbers that respect $z^S=S$ and these numbers are $(\ln(z)+i\frac{2k\pi}{S})$ (19)

We can notice that ***ln(z)*** is the only real number among these ***k*** numbers when ***z*** is a real number and this corresponds to ***k=0***. Hence when ***z*** is a real number, we have only one unique real number ***ln(z)*** for each θ with $0<\theta<\frac{\pi}{2}$.

If ***z*** is a real strictly positive number then $\Im(\frac{\ln(z)}{2})=0$.

Hence we have: $\varepsilon\times\arctan(\frac{2}{\varepsilon})=-2\ln(\varepsilon)+\ln(\varepsilon^2+4)-2\ln(2)$ (20)

And thus, our equation (17) becomes:

$$\frac{\ln(z)}{2}=\frac{-\varepsilon^2}{\varepsilon^2+4}\times(\ln(\varepsilon)-\frac{\ln(\varepsilon^2+4)}{2}+\ln(2))+\arctan(\frac{2}{\varepsilon})\times\frac{2\varepsilon}{\varepsilon^2+4} \qquad (21)$$

Hence:

$$\frac{\ln(z)}{2}=\frac{-\varepsilon^2}{\varepsilon^2+4}\times(\ln(\varepsilon)-\frac{\ln(\varepsilon^2+4)}{2}+\ln(2))+(-2\ln(\varepsilon)+\ln(\varepsilon^2+4)-2\ln(2))\times\frac{2\varepsilon}{\varepsilon^2+4} \qquad (22)$$

Consequently:

$$\frac{\ln(z)}{2}=\frac{-\varepsilon}{\varepsilon^2+4}\times(\varepsilon\times(\ln(\varepsilon)-\frac{\ln(\varepsilon^2+4)}{2}+\ln(2))+\frac{4}{\varepsilon}\times(\ln(\varepsilon)-\ln\frac{(\varepsilon^2+4)}{2}+\ln(2))) \qquad (23)$$

And thus: $\frac{\ln(z)}{2}=\frac{-\varepsilon}{\varepsilon^2+4}\times(\varepsilon+\frac{4}{\varepsilon})\times(\ln(\varepsilon)-\frac{\ln(\varepsilon^2+4)}{2}+\ln(2))$ (24)

We conclude finally that: $\frac{\ln(z)}{2}=-\ln(\varepsilon)+\frac{\ln(\varepsilon^2+4)}{2}-\ln(2)$ (25)

Which is equivalent to: $\ln(z)=\ln(\frac{\varepsilon^2+4}{4\varepsilon^2})$ (26)

Also, we have: $\ln(z)=\ln(\frac{\varepsilon^2+4}{4\varepsilon^2})\Leftrightarrow z=\frac{\varepsilon^2+4}{4\varepsilon^2}\Leftrightarrow z=\frac{1}{4}+\frac{1}{\varepsilon^2}$ (27)

We proved also that: $\varepsilon\times\arctan(\frac{2}{\varepsilon})=-2\ln(\varepsilon)+\ln(\varepsilon^2+4)-2\ln(2)$ (28)

This means that: $\varepsilon\times\theta=\ln(\frac{\varepsilon^2+4}{4\varepsilon^2})$ (29)

And thus: $e^{\varepsilon\times\theta}=\frac{1}{4}+\frac{1}{\varepsilon^2}$ (30)

We conclude that: $z=e^{\varepsilon\times\theta}=e^{\varepsilon\times\arctan(\frac{2}{\varepsilon})}=e^{\frac{2\theta}{\tan(\theta)}}=e^{\frac{\arctan(2b)}{b}}$ (31)

We can also remark that we have: $e^{\varepsilon\times\theta}=\frac{1}{4}+\frac{1}{\varepsilon^2}\Leftrightarrow\frac{\varepsilon^2}{\varepsilon^2\times e^{\varepsilon\times\theta}-1}=4$ (32)

This means that: $\frac{\varepsilon^2}{(\varepsilon\times e^{\frac{\varepsilon}{2}\times\arctan(\frac{2}{\varepsilon})})^2-1}=4$ (33)

