EXISTENCE OF SOLUTION FOR TWO CLASSES OF QUASILINEAR SYSTEMS DEFINED ON A NON-REFLEXIVE ORLICZ-SOBOLEV SPACES

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Abstract

This paper proves the existence of nontrivial solution for two classes of quasilinear systems of the type { - [?] Φ 1 u = F u (x , u , v)+ λ R u (x , u , v) in - [?] Φ 2 v = - F v (x , u , v) - λ R v (x , u , v) in u = v = 0 on [?] where λ>0 is a parameter, is a bounded domain in R N (N[?]2) with smooth boundary *[?]*. The first class we drop the [?] 2 -condition of the functions Φ i (i=1 ,2) and assume that F has a double criticality. For this class, we use a linking theorem without the Palais-Smale condition for locally Lipschitz functionals combined with a concentration–compactness lemma for nonreflexive Orlicz-Sobolev space. The second class, we relax the [?] 2 -condition of the functions Φ i (i=1 ,2). For this class, we consider F=0 and λ=1 and obtain the proof based on a saddle-point theorem of Rabinowitz without the Palais-Smale condition for functionals Fréchet differentiable combined with some properties of the weak * topology.

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ABSTRACT. This paper proves the existence of nontrivial solution for two classes of quasilinear systems of the type

$$\begin{cases} -\Delta_{\Phi_1} u = F_u(x,u,v) + \lambda R_u(x,u,v) \text{ in } \Omega \\ -\Delta_{\Phi_2} v = -F_v(x,u,v) - \lambda R_v(x,u,v) \text{ in } \Omega \\ \ u = v = 0 \text{ on } \partial\Omega \end{cases}$$

where $\lambda > 0$ is a parameter, Ω is a bounded domain in $\mathbb{R}^N (N \geq 2)$ with smooth boundary $\partial\Omega$. The first class we drop the Δ_2-condition of the functions $\tilde{\Phi}_i (i = 1,2)$ and assume that F has a double criticality. For this class, we use a linking theorem without the Palais-Smale condition for locally Lipschitz functionals combined with a concentration–compactness lemma for nonreflexive Orlicz-Sobolev space. The second class, we relax the Δ_2-condition of the functions $\Phi_i (i = 1,2)$. For this class, we consider $F = 0$ and $\lambda = 1$ and obtain the proof based on a saddle-point theorem of Rabinowitz without the Palais-Smale condition for functionals Fréchet differentiable combined with some properties of the weak* topology.

1. INTRODUCTION

In the present paper, we consider the existence of nontrivial solution for a large class of quasilinear systems of the type

$$\begin{cases} -\Delta_{\Phi_1} u = F_u(x,u,v) + \lambda R_u(x,u,v) \text{ in } \Omega \\ -\Delta_{\Phi_2} v = -F_v(x,u,v) - \lambda R_v(x,u,v) \text{ in } \Omega \\ \ u = v = 0 \text{ on } \partial\Omega \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a parameter, Ω is a bounded domain in $\mathbb{R}^N (N \geq 2)$ with smooth boundary $\partial\Omega$, $F : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ and $R : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ are continuous function verifying some conditions which will be mentioned later. It is important to recall that

$$\Delta_{\Phi_i} u = div(\phi_i(|\nabla u|)\nabla u)$$

where $\Phi_i (i = 1,2) : \mathbb{R} \to \mathbb{R}$ is a N-function of the form

$$\Phi_i(t) = \int_0^{|t|} s\phi_i(s)ds, \tag{1.2}$$

and $\phi_i : (0,\infty) \to (0,\infty)$ is a C^1 function verifying some technical assumptions.

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Set $\Phi_1 =: \Phi$, $v = 0$, $F_v(x,t,0) = 0$ and $R_v(x,t,0) = 0$, for all $t \in \mathbb{R}$. Then the system (1.1) reduces to the following quasilinear elliptic equation:

$$\begin{cases} -\Delta_\Phi u = f(x,u) + \lambda r(x,u) \ \text{ in } \Omega \\ \qquad u = 0 \ \text{ on } \partial\Omega \end{cases} \tag{1.3}$$

where Ω is a domain in $\mathbb{R}^N$, $f(x,u) = F_u(x,u,0)$ and $r(x,u) = R_v(x,u,0)$.

The equations like (1.3) have been arousing great interest among scholars. We refer readers to [4,11,13,14,19,26,27,28,33] and reference therein for more information. In all of these works the so called Δ_2-condition has been assumed on Φ and $\tilde{\Phi}$, which ensures that the Orlicz-Sobolev space $W_0^{1,\Phi}(\Omega)$ is a reflexive Banach space. This assertion is used several times in order to get a nontrivial solution for elliptic problems taking into account the weak topology and the classical variational methods to C^1 functionals.

In recent years many researchers have been relaxing the Δ_2-condition of the functions Φ and $\tilde{\Phi}$ to study the equation (1.3). From a mathematical point of view the problem becomes more subtle, because in general the functional energy associated with these problems are in general only continuous and the classical variational methods to C^1 cannot be used. Also, the weak topology cannot be taken into account, since $W_0^{1,\Phi}(\Omega)$ may not be reflective. For example, in [25], García-Huidobro et al. have considered existence of solution for the nonlinear eigenvalue problem like (1.3) where r is a continuous function verifying some other technical conditions. In the first part of that paper the authors consider the function $\Phi = (e^{t^2} - 1)/2$.

More recently, Fukagai et al, in [12], studied the equation (1.3) assuming that the N-function $\tilde{\Phi}$ may not verify the Δ_2-condition. In that paper, the authors assumed that f has a critical Sobolev growth, r is a subcritical term and λ is a real parameter. Using variational arguments along with the second concentration–compactness lemma of P. L. Lions for nonreflexive Orlicz-Sobolev space, they showed that there exists a constant $\lambda_0 > 0$ such that the boundary-value problem (1.3) has a nonnegative nontrivial solution in $W_0^{1,\Phi}(\Omega)$ for any $\lambda > \lambda_0$.

Already in [9], Silva, Gonçalves and Silva, considered existence of multiple solutions for a class of problem like (1.3). In that paper the Δ_2-condition is not also assumed and the main tool used was the truncation of the nonlinearity together with a minimization procedure for the functional energy associated to the quasilinear elliptic problem (1.3). In another paper, Silva, Carvalho, Silva and Gonçalves, in [10], study a class of problem (1.3) where the energy functional satisfies the mountain pass geometry and the N-function $\tilde{\Phi}$ does not satisfy the Δ_2-condition and has a polynomial growth.

Set $\phi_1(t) = |t|^p - 2$, $\phi_2(t) = |t|^q - 2$ $(p, q > 1)$. Then system (1.1) reduces to the following (p, q)-Laplacian system:

$$\begin{cases} -\Delta_p u = F_u(x,u,v) + \lambda R_u(x,u,v) \ \text{ in } \Omega \\ -\Delta_q v = -F_v(x,u,v) - \lambda R_v(x,u,v) \ \text{ in } \Omega \\ u = v = 0 \ \text{ on } \partial\Omega \end{cases} \tag{1.4}$$

For the case where $p = q = 2$, this class of systems is called noncooperative and in recent decades many recent studies have focused on it. For example, in [15], Ding and Figueiredo consider the noncooperative system (1.4) with $\lambda = 1$ allowing that the function $F(x,u,v)$ can assume a supercritical and subcritical growth on v and u respectively. They established the existence of infinitely many solutions to (1.4) provided the nonlinear terms F and R are even in (u, v). Already in [24], Clapp, Ding and Hernández showed that multiple existence of solutions to the noncooperative system (1.4) with some supercritical growth can be established without the symmetry assumption. Motivated by some results found in [24] and [15], Alves and Monari in [5] studied the existence of nontrivial solutions for (1.4) when p and q are different from 2,

$\lambda = 1$ and $F(x, u, v)$ has a supercritical growth on variable v and has a critical growth at infinity on variable u of the type $|u|^{p^*}$ with $p^* = pN/(N-p)$, the critical exponent of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$. The main difficulty in this case case is the lack of compactness of the functional energy associated to system. To overcome this difficulty, they carefully estimate and prove through the concentration–compactness principle due to Lions [32] the existence of a Palais-Smale sequence that has a strongly convergent subsequence.

In a brief bibliographical research, we can mention some contributions devoted to the study of system where Φ_1 and Φ_2 are less trivial functions, as can be seen in [20,23]. We would like to highlight the paper [23], Wang et al. considered the following quasilinear elliptic system in Orlicz-Sobolev spaces:

$$\begin{cases} -\Delta_{\Phi_1} u = R_u(x, u, v) \ \text{ in } \Omega \\ -\Delta_{\Phi_2} v = R_v(x, u, v) \ \text{ in } \Omega \\ \ u = v = 0 \ \text{ on } \partial\Omega \end{cases} \tag{1.5}$$

where Ω is a bounded domain in $R^N (N \geq 2)$ with smooth boundary $\partial\Omega$. In that paper when F satisfies some appropriate conditions including (Φ_1, Φ_2)-superlinear and subcritical growth conditions at infinity as well as symmetric condition, by using the mountain pass theorem and the symmetric mountain pass theorem, they obtained that system (1.5) has a nontrivial weak solution and infinitely many weak solutions, respectively. Some of the results obtained extend and improve those corresponding results in Carvalho et al [26].

In [20], Huentutripay-Manásevich studied an eigenvalue problem to the following system:

$$\begin{cases} -\Delta_{\Phi_1} u = \lambda R_u(x, u, v) \ \text{ in } \Omega \\ -\Delta_{\Phi_2} v = \lambda R_v(x, u, v) \ \text{ in } \Omega \\ \ u = v = 0 \ \text{ on } \partial\Omega \end{cases}$$

where the function R has the form

$$R(x, u, v) = A_1(x, u) + b(x)\Gamma_1(u)\Gamma_2(v) + A_2(x, v).$$

They extended the results of [25] for this system, that is, for a certain λ, translated the existence of a solution into an adequate minimization problem and proved the existence of a solution under some reasonable restriction. It is obvious that in [20], the Orlicz-Sobolev spaces need not be reflexive.

Inspired by the mentioned research works cited above and sharpened by the known difficulty of working with nonreflexive Banach spaces, we intend to consider two new classes of problem (1.1) where $W_0^{1,\Phi}(\Omega)$ can be nonreflexive.

The section 3 of this paper is dedicated to the study of the (1.1) system where the functions $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ may not verify the Δ_2-condition. Inspired by [5], we assume that $F : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ has a supercritical growth on variable v and has a critical growth at infinity on variable u and $R : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function with subcritical term verifying some conditions which will be mentioned in section 3. The first difficulty in studying this case arises from the lack of differentiability of the energy functional $J_\lambda : W_0^{1,\Phi_1}(\Omega) \times W_0^{1,\Phi_2}(\Omega) \to \mathbb{R}$ associated with the system (1.1) given by

$$J_\lambda(u, v) = \int_\Omega \Phi_1(|\nabla u|)dx - \int_\Omega \Phi_2(|\nabla v|)dx - \int_\Omega F(x, u, v)dx - \lambda \int_\Omega R(x, u, v)dx.$$

To get around this difficulty we will use the critical point theory for locally lipschitz fuctionals, here, in particular we apply a version of the linking theorem without Palais-Smale condition for locally lipschitz fuctionals (The version that will be applied in section 3 we took care to enunciate in section 2). A second difficulty of studying this case is the lack of compactness of the energy functional J_λ. To overcome this difficulty, we adapted some arguments presented in the works of Alves and Soares in [5] and from Fukagai et al, in [12]. Here, we carefully estimate and prove through the second concentration–compactness lemma of P. L. Lions for nonreflexive Orlicz-Sobolev space that there exists a constant $\lambda_0 > 0$ such that the boundary-value problem (1.3) has a nonnegative nontrivial solution in $W_0^{1,\Phi}(\Omega)$ for any $\lambda > \lambda_0$.

