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Existence and uniqueness of solutions of differential equations with respect to non-additive measures

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By taking Sugeno-derivative into account, firstly, we investigate the existence of solutions to the initial value problems (IVP) of first-order differential equations with respect to non-additive measure (more precisely, distorted Lebesgue measure). It particularly occurs in the mathematical modeling of biology. We begin by expressing the differential equation in terms of ordinary derivative and the derivative with respect to the distorted Lebesgue measure. Then, by using the fixed point theorem on cones, we construct an operator and prove the existence of positive increasing solutions on cones in semi-order Banach spaces. In addition, we also use Picard's-Lindelöf theorem to prove the existence and uniqueness of the solution of the equation.
Second, we investigate the existence of a solution to the boundary value problem (BVP) on cones with integral boundary conditions of a mix-order differential equation with respect to non-additive measures. Moreover, the Krasnoselskii fixed point theorem is also applied to both BVP and IVP and obtains at least one positive increasing solution. Examples with graphs are provided to validate the results.

Keywords: Initial Value Problem; Boundary Value Problem; Fixed Point Theorem on cone; Choquet Integral; Non-additive Measure; Differential Equation; Semi-order Banach Space; Krasnoselskii fixed point; Picard's-Lindelöf.

1. Introduction

A derivative plays a significant role in several disciplines of mathematics, in both pure and applied mathematics. Its uses include solving practical problems, real-world, theoretical, and non-applied mathematics problems, and so on. On the other hand, a derivative of a function with respect to a set function, which is an extended version of the usual derivative, has a crucial role and applications in measure theory, financial mathematics, and computer science, among other fields, see [8, 29, 30] and references cited therein. Particularly, in probability theory and financial mathematics, derivative of the set function with respect to set, i.e., the Radon–Nikodym derivative is frequently used. It is expressed as $d\mu/d\nu$, where μ and ν are additive measures with the measure μ is absolutely continuous with respect to another measure ν, see Section 6.3 [30]. That is,

$$\mu([0,\varsigma]) = \mu(0) + \int_{[0,\varsigma]} (d\mu/d\nu)\, d\nu. \tag{1.1}$$

In the case of non-additive measure, however, it is not like that. That is, Equation 1.1 does Besides, some authors have also studied the Randon-Nykodym type theorem with respect to strongly subadditive capacities, e.g., see [19, 17] and references cited

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therein, where the Hahn decomposition properties are used to obtain the theorem. On the other hand, as a new research paradigm [34], the author "Sugeno" has defined a new path to Choquet calculus and presented the Choquet integral and derivative with respect to the set function (subadditive and superadditive), specifically, a distorted Lebesgue measures on the non-negative real line. That is, for a continuous and non-decreasing function $p : \mathbb{R}^+ \to \mathbb{R}^+$ and $m \in \mathcal{M}^+$, if we have the Choquet integral equation with respect to μ_m, where μ_m is generated by a monotone transformation ("m") from a Lebesgue measure ("μ"), such that

$$p(\varsigma) = p(0) + (C)\int_{[0,\varsigma]} q\mu_m.$$

Then, the derivative of p with respect to a fuzzy measure μ_m is defined by the solution q as

$$\frac{dp(\varsigma)}{d\mu_m} = q(\varsigma),$$

if $q \in \mathfrak{F}^+$. Here, $\mathfrak{F}^+$ denotes the collection of all positive, continuous and increasing functions on $[0,1]$, the set $\mathcal{M}^+$ denotes a class of non-decreasing and continuously differentiable function: $m : [0,1] \to [0,\infty)$ with $m(0) = 0$. In addition, such a concept of this new derivative, the following initial value problem as model has also been noticed in biology

$$\begin{aligned} \frac{dy(\varsigma)}{d\mu_m} &= 1 - y(\varsigma),\ \varsigma \geq 0 \\ y(0) &= 0. \end{aligned}$$

It represents the transfer dynamics of HIV patients from asymptomatic to the symptomatic stage, for more detail, see [23]. Here, the Choquet integral (see [10]) plays a vital role to solve the equation and it is with respect to distorted Lebesgue measure (μ_m) rather than the Lebesgue integral. In general, the Choquet integral differs from the Lebesgue integral in that it is more general and coincides with the Lebesgue integral (i.e., $m(\varsigma) = \varsigma$ on $[0,1]$) and it has many applications too, see [15, 16, 20, 25, 7] and references cited therein. Many researchers have contributed to the field of Choquet integrals on non-additive measures in continuous and discrete domains. The readers are compelled to move forward in a variety of directions as a result of these. We recommend some books [12, 36, 14] for a basic understanding of the non-additive measure theory.

Inspired by Sugeno's derivative, in this paper, we concern the existence of positive increasing solutions of both the general initial value problem (IVP) and mix-order boundary value problem (BVP). That is, we provide sufficient conditions for existence of solution of both the general equations. To the best of our knowledge, with this derivative, no such problems have been discussed before. Particularly, however, Sugeno [34] has discussed some numerical examples of solutions of first-order homogeneous and non-homogeneous differential equations with respect to μ_m for particular m. But, here, we deal with the existence of a solution of the general differential equations with respect to μ_m for all $m \in \mathcal{M}^+$. For more details about this derivative, readers may consult the paper [34] by Sugeno in which the author has gone over some basic fundamental properties and results with good explanation.

Let us consider a first-order differential equation with respect to a non-additive measure (precisely, distorted Lebesgue measure, i.e., μ_m for $m \in \mathcal{M}^+$) with initial value $y(0) = 0$, of the form

$$\frac{dy(\varsigma)}{d\mu_m} = F(\varsigma, y(\varsigma)),\ \ y(0) = 0, \tag{1.2}$$

for all $\varsigma \in [0,1]$. Here, $F : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is a function. Note that, whenever we use $\frac{dy}{d\mu_m}$, we mean y is differentiable with respect to μ_m, where $m \in \mathcal{M}^+$ and $\mathfrak{F}^+$. Equation 1.2, indeed, is called non-autonomous differential equation with respect to μ_m. Particularly, for $m(\varsigma) = \varsigma$, it is actually a non-autonomous ordinary differential equation.

Moreover, we also discuss the existence of solution of the following mix-order autonomous differential equation boundary value problem (BVP) with integral boundary condition. Let us consider a mix order-$\left(\frac{d}{d\mu_n}, \frac{d}{d\mu_m}\right)$- differential equation of the form

$$\frac{d}{d\mu_n}\left(\frac{dy(\varsigma)}{d\mu_m}\right) = g(y(\varsigma)) \text{ for all } \varsigma \in [0,1] \tag{1.3}$$

$$y(0) = 0, \qquad y(1) = \alpha, \tag{1.4}$$

where $\alpha \in \mathbb{R}^+$, $m, n \in \mathcal{M}^+$ and $g : [0,\infty) \to [0,\infty)$ is a function.

In order to study the existence of their solutions, we use some well-known theorems, like; fixed point theorem on cones in the semi-order Banach space, Krasnoselskii and Picard's-Lindelöf fixed point theorem. Two of them are stated as follows:.

Theorem 1.1 *[38] Let $\mathcal{S}$ and $\mathcal{C}$ be a semi-order Banach space and cone, respectively and $\mathcal{A}$ is a subset of $\mathcal{C}$. Also, let $F : \mathcal{A} \to \mathcal{X}$ be a non-decreasing function. If there exist $\varsigma_1, \varsigma_2 \in \mathcal{A}$ such that $\varsigma_1 \leq \varsigma_2, <\varsigma_1, \varsigma_2> \subset \mathcal{A}$ and ς_1, ς_2 are a lower and upper solution of equation $F(\varsigma) = \varsigma$, then the equation $f(\varsigma) = \varsigma$ has maximum and minimum solution a, b in $<\varsigma_1, \varsigma_2>$, such that $a \leq b$, when one of the following condition holds:*

1. *$\mathcal{C}$ is normal and F is compact continuous,*

2. *$\mathcal{C}$ is normal and F is continuous,*
3. *$\mathcal{X}$ is reflexive, $\mathcal{C}$ is normal, and F is continuous or weak continuous.*

Theorem 1.2 *[9]* **Krasnoselskii fixed point theorem.** *Let $\mathcal{U} \subset \mathcal{X}$ be any non-empty, closed convex subset, where $(\mathcal{X}, \|\cdot\|)$ is Banach space. Suppose $\mathcal{Y}_1$ and $\mathcal{Y}_2$ map $\mathcal{U}$ into $\mathcal{X}$ such that*

1. *$\mathcal{Y}_1 z_1 + \mathcal{Y}_2 z_2 \in \mathcal{U}$ for all $z_1, z_2 \in \mathcal{U}$,*
2. *$\mathcal{Y}_1$ is continuous and compact,*
3. *$\mathcal{Y}_2$ is contraction with constant $\sigma < 1$.*

Then there is a point $z^* \in \mathcal{U}$ with $\mathcal{Y}_1 z^* + \mathcal{Y}_2 z^* = z^*$.

Theorem 1.1 has sparked a surge in interest in the study of existence of solutions to various differential equations on cones in the semi-Banach spaces over the last several decades, see [4, 37, 5, 24] and references cited therein. On the other hand, the Krasnoselskii and Picard's-Lindelöf iterative fixed point theorem have also been noticed in the study of the existence of solutions to various different types of differential equations. See [6, 2, 24, 18, 21, 27, 26, 1] and the reader can also look up the references cited in the papers for more information.

We begin our main section by dealing with the existence of solution of initial value problem (IVP) 1.2 and Theorem 1.1. For this, we first construct a suitable form of an operator for whom we make our differential equation as a combination of an ordinary derivative and derivative with respect to distorted Lebesgue measure. Then after, along with certain conditions, we apply it over Theorem 1.1 and Picard's-Lindelöf theorem to get the desired result. Besides, some possible associated results are also discussed.

In addition, before going to solve BVP 1.3-1.4, we first study similar BVP of it, i.e., consider a function on the right-hand side of Equation 1.3 as only the function of ς rather than $y(\varsigma)$ for all $\varsigma \in [0, 1]$. And then, we obtain an expression of solution in terms of a Choquet integral with respect to a distorted Lebesgue measure. We further illustrate the outcome by putting several examples. Thus, by observing the expression of solution, we prepare an operator in terms of the Choquet integral with respect to a distorted Lebesgue measure for the original BVP 1.3-1.4. Finally, use this operator for the fixed point theorem of cones and along with some conditions, we get at least one positive increasing solution. In the end, for an enhancement in the results, we also study Krasnoselskii fixed point theorem for both the differential equations.