And we know that the maximum of $\frac{\arctan(x)}{x}$ is ***1***. (34)

Hence $\frac{\varepsilon}{2}\times\arctan(\frac{2}{\varepsilon})<1$ (35)

And thus: $e^{\frac{\varepsilon}{2}\times\arctan(\frac{2}{\varepsilon})}<e$ (36)

Finally, we conclude that we should have $\varepsilon\geq\frac{1}{e}$ (37)

which is equivalent to: $\tan(\theta)\leqslant 2e$ (38)

otherwise we will have a contradiction in the equation $\frac{\varepsilon^2}{(\varepsilon\times e^{\frac{\varepsilon}{2}\times\arctan(\frac{2}{\varepsilon})})^2-1}=4$ (39)

because ***4*** can't be equal to any negative value.

3. Second investigation: if z is a strictly negative real number:

Since we have: $e^{S\times\ln(z)}=e^{S\times\ln(z)+i2k\pi}=e^{S\times(\ln(z)+i\frac{2k\pi}{S})}=\frac{e^{i\theta}}{2\cos(\theta)}\quad\forall k\in\mathbb{Z}$ (40)

Then we have ***k*** numbers that respect $z^S=S$ and these numbers are $(\ln(z)+i\frac{2k\pi}{S})$ (41)

If ***z*** is a strictly negative number then ***ln(z)*** is a complex number .

And we have $\exists k'\in\mathbb{Z}$ with $\ln(z)=\ln(-z)+i\pi+i2k'\pi$. (42)

Hence, we have $z^S=e^{S\times(\ln(z)+i\frac{2k\pi}{S})}=e^{S\times(\ln(-z)+i\pi+i2k'\pi+i\frac{2k\pi}{S})}$ (43)

And thus, we should have: $e^{i2k'\pi S+i2k\pi}=1$. (44)

We conclude that: $\exists k''\in\mathbb{Z}$ and $\exists k'''\in\mathbb{Z}$ with $k'\pi S+k\pi=k'''\pi$ (45)

Since ***b*** is a strictly positive number, then we should have obviously:

$k'\pi S=k'\pi(\frac{1}{2}+ib)=0$ Hence, we should have: **k'=0**.

Finally, we conclude that we should investigate about the unique complex number ***ln(z)*** that corresponds to each ***θ*** with $0<\theta<\frac{\pi}{2}$ and that respects: $\ln(z)=\ln(-z)+i\pi$. (46)

In this case, we use $\Im(\frac{\ln(z)}{2})=\frac{\pi}{2}$ (47)

and we get:

$$\arctan(\frac{2}{\varepsilon})\times\frac{\varepsilon^2}{\varepsilon^2+4}+\frac{\varepsilon}{\varepsilon^2+4}\times(2\ln(\varepsilon)-\ln(\varepsilon^2+4))+\frac{2\varepsilon}{\varepsilon^2+4}\times\ln(2)=\frac{\pi}{2} \qquad (48)$$

Hence: $$\arctan(\frac{2}{\varepsilon})\times\frac{\varepsilon^2}{\varepsilon^2+4}=\frac{\pi}{2}-\frac{\varepsilon}{\varepsilon^2+4}\times(2\ln(\varepsilon)-\ln(\varepsilon^2+4))-\frac{2\varepsilon}{\varepsilon^2+4}\times\ln(2) \qquad (49)$$

Consequently: $$\arctan(\frac{2}{\varepsilon})\times\frac{\varepsilon^2}{\varepsilon^2+4}=\frac{\pi}{2}-\frac{\varepsilon}{\varepsilon^2+4}\times(2\ln(\varepsilon)-\ln(\varepsilon^2+4))-\frac{2\varepsilon}{\varepsilon^2+4}\times\ln(2) \qquad (50)$$

And thus: $$\arctan(\frac{2}{\varepsilon})\times\frac{2\varepsilon}{\varepsilon^2+4}=\frac{\pi}{\varepsilon}-\frac{2}{(\varepsilon^2+4)}\times(2\ln(\varepsilon)-\ln(\varepsilon^2+4))-\frac{4}{\varepsilon^2+4}\times\ln(2) \qquad (51)$$