The section 4 of this paper is dedicated to the study of the (1.1) system where the functions Φ_1 and Φ_2 may not verify the Δ_2-condition. Here, we consider $F = 0$, $\lambda = 1$ and R a continuous function verifying some conditions which will be mentioned in section 4. As stated earlier, the first difficulty in relaxing the Δ_2 -condition of the functions Φ_1 and Φ_2 arises from the fact that the energy functional $J : W_0^{1,\Phi_1}(\Omega) \times W_0^{1,\Phi_2}(\Omega) \to \mathbb{R}$ associated with the system (1.1) given by

$$J(u, v) = \int_\Omega \Phi_1(|\nabla u|)dx - \int_\Omega \Phi_2(|\nabla v|)dx - \int_\Omega R(x, u, v)dx.$$

no belongs to $C^1(W_0^{1,\Phi_1}(\Omega) \times W_0^{1,\Phi_2}(\Omega), \mathbb{R})$. Have this in mind, we have decide to work in the space $W_0^1E^{\Phi_i}(\Omega)$, because it is topologically more rich than $W_0^{1,\Phi_i}(\Omega)$, for example, it is possible to prove that the energy functional J is $C^1(W_0^1E^{\Phi_1}(\Omega) \times W_0^1E^{\Phi_2}(\Omega), \mathbb{R})$. Even knowing that the result contained in Proposition 3.7 in [3] presents inconsistency when dropping the Δ_2-condition, we add a "Ambrosetti–Rabinowitz" condition under the function R and we refine part of the technique presented by Alves et al., so that, together with the saddle-point theorem of Rabinowitz without Palais-Smale condition, we can prove the existence of a Palais-Smale bounded sequence. Finally, due to the possible lack of reflexivity of spaces $W_0^{1,\Phi_i}(\Omega)(i = 1, 2)$, we will use some properties of the weak* topology of these spaces to guarantee the existence of nontrivial solutions for the system (1.1).

It is important to stress that, to the best of our knowledge, this is the first paper where the linking theorem for locally lipschitz fuctionals and the saddle-point theorem of Rabinowitz has been used to deal with a quasilinear elliptic system driven by an N-functions may not verify the Δ_2-condition.

2. Preliminaries

We recall a few notations and results on the critical point theory for locally Lipschitz functionals defined on a real Banach space X with norm $\|\cdot\|_X$. This results can be found in [30] and in references therein.

Let X be a Banach space. Let $I : X \to \mathbb{R}$ be a locally Lipschitz functional ($I \in Lip_{loc}(X, \mathbb{R})$, that is, for each $x \in X$, there exist an open neighborhood $N(x)$ of x and a constant $k(x) > 0$, such that

$$|I(y_1) - I(y_2)| \le k(x)\|y_1 - y_2\|,$$

for all y_1 and y_2 in $N(x)$.

A generalized directional derivative of a locally Lipschitz functional $I : X \to \mathbb{R}$ at $x \in X$ in the direction $v \in X$, denoted by $I^0(x; v)$, is defined by

$$I^0(x; v) = \limsup_{h\to 0\ \lambda\to 0^+} \frac{I(x + h + \lambda v) - I(x + h)}{\lambda}$$

and the generalized gradient of I at x is the set

$$\partial I(x) = \{\mu \in X^* : \langle \mu, v\rangle \le I^0(x; v),\ v \in X\}.$$

Let Q be a compact metric space and let Q_* be a nonempty closed subset strictly contained in Q. We set

$$\mathcal{P} = \{p \in C(Q, X) \colon p = p_* \text{ on } Q_*\}, \tag{2.1}$$

where p_* is a fixed continuous map on Q_* and

$$c = \inf_{c\in\mathcal{P}} \max_{x\in Q} I(p(x)). \tag{2.2}$$

So

$$c \geq \max_{x\in Q_*} I(p_*(x)). \tag{2.3}$$

We say that the subset $A \subset X$ *links with the pair* (Q, Q_*) if $p_*(Q_*) \cap A = \emptyset$ and for each $p \in \mathcal{P}$, $p(Q) \cap A \neq \emptyset$.

Theorem 2.1. *Let $I \in Lip_{loc}(X, \mathbb{R})$ and $A \subset I_c = \{x \in X : I(x) \geq c\}$ be a closed subset which links with the pair (Q, Q_*). Then there exists a sequence $(x_n) \subset X$ satisfying*

$$\lim_{n\to\infty} d(x_n, A) = 0, \quad \lim_{n\to\infty} I(x_n) = c \quad e \quad \lim_{n\to\infty} \lambda_I(x_n) = 0,$$

with

$$\lambda_I(x_n) = \min\{\|\mu\|_{X^*} \colon \mu \in \partial I(x_n)\}.$$

Proposition 2.2. *Let $I : X \to \mathbb{R}$ be a continuous and Gateaux-differentiable functional such that $I' : X \to X^*$ is continuous from the norm topology of X to the weak*-topology of X^*. Then $I \in Lip_{loc}(X, \mathbb{R})$ and $\partial I(x) = \{I'(x)\}$, $\forall x \in X$.*

The Theorem 2.1 together with Proposition 2.2 allows us to propose a linking theorem for Gateaux-differentiable functionals. This result will be fundamental to study the class of system proposed in section 3.

Theorem 2.3. *(The linking theorem) Let X be a real Banach space with $X = Y \oplus Z$, where Y is finite dimensional. Suppose that $I : X \longrightarrow \mathbb{R}$ is continuous and Gateaux-differentiable with derivative $I' : X \longrightarrow X^*$ continuous from the norm topology of E to the weak*-topology of X^* satisfying:*

(I_1) *There is $\sigma > 0$ such that if $\mathcal{N} = \{u \in Z : \|u\| \leq \sigma\}$, then $b \doteq \inf_{\partial\mathcal{N}} I > 0$.*

(I_2) *There are $z_* \in Z \cap \partial B_1$ and $\rho > \sigma$ such that*

$$0 = \sup_{\partial\mathcal{M}} I < d \doteq \sup_{\mathcal{M}} I,$$

where

$$\mathcal{M} = \{u = \lambda z_* + y : \|u\| \leq \rho, \ \lambda \geq 0, \ y \in Y\}$$

If

$$c = \inf_{\gamma\in\Gamma} \max_{x\in\mathcal{N}} I(\gamma(t)),$$

with

$$\Gamma = \{\gamma \in C(\mathcal{N}, X) : \gamma|_{\partial\mathcal{N}} = Id_{\partial\mathcal{N}}\}.$$

Then, $b \leq c$ and there is a sequence $(u_n) \subset X$ such that

$$I(u_n) \to c \quad \textit{and} \quad I'(u_n) \to 0.$$

Proof. The result follows from Proposition 2.2 and Theorem 2.1 with $\mathcal{P} = \Gamma$, $Q = \mathcal{N}$, $Q_* = \partial\mathcal{N}$, $p_* = Id_{Q_*}$ e $A = \{x \in Z + Y : I(x) \geq c\}$.

□

For the last section of this paper we will use the already known saddle-point theorem of Rabinowitz without Palais-Smale condition. The proof of this result also follows from Theorem 2.1 along with Proposition 2.2.

Theorem 2.4. *(Saddle-point theorem) Let X be a real Banach space with $X = Y \oplus Z$, where Y is finite dimensional. Suppose that $I : X \to \mathbb{R}$ is continuous and Gateaux-differentiable with derivative $I' : X \to X^*$ continuous from the norm topology of E to the weak*-topology of X^* satisfying:*
(I_1) *there are constants $\rho > 0$ and $\alpha_1 \in \mathbb{R}$ such that if $\mathcal{M} = \{u \in Y : \|u\| \leq \rho\}$, then $I|_{\partial\mathcal{M}} \leq \alpha_1$.*
(I_2) *there is a constant $\alpha_2 > \alpha_1$ such that $I|_Z \geq \alpha_2$. If*

$$c = \inf_{\gamma\in\Gamma} \max_{x\in\mathcal{M}} I(\gamma(t)),$$

with

$$\Gamma = \{\gamma \in C(\mathcal{M}, X) : \gamma|_{\partial\mathcal{M}} = Id|_{\partial\mathcal{M}}\}.$$

Then, there is $(u_n) \subset X$ such that

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0.$$

3. The functions $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ may not verify the Δ_2-condition

In this section, we study the existence of solutions for the following class of quasilinear systems in Orlicz-Sobolev spaces:

$$(S_1) \qquad \begin{cases} -div(\phi_1(|\nabla u|)\nabla u) = F_u(x,u,v) + \lambda R_u(x,u,v) \text{ in } \Omega \\ -div(\phi_2(|\nabla v|)\nabla v) = -F_u(x,u,v) - \lambda H_v(x,u,v) \text{ in } \Omega \\ u = v = 0 \text{ on } \partial\Omega \end{cases}$$

where $\lambda > 0$ is a parameter, Ω is a bounded domain in $\mathbb{R}^N (N \geq 2)$ with smooth boundary $\partial\Omega$, and $\phi_i (i = 1, 2) : (0, \infty) \to (0, \infty)$ are two functions which satisfy:
(ϕ_1) $\phi_i \in C^1(0, +\infty)$ and $t \mapsto t\phi_i(t)$ are stricly increasing.
(ϕ_2) $t\phi_i(t) \to 0$ as $t \to 0$ and $t\phi_i(t) \to +\infty$ as $t \to +\infty$
(ϕ_3) $1 \leq \ell_i = \inf_{t>0} \dfrac{t^2\phi_i(t)}{\Phi_i(t)} \leq \sup_{t>0} \dfrac{t^2\phi_i(t)}{\Phi_i(t)} = m_i < N$, where $\Phi_i(t) = \int_0^{|t|} s\phi_i(s)ds$ and $\ell_i < m_i < \ell_i^*$.

Before continuing this section, consider $\alpha_1, \alpha_2 \in (0, \frac{N}{N-1} - 1)$ such that $\alpha_1 \leq \alpha_2$. We would like to point out that $\Phi_1(t) = |t| \ln(|t|^{\alpha_1} + 1)$ and $\Phi_2(t) = |t| \ln(|t|^{\alpha_2} + 1)$ satisfying $(\phi_1) - (\phi_3)$ with $\ell_1 = \ell_2 = 1$ and $m_1 = 1 + \alpha_1$, $m_2 = 1 + \alpha_2$ respectively. These functions are examples of N-functions whose the complementary functions $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ do not satisfy the Δ_2-condition, consequently $W_0^{1,\Phi_{\alpha_1}}(\Omega) \times W_0^{1,\Phi_{\alpha_2}}(\Omega)$ is nonreflexive.