The rest of the paper is organized as follows: In Section 2, we present definitions of non-additive measures and some essential properties of Sugeno's derivative along with the Choquet integral. Further, some useful theorems and propositions from the papers [33, 38, 34] are presented. For the main results, in Section 3, we first provide sufficient conditions of existence of solution for the first order non-autonomous differential equation of Equation 1.2. Further, a well-known Picard's iteration method is also used to find a unique solution of the differential equation. Then, in Section 4, we also discuss the existence of solutions to the boundary value problem 1.3-1.4. In which, we first present the expression of the solution of a particular mix-order boundary value problem with respect to μ_m. We then present some numerical examples corresponding to various different values of functions m and n. After that, we look for the existence of a solution of the original problem 1.3-1.4 (mix-order autonomous differential equation). Throughout Section 3 and 4, we follow the fixed point theorem on the cone of the semi-order Banach space. By using Krasnoselskii Fixed Point Theorem, we also provide a theorem of the existence of a solution of BVP in Section 5. Finally, in Section 6, we conclude the paper.

2. Preliminaries

In this section, we present all those definitions, theorems and propositions which make our proof easily understandable and readable.

Definition 2.1 *[33] Let a triplet $([0, \infty), \mathcal{S}, \mu)$ be a measurable space, where μ is measure and $\mathcal{S}$ is σ-algebra (smallest) which contains all the closed interval in $[0, \infty)$. Then, the set function $\mu : \mathcal{S} \to [0, \infty)$ is called non-additive measure if and only if*

1. *$\mu(\emptyset) = 0$ (Initial condition),*
2. *$\mu(A) \leq \mu(B)$, if $A \subseteq B$ for $A, B \in \mathcal{S}$(Monotone),*
3. *$\mu(A_k) \downarrow \mu(A)$ if $A_k \downarrow A$ and $\mu(B_k) \uparrow \mu(B)$, if $B_k \uparrow B$(Continuity).*

We can understand it through a particular way, for instance, consider a non-additive measure μ as $\mu = \lambda^2([\varsigma_1, \varsigma_2]) = (\varsigma_2 - \varsigma_1)^2$, where λ is Lebesgue measure.

We next see a specific particular form of non-additive measure (μ), which is known as distorted Lebesgue measure and generalizing the above particular way.

Definition 2.2 *[33] Let us consider a continuous and increasing function $m : [0, \infty) \to [0, \infty)$ such that $m(0) = 0$. Then a non-additive measure μ has a form $\mu = \lambda_m$, which is known as distorted Lebesgue measure, where λ is the Lebesgue measure. It is defined by $\lambda_m(*) = m(\lambda(*))$. Moreover, when $\mu = \lambda_m$ is non-additive measure unless m is linear.*

Let $\mathtt{W}^+$ denotes the class of positive measurable functions on $[0,\infty)$, where $([0,\infty),\mathcal{S})$ is a measurable space. We now define co-monotonicity of functions.

Definition 2.3 *[12] Let $p,q\in\mathtt{W}^+$, then p and q are co-monotonic, if $\varsigma,\varsigma^*\in[0,\infty)$*

$$p(\varsigma)<p(\varsigma^*)\implies q(\varsigma)\leq q(\varsigma^*). \tag{2.5}$$

Moreover, the following also holds

$$(C)\int(p+q)d\mu=(C)\int pd\mu+(C)\int qd\mu. \tag{2.6}$$

Choquet integral in terms of usual Lebesgue integral on continuous domain.

Theorem 2.4 *[33] If $f\in\mathfrak{F}^+$, then the Choquet integral of f with respect to a fuzzy measure, say $\bar{\mu}$ on $[0,\varsigma]$ is represented as*

$$(C)\int_{[0,\varsigma]}f(\tau)d\bar{\mu}(\tau)=-\int_0^{\varsigma}\bar{\mu}'([\tau,\varsigma])f(\tau)d\tau.$$

In particular, for $\bar{\mu}=\lambda_m$,

$$(C)\int_{[0,\varsigma]}f(\tau)d\bar{\mu}(\tau)=\int_0^{\varsigma}m'(\varsigma-\tau)f(\tau)d\tau. \tag{2.7}$$

Commutativity of derivative with respect to distorted Lebesgue measures.

Propostion 2.5 *[34] For m and n in $\mathcal{M}^+$ such that $m\neq n$. Then, for $f\in\mathfrak{F}^+$, we have*

$$\frac{d}{d\mu_m}\frac{df}{d\mu_n}=\frac{d}{d\mu_n}\frac{df}{d\mu_m}, \tag{2.8}$$

provided $\frac{df}{d\mu_m}(0)=\frac{df}{d\mu_n}(0)=0$.

Propostion 2.6 *[34] For $p,q\in\mathfrak{F}^+$ and $m,n\in\mathcal{M}^+$, we have*

1. *if $p\leq q$, then $(C)\int_{[0,\varsigma]}pd\mu_m\leq(C)\int_{[0,\varsigma]}qd\mu_m$,*
2. *if $m\leq n$, then $(C)\int_{[0,\varsigma]}pd\mu_m\leq(C)\int_{[0,\varsigma]}pd\mu_n$,*
3. *if $\varsigma_1\leq\varsigma_2$, then $(C)\int_{[0,\varsigma_1]}pd\mu_m\leq(C)\int_{[0,\varsigma_2]}pd\mu_m$.*

Propostion 2.7 *[34] For $p,q\in\mathfrak{F}^+$ and $m,n\in\mathcal{M}^+$, we have*

1. *if $m\leq n$, then $\frac{dp}{d\mu_n}\leq\frac{dp}{d\mu_m}$,*
2. *if $p\leq q$, then $\frac{dp}{d\mu_m}\leq\frac{dq}{d\mu_m}$.*

Following definition follows from Definition 17 [34].

Definition 2.8 *For m and n in $\mathcal{M}^+$, then define a binary relation "$\gg$" on $\mathcal{M}^+$ by*

$$m\gg n\iff\frac{m}{n}\text{ exists, i.e., }\frac{m}{n}\in\mathcal{M}^+. \tag{2.9}$$

We immediately define a result from [34] which plays an important role in one of our main theorems.

Corollary 2.9 *For $m\gg n$ and $f\in\mathfrak{F}^+$, we have*

$$\frac{d}{d\mu_m}\left((C)\int_{[0,\varsigma]}f\mu_n\right)=\frac{df}{d\mu_{m/n}}. \tag{2.10}$$

Propostion 2.10 *[34] For $m,n\in\mathcal{M}^+$, we have*

$$\frac{d}{d\mu_m}\frac{df}{d\mu_n}=\frac{df}{d\mu_r},\quad\text{if }\frac{df}{d\mu_n}(0)=0, \tag{2.11}$$

where $r=(C)\int md\mu_n$.

Theorem 2.11 *[11] A function $p : A \subset \mathbb{R} \to \mathbb{R}$ is said to be Lipschitz continuous at $\varsigma \in A$ if there exists a constant L such that $|p(\varsigma_1) - p(\varsigma_2)| \leq L|\varsigma_1 - \varsigma_2|$ for all ς_1, ς_2.*

Theorem 2.12 *[11] Let $A \subset \mathbb{R}$ be any interval and let $p : A \to \mathbb{R}$ be continuous on A and differentiable on the interior $int(A)$ of A. Then function p is Lipschitz $\iff$ p' is bounded on $int(A)$.*

Let us denote the set $\mathcal{X} = C[0,1]$ as the class of all continuous functions defined on $[0,1]$ with the sup norm. Also, the set $\mathcal{C}$ denotes the collection of all non-negative functions of $\mathcal{X}$, clearly, $\mathcal{C} \subset \mathcal{X}$.

Definition 2.13 *[32] A nonempty, closed, convex set $\mathcal{C} \subset \mathcal{X}$ is said to be a cone provided the following are satisfied*

1. *if $y \in \mathcal{C}$ and $\alpha \geq 0$, then $\alpha y \in \mathcal{C}$;*
2. *if $y \in \mathcal{C}$ and $-y \in \mathcal{C}$, then $y = 0$.*

For a given cone $\mathcal{C}$, define a partial ordering ("$\leq$") with respect to cone $\mathcal{C}$ by $y_1 \leq y_2 \iff y_2 - y_1 \in \mathcal{C}$.

Definition 2.14 *[32] A cone $\mathcal{C}$ is said to be "Normal" if $\exists$ a number $\kappa > 0$ such that*

$$0 \leq y_1 \leq y_2 \text{ implies } \|y_1\| \leq \kappa\|y_2\|, \text{ for all } y_1, y_2 \in \mathcal{C}. \tag{2.12}$$

Before going to start our main sections, let us first define a set

$$\mathcal{A} = \{y \in \mathcal{X} | y \geq 0 \text{ is non-decreasing on } [0,1]\}. \tag{2.13}$$

It will not difficult to see that the set $\mathcal{A}$ is non-empty closed subset of cone $\mathcal{C}$.

3. Existence of solution of initial value problem 1.2

This section contains some results of the existence of the solution of the initial and boundary value problem of the mix-order differential equation with respect to non-additive measure (or distorted Lebesgue measure) on continuous domain.

Following definition presents the lower and upper solution for both problems 1.2 (IVP) and 1.3-1.4 (BVP), it can also see in [39].

Definition 3.1 *Function $q \in \mathcal{X}$ will be called a lower solution for Equation 1.2(also with respect to operator $\mathcal{Y}$) if*

$$\frac{dq(\varsigma)}{d\mu_m} \leq F(\varsigma, q(\varsigma)), \tag{3.14}$$

and $q(\varsigma) \leq \mathcal{Y}q(\varsigma)$ for all $\varsigma \in [0,1]$. Similarly, we upper solution for Equation 1.2 and operator $\mathcal{Y}$ such that

$$\frac{dq(\varsigma)}{d\mu_m} \geq F(\varsigma, q(\varsigma)), \tag{3.15}$$

and $q(\varsigma) \geq \mathcal{Y}q(\varsigma)$ for all $\varsigma \in [0,1]$.

In a similar manner, this definition can also be used for problem 1.3-1.4. Further, it is also noted that it will also hold for the general semi-Banach space, see [3].