We conclude that: $$\theta\times\frac{2\varepsilon}{\varepsilon^2+4}=\frac{\pi}{\varepsilon}-\frac{2}{\varepsilon^2+4}\times\ln(\frac{4\varepsilon^2}{\varepsilon^2+4}) \qquad (52)$$

which is equivalent to: $$\ln(\frac{4\varepsilon^2}{\varepsilon^2+4})=(-\theta\times\frac{2\varepsilon}{\varepsilon^2+4}+\frac{\pi}{\varepsilon})\times\frac{\varepsilon^2+4}{2}=-\varepsilon\theta+\frac{\varepsilon\pi}{2}+\frac{2\pi}{\varepsilon} \qquad (53)$$

Consequently, we get: $$\frac{4\varepsilon^2}{\varepsilon^2+4}=e^{-\varepsilon\theta}\times e^{\frac{\varepsilon\pi}{2}}\times e^{\frac{2\pi}{\varepsilon}} \qquad (54)$$

Hence: $$\frac{4\varepsilon^2}{e^{-\varepsilon\theta}\times e^{\frac{\varepsilon\pi}{2}}\times e^{\frac{2\pi}{\varepsilon}}}=\varepsilon^2+4 \qquad (55)$$

And thus: $$4=\varepsilon^2\times(\frac{4}{e^{-\varepsilon\theta}\times e^{\frac{\varepsilon\pi}{2}}\times e^{\frac{2\pi}{\varepsilon}}}-1) \qquad (56)$$

Hence, in order to avoid the contradiction, we should have: $$\frac{4}{e^{-\varepsilon\theta}\times e^{\frac{\varepsilon\pi}{2}}\times e^{\frac{2\pi}{\varepsilon}}}>1 \qquad (57)$$

And this is equivalent to: $$\frac{(2e^{\frac{\varepsilon}{2}\theta})^2}{e^{\frac{\varepsilon\pi}{2}}\times e^{\frac{2\pi}{\varepsilon}}}>1\Leftrightarrow\frac{(2e^{\frac{\varepsilon}{2}\arctan(\frac{2}{\varepsilon})})^2}{e^{\frac{\varepsilon\pi}{2}}\times e^{\frac{2\pi}{\varepsilon}}}>1 \qquad (58)$$

And we proved that: $$e^{\frac{\varepsilon}{2}\times\arctan(\frac{2}{\varepsilon})}<e \qquad (59)$$

And thus, in order to avoid the contradiction, we should have: $$e^{\frac{\varepsilon\pi}{2}}\times e^{\frac{2\pi}{\varepsilon}}<(2e)^2 \qquad (60)$$

Now let's check if this inequality is possible:

We can notice that: $$\frac{\varepsilon\pi}{2}+\frac{2\pi}{\varepsilon}-4=\frac{\varepsilon^2\pi-8\varepsilon+4\pi}{2\varepsilon} \qquad (61)$$

and that the discriminant of the quadratic equation $\varepsilon^2\pi-8\varepsilon+4\pi$ is negative, Hence $\frac{\varepsilon^2\pi-8\varepsilon+4\pi}{2\varepsilon}>0$ (62)

And thus: $\frac{\varepsilon\pi}{2}+\frac{2\pi}{\varepsilon}>4$ (63)

Consequently: $e^{\frac{\varepsilon\pi}{2}}\times e^{\frac{2\pi}{\varepsilon}}>e^4$ (64)

And we should have: $e^{\frac{\varepsilon\pi}{2}}\times e^{\frac{2\pi}{\varepsilon}}<4e^2$ (65)

However $e^4>4e^2$ (66)

so we conclude that we can never have $e^{\frac{\varepsilon\pi}{2}}\times e^{\frac{2\pi}{\varepsilon}}<(2e)^2$ (67)

and this leads to the contradiction because we conclude that: $\varepsilon^2\times(\frac{4}{e^{-\varepsilon\theta}\times e^{\frac{\varepsilon\pi}{2}}\times e^{\frac{2\pi}{\varepsilon}}}-1)$ can't be equal to ***4***.