In this section, we would like to recall that $(u, v) \in W_0^{1,\Phi_1}(\Omega) \times W_0^{1,\Phi_2}(\Omega)$ is a weak solution of (S_1) whenever

$$\int_\Omega \phi_1(|\nabla u|)\nabla u\nabla w_1 dx - \int_\Omega \phi_2(|\nabla v|)\nabla v\nabla w_2 dx = \int_\Omega H_u(x,u,v)w_1 dx + \int_\Omega H_v(x,u,v)w_2 dx,$$

for all $(w_1, w_2) \in W_0^{1,\Phi_1}(\Omega) \times W_0^{1,\Phi_2}(\Omega)$. Here, let us consider the H function as follows:

$$H(x,u,v) = F(x,u,v) + \lambda R(x,u,v)$$

where $F(x,u,v) = \Phi_{1*}(u) + G(v)$, $\lambda > 0$ is a real parameter and Φ_{1*} denotes the Sobolev conjugate function of Φ_1 defined by

$$\Phi_{1*}^{-1}(t) = \int_0^t \frac{\Phi_1^{-1}(s)}{s^{(N+1)/N}} ds \text{ for } t > 0 \quad \text{when} \quad \int_1^{+\infty} \frac{\Phi_1^{-1}(s)}{s^{(N+1)/N}} ds = +\infty.$$

Furthermore, the functions G and R satisfy the following conditions:
(G_1) There are $C > 0$, $G \in C^1(\mathbb{R}, \mathbb{R})$, $a_1, a_2 \in (1, \infty)$ and a N-function $A(t) = \int_0^{|t|} sa(s)ds$ satisfying

$$(i) \qquad m_2 < a_1 \leq \frac{a(t)t^2}{A(t)} \leq a_2, \quad \forall t > 0$$

and

$$(ii) \qquad |g(s)| \leq a_1 C a(|s|)|s|, \quad \text{for all } s \in \mathbb{R}$$

where $g(s) = G'(s)$. If $a_2 \geq \ell_2^*$, we add that

$$(iii) \qquad (g(t) - g(s))(t - s) \geq C a(|t - s|)|t - s|^2, \quad \text{for all } t, s \in \mathbb{R}.$$

(G_2) There exists $\nu \in (0, \ell_1)$ such that

$$0 \leq \nu G(s) \leq s g(s), \quad \text{for all } s \in \mathbb{R}.$$

(R_1) $R \in C^1(\overline{\Omega} \times \mathbb{R}^2)$, $R_u(x, 0, 0) = 0$, $R_v(x, 0, 0) = 0$, $R(x, u, v) \geq 0$ and $R_u(x, u, v)u \geq 0$, for all $(x, u, v) \in \overline{\Omega} \times \mathbb{R}^2$.
(R_2) There are N-functions $B(t) = \int_0^{|t|} sb(s)ds$, $P(t) = \int_0^{|t|} sp(s)ds$, $Q(t) = \int_0^{|t|} sq(s)ds$ and $Z(t) = \int_0^{|t|} sz(s)ds$ satisfying

$$(i) \qquad m_1 < p_1 \leq \frac{p(t)t^2}{P(t)} \leq p_2 < \ell_1^*$$

$$(ii) \qquad m_1 < b_1 \leq \frac{b(t)t^2}{B(t)} \leq b_2 < \ell_1^*$$

$$(iii) \qquad m_2 < q_1 \leq \frac{q(t)t^2}{Q(t)} \leq q_2 < \ell_2^*$$

$$(iv) \qquad m_2 < z_1 \leq \frac{z(t)t^2}{Z(t)} \leq z_2 < \ell_2^*,$$

with $\max\{b_2, q_2\} < \min\{\ell_1^*, \ell_2^*\}$ so that

$$(3.1) \qquad |R_u(x, u, v)| \leq C(p(|u|)u + q(|v|)v) \quad \text{and} \quad |R_v(x, u, v)| \leq C(b(|u|)u + z(|v|)v),$$

for all $(x, u, v) \in \Omega \times \mathbb{R}^2$ and for some constant $C > 0$.
(R_3) There exists $\mu \in (m_1, \ell_1^*)$ such that

$$\frac{1}{\mu} R_u(x, u, v) + \frac{1}{\nu} R_v(x, u, v) - R(x, u, v) \geq 0, \quad \text{for all } x \in \Omega \text{ and } (u, v) \in \mathbb{R}^2,$$

where ν is given by condition (G_2).
(R_4) There exists $s \in (m_1, \max\{p_2, b_2\}]$, a nonempty open subset $\Omega_0 \subset \Omega$ and a constant $\omega > 0$ such that

$$R(x, u, v) \geq \omega |u|^s \quad \text{for all } x \in \Omega_0 \text{ and } (u, v) \in \mathbb{R}^2.$$

The main result of this section is the following.

Theorem 3.1. *If $(\phi_1) - (\phi_3)$, $(G_1) - (G_2)$, $(R_1) - (R_4)$ hold, then there exists $\lambda_0 > 0$ such that (S_1) possesses a nontrivial solution for all $\lambda > \lambda_0$.*

Fix $p \in (m_1, \ell_1^*)$ and $q \in (m_2, \ell_2^*)$. The function $R(u,v) = |u|^p + C|v|^q + \varepsilon \sin|u|^p \sin|v|^q$ satisfies $(R_1)-(R_4)$ with $P(t) = B(t) = |t|^p/p$, $Q(t) = Z(t) = |t|^q/q$, $C > 0$ and $\varepsilon > 0$ small enough.

Before proving the above theorem, we have to fix some notations. In the sequel V_A stands for the space $W_0^{1,\Phi_2}(\Omega) \cap L^A(\Omega)$ endowed with the norm

$$\|v\|_A = \|v\|_{W_0^{1,\Phi_2}(\Omega)} + |v|_A,$$

where $\|v\|_{W_0^{1,\Phi_2}(\Omega)}$ and $|v|_A$ denote the usual norms in in $W_0^{1,\Phi_2}(\Omega)$ and $L^A(\Omega)$, respectively.

We write X for the space $W_0^{1,\Phi_1}(\Omega) \times V_A$ endowed with the norm

$$\|(u,v)\|^2 = \|u\|^2_{W_0^{1,\Phi_1}(\Omega)} + \|v\|_A^2,$$

where $\|u\|_{W_0^{1,\Phi_1}(\Omega)}$ denotes the usual norm in $W_0^{1,\Phi_1}(\Omega)$. Under the assumptions (G_1) and (R_2), the functional $\mathcal{H}_\lambda$ given by

$$\mathcal{H}_\lambda(u,v) = \int_\Omega H(x,u,v)dx. \tag{3.2}$$

is well defined, belongs to $C^1(X,\mathbb{R})$ and

$$\mathcal{H}'_\lambda(u,v)(w_1,w_2) = \int_\Omega H_u(x,u,v)w_1 dx + \int_\Omega H_v(x,u,v)w_2 dx, \tag{3.3}$$

for all $(u,v),(w_1,w_2) \in X$. Now, we consider the functional $Q : X \to \mathbb{R}$ which is given by

$$Q(u,v) = \int_\Omega \Phi_1(|\nabla u|)dx - \int_\Omega \Phi_2(|\nabla v|)dx, \tag{3.4}$$

It is well known in the literature that $Q \in C^1(E,\mathbb{R})$ when Φ_1, Φ_2, $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ satisfy the Δ_2-condition and this occurs when we have the condition satisfied to $\ell_1 > 1$ and $\ell_2 > 1$. When $\ell_1 = 1$ (or $\ell_2 = 1$), we know that $\tilde{\Phi}_1 \notin (\Delta_2)$ (or $\tilde{\Phi}_2 \notin (\Delta_2)$) and therefore cannot guarantee the differentiability of functional Q. However, following the ideas presented in [22], it is clear that the functional Q is continuous and Gateaux-differentiable with derivative $Q' : X \to X^*$ given by

$$Q'(u,v)(w_1,w_2) = \int_\Omega \phi_1(|\nabla u|)\nabla u \nabla w_1 dx - \int_\Omega \phi_2(|\nabla v|)\nabla v \nabla w_2 dx,$$

continuous from the norm topology of X to the weak*-topology of X^*. Therefore, we can conclude that the energy functional $J_\lambda : X \to \mathbb{R}$ associated with the system (S_1) given by

$$J_\lambda(u,v) = \int_\Omega \Phi_1(|\nabla u|)dx - \int_\Omega \Phi_2(|\nabla v|)dx - \int_\Omega H(x,u,v)dx.$$

is continuous and Gateaux-differentiable with derivative $J'_\lambda : X \to X^*$ defined by

$$J'_\lambda(u,v)(w_1,w_2) = \int_\Omega \phi_1(|\nabla u|)\nabla u \nabla w_1 dx - \int_\Omega \phi_2(|\nabla v|)\nabla v \nabla w_2 dx - \int_\Omega H_u(x,u,v)w_1 dx - \int_\Omega H_v(x,u,v)w_2 dx$$

continuous from the norm topology of X to the weak*-topology of X^*. Since $J'_\lambda(0,0) = 0$, we say that (u,v) is a nontrivial solution of (S_1) when $J'_\lambda(u,v)(w_1,w_2) = 0$, for all $(w_1,w_2) \in X$ and satisfies $J_\lambda(u,v) \neq 0$.

In order to apply the linking theorem 2.3, we introduce one more piece of notation. Since $(V_A, \|\cdot\|_A)$ is separable, then there exists a sequence $(e_n) \subset V_A$ such

$$V_A = \overline{span\{e_n : n \in \mathbb{N}\}}. \tag{3.5}$$

Hereafter, for each $n \in \mathbb{N}$ we denote by V_A^n and X_n the following spaces

$$V_A^n = span\{e_j : j = 1, \cdots, n\} \quad \text{and} \quad X_n = W_0^{1,\Phi_1}(\Omega) \times V_A^n.$$

The restriction of J_λ to X_n will be denoted by $J_{\lambda,n}$. Then $J_{\lambda,n} : X_n \longrightarrow \mathbb{R}$ is the functional given by

$$J_{\lambda,n}(u, v) = \int_\Omega \Phi_1(|\nabla u|)dx - \int_\Omega \Phi_2(|\nabla v|)dx - \int_\Omega H(x, u, v)dx.$$

is continuous and Gateaux-differentiable with derivative $J'_{\lambda,n} : X_n \longrightarrow X_n^*$ given by

$$J'_{\lambda,n}(u,v)(w_1,w_2) = \int_\Omega \phi_1(|\nabla u|)\nabla u \nabla w_1 dx - \int_\Omega \phi_2(|\nabla v|)\nabla v \nabla w_2 dx - \int_\Omega H_u(x,u,v)w_1 dx - \int_\Omega H_v(x,u,v)w_2 dx$$

continuous from the norm topology of X_n to the weak*-topology of X_n^*.

In the following, we prove that $J_{\lambda,n}$ satisfies the hypotheses of Theorem 2.3.