Theorem 3.2 *Assume $m \in \mathcal{M}^+$ with $m(\varsigma) < \varsigma$ for $\varsigma \in [0,1]$, and $m \gg i_{id}$, where i_{id} is identity function. Moreover, if the following hold:*

1. *$F : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous and $F(\varsigma, \cdot)$ is an increasing function with respect to the second variable for all $\varsigma \in [0,1]$ with $F(0,0) = 0$. Moreover, $F(\varsigma, \cdot)$ is strictly increasing on $[0,\eta]$ for all $\varsigma \in [0,1]$, where η is a positive real number.*
2. *m' is continuously differentiable on $[0,1]$,*
3. *y^* and y^{**} are lower and upper solution of Equation 1.2 such that $y^*(\varsigma) \leq y^{**}(\varsigma)$ for $\varsigma \in [0,1]$,*

then Equation 1.2 has at least one positive solution.

Proof 3.1 *For $\varsigma \in [0,1]$, Equation 1.2 can be written as*

$$\frac{d}{d\mu_m}\left(\frac{dy_1(\varsigma)}{d\varsigma}\right) = F(\varsigma, y_1'(\varsigma)), \tag{3.16}$$

where $y_1(\varsigma) = \int_0^\varsigma y(\tau)d\tau$ for a function $y \in \mathfrak{F}^+$. Now, in order to interchange the order of derivative in the later equation, we first claim $\frac{dy_1(\varsigma)}{d\mu_m}\Big|_{\varsigma=0} = 0$.

Since $m \gg i_{id}$ on $[0,1]$, where i_{id} is identity function, then we can obtain the following

$$\begin{aligned}\frac{dy_1(\varsigma)}{d\mu_m}\Big|_{\varsigma=0} &= \frac{d}{d\mu_m}\left((C)\int_{[0,\varsigma]} y\, d\mu_{i_{id}}\right)\Big|_{\varsigma=0} \\ &\overset{Eq.\text{-}2.10}{=} \frac{dy}{d\mu_{m/i_{id}}}\Big|_{\varsigma=0} \\ &\overset{Prop.\text{-}2.7(2)}{\leq} \frac{dy}{d\mu_m}\Big|_{\varsigma=0} = F(0,0) = 0. \end{aligned} \tag{3.17}$$

We also have $m(\varsigma) < \varsigma$ for all $\varsigma \in [0,1]$. Then, we get

$$\frac{dy_1(\varsigma)}{d\mu_m}\Big|_{\varsigma=0} \overset{Prop.\text{-}2.7(1)}{\geq} \frac{dy_1(\varsigma)}{d\varsigma}\Big|_{\varsigma=0} = y(0) = 0. \tag{3.18}$$

Consequently, from Equations 3.17 and 3.18, we get

$$\frac{dy_1(\varsigma)}{d\mu_m}\Big|_{\varsigma=0} = 0. \tag{3.19}$$

After setting $n(\varsigma) = \varsigma$ in Proposition 2.5, we can write Equation 3.16 as follows

$$\frac{d}{d\varsigma}\left(\frac{dy_1(\varsigma)}{d\mu_m}\right) = F(\varsigma, y_1'(\varsigma)). \tag{3.20}$$

Integrating equation 3.20 from 0 to ς, we obtain

$$\frac{dy_1(\varsigma)}{d\mu_m} = \int_0^\varsigma F(\tau, y_1'(\tau))d\tau, \tag{3.21}$$

then by Equation 2.6, we have Equation 3.21 in the form

$$y_1(\varsigma) = (C)\int_{[0,\varsigma]}\left(\int_0^{(\cdot)} F(s, y_1'(s))ds\right)d\mu_m$$

We also have

$$\begin{aligned} y(\varsigma) &= \int_0^\varsigma m''(\varsigma-\tau)\left(\int_0^\tau F(s,y(s))ds\right)d\tau + m'(0)\int_0^\varsigma F(s,y(s))ds \\ &= \int_0^\varsigma m'(\varsigma-\tau)F(\tau,y(\tau))d\tau. \end{aligned}$$

Now, let us consider an operator $\mathcal{Y} : \mathcal{A} \to \mathcal{C}$ such that

$$\mathcal{Y}y(\varsigma) = \int_0^\varsigma m'(\varsigma-\tau)F(\tau,y(\tau))d\tau, \tag{3.22}$$

It is not difficult to show that $\mathcal{Y}$ is positive and continuous because of positiveness and continuity of functions m', m'', y and F in the expression of operator $\mathcal{Y}$.

In order to apply Theorem 1.1, it suffices to check the operator $\mathcal{Y}$ for compact continuity and increasingness because we already have $\mathcal{Y}y^{**} \leq y^{**}$ and $\mathcal{Y}y^* \geq y^*$ from condition (2) and Definition 3.1. For compact continuity, we will use a well-known Arzela-Ascoli Theorem [35].

We now claim the operator $\mathcal{Y}$ is bounded and completely continuous. Therefore, let $\mathcal{B}$ be any bounded subset of the $\mathcal{A}$. Then, for all $y \in \mathcal{B}$, from equation 3.22 we conclude the following

$$|\mathcal{Y}y(\varsigma)| \leq \int_0^\varsigma m'(\varsigma-\tau)|F(\tau,y(\tau))|d\tau, \ \forall\varsigma \in [0,1]. \tag{3.23}$$

Let Θ be a positive constant and there exits $\theta > 0$ such that $\Theta > \max_{s\in[0,1], y\in[0,\theta]} F(s,y(s))$. Then, from Equation 3.23, we arrive at

$$\|\mathcal{Y}y(\varsigma)\| \leq \Theta m(\varsigma) \leq \Theta m(1) := \Theta^*. \tag{3.24}$$

Thus, $\mathcal{Y}(\mathcal{B})$ is uniformly bounded.

Next, for each $y \in \mathcal{B}$ and $\varsigma_1, \varsigma_2 \in [0,1]$ such that $\varsigma_1 < \varsigma_2$, let us consider

$$\begin{aligned} |\mathcal{Y}y(\varsigma_1) - \mathcal{Y}y(\varsigma_2)| &= \left| \int_0^{\varsigma_1} m'(\varsigma_1 - \tau) F(\tau, y(\tau)) d\tau - \int_0^{\varsigma_2} m'(\varsigma_2 - \tau) F(\tau, y(\tau)) d\tau \right| \\ &\leq \int_0^{\varsigma_1} |m'(\varsigma_1 - \tau) - m'(\varsigma_2 - \tau)||F(\tau, y(\tau))| d\tau + \int_{\varsigma_1}^{\varsigma_2} m'(\varsigma_2 - \tau)|F(\tau, y(\tau))| d\tau. \end{aligned} \tag{3.25}$$

Since the function m' continuously differentiable implies Lipschitz continuous with Lipchitz constant L and $m(\varsigma) \leq \varsigma$ for all $\varsigma \in [0,1]$, then Equation 3.25 becomes

$$|\mathcal{Y}y_1(\varsigma_1) - \mathcal{Y}y_1(\varsigma_2)| \leq \Theta(L+1)|\varsigma_2 - \varsigma_1|. \tag{3.26}$$

Further, whenever $|\varsigma_1 - \varsigma_2| < \delta$, where $\delta > 0$, that is, for all $\epsilon > 0$, δ is taken as $\delta = \dfrac{\epsilon}{\Theta(L+2)} > 0$. Thus, Equation 3.26 can be written as

$$|\mathcal{Y}y_1(\varsigma_1) - \mathcal{Y}y_1(\varsigma_2)| \leq \epsilon, \tag{3.27}$$

which implies the equicontinuity of $\mathcal{Y}(\mathcal{B})$. Therefore, by using Arzela-Ascoli Theorem [35], $\overline{\mathcal{Y}(\mathcal{B})}$ is compact.

Next, our aim is to check the operator $\mathcal{Y}$ is increasing. Therefore, let y_1, y_2 be any two elements of $\mathcal{A}$ such that $y_1 \leq y_2$. Then, from Equation 3.22 and condition (1), we have

$$\begin{aligned} \mathcal{Y}y_1(\varsigma) &= \int_0^{\varsigma} m'(\varsigma - \tau) F(\tau, y_1(\tau)) d\tau \\ &\leq \int_0^{\varsigma} m'(\varsigma - \tau) F(\tau, y_2(\tau)) d\tau = \mathcal{Y}y_2(\varsigma). \end{aligned} \tag{3.28}$$

for all $\varsigma \in [0,1]$. Thus, the operator $\mathcal{Y} : \langle y_*, y^* \rangle \to \langle y_*, y^* \rangle$ is compact continuous, and $\mathcal{C}$ is normal cone. Therefore, Theorem 1.1 is applicable for the operator $\mathcal{Y}$ and then it preserves fixed point in $\langle y_*, y^* \rangle$. Hence, we complete the proof.

Remark 3.3 *It is noticed that the Theorem 3.2 does not apply to the ordinary case $m(\varsigma) = \varsigma$ for all $\varsigma \in [0,1]$ because $m \gg i_{id}$ does not hold, i.e., $\frac{m(\varsigma)}{\varsigma} = 1 \notin \mathcal{M}^+$.*

The following theorem answers the Remark 3.3.

Theorem 3.4 *Assume $m \in \mathcal{M}^+$ with $m(\varsigma) \leq \varsigma$ for $\varsigma \in [0,1]$. Moreover, if the following hold:*

1. *$F : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous and $F(\varsigma, \cdot)$ is increasing function with respect to the second variable for all $\varsigma \in [0,1]$ with $F(0,0) = 0$. Moreover, $F(\varsigma, \cdot)$ is strictly increasing on $[0,\eta]$ for all $\varsigma \in [0,1]$, where η is positive real number.*
2. *m' is continuously differentiable on $[0,1]$,*
3. *y^* and y^{**} are lower and upper solution of Equation 1.2 such that $y^*(\varsigma) \leq y^{**}(\varsigma)$ for $\varsigma \in [0,1]$,*

then Equation 1.2 has at least one positive solution.

Proof 3.2 *Since we have $y_1(\varsigma) = \int_0^{\varsigma} y(\tau) d\tau$ for $y \in \mathfrak{F}^+$, then obviously, $y_1(\varsigma) \leq y(\varsigma)$ for all $\varsigma \in [0,1]$. It is easy to get $\frac{dy_1(\varsigma)}{d\mu_m}\Big|_{\varsigma=0} \leq 0$ and then after keeping Equation 3.18 in hand, we must have $\frac{dy_1(\varsigma)}{d\mu_m}\Big|_{\varsigma=0} = 0$. Now, we proceed the proof of Theorem 3.2. Hence, proof is done.*

For a special case of Theorem 3.4, consider $m(\varsigma) = \varsigma$ for $\varsigma \in [0,1]$. then, we have the following result of a existence of a solution of the differential equation.