And thus we have always:

$$\arctan(\frac{2}{\varepsilon})\times\frac{\varepsilon^2}{\varepsilon^2+4}+\frac{\varepsilon}{\varepsilon^2+4}\times(2\ln(\varepsilon)-\ln(\varepsilon^2+4))+\frac{2\varepsilon}{\varepsilon^2+4}\times\ln(2)\neq\frac{\pi}{2} \quad (68)$$

Finally we conclude that if z<***0*** then we have always $z^S\neq S$.

4. First conclusion:

We could prove that we have the following implications:

First of all: $z^S=S$ and $z\in\mathbb{R}$ and $S=\frac{1}{2}+ib\Rightarrow z>0$ (69)

And also: $z^S=S$ and $z\in\mathbb{R}$ and $S=\frac{1}{2}+ib\Rightarrow z=e^{\varepsilon\times\theta}=e^{\varepsilon\times\arctan(\frac{2}{\varepsilon})}=e^{\frac{2\theta}{\tan(\theta)}}=e^{\frac{\arctan(2b)}{b}}$ (70)

And we have also: $z^S=S$ and $z\in\mathbb{R}$ and $S=\frac{1}{2}+ib\Rightarrow b\leqslant e$ (71)

Since the function ***f*** admits a fixed-point ***z=r*** and since we proved that ***z*** can only be strictly positive when ***z*** is a real number, we have also:

$$f(z)=z\Leftrightarrow S=z^S\Leftrightarrow\frac{1}{2}+ib=r^S\Leftrightarrow\frac{1}{2}+ib=r^{\frac{1}{2}}\times r^{ib} \quad (72)$$

And we have: $r^{\frac{1}{2}}\times r^{ib}=r^{\frac{1}{2}}\times e^{ib\times\ln(r)}=r^{\frac{1}{2}}\times(\cos(b\times\ln(r))+i\sin(b\times\ln(r)))$ (73)

Hence, we have: $\sqrt{r}\times\cos(b\times\ln(r))=\frac{1}{2}$ (74)

And we have: $\sqrt{r}\times\sin(b\times\ln(r))=b$ (75)

And thus, we have: $r=\frac{1}{4}+b^2$ (76)

This means that: $e^{\frac{\arctan(2b)}{b}}=\frac{1}{4}+b^2$ (77)

And thus, the strictly positive real number ***b*** can only be the solution of the equation:

$$e^{\frac{\arctan(2b)}{b}}-b^2=\frac{1}{4} \quad (78)$$

Analytically from the curve of the equation, we have: $b\approx\pm 1{,}449$ (79)

But since we considered that the real number ***b*** is strictly positive, then we have :

$$z^S=S \quad \text{and} \quad z\in\mathbb{R} \quad \text{and} \quad S=\frac{1}{2}+ib\Rightarrow b\approx 1{,}449 \quad \text{and} \quad z=e^{\frac{\arctan(2b)}{b}}\approx 2{,}323 \quad (80)$$

5. Third investigation: if $z=e^{iB}$ with B a real number and $\forall k'\in\mathbb{Z} \quad B\neq k'\pi$

The statement $\forall k'\in\mathbb{Z} \quad B\neq k'\pi$ means that $z\notin\mathbb{R}$. (81)

We have $z=e^{iB}\Rightarrow\exists k\in\mathbb{Z}$ with $\ln(z)=i\times(B+2k\pi)$ with $z^S=S$. (82)

We proved that: $\frac{\ln(z)}{2}=\frac{-\varepsilon^2}{\varepsilon^2+4}\times(\ln(\varepsilon)-\frac{\ln(\varepsilon^2+4)}{2}+\ln(2))+\arctan(\frac{2}{\varepsilon})\times\frac{2\varepsilon}{\varepsilon^2+4}$ +

$$i(\arctan(\frac{2}{\varepsilon})\times\frac{\varepsilon^2}{\varepsilon^2+4}+\frac{\varepsilon}{\varepsilon^2+4}\times(2\ln(\varepsilon)-\ln(\varepsilon^2+4))+\frac{2\varepsilon}{\varepsilon^2+4}\times\ln(2)) \quad (83)$$

We know that if $z=e^{iB}$ then the real part of ***ln(z)*** is null.