Lemma 3.2. *Assume that $(G_1) - (G_2)$ and $(R_1) - (R_4)$ hold. For every $\lambda > 0$, there exist $\sigma > 0$ and $\rho > \sigma$ such that if $u_* \in W_0^{1,\Phi_1}(\Omega)$ satisfies $\|u_*\|_{W_0^{1,\Phi_1}(\Omega)} = 1$, then*

$$d_n := \sup_{\mathcal{M}_{u_*}^n} J_{\lambda,n} \geq b_n := \inf_{\mathcal{N}_n} J_{\lambda,n} > 0 = \max_{\partial \mathcal{M}_{u_*}^n} J_{\lambda,n},$$

where

$$\mathcal{M}_{u_*}^n = \{(\theta u_*, v) \in X_n : \|(\theta u_*, v)\|^2 \leq \rho^2, \ \theta \geq 0\} \ \text{ and } \ \mathcal{N}_n = \{(u, 0) \in X_n : \|u\|_{W_0^{1,\Phi_1}(\Omega)} = \sigma\}.$$

Proof. By (R_1) and (R_2),

$$J_{\lambda,n}(u, 0) \geq \xi_\Phi^0(\|u\|_{1,\Phi}) - \xi_{\Phi_*}^1(\|u\|_{\Phi_*}) - C\lambda \xi_P^1(\|u\|_P),$$

where

$$\xi_{\Phi_1}^0(t) = \min\{t^{\ell_1}, t^{m_1}\}, \quad \xi_{\Phi_{1*}}^1(t) = \max\{t^{\ell_1^*}, t^{m_1^*}\} \quad \text{and} \quad \xi_P^1(t) = \max\{t^{p_1}, t^{p_2}\}.$$

Now, remember that by the assumption $(R_2)(i)$ it is possible to show the following limits:

$$\lim_{t \to 0} \frac{P(|t|)}{\Phi_1(|t|)} = 0 \quad \text{and} \quad \lim_{|t| \to +\infty} \frac{P(|t|)}{\Phi_{1*}(|t|)} = 0.$$

Through these two limits we can guarantee the existence of a constant $C_1 > 0$ that does not depend on u so that

$$\|u\|_P \leq C_1 \|u\|_{1,\Phi_1}, \quad \forall u \in W_0^{1,\Phi_1}(\Omega).$$

Another important inequality was proved by Donaldson and Trudinger [34], which establishes the existence of a constant $S_0 > 0$ that depends on N such that

$$\|u\|_{\Phi_{1*}} \leq S_0 \|u\|_{1,\Phi_1}, \quad \forall u \in W_0^{1,\Phi_1}(\Omega). \tag{3.6}$$

Thus,

$$J_{\lambda,n}(u, 0) \geq \|u\|_{1,\Phi_1}^{m_1} - \|u\|_{1,\Phi_1}^{\ell_1^*} - C\lambda \|u\|_{1,\Phi_1}^{p_1}, \quad \text{for } \|u\|_{1,\Phi} < 1$$

Since $m_1 < \ell_1^*$ and $m_1 < p_1$, choose $\rho > 0$ sufficiently small such that

$$J_{\lambda,n}(u, 0) \geq C\sigma^{\ell_2}, \quad \text{for } \|u\|_{1,\Phi} = \rho, \tag{3.7}$$

therefore

$$b_n := \inf_{\mathcal{N}_n} J_{\lambda,n} \geq C\sigma^{\ell_2}, \ \ \forall n \in \mathbb{N}. \tag{3.8}$$

Now, from (G_2) and (R_1),

$$J_{\lambda,n}(0, v) \leq 0, \ \ \forall v \in V_A^n. \tag{3.9}$$

Consider $u_* \in W_0^{1,\Phi_1}(\Omega)$ with $\|u_*\|_{1,\Phi_1}= 1$, by assumptions (G_2) and (R_1),

$$\begin{aligned} J_{\lambda,n}(\theta u_*, v) &\le \int_\Omega \Phi_1(|\nabla(\theta u_*)|)dx - \int_\Omega \Phi_2(|\nabla v|)dx - \int_\Omega \Phi_{1*}(|\theta u_*|)dx \\ &\le \xi^1_{\Phi_1}(\theta)\xi^1_{\Phi_1}(\|u_*\|_{1,\Phi_1}) - \xi^0_{\Phi_2}(\|v\|_{1,\Phi_2}) - \xi^0_{\Phi_{1*}}(\theta)\xi^0_{\Phi_{1*}}(\|u_*\|_{\Phi_{1*}}) \end{aligned}$$

for each $\theta > 0$ and $v \in V_A^n$, where

$$\xi^1_{\Phi_1}(t) = \max\{t^{\ell_1}, t^{m_1}\}, \quad \xi^0_{\Phi_{1*}}(t) = \min\{t^{\ell_1^*}, t^{m_1^*}\} \quad \text{and} \quad \xi^0_{\Phi_2}(t) = \max\{t^{\ell_2}, t^{m_2}\}.$$

If $a_2 < m_1^*$, then A increases essentially more slowly than Φ_{2*} near infinity. From Theorem 8.35 in [1] it follows that $L^{\Phi_{2*}}(\Omega)$ is continuously embedded in $L^A(\Omega)$, consequently $W_0^{1,\Phi_1}(\Omega) = V_A$ and as norms $\|\cdot\|_A$ and $\|\cdot\|_{1,\Phi_2}$ are equivalent. Given this, there is a constant $C > 0$ such that

$$J_{\lambda,n}(\theta u_*, v) \le \xi^1_{\Phi_1}(\theta) - C\xi^0_{\Phi_2}(\|v\|_A) - \xi^0_{\Phi_*}(\theta)\xi^0_{\Phi_{1*}}(\|u_*\|_{\Phi_{1*}}),$$

for each $\theta > 0$ and $v \in V_A^n$.

Note that $\|(\theta u_*, v)\|^2= \theta^2 + \|v\|_A^2= \rho^2$ implies that

$$\theta^2 \ge \frac{\rho^2}{2} \quad \textit{or} \quad \|v\|_A^2 \ge \frac{\rho^2}{2}.$$

Assume that $\theta^2 \ge \rho^2/2$ occurs, then for $\rho > 0$ large enough, we have

$$\xi^1_{\Phi_1}(\theta) - C\xi^0_{\Phi_2}(\|v\|_A) - \xi^0_{\Phi_{1*}}(\theta)\xi^0_{\Phi_{1*}}(\|u_*\|_{\Phi_{1*}}) = \theta^{m_1} - C\xi^0_{\Phi_2}(\|v\|_A) - \theta^{\ell_1^*}\xi^0_{\Phi_*}(\|u_*\|_{\Phi_*}) < 0,$$

because $m_1 < \ell_1^*$. Similar property happens when $\|v\|_A^2 \ge \rho^2/2$. Therefore, we conclude that there exists $\rho > \sigma$ such that

$$J_{\lambda,n}(\theta u_*, v) \le 0, \tag{3.10}$$

for all $(\theta u_*, v) \in X_n$ so that $\|\theta u_*\|_{1,\Phi_1} + \|v\|_A^2= \rho^2$ and $\theta > 0$. By (3.9) and (3.10), we have $\max_{\partial \mathcal{M}^n_{u_*}} J_n = 0$, since $(0,0) \in \partial\mathcal{M}^n_{u_*}$, and the proof is complete in this case.

Now, if $a_2 \ge m_1^*$, from $(G_1)(ii)$ there is a positive constant C such that

$$\begin{aligned} J_{\lambda,n}(\theta u_*, v) &\le \int_\Omega \Phi_1(|\nabla(\theta u_*)|)dx - \int_\Omega \Phi_2(|\nabla v|)dx - \int_\Omega \Phi_{1*}(|\theta u_*|)dx - C\int_\Omega A(|v|)dx \\ &\le \xi^1_{\Phi_1}(\theta)\xi^1_{\Phi_1}(\|u_*\|_{1,\Phi}) - \xi^0_{\Phi_2}(\|v\|_{1,\Phi_2}) - \xi^0_{\Phi_{1*}}(\theta)\xi^0_{\Phi_{1*}}(\|u_*\|_{\Phi_{1*}}) - \xi^0_A(|v|_A), \end{aligned}$$

where

$$\xi^0_A(t) = \min\{t^{a_1}, t^{a_2}\}.$$

Observing that $\|(\theta u_*, v)\|^2= \theta^2 + \|v\|_A^2= \rho^2$ implies that

$$\theta^2 \ge \frac{\rho^2}{2}, \ \ \|v\|^2_{1,\Phi_2} \ge \frac{\rho^2}{4} \quad \textit{or} \quad |v|_A^2 \ge \frac{\rho^2}{4},$$

the same argument used in the former case implies that for $\rho > 0$ large enough

$$J_n(\theta u_*, v) \le 0, \tag{3.11}$$

for all $(\theta u_*, v) \in X_n$ so that $\|\theta u_*\|_{1,\Phi_1} + \|v\|_A^2= \rho^2$ and $\theta > 0$. Therefore, the lemma is proved. □

In order to prove the Theorem 3.1, we need to consider that $\Omega_0 \subset \Omega$ be an open set satisfying (R_4) and $u_0 \in W_0^{1,\Phi_1}(\Omega)$ such that

$$u_0 \geq 0, \quad u_0 \neq 0, \quad supp(u_0) \subset \Omega_0 \quad \text{and} \quad \|u_0\|_{W_0^{1,\Phi_1}(\Omega)} = 1. \tag{3.12}$$

Then, by Lemma 3.2, we can apply the linking theorem 2.3 to functional $J_{\lambda,n}$ using a point $z_n = (u_0, 0)$ and the sets

$$Y_n = \{0\} \times V_r^n, \quad Z = W_0^{1,\Phi_1}(\Omega) \times \{0\} \quad \text{and} \quad \mathcal{N}_n = \{(u,0) \in X_1 : \|u\|_{1,\Phi_1} = \sigma\}.$$

Then, there are sequences $(u_k, v_k) \subset X_n$ such that

$$J_{\lambda,n}(u_k, v_k) \to c_{\lambda,n} \quad \text{and} \quad J'_{\lambda,n}(u_k, v_k) \to 0 \ \text{ as } k \to \infty \tag{3.13}$$

where

$$b_n \leq c_{\lambda,n} := \inf_{\gamma \in \Gamma} \max_{u \in \mathcal{M}_{u_0}^n} J_{\lambda,n}(\gamma(u)), \tag{3.14}$$

$$\Gamma = \{\gamma \in C(\mathcal{M}_{u_0}^n, X_n) : \ \gamma|_{\partial \mathcal{M}_{u_0}^n} = Id_{\partial \mathcal{M}_{u_0}^n}\}.$$

Lemma 3.3. *The sequence (u_k, v_k) is bounded in X_n.*

Proof. From (3.13)

$$J_{\lambda,n}(u_k, v_k) - J'_n(u_k, v_k)(\frac{1}{\mu}u_k, \frac{1}{\nu}v_k) = c_{\lambda,n} + o_k(1)\|(u_k, v_k)\|$$