Corollary 3.5 *If the following conditions*

1. *$F : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous and $F(\varsigma, \cdot)$ is increasing function with respect to the second variable for all $\varsigma \in [0,1]$ with $F(0,0) = 0$. Moreover, $F(\varsigma, \cdot)$ is strictly increasing on $[0,\eta]$ for all $\varsigma \in [0,1]$, where η is positive real number,*
2. *y^* and y^{**} are lower and upper solution of Equation 1.2 such that $y^*(\varsigma) \leq y^{**}(\varsigma)$ for $\varsigma \in [0,1]$,*

hold, then the first-order non-autonomous differential equation $y'(\varsigma) = F(\varsigma, y(\varsigma))$ for $\varsigma \in [0,1]$, has at least one positive solution.

A proposition is presented for the next main theorem.

Propostion 3.6 *If $m, m' \in \mathcal{M}^+$, then $m(\varsigma) = (C)\int_{[0,\varsigma]} i_{id}\, d\mu_{m'}$, where "$i_{id}$" is an identity function.*

Proof 3.3 *Since $m, m' \in \mathcal{M}^+$, we have*

$$m(\varsigma) \;=\; \int_0^\varsigma m'(\tau)d\tau = \int_0^\varsigma m'(\varsigma-\tau)d\tau = \int_0^\varsigma m''(\varsigma-\tau)\tau d\tau = (C)\int_{[0,\varsigma]} \tau d\mu_{m'}. \tag{3.29}$$

Thus, we arrive at the required expression.

Example 3.7 *For $\varsigma \in [0,\infty)$, we have $\dfrac{\varsigma^{k+1}}{k+1} = (C)\displaystyle\int_{[0,\varsigma]} \tau d\mu_{\varsigma^k}$, where k is a real positive constant. Particularly, for $k=1$, $\displaystyle\int_0^\varsigma \tau d\tau = \varsigma^2/2$.*

Following theorem contains less sufficient conditions than the Theorem 3.2. That is, here condition on m such that $m(\varsigma) \le \varsigma$ for $\varsigma \in [0,1]$, and $m \gg i_{id}$, where i_{id} is identity function, are not necessary.

Theorem 3.8 *Assume conditions (1) and (3) in Theorem 3.2 hold, also if $\frac{dy(\varsigma)}{d\mu_{m'}}\Big|_{\varsigma=0} = 0$ for $m' \in \mathcal{M}^+$. Then, Equation 1.2 has at least one positive solution.*

Proof 3.4 *From Equation 1.2, Proposition 2.10 and 3.6, we easily conclude $\frac{dy(\varsigma)}{d\mu_m} = \frac{d}{d\varsigma}\left(\frac{dy(\varsigma)}{d\mu_{m'}}\right) = F(\varsigma, y(\varsigma))$ for $\varsigma \in [0,1]$. Now, follow the steps from 3.20 to 3.28 in the proof of Theorem 3.2. Hence, we reached the desired conclusion and have at least one positive solution of Equation 1.2.*

Next, we discuss a well-known Picard's method for the existence and uniqueness of Equation 1.2.

3.1. Picard-Lindelöf

Through this section, our motive is to provide another way of finding the sufficient conditions of existence and uniqueness of Equation 1.2.

Theorem 3.9 *Let $\mathcal{W} = \{(\varsigma, y) |\ 0 \le \varsigma \le \alpha, 0 \le y \le \beta\}$, and assume $m \in \mathcal{M}^+$ such that $m(\varsigma) < \varsigma$ for $\varsigma \in [0,1]$, and $m \gg i_{id}$, where i_{id} is identity function. Moreover, if the following conditions hold:*

1. *$F : [0,1] \times [0,\infty) \to [0,\infty)$ is Lipschitz continuous with respect to the second variable with Lipschitz constant Λ. Also, there exists a positive real constant λ such that $|F(\varsigma, y(\varsigma))| \le \lambda$ on [0,1],*
2. *$m(1) \le \min\left\{\alpha, \frac{\beta}{\lambda}\right\}$, $\lambda m(1) < 1$ and $m(1) < \frac{1}{\lambda\Lambda}$,*

then Equation 1.2 has exactly one positive increasing solution on $[0, m(1)]$.

Proof 3.5 **Existence:** *We proceed with the proof of Theorem 3.2 till Equation 3.22. Then, for $(\varsigma, y(\varsigma)) \in \mathcal{W}$ and from the expression of Equation 3.22, we construct the iterations as follows*

$$\begin{aligned} y_0(\varsigma) &= 0 \\ y_1(\varsigma) &= \int_0^\varsigma m'(\varsigma-\tau)F(\tau, y_0(\varsigma))d\tau \\ &\vdots \\ y_k(\varsigma) &= \int_0^\varsigma m'(\varsigma-\tau)F(\tau, y_{k-1}(\varsigma))d\tau. \end{aligned} \tag{3.30}$$

For $\varsigma \in [0, m(1)]$ for all $m \in \mathcal{M}^+$, then first we show $(\varsigma, y_k(\varsigma)) \in \mathcal{W}$. It is clearly seen that $|y_1(\varsigma)| \le \Lambda m(1) \le \beta$, i.e., $(\varsigma, y_1(\varsigma)) \in \mathcal{W}$. Now, if we assume $(\varsigma, y_{k-1}(\varsigma)) \in \mathcal{W}$, i.e., the relation $|y_{k-1}(\varsigma)| \le \beta$ for all $\varsigma \in [0,1]$ holds, then it is easy to have the following

$$|y_k| \;\le\; \int_0^\varsigma m'(\varsigma-\tau)\,|F(\tau, y_{k-1}(\tau))|\,d\tau \le \Lambda m(1) \le \beta.$$

Thus, by mathematical induction, the above expression leads to $(\varsigma, y_k(\varsigma)) \in \mathcal{W}$ *for all* $k \in \mathbb{N}$ *and* $\varsigma \in [0, m(1)]$. *Next, we also have*

$$\begin{aligned}|y_k(\varsigma) - y_{k-1}(\varsigma)| &\leq \int_0^{\varsigma} m'(\varsigma-\tau)|F(\tau, y_{k-1}(\tau)) - F(\tau, y_{k-2}(\tau))|d\tau \\ &\leq \lambda \int_0^{\varsigma} m'(\varsigma-\tau)|y_{k-1}(\tau) - y_{k-2}(\tau)|d\tau \\ &\leq \lambda^{k-1} \int_0^{\varsigma} m'(\varsigma-\tau)\int_0^{\tau} m'(\tau-s)\cdots\int_0^{\eta} m'(\eta-\xi)|y_1(\xi)|d\tau ds\cdots d\xi dz \\ &\leq \lambda^{k-1}\Lambda m^k(1). \end{aligned} \tag{3.31}$$

That implies,

$$\sum_{k=1}^{\infty}|y_k(\varsigma)-y_{k-1}(\varsigma)| \leq \Lambda \sum_{k=1}^{\infty}\lambda^{k-1}m^k(1),$$

which converges absolutely as $m(1) \leq 1/\lambda\Lambda$ *and also straightforward to see that the series converges uniformly on the interval* $[0, m(1)]$. *Consequently,* $y_k(\varsigma) = \sum_{j=1}^{k}(y_j(\varsigma) - y_{j-1}(\varsigma))$ *is convergent on* $[0, m(1)]$. *Further, being the series converges uniformly and function* F *is Lipschitz continuous, we must have an expression of solution* y *of Equation 1.2 as* $y(\varsigma) = \lim_{k\to\infty} y_k(\varsigma) = \int_0^{\varsigma} m'(\varsigma-\tau)F(\tau, y(\tau))$. *That ensures the existence of a solution of a given differential equation with respect to* μ_m, *see 1.2.*

Uniqueness: *Suppose* z_1 *and* z_2 *be two different solutions of Equation 1.2 on* $[0, m(1)]$. *Clearly,* $z_1(0) - z_2(0) = 0$, *and for* $\varsigma \in [0, m(1)]$, *we have*

$$\begin{aligned}|z_1(\varsigma) - z_2(\varsigma)| &\leq \int_0^{\varsigma} m'(\varsigma-\tau)|F(\tau, z_1(\tau)) - F(\tau, z_2(\tau))|d\tau \\ &\leq \Lambda\int_0^{\varsigma} m'(\varsigma-\tau)|z_1(\tau) - z_2(\tau)|d\tau. \end{aligned} \tag{3.32}$$

From the Gronwall inequality (see [31]), if $y(\varsigma) \leq y_0 + \int_0^{\varsigma} a(\tau)y(\tau)d\tau$, *for all* $\varsigma \in [0, \infty)$ *holds, then*

$$y(\varsigma) \leq y_0 \exp\left(\int_0^{\varsigma} y(\tau)d\tau\right), \tag{3.33}$$

where y_0 *is a non-negative real constant, and* a *and* y *are two non-negative function on* $[0, \infty)$. *Hence, from Equation 3.32 and 3.33, we conclude* $y_0 = 0$ *and then we have* $|z_1 - z_2| \leq 0$ *implies* $z_1 = z_2$.

Example 3.10 *Consider a first order differential equation with respect to distorted Lebesgue measure* (μ_m) *on* $[0, 0.25]$, *of the form*

$$\frac{dy}{d\mu_m} = \log(3-y) \text{ with } y(0) = 0, \tag{3.34}$$

where $\alpha = 1$ *and* $\beta = 2$ *such that* $\mathcal{W} = \{(\varsigma, y(\varsigma)) | \ 0 \leq \varsigma \leq 1, \ 0 \leq y(\varsigma) \leq 2\}$, *and* $m(\varsigma) = \frac{\varsigma^4}{4}$ *such that* $m(\varsigma) \leq \varsigma$ *and* $\frac{m(\varsigma)}{\varsigma} \in \mathcal{M}^+$ *for all* $\varsigma \in [0, 1]$. *Clearly, we have* $0 \leq \mathcal{F} = \log(3 - y(\varsigma)) \leq \log(3) = \lambda$ *and* $|\mathcal{F}(\varsigma, y_1(\varsigma)) - \mathcal{F}(\varsigma, y_2(\varsigma))| \leq |y_1(\varsigma) - y_2(\varsigma)|$ *for all* y_1 *and* y_2, *implies, Lipschitz constant* $\Lambda = 1$. *It is now seen that the conditions of Theorem 3.2 for the existence and uniqueness of solution of Equation 3.34 are satisfied. Hence, Equation 3.34* has exactly one positive increasing solution on $[0, 0.25]$.