Hence we get: $\frac{\varepsilon^2}{\varepsilon^2+4}\times(\ln(\varepsilon)-\frac{\ln(\varepsilon^2+4)}{2}+\ln(2))=\arctan(\frac{2}{\varepsilon})\times\frac{2\varepsilon}{\varepsilon^2+4}$ (84)

Consequently: $\frac{\varepsilon}{2}\times(\ln(\varepsilon)-\frac{\ln(\varepsilon^2+4)}{2}+\ln(2))=\arctan(\frac{2}{\varepsilon})$ (85)

We can use this result in the imaginary part and we get:

$$\frac{\ln(z)}{2}=i(\arctan(\frac{2}{\varepsilon})\times\frac{\varepsilon^2}{\varepsilon^2+4}+\frac{\varepsilon}{\varepsilon^2+4}\times(2\ln(\varepsilon)-\ln(\varepsilon^2+4))+\frac{2\varepsilon}{\varepsilon^2+4}\times\ln(2)) \quad (86)$$

Hence:

$$\frac{\ln(z)}{2}=i(\frac{\varepsilon}{2}\times(\ln(\varepsilon)-\frac{\ln(\varepsilon^2+4)}{2}+\ln(2))\times\frac{\varepsilon^2}{\varepsilon^2+4}+\frac{\varepsilon}{\varepsilon^2+4}\times(2\ln(\varepsilon)-\ln(\varepsilon^2+4))+\frac{2\varepsilon}{\varepsilon^2+4}\times\ln(2)) \quad (87)$$

Consequently:

$$\frac{\ln(z)}{2}=i(\frac{\varepsilon}{4}\times(2\ln(\varepsilon)-\ln(\varepsilon^2+4)+2\ln(2))\times\frac{\varepsilon^2}{\varepsilon^2+4}+\frac{\varepsilon}{\varepsilon^2+4}\times(2\ln(\varepsilon)-\ln(\varepsilon^2+4)+2\ln(2))) \quad (88)$$

And thus:

$$\frac{\ln(z)}{2}=i(2\ln(\varepsilon)-\ln(\varepsilon^2+4)+2\ln(2))\times\frac{\varepsilon}{\varepsilon^2+4}\times(1+\frac{\varepsilon^2}{4}) \quad (89)$$

We conclude that: $\frac{\ln(z)}{2}=i\frac{\varepsilon}{4}\times(2\ln(\varepsilon)-\ln(\varepsilon^2+4)+2\ln(2))$ (90)

which means that: $\frac{\ln(z)}{2}=i\frac{\varepsilon}{4}\times\ln(\frac{4\varepsilon^2}{\varepsilon^2+4})$ (91)

Also, we proved that: $\frac{\varepsilon}{2}\times(\ln(\varepsilon)-\frac{\ln(\varepsilon^2+4)}{2}+\ln(2))=\arctan(\frac{2}{\varepsilon})=\theta$ (92)

Hence: $\frac{\varepsilon}{4}\times\ln(\frac{4\varepsilon^2}{\varepsilon^2+4})=\theta$ (93)

And thus: $\frac{\ln(z)}{2}=i\theta$ (94)

We conclude that: $z=e^{i2\theta}$ (95)

And from: $\frac{\varepsilon}{4}\times\ln(\frac{4\varepsilon^2}{\varepsilon^2+4})=\theta$ we conclude that: $e^{4\theta}=(\frac{4\varepsilon^2}{\varepsilon^2+4})^{\varepsilon}$ (96)

We know that: $\cos(\theta)^2=\frac{\varepsilon^2}{\varepsilon^2+4}$ Hence we have: $4\times\cos(\theta)^2=e^{\frac{\theta\times4}{\varepsilon}}$ (97)

We remark that: $0<\theta<\frac{\pi}{2}\Rightarrow 1<e^{4\frac{\theta}{\varepsilon}}<e^{2\frac{\pi}{\varepsilon}}$ (98)