By (G_2), (R_3), (ϕ_3),

$$J_{\lambda,n}(u_k, v_k) - J'_{\lambda,n}(u_k, v_k)(\frac{1}{\mu}u_k, \frac{1}{\nu}v_k) \geq \left(1 - \frac{m_1}{\mu}\right)\xi^0_{\Phi_1}(\|u_k\|_{1,\Phi_1}) + \left(\frac{\ell_2}{\nu} - 1\right)\xi^0_{\Phi_2}(\|v_k\|_{1,\Phi_2}),$$

where $\xi^0_{\Phi_1}(t) = \min\{t^{\ell_1}, t^{m_1}\}$ and $\xi^0_{\Phi_2}(t) = \min\{t^{\ell_2}, t^{m_2}\}$. Since V_A^n is a finite dimensional space, the norms $\|\cdot\|_{1,\Phi_2}$ and $\|\cdot\|_A$ are equivalent, hence, from the above inequalities

$$c_{\lambda,n} + o_k(1)\|(u_k, v_k)\| \geq \left(1 - \frac{m_1}{\mu}\right)\xi^0_{\Phi_1}(\|u_k\|_{1,\Phi_1}) + \left(\frac{\ell_2}{\nu} - 1\right)\xi^0_{\Phi_2}(C\|v_k\|_A), \tag{3.15}$$

for some $C = C(n) > 0$. Suppose for contradiction that, up to a subsequence, $\|(u_k, v_k)\| \to +\infty$ as $k \to +\infty$. This way, we need to study the following situations:
(i) $\|u_k\|_{1,\Phi_1} \to +\infty$ and $\|v_k\|_A \to \infty$
(ii) $\|u_k\|_{1,\Phi_1} \to +\infty$ and $\|v_k\|_A$ is bounded
(iii) $\|v_k\|_A \to \infty$ and $\|u_k\|_{1,\Phi_1}$ is bounded

In the first case, the inequality (3.15) implies that

$$2c^2_{\lambda,n} + o_k(1)\|(u_k, v_k)\|^2 \geq \left(1 - \frac{m_1}{\mu}\right)^2 \|u_k\|^{2\ell_1}_{1,\Phi_1} + \left(\frac{\ell_2}{\nu} - 1\right)^2 \|v_k\|^{2\ell_2}_A.$$

for k large enough. Which is absurd, because $\ell_1 \geq 1$, $\ell_2 \geq 1$ and $o_k(1) \to 0$.

In case (ii), we have for k large enough

$$2c^2_{\lambda,n} + C_1 + o_k(1)\|u_k\|_{1,\Phi} \geq \left(1 - \frac{m_1}{\mu}\right)^2 \|u_k\|^{2\ell_1}_{1,\Phi_1}$$

an absurd. The last case is similar to the case (iii).

□

Corollary 3.4. *The following sequences*

$$\{\|u_k\|_{\Phi_{1*}}\}_{k\in\mathbb{N}}, \quad \left\{\int_\Omega \Phi_1(|\nabla u_k|)dx\right\} \quad \textit{and} \quad \left\{\int_\Omega \Phi_{1*}(|\nabla u_k|)dx\right\}$$

are bounded.

From Lemma 4.12, we may assume that there exists a subsequence of (u_k, v_k), still denoted by itself, and $(w_n, y_n) \in X_n$ such that

$$u_k \overset{*}{\rightharpoonup} w_n \text{ weakly in } W_0^{1,\Phi_1}(\Omega) \quad \text{and} \quad v_k \overset{*}{\rightharpoonup} y_n \text{ weakly in } V_n, \quad \text{as} \quad k\to\infty \tag{3.16}$$

$$u_k \longrightarrow w_n \quad \text{in } L^{\Phi_{1*}}(\Omega) \tag{3.17}$$

$$\frac{\partial u_k}{\partial x_i} \overset{*}{\rightharpoonup} \frac{\partial w_n}{\partial x_i} \quad \text{in } L^{\Phi_1}(\Omega),\ i\in\{1,\cdots,N\}. \tag{3.18}$$

$$u_k \longrightarrow w_n \quad \text{in } L^{\Phi_1}(\Omega), \tag{3.19}$$

and

$$u_k(x) \longrightarrow w_n(x) \quad \text{a.e. } \Omega. \tag{3.20}$$

In view of (3.16) and Corollary 3.4, for each $n\in\mathbb{N}$ we may assume that there exist nonnegative functions $\mu_n, \nu_n \in \mathcal{M}(\mathbb{R}^N)$, the space of Radon measures, such that

$$\Phi(|\nabla u_k|) \overset{*}{\rightharpoonup} \mu_n \quad \text{in } \mathcal{M}(\mathbb{R}^N) \ \text{ and } \ \Phi_*(|u_k|) \overset{*}{\rightharpoonup} \nu_n \quad \text{in } \mathcal{M}(\mathbb{R}^N) \ \text{ as } k\to\infty. \tag{3.21}$$

The result below is known as second concentration-compactness lemma of P. L. Lions. We would like to point out that also this lemma holds for nonreflexive Orlicz-Sobolev space. The proof of this fact can be seen in Proposition 4.3. in [11].

Lemma 3.5. (i) *For every $n\in\mathbb{N}$ and $\lambda>0$, there exist an at most countable set I_λ, a family $\{x_i\}_{i\in I_\lambda}$ of distinct points in $\mathbb{R}^N$ and a family $\{\nu_i\}_{i\in I_\lambda}$ of constant $\nu_i>0$ such that*

$$\nu_n = \Phi_{1*}(|w_n|) + \sum_{i\in I_\lambda} \nu_i \delta_{x_i}. \tag{3.22}$$

(ii) *In addition we have*

$$\mu_n \geq \Phi_1(|\nabla w_n|) + \sum_{i\in I_\lambda} \mu_i \delta_{x_i}, \tag{3.23}$$

for some $\mu_j>0$ satisfying

$$0<\nu_j \leq \max\{S_0^{\ell_1^*}\mu_i^{\ell_1^*/\ell_1}, S_0^{m_1^*}\mu_i^{m_1^*/\ell_1}, S_0^{\ell_1^*}\mu_i^{\ell_1^*/m_1}, S_0^{m_1^*}\mu_i^{m_1^*/m_1}\} \tag{3.24}$$

for all $i\in I_\lambda$, where δ_{x_i} is the Dirac measure of mass 1 concentrated at x_i.

Lemma 3.6. *The set $\{x_i\}_{i\in I_\lambda}$ in Lemma 3.5 is a finite set.*

Proof. Let an x_i be fixed. Take $\varphi\in C_0^\infty(\mathbb{R}^N)$ such that

$$0\leq\varphi\leq 1, \quad \varphi(x)=1 \text{ in } B_1(0) \quad \text{and} \quad \varphi(x)=0 \text{ in } \mathbb{R}^N\backslash B_2(0)$$

and put $\varphi_\varepsilon(x)=\varphi((x-x_i)/\varepsilon)$ for $\varepsilon>0$. It is clear that $\{\varphi_\varepsilon u_k\}_{k\in\mathbb{N}}$ is bounded in $W_0^{1,\Phi_1}(\Omega)$, thus $J'_{\lambda,n}(u_k,v_k)(\varphi_\varepsilon u_k,0)=o_k(1)$, that is,

$$\int_\Omega \phi_1(|\nabla u_k|)\nabla u_k\cdot\nabla(\varphi_\varepsilon u_k)dx = \int_\Omega \phi_{1*}(|u_k|)u_k(\varphi_\varepsilon u_k)dx + \lambda\int_\Omega R_u(x,u_k,v_k)(\varphi_\varepsilon u_k)dx + o_k(1)$$

Knowing that

$$\int_\Omega R_u(x,u_k,v_k)(\varphi_\varepsilon u_k)dx \longrightarrow \int_\Omega R_u(x,w_n,y_n)(\varphi_\varepsilon w_n)dx \text{ as } k\to\infty$$

We can conclude that

$$\int_\Omega \phi_1(|\nabla u_k|)\nabla u_k\cdot\nabla(\varphi_\varepsilon u_k)dx \le m_1^*\int_\Omega \Phi_{1*}(|u_k|)\varphi_\varepsilon dx+\lambda\int_\Omega R_u(x,w_n,y_n)(\varphi_\varepsilon w_n)dx+o_k(1) \tag{3.25}$$

By (ϕ_3),

$$\int_\Omega \phi_1(|\nabla u_k|)\nabla u_k\cdot\nabla(\varphi_\varepsilon u_k)dx \ge \ell_1\int_\Omega \Phi_1(|\nabla u_k|)\varphi_\varepsilon dx+\int_\Omega \phi_1(|\nabla u_k|)u_k\nabla u_k\nabla\varphi_\varepsilon dx.$$

Therefore,

$$\ell_1\int_\Omega \Phi_1(|\nabla u_k|)\varphi_\varepsilon dx+\int_\Omega \phi_1(|\nabla u_k|)u_k\nabla u_k\nabla\varphi_\varepsilon dx \le m_1^*\int_\Omega \Phi_{1*}(|u_k|)\varphi_\varepsilon dx+\lambda\int_\Omega R_u(x,w_n,y_n)(\varphi_\varepsilon w_n)dx+o_k(1) \tag{3.26}$$

Since the sequence $(\phi_1(|\nabla u_k|)\dfrac{\partial u_k}{\partial x_j})_{k\in\mathbb{N}}$ is limited to $L^{\tilde{\Phi}_1}(\Omega)$, there is $\omega_j\in L^{\tilde{\Phi}_1}(\Omega)$ such that

$$\phi_1(|\nabla u_k|)\frac{\partial u_k}{\partial x_j} \overset{*}{\rightharpoonup} \omega_j \text{ in } L^{\tilde{\Phi}_1}(\Omega),\ \ j\in\{1,\cdots,N\} \tag{3.27}$$

for some subsequence. Knowing that

$$u_k\frac{\partial\varphi_\varepsilon}{\partial x_j}\longrightarrow w_n\frac{\partial\varphi_\varepsilon}{\partial x_j} \text{ in } L^{\Phi_1}(\Omega),\ \ j\in\{1,\cdots,N\}$$

we conclude that

$$\int_\Omega u_k\phi_1(|\nabla u_k|)\frac{\partial u_k}{\partial x_j}\frac{\partial\varphi_\varepsilon}{\partial x_j}dx\longrightarrow\int_\Omega w_n\phi_1(|\nabla w_n|)\frac{\partial w_n}{\partial x_j}\frac{\partial\varphi_\varepsilon}{\partial x_j}dx,\ \ j\in\{1,\cdots,N\}. \tag{3.28}$$

Therefore, considering $\omega=(\omega_1,\cdots,\omega_N)$ we get

$$\int_\Omega \phi_1(|\nabla u_k|)u_k\nabla u_k\nabla\varphi_\varepsilon dx-\int_\Omega u_n\omega\nabla\varphi_\varepsilon dx=o_k(1) \tag{3.29}$$

From (3.26) and (3.29),

$$m_1^*\int_\Omega \Phi_{1*}(|u_k|)\varphi_\varepsilon dx+\lambda\int_\Omega R_u(x,u_n,v_n)(\varphi_\varepsilon u_n)dx\ge\ell_1\int_\Omega \Phi_1(|\nabla u_k|)\varphi_\varepsilon dx+\int_\Omega u_n\omega\nabla\varphi_\varepsilon dx+o_k(1) \tag{3.30}$$