4. Existence of solution of boundary value problem 1.3-1.4

In this section, we first discuss the solution of following mix order-$\left(\frac{d}{d\mu_n}, \frac{d}{d\mu_m}\right)$, where $m, n \in \mathcal{M}^+$. This result will be very helpful in the study of existence of solution of boundary value problem 1.3-1.4. Throughput this section, we use the notation g_c as $g_c(s) = (C)\int_{[0,s]} g d\mu_n$, for all $s \in [0, 1]$ and $n \in \mathcal{M}^+$.

Theorem 4.1 *If* $n, m \in \mathcal{M}^+$ *with* $m' : [0, 1] \to [0, \infty)$ *an increasing function, and* $g \in \mathfrak{F}^+$. *Then, the boundary value problem 1.3-1.4 with* $\alpha = (C)\int_{[0,1]}\left(\frac{1}{m'(1-\tau)} + 1\right) g_c d\mu_m < \infty$, *has a solution of the form*

$$y(\varsigma) = (C)\int_{[0,\varsigma]} g_c d\mu_{m_1} + (C)\int_{[\varsigma,1]} g_c d\mu_{m_2}, \tag{4.35}$$

where$m_1(s) = \frac{m(\varsigma)}{m(1)}s + m(s)$ *and* $m_2(s) = \frac{m(\varsigma)}{m(1)}s$ *for* $s, \varsigma \in [0, 1]$.

Proof 4.1 *Let us consider* $\frac{d}{d\mu_n}\left(\frac{dy}{d\mu_m}\right) = 0$ *for* $\varsigma \in [0,1]$, *then it has solution* $y(\varsigma) = am(\varsigma) + b$, *where* $a, b \in \mathbb{R}^+$. *Thus, in view of boundary value problem 1.3-1.4, it is not difficult to get the desire expression as solution along with two arbitrary constants*

$$y(\varsigma) = am(\varsigma) + b + (C)\int_{[0,\varsigma]} g_c d\mu_m. \tag{4.36}$$

Putting $y(0) = 0$ *in Equation 4.36, we obtain* $b = 0$. *Similarly, by second condition of Equation 1.4, we have* $a = \frac{1}{m(1)}(C)\int_{[0,1]} \frac{1}{m'(1-(\cdot))} g_c(\cdot) d\mu_m$. *Now, using these values together in Equation 4.36, we yield*

$$\begin{aligned} y(\varsigma) &= \frac{m(\varsigma)}{m(1)}(C)\int_{[0,1]} \frac{g_c}{m'(1-(\cdot))} d\mu_m + (C)\int_{[0,\varsigma]} g_c(\cdot) d\mu_m \\ &= \int_0^1 \frac{m(\varsigma)}{m(1)} g_c(\tau)d\tau + \int_0^\varsigma m'(\varsigma-\tau) g_c(\tau)d\tau \\ &= \int_0^\varsigma \frac{m(\varsigma)}{m(1)} g_c(\tau)d\tau + \int_\varsigma^1 \frac{m(\varsigma)}{m(1)} g_c(\tau)d\tau + \int_0^\varsigma m'(\varsigma-\tau) g_c(\tau)d\tau \\ &= \int_0^\varsigma \left(\frac{m(\varsigma)}{m(1)} + m'(\varsigma-\tau)\right) g_c(\tau)d\tau + \int_\varsigma^1 \frac{m(\varsigma)}{m(1)} g_c(\tau)d\tau \\ &= (C)\int_{[0,\varsigma]} g_c d\mu_{m_1} + (C)\int_{[\varsigma,1]} g_c d\mu_{m_2}, \end{aligned} \tag{4.37}$$

where $m_1(s) = \frac{m(\varsigma)}{m(1)}s + m(s)$ *and* $m_2(s) = \frac{m(\varsigma)}{m(1)}s$ *for* $s \in [0,1]$. *Thus, we complete the proof.*

Remark 4.2 *Results on* $[a,b]$ *for all* $a, b \in \mathbb{R}^+$ *also hold.*

The special case (i.e., $m(\varsigma) = \varsigma = n(\varsigma)$) of Theorem 4.1 is given.

Corollary 4.3 *If* $g : [0,\infty) \to [0,\infty)$ *is continuous function, then the boundary value problem*

$$y''(\varsigma) = g(\varsigma) \text{ for all } \varsigma \in [0,1] \tag{4.38}$$

$$y(0) = 0, \qquad y(1) = 2(C)\int_{[0,1]} g_c d\tau, \tag{4.39}$$

has solution

$$y(\varsigma) = (\varsigma+1)\int_0^\varsigma g_c(\tau)d\tau + \varsigma\int_\varsigma^1 g_c(\tau)d\tau, \tag{4.40}$$

where $g_c(\varsigma) = \int_0^\varsigma g(s)ds$.

Corollary 4.4 *If* $n, m \in \mathcal{M}^+$ *with* $m(\varsigma) = \varsigma$, *then the problem*

$$\frac{d}{d\mu_n}\left(\frac{dy(\varsigma)}{d\varsigma}\right) = g(\varsigma) \text{ for all } \varsigma \in [0,1] \tag{4.41}$$

$$y(0) = 0, \qquad y(1) = 2(C)\int_{[0,1]} g_c d\mu_m, \tag{4.42}$$

has solution

$$y(\varsigma) = (\varsigma+1)\int_0^\varsigma g_c(\tau)d\tau + \varsigma\int_\varsigma^1 g_c(\tau)d\tau, \tag{4.43}$$

where $g_c(\varsigma) = (C)\int_{[0,\varsigma]} g d\mu_n$.

The following examples illustrate Theorem 4.1 on various aspects of the parameters in the theorem.

Example 4.5 *For* $\varsigma \in [0,1]$, *let us consider mix order-*$\left(\frac{d}{d\mu_n}, \frac{d}{d\mu_m}\right)$ *differential equation*

$$\frac{d}{d\mu_n}\left(\frac{dy(\varsigma)}{d\mu_m}\right) = e^\varsigma \text{ for all } \varsigma \in [0,1] \tag{4.44}$$

$$y(0) = 0, \qquad y(1) = \frac{7e}{6}, \tag{4.45}$$

where $g(\varsigma) = e^\varsigma$, $m(\varsigma) = e^\varsigma - 1$ *and* $n(\varsigma) = \varsigma e^\varsigma$ *for all* $\varsigma \in [0,1]$. *It is concluded that* $g_c(\varsigma) = \frac{e^\varsigma \varsigma(\varsigma+2)}{2}$ *for all* $\varsigma \in [0,1]$. *Therefore, from Theorem 4.1, we have* y *as solution of problem 4.44-4.45 such that*

$$\begin{aligned} y(\varsigma) &= \frac{1}{e-1}\int_0^\varsigma (e^{\varsigma-\tau+1} - e^{\varsigma-\tau} + e^\varsigma - 1) g_c(\tau)d\tau + \int_\varsigma^1 \frac{e^\varsigma - 1}{e-1} g_c(\tau)d\tau \\ &= \frac{\varsigma^2 e^\varsigma}{6(e-1)}(3e^\varsigma + (e-1)\varsigma + 3e - 6) + \frac{1}{2(e-1)}(e^\varsigma - 1)(e - \varsigma^2 e^\varsigma) \\ &= \frac{1}{6(e-1)}(e^{\varsigma+1}(\varsigma^3 + 3(\varsigma^2+1)) - \varsigma^2 e^\varsigma(\varsigma+3) - 3e). \end{aligned} \tag{4.46}$$

We now consider mix order-$\left(\frac{d}{d\varsigma}, \frac{d}{d\mu_m}\right)$ boundary value problem.

Example 4.6 *For* $\varsigma \in [0,1]$, *let us consider mix order-*$\left(\frac{d}{d\varsigma}, \frac{d}{d\mu_m}\right)$ *differential equation*

$$\frac{d}{d\varsigma}\left(\frac{dy(\varsigma)}{d\mu_m}\right) = \log(1+\varsigma) \text{ for all } \varsigma \in [0,1] \tag{4.47}$$

$$y(0) = 0, \qquad y(1) = \frac{(96\log(2) - 6)}{18}, \tag{4.48}$$

where $g(\varsigma) = \log(1+\varsigma)$, $m(\varsigma) = \varsigma + \frac{\varsigma^2}{2}$ *and* $n(\varsigma) = \varsigma$ *for all* $\varsigma \in [0,1]$. *Thus, we have* $g_c(\varsigma) = (\varsigma+1)\log(1+\varsigma) - \varsigma$ *for all* $\varsigma \in [0,1]$. *Therefore, by Theorem 4.1, we obtain*

$$\begin{aligned} y(\varsigma) &= \frac{1}{3}\int_0^\varsigma (\varsigma^2 + 5\varsigma + 3(1-\tau))((\tau+1)\log(1+\tau) - \tau)d\tau + \frac{1}{3}\int_\varsigma^1 (\varsigma^2 + 2\varsigma)((\tau+1)\log(1+\tau) - \tau)d\tau \\ &= \frac{1}{36}(\log(1+\varsigma)(6\varsigma^3 + 36\varsigma^2 + 54\varsigma + 24) + 24\varsigma(\varsigma+2)\log(2) - 11\varsigma^3 - 57\varsigma^2 - 54\varsigma). \end{aligned} \tag{4.49}$$

The following graphs represent the solutions of their corresponding boundary value problems.

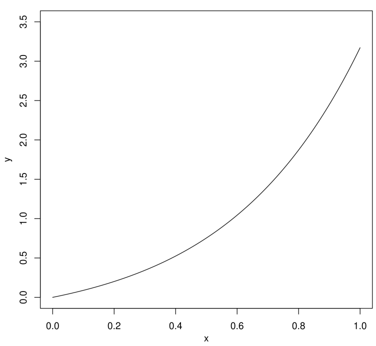

Figure 1. Solution of problem 4.44-4.45.