Consequently we have: $1<4\times\cos(\theta)^2<e^{2\frac{\pi}{\varepsilon}}$ (99)

And since: $0<\theta<\frac{\pi}{2}$ then we have: $\cos(\theta)^2>\frac{1}{4}\Rightarrow\cos(\theta)>\frac{1}{2}$ (100)

We conclude that: $\theta<\frac{\pi}{3}$ and since the function ***tangent*** is a monotonic increasing function in

$0<\theta<\frac{\pi}{2}$, then we get: $b=\frac{\tan(\theta)}{2}<\frac{\tan(\frac{\pi}{3})}{2}=\frac{\sqrt{3}}{2}$ (101)

We have also: $\cos(\theta)^2<\frac{e^{2\frac{\pi}{\varepsilon}}}{4}$ and this case should be also studied since we can have:

$$\frac{e^{2\frac{\pi}{\varepsilon}}}{4}\leq 1\Leftrightarrow 2\frac{\pi}{\varepsilon}\leq\ln(4)\Leftrightarrow b\leq\frac{\ln(2)}{\pi} \quad (102)$$

with: $\frac{\ln(2)}{\pi}\approx 0{,}2206$ (103)

in this case, since: $0<\theta<\frac{\pi}{2}$, we have: $\theta>\arccos(\frac{e^{\frac{\pi}{\varepsilon}}}{2})$ (104)

Hence, we have: $b=\frac{\tan(\theta)}{2}>\frac{\tan(\arccos(\frac{e^{\frac{\pi}{\varepsilon}}}{2}))}{2}$ (105)

And thus, we have: $b>\frac{2\times\sqrt{1-\frac{e^{2\frac{\pi}{\varepsilon}}}{4}}}{e^{\frac{\pi}{\varepsilon}}}$ (106)

This means that we have: $\frac{b\times e^{b\times\pi}}{2}>\sqrt{1-\frac{e^{2\pi\times b}}{4}}$ (107)

Which means that: $\frac{b^2\times e^{2\times b\times\pi}}{4}+\frac{e^{2\pi\times b}}{4}>1$ (108)

And this is equivalent to: $\frac{e^{2\times b\times\pi}}{4}\times(b^2+1)>1$ (109)

And since for any strictly positive real number ***x*** we have the function $g(x)=\frac{e^{2\times x\times\pi}}{4}\times(x^2+1)$ is a monotonic increasing function then we can deduce analytically from the curve of the function ***g*** that: $b>0{,}214$. (110)

6. Second conclusion:

We know that: $z \notin \mathbb{R}$ causes that $\forall k' \in \mathbb{Z} \quad B \neq k'\pi$. (111)

We conclude that we could prove the following implications:

If we have:

$$\exists B \in \mathbb{R} \quad \text{with} \quad z = e^{iB} \quad \text{and} \quad \forall k' \in \mathbb{Z} \quad B \neq k'\pi \quad \text{and} \quad S = \frac{1}{2} + ib \quad \text{and} \quad z^S = S \qquad (112)$$

Then we have: $\exists k \in \mathbb{Z}$ with $B = 2\theta + 2k\pi$ (113)

Which is equivalent to: $\exists k \in \mathbb{Z}$ with $B = 2\arctan(2b) + 2k\pi$ (114)

And we have also: $0{,}214 < b < \frac{\sqrt{3}}{2}$ (115)

Since the function ***f*** admits a fixed-point $z = e^{iB}$, we have also:

$$f(z) = z \Leftrightarrow S = z^S \Leftrightarrow \frac{1}{2} + ib = e^{S \times iB} \Leftrightarrow \frac{1}{2} + ib = e^{-Bb} \times e^{\frac{iB}{2}} \qquad (116)$$

And we have: $e^{-Bb} \times e^{\frac{iB}{2}} = e^{-Bb} \times (\cos(\frac{B}{2}) + i\sin(\frac{B}{2}))$ (117)

Hence, we have: $e^{-Bb} \times \cos(\frac{B}{2}) = \frac{1}{2}$ (118)

And we have: $e^{-Bb} \times \sin(\frac{B}{2}) = b$ (119)