By (3.21), taking the limit of $k\to+\infty$, we get

$$m_1^*\int_\Omega \varphi_\varepsilon d\nu_n+\lambda\int_\Omega R_u(x,u_n,v_n)\varphi_\varepsilon u_n dx\ge\ell_1\int_\Omega \varphi_\varepsilon d\mu_n+\int_\Omega u_n w\nabla\varphi_\varepsilon dx+o_k(1). \tag{3.31}$$

On the other hand, given $v\in W_0^{1,\Phi_1}(\Omega)$, it follows from (3.13) that

$$o_k(1)=\int_\Omega \phi_1(|\nabla u_k|)\nabla u_k\nabla v dx-\int_\Omega \phi_{1*}(|u_k|)u_k v dx-\lambda\int_\Omega R_u(x,u_k,v_k)v dx,$$

since the sequence $(\phi_{1*}(|u_k|)u_k)$ is bounded in $L^{\tilde{\Phi}_{1*}}(\Omega)$, there is $\eta_n\in L^{\tilde{\Phi}_{1*}}(\Omega)$ such that

$$\phi_{1*}(|u_k|)u_k\rightharpoonup\eta_n \text{ in } L^{\tilde{\Phi}_{1*}}(\Omega) \text{ as } k\to\infty \tag{3.32}$$

so, from (3.27) and (3.32),

$$\int_\Omega \omega\nabla v dx-\int_\Omega \eta_n v dx-\lambda\int_\Omega R_u(x,u_n,v_n)v dx=0,\ \ \forall v\in W_0^{1,\Phi_1}(\Omega).$$

In particular, for $v = u_n\varphi_\varepsilon$, we have

$$\int_\Omega u_n\omega\nabla\varphi_\varepsilon dx = \int_\Omega \eta_n u_n\varphi_\varepsilon dx + \lambda\int_\Omega R_u(x,u_n,v_n)u_n\varphi_\varepsilon dx - \int_\Omega \varphi_\varepsilon\omega\nabla u_n dx, \tag{3.33}$$

Therefore,

$$\lim_{\varepsilon\to 0}\int_\Omega u_n(\omega\nabla\varphi_\varepsilon)dx = 0 \tag{3.34}$$

It follows from (3.31) and (3.34) that

$$\ell_1\mu_i \le m_1^*\nu_i, \quad i\in I_\lambda, \tag{3.35}$$

and by Lemma 3.5,

$$\ell_1\mu_i \le m_1^* S_0^\beta \mu_i^\alpha, \tag{3.36}$$

for some α and β verifying

$$1<\alpha\in\left\{\frac{\ell_1^*}{\ell_1},\frac{m_1^*}{\ell_1},\frac{\ell_1^*}{m_1},\frac{m_1^*}{m_1}\right\} \quad\text{and}\quad \beta\in\{\ell_1^*,m_1^*\}. \tag{3.37}$$

Thereby,

$$0<\frac{\ell_1}{m_1^*S_0^\beta}\le\mu_i^{\alpha-1}, \quad i\in I_\lambda,$$

showing that

$$\mu_i \ge \left(\frac{\ell_1}{m_1^*S_0^\beta}\right)^{\frac{1}{\alpha-1}}, \quad i\in I_\lambda. \tag{3.38}$$

By (3.35) and (3.38)

$$\nu_i \ge \left(\frac{\ell_1}{m_1^*}\right)^{\frac{\alpha}{\alpha-1}} S_0^{-\frac{\beta}{\alpha-1}}, \quad i\in I_\lambda.$$

Since ν_n is a finite measure, the last inequality yields I_λ is finite.

□

In fact we will see that the set $\{x_i\}_{i\in I_\lambda}$ is an empty set for $\lambda>0$ large enough. For this, we will establishes an important estimate from above of the mountain level of functional $J_{\lambda,n}$.

Lemma 3.7. *Let $n\in\mathbb{N}$ be arbitrary and consider u_0 given in (3.12). Then, there is $\lambda_0>0$ such that if $\lambda>\lambda_0$, we have that*

$$c_{\lambda,n} < \left(1-\frac{m_1}{\mu}\right)\min\left\{\left(\frac{\ell_1}{m_1^*S_0^\beta}\right)^{\frac{1}{\alpha-1}} : \alpha\in\left\{\frac{\ell_1^*}{\ell_1},\frac{m_1^*}{\ell_1},\frac{\ell_1^*}{m_1},\frac{m_1^*}{m_1}\right\} \ \ \textit{and}\ \ \beta\in\{\ell_1^*,m_1^*\}\right\}, \tag{3.39}$$

where $c_{\lambda,n}$ is given in (3.14).

Proof. By (R_4), given $\theta\ge 0$ and $v\in V_A^n$, we have

$$\begin{aligned} J_{\lambda,n}(\theta u_0,v) &\le \xi^1_{\Phi_1}(\theta)\xi^1_{\Phi_1}(\|u_0\|_{1,\Phi_1}) - \xi^0_{\Phi_{1*}}(\theta)\xi^1_{\Phi_1}(\|u_0\|_{\Phi_1}) - \lambda\int_{\Omega_0}R(x,\theta u_0,v)dx \\ &\le \xi^1_{\Phi_1}(\theta)\xi^1_{\Phi_1}(\|u_0\|_{1,\Phi_1}) - \xi^0_{\Phi_{1*}}(\theta)\xi^1_{\Phi_1}(\|u_0\|_{\Phi_1}) - \lambda\omega\int_{\Omega_0}|\theta u_0|^s dx \end{aligned}$$

This inequality implies that

$$0<b_n\le\inf_{\gamma\in\Gamma}\max_{u\in\mathcal{M}^n_{u_0}} J_{\lambda,n}(\gamma(u)) \le \max_{\mathcal{M}^n_{u_0}} J_{\lambda,n} \le \max_{\theta\ge 0}\mathcal{V}_\lambda(\theta) \tag{3.40}$$

where

$$\mathcal{V}_\lambda(\theta) = \xi^1_{\Phi_1}(\theta)\xi^1_{\Phi_1}(\|u_0\|_{1,\Phi_1}) - \xi^0_{\Phi_{1*}}(\theta)\xi^1_{\Phi_1}(\|u_0\|_{\Phi_1}) - \lambda\omega \int_{\Omega_0} |\theta u_0|^s dx. \tag{3.41}$$

In what follows, we denote by $\theta_\lambda > 0$ the real number verifying

$$\max_{\theta \geq 0} \mathcal{V}_\lambda(\theta) = \mathcal{V}_\lambda(\theta_\lambda) \tag{3.42}$$

Let us see that $\mathcal{V}_\lambda(\theta_\lambda) \to 0$ as $\lambda \to \infty$. For that, consider (λ_m) a sequence verifying

$$\lambda_m \longrightarrow \infty \quad \text{as} \quad m \to \infty.$$

We claim that (θ_{λ_m}) is a bounded sequence. Indeed, assuming by contradiction that (θ_{λ_m}) is not bounded, we have that for a subsequence, still denote by itself,

$$\theta_{\lambda_m} \to \infty \quad \text{as} \quad m \to \infty.$$

Por (3.41), (3.40) and (3.42),

$$0 < \max_{\theta \geq 0} \mathcal{V}_{\lambda_m}(\theta) = \mathcal{V}_{\lambda_m}(\theta_{\lambda_m}) \leq \theta^{m_1}_{\lambda_m} \xi^1_{\Phi_1}(\|u_0\|_{1,\Phi_1}) - \theta^{\ell_1^*}_{\lambda_m} \xi^1_{\Phi_1}(\|u_0\|_{\Phi_1}) \to -\infty \quad \text{as} \quad m \to \infty,$$

which is an absurd, and so, (θ_{λ_m}) is bounded. We claim that $\theta_{\lambda_m} \to 0$ as $m \to \infty$. If the above limit does not hold, we can assume by contradiction, that for some subsequence, still denote by (θ_{λ_m}), there is $k_0 > 0$ such that

$$\theta_{\lambda_m} > k_0 > 0, \quad \forall n \in \mathbb{N}$$

Then, by (3.41) and (3.42),

$$0 < \max_{\theta \geq 0} \mathcal{V}_{\lambda_m}(\theta) = \mathcal{V}_{\lambda_m}(\theta_{\lambda_m}) \leq \xi^1_{\Phi_1}(\theta_{\lambda_m})\xi^1_{\Phi_1}(\|u_0\|_{1,\Phi_1}) - \xi^0_{\Phi_{1*}}(\theta_{\lambda_m})\xi^1_{\Phi_1}(\|u_0\|_{\Phi_1}) - \lambda_m\omega \int_{\Omega_0} |\theta_{\lambda_m} u_0|^s dx,$$

thus

$$0 < \max_{\theta \geq 0} \mathcal{V}_{\lambda_m}(\theta) \to -\infty \quad \text{as} \quad m \to \infty,$$

which is a contradiction. Hence,

$$t_{\lambda_m} \to 0 \quad \text{as} \quad m \to \infty,$$

which leads to

$$\max_{\theta \geq 0} \mathcal{V}_{\lambda_m}(\theta) = \mathcal{V}_{\lambda_m}(\theta_{\lambda_m}) \to 0 \quad \text{as} \quad m \to \infty,$$

and by (3.40) it follows that

$$c_{\lambda_m,n} \to 0 \quad \text{as} \quad m \to \infty.$$

□

Lemma 3.8. *For every $n \in \mathbb{N}$ and $\lambda > \lambda_0$ the set I_λ is empty, where λ_0 was given in Lemma 3.7.*

Proof. Let $(u_k, u_k) \subset X_n$ the $(PS)_{c_{\lambda,n}}$ sequence obtained in (3.13). By the assumptions (G_2), (R_3), (ϕ_3), we have that

$$c_{\lambda,n} + o_k(1) \geq J_n(u_k, v_k) - J'_{\lambda,n}(u_k, v_k)(\frac{1}{\mu}u_k, \frac{1}{\nu}v_k) \geq \left(1 - \frac{m_1}{\mu}\right) \int_\Omega \Phi_1(|\nabla u_k|)dx.$$

Fixing a function $\varphi \in C_0^\infty(\mathbb{R}^N)$ with $\varphi(x) = 1$ on $\overline{\Omega}$, we derive that

$$c_{\lambda,n} + o_k(1) \geq \left(1 - \frac{m_1}{\mu}\right) \int_{\mathbb{R}^N} \Phi_1(|\nabla u_k|)\varphi dx.$$

Taking the limit of $k \to +\infty$, we get

$$c_{\lambda,n} \geq \left(1 - \frac{m_1}{\mu}\right) \int_{\mathbb{R}^N} \varphi d\mu_n \geq \left(1 - \frac{m_1}{\mu}\right) \mu_n(\overline{\Omega}).$$

Supposing that I_λ is not empty, there is $i \in I_\lambda$, and so,

$$c_{\lambda,n} \geq \left(1 - \frac{m_1}{\mu}\right) \mu_i.$$

In (3.38) we show that

$$\mu_i \geq \Big(\frac{\ell_1}{m_1^* S_0^\beta}\Big)^{\frac{1}{\alpha-1}}, \ i \in I_\lambda$$

where α is given in (3.36) and (3.37). Therefore, we can conclude that

$$c_{\lambda,n} \geq \left(1 - \frac{m_1}{\mu}\right) \min \left\{ \left(\frac{\ell_1}{m_1^* S_0^\beta}\right)^{\frac{1}{\alpha-1}} : \alpha \in \left\{\frac{\ell_1^*}{\ell_1}, \frac{m_1^*}{\ell_1}, \frac{\ell_1^*}{m_1}, \frac{m_1^*}{m_1}\right\} \text{ and } \beta \in \{\ell_1^*, m_1^*\} \right\}.$$

Then, if $\lambda \geq \lambda_0$, the last inequality together with Lemma 3.7 yields $I_\lambda = \emptyset$ is empty.