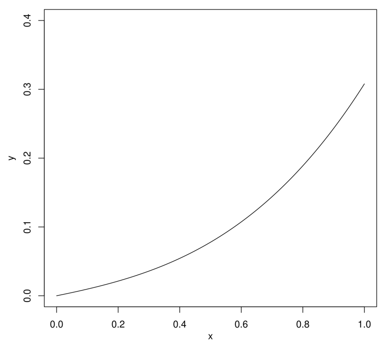

Figure 2. Solution of problem 4.47-4.48.

In the next example, we validate Theorem 4.1 by considering various order of BVP with $m(\varsigma) = n(\varsigma) = \varsigma^k$, $k \in \mathbb{N}$ and for all $\varsigma \in [0,1]$.

Example 4.7 *Let us consider mix order-*$\left(\frac{d}{d\mu_n}, \frac{d}{d\mu_m}\right)$ *or second-order differential equation*

$$\frac{d}{d\mu_n}\left(\frac{dy(\varsigma)}{d\mu_m}\right) = \sin(\varsigma) \text{ for all } \varsigma \in [0,1] \tag{4.50}$$

$$y(0) = 0, \qquad y(1) = \alpha, \tag{4.51}$$

where α *is as in Theorem 4.1,* $m(\varsigma) = n(\varsigma) = \varsigma^a$, $a > 0$ *and* $g(\varsigma) = \sin(\varsigma)$ *for all* $\varsigma \in [0,1]$. *Thus, in view of Theorem 4.1, we have*

$$y(\varsigma) = (C)\int_{[0,\varsigma]} g_c d\mu_{m_1} + (C)\int_{[\varsigma,1]} g_c d\mu_{m_2}, \tag{4.52}$$

where $g_c(s) = (C)\int_{[0,s]} \sin(s)d\mu_n$, $m_1(s) = s(\varsigma^a + s^{a-1})$ *and* $m_2(s) = \varsigma^a s$ *for* $s, \varsigma \in [0,1]$.

Particularly, for $a = 1$*, we have the following second-order boundary value problem*

$$y'' = \sin(\varsigma) \text{ for all } \varsigma \in [0, 1] \tag{4.53}$$
$$y(0) = 0, \quad y(1) = 2(1 - \sin(1)), \tag{4.54}$$

where $m(\varsigma) = n(\varsigma) = \varsigma$ *for all* $\varsigma \in [0, 1]$*. It is to have* $g_c(\varsigma) = 1 - \cos(\varsigma)$ *for all* $\varsigma \in [0, 1]$*. Thus, from Corollary 4.3, we have* y *as solution of problem 4.53-4.54 such that*

$$\begin{aligned} y(\varsigma) &= \int_0^{\varsigma} (\varsigma + 1)(1 - \cos(\tau))d\tau + \int_{\varsigma}^1 \varsigma(1 - \cos(\tau))d\tau \\ &= (\varsigma + 1)(\varsigma - \sin(\varsigma)) + \varsigma(\sin(\varsigma) - \varsigma - \sin(1) + 1) \\ &= 2\varsigma(1 - 0.5\sin(1)) - \sin(\varsigma). \end{aligned} \tag{4.55}$$

The figures below show the behavior of solutions to the problem 4.50-4.51 corresponding to $m(\varsigma) = n(\varsigma) = \varsigma^k$, for $k \in \mathbb{N}$.

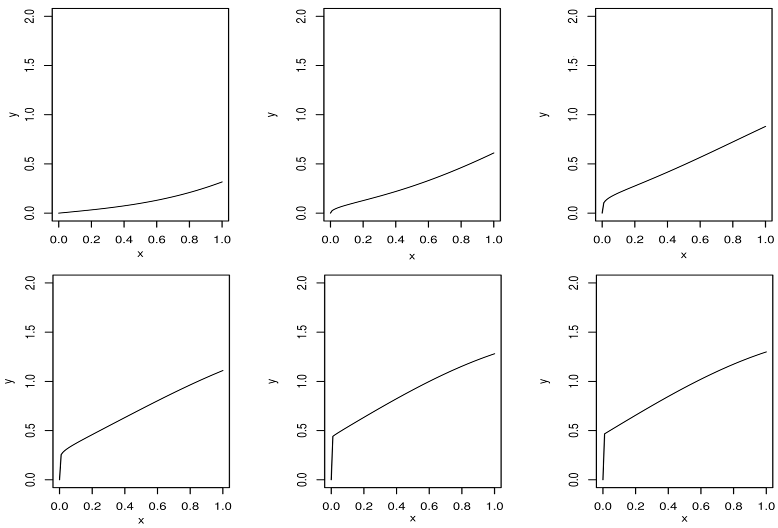

Figure 3. Solutions of problem 4.50-4.51 with $m(\varsigma) = n(\varsigma) = \varsigma^k$ for $k = 1, 2, 3, 10, 100$ and 1000.

In a similar manner, for $a = 1/k$, we have $m(\varsigma) = n(\varsigma) = \varsigma^{1/k}$ for $k \in \mathbb{N}$.

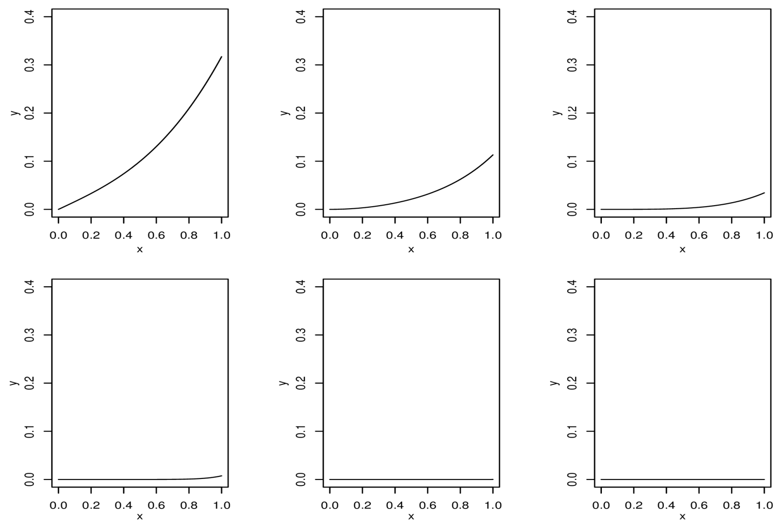

Figure 4. Solutions of problem 4.50-4.51 with $m(\varsigma) = n(\varsigma) = \varsigma^{1/k}$ for $k = 1, 2, 3, 10, 100$ and 1000.

Remark 4.8 *From Fig. 3, we observe that the solution* y *goes upwards when* $a = k = 1, 2, 3, \cdots, 1000$*. In contrast, in Fig. 4, solution* y *is going downwards corresponding to each* $a = 1/k = 1, 1/2, 1/3, \cdots, 1/1000$.

We have sufficient supporting results to the study of existence of solution of boundary value problem 1.3-1.4.

Theorem 4.9 *Assume $m, n \in \mathcal{M}^+$ with $m(\varsigma) \leq \varsigma$ for all $\varsigma \in [0,1]$. If the following hold:*

1. *$m' : [0,1] \to (0,\infty)$ is Lipschitz continuous and increasing function and $g : [0,\infty) \to [0,\infty)$ is continuous and increasing function,*
2. *y^* and y_* are a upper and lower solution of problem 1.3-1.4 along with condition $y^*(\varsigma) \geq y_*(\varsigma)$ for all $\varsigma \in [0,1]$.*

Then, for $\alpha = (C)\int_{[0,1]} \left(\frac{1}{m'(1-\tau)} + 1\right)(g \circ y)_c d\mu_m > 0$, the problem 1.3-1.4 has at least one positive solution.

Proof 4.2 *In view of Theorem 4.1, let us consider a operator $\mathcal{Y} : \mathcal{A} \to \mathcal{C}$ such that*

$$\mathcal{Y}y(\varsigma) = (C)\int_{[0,\varsigma]} (g \circ y)_c d\mu_{m_1} + (C)\int_{[\varsigma,1]} (g \circ y)_c d\mu_{m_2}, \tag{4.56}$$

where $(g \circ y)_c(s) = (C)\int_{[0,s]} g \circ y d\mu_n$, $m_1(s) = \frac{m(\varsigma)}{m(1)}s + m(s)$ and $m_2(s) = \frac{m(\varsigma)}{m(1)}s$ for $s, \varsigma \in [0,1]$. It is noted that the operator $\mathcal{Y}$ is continuous and positive because the functions g and G are positive and continuous. In order to apply Theorem 1.1 for the existence of positive solution of problem 1.3-1.4, we define some steps.

Step-1, show that the operator $\mathcal{Y}$ is bounded. Let us consider a bounded subset, say $\mathcal{B}$, of $\mathcal{C}$, i.e., there exists $\lambda_1 > 0$, such that $\|y\| \leq \lambda_1$ for all $y \in \mathcal{B}$. Furthermore, let λ_2 be any positive constant then $|g(y(\varsigma))| < \lambda_2$ for all $\varsigma \in [0,1]$ and $y \in \mathcal{B}$.

Now, it follows from Equation 4.56 that

$$\begin{aligned}
|\mathcal{Y}y(\varsigma)| &= \left|(C)\int_{[0,\varsigma]} (g \circ y)_c d\mu_{m_1} + (C)\int_{[\varsigma,1]} (g \circ y)_c d\mu_{m_2}\right| \\
&\leq \left|(C)\int_{[0,\varsigma]} (g \circ y)_c d\mu_{m_1}\right| + \left|(C)\int_{[\varsigma,1]} (g \circ y)_c d\mu_{m_2}\right| \\
&\leq \lambda_2(m_1(\varsigma) + m_2(1-\varsigma)) \\
&\leq \lambda_2(m_1(1) + m_2(1)) \\
&\leq \lambda_2(2 + m(1)),
\end{aligned} \tag{4.57}$$

thus, we conclude $\mathcal{Y}(\mathcal{B})$ is uniformly bounded.