And thus, we have: $e^{-2Bb} = \frac{1}{4} + b^2$ (120)

Consequently, we have: $B = \dfrac{-\ln(\frac{1}{4} + b^2)}{2b}$ (121)

This means that: $\exists k \in \mathbb{Z}$ with $2\arctan(2b) + 2k\pi = \dfrac{-\ln(\frac{1}{4} + b^2)}{2b}$ (122)

And thus, the strictly positive real number ***b*** can only be the solution of the equation:

$$\exists k \in \mathbb{Z} \quad \text{with} \quad \frac{\arctan(2b)}{\pi} + \frac{\ln(\frac{1}{4} + b^2)}{4b\pi} = -k \qquad (123)$$

Since we considered that the real number ***b*** is strictly positive, then this equation has analytically from the curve of the equation one solution for each integer ***k*** with $-k\leq 0 \Leftrightarrow k\geq 0$ (124)

But since we proved that $0{,}214<b<\frac{\sqrt{3}}{2}$, then we notice analytically from the curve of the equation that this equation has only one solution with:

$$\boldsymbol{k=0} \quad \text{and} \quad b\approx 0{,}371 \quad \text{consequently:} \quad B=\frac{-\ln\left(\frac{1}{4}+b^2\right)}{2b}\approx 1{,}277 \tag{125}$$

Finally, we conclude that if we have:

$$\exists B\in\mathbb{R} \quad \text{with} \quad z=e^{iB} \quad \text{and} \quad \forall k'\in\mathbb{Z} \quad B\neq k'\pi \quad \text{and} \quad S=\frac{1}{2}+ib \quad \text{and} \quad z^S=S \tag{126}$$

Then we have: $b\approx 0{,}371$ and $B\approx 1{,}277$ (127)

7. Summary and final conclusion:

We considered in this work a complex number ***S*** that respects that $S=\frac{1}{2}+ib$ where ***b*** is a strictly positive real number. Also, we considered a function ***f*** of complex numbers ***z*** defined from $\mathbb{C}$ in $\mathbb{C}$ as: $f(z)=\frac{z^{S+1}}{S}$. We studied in this article in two cases the fixed points of this function which are the solution of the equation $z^S=S$:

- The first case is when ***z*** is a real number and we demonstrated that the unique fixed point of the function ***f*** in this case is with: $b\approx 1{,}449$ and $z=e^{\frac{\arctan(2b)}{b}}\approx 2{,}323$.

- The second case is when ***z*** is a complex number respecting:

$$\exists B\in\mathbb{R} \quad \text{with} \quad z=e^{iB} \quad \text{and} \quad \forall k'\in\mathbb{Z} \quad B\neq k'\pi$$

And we demonstrated that the unique fixed point of the function ***f*** in this case is with:

$$b\approx 0{,}371 \quad \text{and} \quad B=\frac{-\ln\left(\frac{1}{4}+b^2\right)}{2b}\approx 1{,}277$$

Hence, let’s accept in this case that: $z\approx e^{i\times 1{,}277}$

And thus, we conclude that the complex function $f(z)=\frac{z^{S+1}}{S}$ has indeed two special different fixed points depending on the nature of the complex inputs when $S=\frac{1}{2}+ib$. However, in order to definitely know all the fixed points of this function when $S=\frac{1}{2}+ib$, we should study another case which needs a future longer proof. It is the case of:

$\exists B\in\mathbb{R}$ with $z=x\times e^{iB}$ and $\forall k'\in\mathbb{Z}$ $B\neq k'\pi$

where ***x*** is a real number with $x\neq 0$ and $x\neq 1$.

Finally, I hope that this work can be useful in the study of the characteristics of the special complex numbers which have the form $S=\frac{1}{2}+ib$ since this would be useful for the study of the case of Landau-Siegel Zeros and Riemann Hypothesis.

Declarations:

Ethical Approval
this declaration is “not applicable”.

Competing interests
this declaration is “not applicable”.

Authors' contributions
this declaration is “not applicable”.

Funding
this declaration is “not applicable”.

Availability of data and materials
this declaration is “not applicable”.

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