□

Lemma 3.9. *For $\lambda \geq \lambda_0$, the sequence (u_k) is strongly convergent for its weak limit w_n in $L^{\Phi_{1*}}(\Omega)$ as $k \to \infty$.*

Proof. Let $\varphi \in C^\infty(\mathbb{R}^N)$ be a function verifying $\varphi(x) = 1$, for $x \in \Omega$. In this case,

$$\lim_{k\to\infty} \int_\Omega \Phi_{1*}(u_k)dx = \lim_{k\to\infty} \int_{\mathbb{R}^N} \Phi_{1*}(u_k)\varphi dx = \int_{\mathbb{R}^N} \varphi d\nu_n$$

The Lemma 3.5(i) combined with Lemma 3.8 gives

$$\lim_{k\to\infty} \int_\Omega \Phi_{1*}(u_k)dx = \int_{\mathbb{R}^N} \Phi_{1*}(u_n)\varphi dx = \int_\Omega \Phi_{1*}(u_n)dx.$$

Since Φ_{1*} is a convex function, it follows from a result found in [17] that

$$\lim_{k\to\infty} \int_\Omega \{\Phi_{1*}(|u_k|) - \Phi_{1*}(|u_k - u_n|) - \Phi_{1*}(|u_n|)\}dx = 0.$$

Then,

$$\lim_{k\to\infty} \int_\Omega \Phi_{1*}(|u_k - u_n|)dx = 0,$$

we can conclude that (u_k) converges strongly for u_n in $L^{\Phi_{1*}}(\Omega)$.

□

Lemma 3.10. *Consider $\lambda > \lambda_0$ and $(u_k) \subset W_0^{1,\Phi_1}(\Omega)$ the sequence obtained in (3.13). Then, for some subsequence, still denoted by itself,*

$$u_k \to w_n \ \text{ in } W_0^{1,\Phi_1}(\Omega) \ \text{ as } \ k \to \infty.$$

Proof. Since (u_k) is a bounded sequence in $W_0^{1,\Phi_1}(\Omega)$, then

$$o_k(1) = \int_\Omega \phi_1(|\nabla u_k|)\nabla u_k \nabla(v - u_k)dx - \int_\Omega \phi_{1*}(|u_k|)u_k(v - u_k)dx - \lambda \int_\Omega R_u(x, u_k, v_k)(v - u_k)dx.$$

Given $v \in W_0^{1,\Phi_1}(\Omega)$, by the convexity of Φ_1 it follows that

$$\Phi_1(|\nabla v|) - \Phi_1(|\nabla u_k|) \geq \phi_1(|\nabla u_k|)\nabla u_k \nabla(v - u_k),$$

thus,

$$\int_\Omega \Phi_1(|\nabla v|)dx - \int_\Omega \Phi_1(|\nabla u_k|)dx \geq \int_\Omega \phi_{1*}(|u_k|)u_k(v-u_k)dx - \lambda \int_\Omega R_u(x,u_k,v_k)(v-u_k)dx + o_k(1). \tag{3.43}$$

Through the sequence (u_k) in $W_0^{1,\Phi_1}(\Omega)$ together with the limits

$$u_k(x) \longrightarrow w_n \quad \text{a.e. in } \Omega \quad \text{and} \quad \frac{\partial u_k}{\partial x_i} \rightharpoonup \frac{\partial w_n}{\partial x_i} \quad \text{in } L^1(\Omega),$$

we can apply [18, Theorem 2.1, Chapter 8] to get

$$\liminf_{k\to\infty} \int_\Omega \Phi_1(|\nabla u_k|)dx \geq \int_\Omega \Phi_1(|\nabla w_n|)dx,$$

Furthermore, since (u_k) strongly converges to u_n in $L^{\Phi_{1*}}(\Omega)$,

$$\int_\Omega \phi_{1*}(|u_k|)u_k(v-u_k)dx \to \int_\Omega \phi_{1*}(|w_n|)w_n(v-w_n)dx, \quad \text{as} \quad k\to\infty.$$

Therefore, it follows from (3.43) that

$$\int_\Omega \Phi_1(|\nabla v|)dx - \int_\Omega \Phi_1(|\nabla w_n|)dx \geq \int_\Omega \phi_{1*}(|w_n|)w_n(v-w_n)dx + \lambda \int_\Omega R_u(x,w_n,y_n)(v-w_n)dx.$$

By arbitrariness v we can conclude that

$$\int_\Omega \phi_1(|\nabla w_n|)\nabla w_n \nabla(w_n-u_k)dx = \int_\Omega \phi_{1*}(|w_n|)w_n(w_n-u_k)dx + \lambda \int_\Omega R_u(x,w_n,y_n)(w_n-u_k)dx,$$

implying that

$$\int_\Omega \phi_1(|\nabla w_n|)\nabla w_n \nabla(w_n-u_k)dx = o_k(1). \tag{3.44}$$

On the other hand, since (u_k,v_k) is a sequence $(PS)_{c_{\lambda,n}}$,

$$\begin{aligned} o_k(1) = J'_{\lambda,n}(u_k,v_k)(w_n-u_k,0) = &\int_\Omega \phi_1(|\nabla u_k|)\nabla u_k \nabla(w_n-u_k)dx - \int_\Omega \phi_{1*}(|u_k|)u_k(w_n-u_k)dx \\ &- \lambda \int_\Omega R_u(x,u_k,v_k)(w_n-u_k)dx, \end{aligned}$$

Therefore,

$$\int_\Omega \phi_1(|\nabla u_k|)\nabla u_k \nabla(w_n-u_k)dx = o_k(1). \tag{3.45}$$

From (3.44) and (3.45),

$$\int_\Omega \big(\phi_1(|\nabla u_k|)\nabla u_k - \phi_1(|\nabla w_n|)\nabla w_n\big)\big(\nabla u_k - \nabla w_n\big)dx \longrightarrow 0 \quad \text{as } k\to\infty.$$

Now, applying a result due to Dal Maso and Murat [8], it follows that

$$\nabla u_k(x) \longrightarrow \nabla w_n(x) \quad \text{a.e. in } \Omega \quad \text{as } k\to\infty. \tag{3.46}$$

Since (u_k) is bounded in $W_0^{1,\Phi_1}(\Omega)$ and $\Phi_1 \in (\Delta_2)$, then the sequence $(\phi_1(|\nabla u_k|)\dfrac{\partial u_k}{\partial x_j})_{k\in\mathbb{N}}$ is bounded by $L^{\tilde{\Phi}_1}(\Omega)$, for each $j\in\{1,\cdots,N\}$. Furthermore, by (3.46), it follows that

$$\phi_1(|\nabla u_k(x)|)\frac{\partial u_k(x)}{\partial x_j} \to \phi_1(|\nabla w_n(x)|)\frac{\partial w_n(x)}{\partial x_j} \quad \text{a.e. in } \Omega \quad \text{as} \quad k\to\infty.$$

Thus, by Lemma 2.5 in [2],

$$\int_\Omega \phi_1(|\nabla u_k|)\nabla u_k \nabla v dx \to \int_\Omega \phi_1(|\nabla u_n|)\nabla u_n \nabla v dx, \ \ v \in C_0^\infty(\Omega) \ \text{ as } \ k \to \infty. \tag{3.47}$$

Still due to the boundedness of the sequence $(\phi_1(|\nabla u_k|)\dfrac{\partial u_k}{\partial x_j})_{k\in\mathbb{N}}$ in $L^{\tilde{\Phi}_1}(\Omega)$, there will be $v_j \in L^{\tilde{\Phi}_1}(\Omega)$ such that

$$\phi_1(|\nabla u_k|)\frac{\partial u_k}{\partial x_j} \overset{*}{\longrightarrow} v_j \ \text{ in } L^{\tilde{\Phi}_1}(\Omega) \ \text{ as } \ k \to \infty.$$

i.e,

$$\int_\Omega \phi_1(|\nabla u_k|)\frac{\partial u_k}{\partial x_j} w dx \to \int_\Omega v_j w dx, \ \forall w \in E^{\Phi_1}(\Omega) = L^{\Phi_1}(\Omega) \ \text{ as } \ k \to \infty. \tag{3.48}$$

By (3.47) and (3.48), it follows that $v_j = \phi_1(|\nabla u_n|)\dfrac{\partial u_n}{\partial x_j}$, for each $j \in \{1, \cdots, N\}$. Still from (3.48),

$$\int_\Omega \phi_1(|\nabla u_k|)\nabla u_k \nabla w dx \to \int_\Omega \phi_1(|\nabla u_n|)\nabla u_n \nabla w dx, \ \ \forall w \in W_0^{1,\Phi_1}(\Omega) \ \text{ as } \ k \to \infty.$$

We know from (3.45) that

$$\int_\Omega \phi_1(|\nabla u_k|)\nabla u_k \nabla (u_n - u_k) dx = o_k(1),$$

then

$$\int_\Omega \phi_1(|\nabla u_k|)|\nabla u_k|^2 dx \to \int_\Omega \phi_1(|\nabla u_n|)|\nabla u_n|^2 dx \ \text{ as } \ k \to \infty.$$

Given this, we can conclude that

$$\phi_1(|\nabla u_k|)|\nabla u_k|^2 \to \phi_1(|\nabla u_n|)|\nabla u_n|^2 \ \text{ in } L^1(\Omega) \ \text{ as } \ k \to \infty.$$

By (ϕ_3) together with the Δ_2-condition, it follows that

$$u_k \to u_n \quad \text{in } W_0^{1,\Phi_1}(\Omega) \ \text{ as } \ k \to \infty.$$

This finishes the proof.