Step-2, show that $\mathcal{Y}(\mathcal{B})$ is equicontinuous. Therefore, for $\varsigma_1, \varsigma_2 \in [0,1]$ such that $\varsigma_1 \leq \varsigma_2$, let us consider

$$\begin{aligned}
|\mathcal{Y}y(\varsigma_1) - \mathcal{Y}y(\varsigma_2)| &= \Bigg|(C)\int_{[0,\varsigma_1]} (g \circ y)_c d\mu_{m_1} + (C)\int_{[\varsigma_1,1]} (g \circ y)_c d\mu_{m_2} - (C)\int_{[0,\varsigma_2]} (g \circ y)_c d\mu_{m_1} \\
&- (C)\int_{[\varsigma_2,1]} (g \circ y)_c d\mu_{m_2}\Bigg| \\
&\leq \left|(C)\int_{[0,\varsigma_1]} (g \circ y)_c d\mu_{m_1} - (C)\int_{[0,\varsigma_2]} (g \circ y)_c d\mu_{m_1}\right| + \Bigg|(C)\int_{[\varsigma_1,1]} (g \circ y)_c d\mu_{m_2} \\
&\quad -(C)\int_{[\varsigma_2,1]} (g \circ y)_c d\mu_{m_2}\Bigg| \\
&\leq \left|\int_0^{\varsigma_1} (m_1'(\varsigma_2 - \tau) - m_1'(\varsigma_1 - \tau))(g \circ y)_c(\tau) d\tau\right| + \left|\int_{\varsigma_1}^{\varsigma_2} m_1'(\varsigma_2 - \tau)(g \circ y)_c(\tau) d\tau\right| \\
&+ \left|\int_{\varsigma_1}^{\varsigma_2} m_2'(1 - \tau)(g \circ y)_c(\tau) d\tau\right| \\
&\leq \lambda_2\left[\int_0^{\varsigma_1} |m_1'(\varsigma_2 - \tau) - m_1'(\varsigma_1 - \tau)| d\tau + m_1(\varsigma_2 - \varsigma_1) + |m_2(1-\varsigma_2) - m_2(1-\varsigma_1)|\right] \\
&\leq \lambda_2\Bigg[\int_0^{\varsigma_1} |m'(\varsigma_2 - \tau) - m'(\varsigma_1 - \tau)| d\tau + \frac{m(\varsigma)}{m(1)}|\varsigma_2 - \varsigma_1| + m(\varsigma_2 - \varsigma_1) \\
&+ \frac{m(\varsigma)}{m(1)}|\varsigma_2 - \varsigma_1|\Bigg].
\end{aligned} \tag{4.58}$$

Since m' is Lipschitz continuous, so there exits a positive real constant L such that

$$|m'(\varsigma_1) - m'(\varsigma_2)| \leq L|\varsigma_1 - \varsigma_2|, \text{ for all } \varsigma, \varsigma_2 \in [0,1]. \tag{4.59}$$

Thus, from Equation 4.58 and 4.59, we obtain

$$\begin{aligned}|\mathcal{Y}y(\varsigma_1)-\mathcal{Y}y(\varsigma_2)| &\le \lambda_2\left[L\int_0^{\varsigma_1}|\varsigma_2-\varsigma_1|d\tau+\frac{m(\varsigma)}{m(1)}|\varsigma_2-\varsigma_1|+|\varsigma_2-\varsigma_1|\right.\\ &+ \left.\frac{m(\varsigma)}{m(1)}|\varsigma_2-\varsigma_1|\right]\\ &\le \lambda_2(L+3)|\varsigma_2-\varsigma_1|. \end{aligned} \tag{4.60}$$

Whenever $|\varsigma_1-\varsigma_2|<\delta$, where for all $\epsilon>0$ we consider δ as $\delta=\lambda_2^{-1}(L+3)^{-1}$. Finally, with Equation 4.60, we reach to the following relation

$$|\mathcal{Y}y(\varsigma_1)-\mathcal{Y}y(\varsigma_2)|<\epsilon,\ \text{for all } \varsigma_1,\varsigma_2\in[0,1]. \tag{4.61}$$

Hence, $\mathcal{Y}(\mathcal{B})$ is equicontinuous. Therefore, by using Arzela-Ascoli theorem [35], we have $\overline{\mathcal{Y}(\mathcal{B})}$ is compact.

Step-3, we now show the operator $\mathcal{Y}$ is increasing with respect to $y\in\mathcal{C}$. So, if $y_1,y_2\in\mathcal{C}$ such that $y_1\le y_2$, and since g is increasing, then we conclude the following

$$\begin{aligned}\mathcal{Y}y_1(\varsigma) &= (C)\int_{[0,\varsigma]}(g\circ y_1)_c d\mu_{m_1}+(C)\int_{[\varsigma,1]}(g\circ y_1)_c d\mu_{m_2}\\ &\le (C)\int_{[0,\varsigma]}(g\circ y_2)_c d\mu_{m_1}+(C)\int_{[\varsigma,1]}(g\circ y_2)_c d\mu_{m_2}=\mathcal{Y}y_2(\varsigma).\end{aligned} \tag{4.62}$$

for all $\varsigma\in[0,1]$. Thus, the operator $\mathcal{Y}:\langle y_*,y^*\rangle\to\langle y_*,y^*\rangle$ is compact continuous, and $\mathcal{C}$ is normal cone. Therefore, Theorem 1.1 is applicable for the operator $\mathcal{Y}$ (see Equation 4.56) and then it preserves fixed point in $\langle y_*,y^*\rangle$. Hence, we complete the proof.

Next theorem is immediately obtained from the observation of the above theorem.

Theorem 4.10 *Assume $m\in\mathcal{M}^+$, and let us consider boundary value problem*

$$\frac{d}{d\varsigma}\left(\frac{dy(\varsigma)}{d\mu_m}\right) = F(\varsigma,y(\varsigma)) \text{ for all } \varsigma\in[0,1] \tag{4.63}$$

$$y(0)=0,\qquad y(1)=(C)\int_{[0,1]}\left(\frac{1}{m'(1-\tau)}+1\right)F_c d\mu_m, \tag{4.64}$$

where $F_c(\varsigma)=\int_0^{\varsigma}F(\tau,y(\tau))d\tau$. If the following hold:

1. *$F:[0,1]\times[0,\infty)\to[0,\infty)$ is continuous and $F(\varsigma,\cdot)$ is increasing function with respect to the second variable for all $\varsigma\in[0,1]$ with $F(0,0)=0$. Moreover, $F(\varsigma,\cdot)$ is strictly increasing on $[0,\eta]$ for all $\varsigma\in[0,1]$, where η is positive real number,*
2. *$m':[0,1]\to(0,\infty)$ is Lipschitz continuous and increasing function,*
3. *y^* and y_* are a upper and lower solution of problem 1.3-1.4 along with condition $y^*(\varsigma)\ge y_*(\varsigma)$ for all $\varsigma\in[0,1]$,*

then, problem 4.63-4.64 has at least one positive solution.

Proof 4.3 *The proof of this theorem can be followed by the proof of the Theorem 4.9, therefore, it is omitted.*

Remark 4.11 *With the help of Theorem 3.2, we can also discuss the existence of solution of the problem 4.63-4.64. Indeed, if we replace condition (2) of Theorem 4.10 by the condition (2) of Theorem 3.2 along with $m'\in\mathcal{M}^+$ and $m''\overset{a.e}{=}\dfrac{1+m'}{m'}$ on $[0,1]$, then we have at least one positive and increasing solution of the BVP 4.63-4.64.*

In the coming theorem, we present some new sufficient conditions for existence of given boundary value problems, i.e., Eq. 1.2 and 1.3-1.4.

Theorem 4.12 *Assume $m,n\in\mathcal{M}^+$ with $m(\varsigma)\le\varsigma$ for all $\varsigma\in[0,1]$. If the following hold:*

1. *$m':[0,1]\to(0,\infty)$ is Lipschitz continuous and increasing function,*
2. *$g:[0,\infty)\to[0,\infty)$ is continuous and increasing function such that $0<\lim_{y\to\infty}g(y(\varsigma))<\infty$.*

Then, for $\alpha=(C)\int_{[0,1]}\left(\dfrac{1}{m'(1-\tau)}+1\right)(g\circ y)_c d\mu_m>0$, the problem 1.3-1.4 has at least one positive solution.

Proof 4.4 *In view of statement of Theorem 4.9, we just need to have lower and upper solution of Equation 1.3. From condition (2), let α, β be two positive constants such that for $0 \le y \le \alpha, \gamma = \max_{y\in[0,\alpha]} g(y) > 0$, and for $y \ge \alpha$, we have $g \le \beta$. Clearly, we have $g(y) \le \alpha + \beta$ for all $y \ge 0$.*

Let us consider the mix order-$\left(\frac{d}{d\mu_n}, \frac{d}{d\mu_m}\right)$ differential equation

$$\frac{d}{d\mu_n}\left(\frac{dz_1(\varsigma)}{d\mu_m}\right) = \alpha + \beta \text{ for } 0 < \varsigma < 1, \tag{4.65}$$

Note that, Equation 4.65 has solution

$$z_1(\varsigma) = (\alpha + \beta)(C)\int_{[0,\varsigma]} n d\mu_m \text{ for } 0 < \varsigma < 1, \tag{4.66}$$

clearly, z_1 is an upper solution of Equation 4.65, that is, $\frac{d}{d\mu_n}\left(\frac{dz_1}{d\mu_m}\right) \ge g(z_1)$. Similarly, let $z_2 = 0$ be a lower solution of Equation 4.65, and then we have $z_1(\varsigma) \ge z_2(\varsigma) = 0$ for all $\varsigma \in [0, 1]$. Thus, conditions (1) and (2) of Theorem 4.9 hold. Hence, proof is done.

Theorem 4.13 *Assume that the conditions of Theorem 3.2 hold except condition (3). Moreover, if the following condition*

$(\mathbb{E}):\ 0 < \lim_{y\to\infty} F(\varsigma, y(\varsigma)) < \infty.$

holds, then the problem 1.2 has at least one positive solution.

Proof 4.5 *Proceeding the proof of Theorem 4.12 for equation 1.2 and then by using Theorem 3.2. Thus, we compete the proof.*

Remark 4.14 *In all above theorems, we can also consider m' as continuously differentiable function on $[0, 1]$ instead of conditions on m' because every continuously differentiable function is Lipschitz continuous on $[0, 1]$.*

Remark 4.15 *In view of Theorem 4.13, similar theorem can be done for problem 1.2 with some different conditions.*

5. Krasnoselskii Fixed Point Theorem

In this section, we are going to discuss a well-known Krasnoselskii fixed point theorem for the considered boundary value problem (BVP) 1.3-1.4. In the end, one simple example is given to illustrate the finding.

Theorem 5.1 *Assume $m, n \in \mathcal{M}^+$ with m' is positive increasing function on $[0, 1]$, and if the following*

1. *$g : [0, \infty) \to [0, \infty)$ is continuous and increasing function, and there exists a constant $\mathcal{L} > 0$ such that*

$$|g(z_1) - g(z_2)| \le \mathcal{L}\|z_1 - z_2\|_{\mathcal{X}}. \tag{5.67}$$

where $z_1, z_2 \in \mathcal{X}$ with $\mathcal{L} \le \dfrac{1}{m(1)n(1)}$,

holds. Then, for $\alpha = (C)\int_{[0,1]}\left(\dfrac{1}{m'(1-\tau)} + 1\right)(g \circ y)_c d\mu_m > 0$, the BVP 1.3-1.4 has at least one positive increasing solution in $\mathcal{U}$.

Proof 5.1 *Let $\|g \circ y\|_{\mathcal{X}} \le \lambda_2$, and fix $\lambda_2(1 + m(1)) \le \gamma$, where γ is positive real number. In order to apply Theorem 1.2 to the given boundary value problem, we first need to consider a bounded set*

$$\mathcal{U} = \{y \in \mathcal{X}|\ y \text{ is non-negative and increasing with } \|y\|_{\mathcal{X}} \le \gamma,\ \gamma > 0\}. \tag{5.68}$$

Implies, $\mathcal{U}$ is non-empty closed bounded convex subset of $\mathcal{X}$, and then define the operators $\mathcal{Y}_1$ and $\mathcal{Y}_2$ on $\mathcal{U}$ such that

$$(\mathcal{Y}_1 y)(\varsigma) = (C)\int_{[0,1]}(g \circ y)_c d\mu_{m_2}, \tag{5.69}$$

and

$$(\mathcal{Y}_2 y)(\varsigma) = (C)\int_{[0,\varsigma]}(g \circ y)_c d\mu_m, \tag{5.70}$$

where $m_2(s) = \frac{m(\varsigma)}{m(1)}s$ *for all* $s \in [0,1]$ *(see Equation 4.56). Clearly,* $\mathcal{Y}_1 y$ *and* $\mathcal{Y}_2 y$ *both are non-negative and increasing, then so* $\mathcal{Y}_1 y_1 + \mathcal{Y}_2 y_2$ *for all* $y_1, y_2 \in \mathcal{U}$*. Now, for* $y_1, y_2 \in \mathcal{U}$*, we have*

$$\begin{aligned} |\mathcal{Y}_1 y_1 + \mathcal{Y}_2 y_2| &\leq \int_0^1 \frac{m(\varsigma)}{m(1)} |(g \circ y_1)_c(\tau)| d\tau + \int_0^{\varsigma} m'(\varsigma - \tau) |(g \circ y_2)_c(\tau)| d\tau \\ &\leq \lambda_2 (1 + m(1)) \leq \gamma. \end{aligned} \tag{5.71}$$

Hence, $\|\mathcal{Y}_1 y_1 + \mathcal{Y}_2 y_2\|_{\mathcal{X}} \leq \lambda_2$ *implies* $\mathcal{Y}_1 y_1 + \mathcal{Y}_2 y_2 \in \mathcal{U}$*. Thus, condition* (1) *of Theorem 1.2 holds.*

Being the continuity of the functions g, y *and* m*, we have the continuity of* $\mathcal{Y}_1$*. It is also uniformly bounded as for all* $y \in \mathcal{U}$ *and* $\varsigma \in [0,1]$*, we have*

$$|(\mathcal{Y}_1 y)\varsigma| \leq \left| \int_0^1 \frac{m(\varsigma)}{m(1)} (g \circ y)_c(\tau) d\tau \right| \leq \lambda_2, \tag{5.72}$$

that is, $\|\mathcal{Y}_1 y\|_{\mathcal{X}} \leq \lambda_2$*. Therefore,* $\mathcal{Y}_1$ *is uniformly bounded operator. Next to show the equicontinuity of the operator* $\mathcal{Y}_1$*, so, let* $y \in \mathcal{U}$ *and* $\varsigma_1, \varsigma_2 \in [0,1]$ *such that* $\varsigma_1 < \varsigma_2$*. Then, we conclude*

$$\begin{aligned} |\mathcal{Y}_1 y(\varsigma_1) - \mathcal{Y}_1 y(\varsigma_2)| &= \left| \int_0^1 \frac{m(\varsigma_1)}{m(1)} (g \circ y)_c(\tau) d\tau - \int_0^1 \frac{m(\varsigma_2)}{m(1)} (g \circ y)_c(\tau) d\tau \right| \\ &\leq \frac{\lambda_2}{m(1)} |\varsigma_1 - \varsigma_2| \leq \epsilon, \end{aligned} \tag{5.73}$$

whenever $|\varsigma_1 - \varsigma_2| < \delta$*, where* $\delta = \frac{\epsilon m(1)}{\lambda_2}$ *for each* $\epsilon > 0$*. Hence,* $\mathcal{Y}_1(\mathcal{U})$ *is equicontinuous and then by Arzela-Ascoli Theorem, we obtain the compactness of* $\mathcal{Y}_1$*. It is now enough to show the contraction of the operator* $\mathcal{Y}_2$*. For* $y_1, y_2 \in \mathcal{U}$ *and from condition* (1) *of Theorem 5.1, we obtain*

$$\begin{aligned} |(\mathcal{Y}_1 y_1)(\varsigma) - (\mathcal{Y}_1 y_2)(\varsigma)| &\leq \left| \int_0^{\varsigma} m'(\varsigma - \tau)(g \circ y_1)_c(\tau) d\tau - \int_0^{\varsigma} m'(\varsigma - \tau)(g \circ y_2)_c(\tau) d\tau \right| \\ &\leq \int_0^{\varsigma} m'(\varsigma - \tau) \left| (g \circ y_1)_c(\tau) - (g \circ y_2)_c(\tau) \right| d\tau \\ &\leq \int_0^{\varsigma} m'(\varsigma - \tau) \left(\int_0^{\tau} n'(\tau - s) \left| g(y_1(s)) - g(y_2(s)) \right| ds \right) d\tau \\ &\leq \mathcal{L} n(1) m(1) |y_1 - y_2|. \end{aligned} \tag{5.74}$$

Implies $\|(\mathcal{Y}_1 y_1)(\varsigma) - (\mathcal{Y}_1 y_2)(\varsigma)\|_{\mathcal{X}} \leq \mathcal{L} n(1) m(1) \|y_1 - y_2\|_{\mathcal{X}}$*. Since* $\mathcal{L} \leq \frac{1}{m(1)n(1)}$*, therefore, operator* $\mathcal{Y}$ *is a contraction. Finally, by using Krasnoselskii Fixed Point Theorem 1.2, there is a point* $y \in \mathcal{U}$ *such that*

$$\begin{aligned} y &= \mathcal{L}_1 y + \mathcal{Y}_2 y \\ &= (C) \int_{[0,\varsigma]} (g \circ y)_c d\mu_{m_1} + (C) \int_{[\varsigma,1]} (g \circ y)_c d\mu_{m_2}, \end{aligned} \tag{5.75}$$

where m_1 *and* m_2 *are in Equation 4.56. It is a positive increasing solution of BVP 1.3-1.4, i.e., it has at least one solution. Hence, proof is done.*

Example 5.2 *Consider the following boundary value problem (BVP)*

$$\frac{d}{d\mu_n} \left(\frac{dy(\varsigma)}{d\mu_m} \right) = \frac{e^y}{3} \text{ for all } \varsigma \in [0,1] \tag{5.76}$$

$$y(0) = 0, \qquad y(1) = \frac{2}{3} \int_0^1 \int_0^{\tau} (1 + e^{1-\tau})(\tau - s) e^{y(s)} ds d\tau, \tag{5.77}$$

here, $m(\varsigma) = e^{\varsigma} - 1$*, and* $n(\varsigma) = \varsigma^2$ *for all* $\varsigma \in [0,1]$*, and* $g(\varsigma) = e^{\varsigma}$ *for* $\varsigma \geq 0$*. Therefore, it is easy to see that the functions* $m, n \in \mathcal{M}^+$ *and* g *is positive, increasing and satisfy equation 5.67 such that*

$$\frac{1}{3} |e^{y_1(\varsigma)} - e^{y_2(\varsigma)}| \leq \frac{1}{3} \|y_1 - y_2\|_{\mathcal{X}}, \text{ for each } y_1, y_2 \in [0, \infty), \varsigma \in [0,1], \tag{5.78}$$

where $\mathcal{L} = \frac{1}{3}$ *and* $\mathcal{L} m(1) n(1) = \frac{(e-1)}{3} < 1$*. Thus, from Theorem 5.1, it is straightforward to conclude that the problem 5.76-5.77 has at least one positive solution in* $\mathcal{U}$*.*

Remark 5.3 *Similarly, Theorem 1.2 can also be used for IVP 1.2, where the two important operators can be taken as* $\mathcal{Y}_1 y(\varsigma) = (C) \int_{[0,\varsigma]} \left(\int_0^{\tau} F(s, y(s)) ds \right) d\mu_{m'}$ *and* $\mathcal{Y}_2 y(\varsigma) = m'(0) \int_0^{\varsigma} F(s, y(s)) ds$*. It is not difficult to show that* $\mathcal{Y}_1 y_1 + \mathcal{Y}_2 y_2 \in \mathcal{U}$ *for all* $y_1, y_2 \in \mathcal{U}$*. Also, under certain conditions on* m'' *and* F*, we can have the continuity and compactness of* $\mathcal{Y}_1$*, and contraction of operator* $\mathcal{Y}_2$*.*

6. Conclusion

Motivated by the Sugeno derivative, we have studied the existence of solutions of the new first and mix-order differential equation with respect to distorted Lebesgue measure. These equations are more general than the ordinary differential equations (IVP and BVP), and both coincide with particular distorted Lebesgue measure μ_m (i.e., $m(\varsigma) = \varsigma$ for all $\varsigma \in [0, 1]$). Working with these equations is more restrictive than working with ordinary differential equations because we keep the non-negative and non-decreasing conditions on the unknown function throughout the investigation. Importantly, in order to apply a well-known fixed point theorem on cones in the semi-order Banach space, we have constructed suitable corresponding operators for the problems, respectively in this paper. Further, for the existence and uniqueness of first-order differential equation, we use Picard's method. In addition, we also applied the Krasnoselskii fixed point theorem for the IVP and BVP. We have also demonstrated the expression of the solutions of corresponding equations by giving various variety examples with graphical representations to help readers understand.

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8. Data availability statements

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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