□

Lemma 3.11. *For $\lambda > \lambda_0$, the sequence (w_n, y_n) is bounded in X. Moreover*

$$J_{\lambda,n}(w_n, y_n) = c_{\lambda,n} \quad \textit{and} \quad J'_{\lambda,n}(w_n, y_n) = 0 \ \textit{in} \ X_n^*. \tag{3.49}$$

Proof. Since V_A^n is a finite dimensional space, (v_k) converges strongly to (y_n) in V_A^n. Therefore,

$$(u_k, v_k) \longrightarrow (w_n, y_n) \quad \text{in } X_n \ \text{ as } \ k \to \infty.$$

which implies

$$J_{\lambda,n}(w_n, y_n) = c_{\lambda,n} \in [b_n, d_n] \quad \text{and} \quad J'_{\lambda,n}(w_n, y_n) = 0 \quad \text{in } X_n^*.$$

In a first moment, let us assume that $a_2 < \ell_2^*$. By hypothesis $m_2 < a_1$, then $W_0^{1,\Phi_2}(\Omega)$ is continuously embedded in $L^A(\Omega)$, thus, there will be $C > 0$ such

$$\|v\|_A \leq C\|v\|_{1,\Phi_2}, \ \ \forall v \in V_A^n \tag{3.50}$$

By $(G_2)(ii)$, (R_3) and (3.50),

$$\begin{aligned} c_{\lambda,n} =& J_{\lambda,n}(w_n, y_n) - J'_{\lambda,n}(w_n, y_n)(\frac{1}{\mu}w_n, \frac{1}{\nu}y_n) \\ \geq& \left(1 - \frac{m_1}{\mu}\right)\int_\Omega \Phi_1(|\nabla w_n|)dx + \left(\frac{\ell_2}{\nu} - 1\right)\int_\Omega \Phi_2(|\nabla y_n|)dx + \left(\frac{\ell_1^*}{\mu} - 1\right)\int_\Omega \Phi_{1*}(|\nabla w_n|)dx \\ \geq& \left(1 - \frac{m_1}{\mu}\right)\xi^0_\Phi(\|w_n\|_{1,\Phi_1}) + \left(\frac{\ell_2}{\nu} - 1\right)\xi^0_{\Phi_2}(\frac{1}{C}\|y_n\|_A). \end{aligned} \tag{3.51}$$

It follows from the inequalities (3.39) and (3.51) that (w_n, y_n) is bounded in X.

Now, let us assume that $a_2 \geq \ell_2^*$. By (R_3), (G_2), (3.50) and from items $(i) - (iii)$ of (G_1), it follows that

$$\begin{aligned} c_{\lambda,n} =& J_{\lambda,n}(u_n, v_n) - J'_{\lambda,n}(u_n, v_n)(\frac{1}{\mu}u_n, \frac{1}{\nu}v_n) \\ \geq& \left(1 - \frac{m_1}{\mu}\right)\int_\Omega \Phi_1(|\nabla u_n|)dx + \left(\frac{\ell_2}{\nu} - 1\right)\int_\Omega \Phi_2(|\nabla v_n|)dx + \frac{C}{\nu}\int_\Omega a(|v_n|)|v_n|^2dx - Ca_1\int_\Omega A(v_n)dx \\ \geq& \left(1 - \frac{m_1}{\mu}\right)\int_\Omega \Phi_1(|\nabla u_n|)dx + \left(\frac{\ell_2}{\nu} - 1\right)\int_\Omega \Phi_2(|\nabla v_n|)dx + \left(\frac{a_1C}{\nu} - a_1C\right)\int_\Omega A(|v_n|)dx \\ \geq& \left(1 - \frac{m_1}{\mu}\right)\xi^0_{\Phi_1}(\|u_n\|_{1,\Phi_1}) + \left(\frac{\ell_2}{\nu} - 1\right)\xi^0_{\Phi_2}(\|v_n\|_{1,\Phi_2}) + \left(\frac{a_1C}{\nu} - a_1C\right)\xi^0_A(|v_n|_A). \end{aligned}$$

From the above inequality together with (3.39), we can conclude that (w_n, y_n) is bounded by X.

□

3.1. **Proof of Theorem 3.1.** The proof of Theorem 3.1 will be carried out in three lemmas. We start observing that since (w_n, y_n) is bounded, there is no loss of generality in assuming that

$$(w_n, y_n) \overset{*}{\rightharpoonup} (u, v) \quad \text{in } X \quad \text{as} \quad n \to \infty. \tag{3.52}$$

The same arguments used in the proof of Lemma 3.10 can be repeated to show that

$$u_n \to u \quad \text{in } W_0^{1,\Phi_1}(\Omega) \quad \text{as} \quad n \to \infty. \tag{3.53}$$

By the limit (3.52), it follows that

$$y_n \overset{*}{\rightharpoonup} v \quad \text{in } L^A(\Omega) \tag{3.54}$$

and

$$y_n \overset{*}{\rightharpoonup} v \quad \text{in } W_0^{1,\Phi_2}(\Omega) \tag{3.55}$$

Lemma 3.12. *For $\lambda > \lambda_0$, the sequence (y_n) verifies the following limit $y_n \to v$ in $L^A(\Omega)$.*

Proof. From (3.5), there is $(\xi_k) \subset V_A$ such that

$$\xi_k \to v \quad \text{in} \quad V_A \tag{3.56}$$

and

$$\xi_k = \sum_{i=1}^{j(k)} \alpha_i e_i \in V_A^{j(k)}$$

where $j(k) \in \mathbb{N}$ for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, it follows that

$$V_A^{j(k)} \subset V_A^n \quad \text{for all} \quad n \geq n_0$$

for some $n_0 \geq j(k)$.

If $a_2 \geq \ell_2^*$, from (G_1), we have that there is $C > 0$ such that

$$a_1 C \int_\Omega A(|y_n - \xi_k|)dx \leq C \int_\Omega a(|y_n - \xi_k|)|y_n - \xi_k|^2 dx \leq \int_\Omega (g(y_n) - g(\xi_k))(y_n - \xi_k)dx. \tag{3.57}$$

Since $J'_{\lambda,n}(u_n, v_n) = 0$ in X_n^*, we derive that

$$\begin{aligned} \int_\Omega (g(y_n) - g(\xi_k))(y_n - \xi_k)dx &= \int_\Omega \phi_2(|\nabla y_n|)\nabla y_n(\nabla \xi_k - \nabla y_n)dx - \lambda \int_\Omega R_v(x, w_n, y_n)y_n dx \\ &\quad + \lambda \int_\Omega R_v(x, w_n, y_n)\xi_k dx - \int_\Omega g(\xi_k)(y_n - \xi_k)dx \end{aligned} \tag{3.58}$$

Due to the convexity of Φ_2, we have

$$\Phi_2(|\nabla \xi_k|) - \Phi_2(|\nabla y_n|) \geq \phi_2(|\nabla y_n|)\nabla y_n \nabla(\xi_k - y_n), \quad n \in \mathbb{N} \tag{3.59}$$

It follows from the above inequalities that

$$\begin{aligned} a_1 C \int_\Omega A(|y_n - \xi_k|)dx &\leq \int_\Omega \Phi_2(|\nabla \xi_k|)dx - \int_\Omega \Phi_2(|\nabla w_n|)dx - \lambda \int_\Omega R_v(x, w_n, y_n)y_n dx \\ &\quad + \lambda \int_\Omega R_v(x, w_n, y_n)\xi_k dx - \int_\Omega g(\xi_k)(y_n - \xi_k)dx \end{aligned} \tag{3.60}$$

Knowing that

$$y_n(x) \longrightarrow v(x) \text{ a.e. in } \Omega \quad \text{and} \quad \frac{\partial y_n}{\partial x_i} \rightharpoonup \frac{\partial v}{\partial x_i} \text{ in } L^1(\Omega),$$

we can apply [18, Theorem 2.1, Chapter 8] to get

$$\liminf_{n\to\infty} \int_\Omega \Phi_2(|\nabla v_n|)dx \geq \int_\Omega \Phi_2(|\nabla v|)dx. \tag{3.61}$$

Taking as limit $n \to \infty$, it follows that

$$\begin{aligned} \limsup_{n\to\infty} \left(a_1 C \int_\Omega A(|y_n - \xi_k|)dx\right) &\leq \int_\Omega \Phi_2(|\nabla \xi_k|)dx - \int_\Omega \Phi_2(|\nabla v|)dx - \int_\Omega R_v(x, u, v)v dx \\ &\quad + \int_\Omega R_v(x, u, v)\xi_k dx - \int_\Omega g(\xi_k)(v - \xi_k)dx. \end{aligned} \tag{3.62}$$

By the limit (3.56), given $\delta > 0$ there is $k_0 \in \mathbb{N}$ such that

$$\frac{1}{a_1 C}\left[\int_\Omega \Phi_2(|\nabla \xi_k|)dx - \int_\Omega \Phi_2(|\nabla u|)dx - \int_\Omega R_v(x, u, v)v dx + \int_\Omega R_v(x, u, v)\xi_k dx - \int_\Omega g(\xi_k)(v - \xi_k)dx\right] < \frac{\delta}{2},$$

for each $k \geq k_0$. Hence,

$$\limsup_{n\to\infty} \int_\Omega A(|y_n - \xi_k|)dx \leq \frac{\delta}{2}, \quad \text{for all } k \geq k_0. \tag{3.63}$$

Given $0 < \varepsilon < 4$, for δ sufficiently small, it follows that

$$\limsup_{n\to\infty} \int_\Omega A(|y_n - \xi_k|)dx \leq \frac{\varepsilon}{4}, \quad \text{for all } k \geq k_0. \tag{3.64}$$

Fixing $k \geq k_0$ sufficiently large such that

$$|\xi_k - v|_A < \left(\frac{\varepsilon}{4}\right)^{1/a_1} \tag{3.65}$$

follows from $(G_1)(i)$ that

$$\int_\Omega A(|y_n - v|)\,dx \leq C \int_\Omega A(|y_n - \xi_k|)\,dx + C|\xi_k - v|_A^{a_1} \leq C \int_\Omega A(|y_n - \xi_k|)\,dx + \frac{\varepsilon C}{4}, \tag{3.66}$$

for some constant $C > 0$ that does not depend on n and k. By (3.64) and (3.66), we have

$$\limsup_{n\to\infty} \int_\Omega A\left(|y_n - v|\right) dx < \frac{\varepsilon C}{2}$$

and by the arbitrariness of $\varepsilon > 0$,

$$\lim_{n\to\infty} \int_\Omega A\left(|y_n - v|\right) dx = 0.$$

Therefore,

$$y_n \to v \quad \text{in } L^A(\Omega).$$

Now, let us consider $a_2 < \ell_2^*$, then A increases essentially more slowly than Φ_{2*} near infinity. In this case, the space $W_0^{1,\Phi_2}(\Omega)$ is compactly embedded in $L^A(\Omega)$, therefore, the desired limit follows directly from that compact embedding.

□

The following lemma is made using similar arguments to those given in Lemma 3.10. Therefore, we will omit its proof.

Lemma 3.13. *For $\lambda > \lambda_0$, the sequence (y_n) verifies the following limit $y_n \to v$ in $W_0^{1,\Phi_2}(\Omega)$.*

From the above lemmas, we can conclude that

$$y_n \to v \;\text{ in }\; V_A. \tag{3.67}$$

In view of the above facts, it is possible to obtain the following result.

Lemma 3.14. *For $\lambda > \lambda_0$, the pair (u, v) satisfies $J'(u, v) = 0$ in X and $J(u, v) \neq 0$.*

Proof. Fixing $k, n \in \mathbb{N}$ with $n \geq k$, we have $X_k \subset X_n$. Thus, for $(\varphi_1, \varphi_2) \in X_k$, it follows that

$$J'_{\lambda,n}(w_n, y_n)(\varphi_1, \varphi_2) = 0, \quad \forall n \geq k,$$

because, by Lemma 3.11, $J'_{\lambda,n}(w_n, y_n) = 0$. Combining (3.67) with (3.53) we get

$$J'_\lambda(u, v)(\varphi_1, \varphi_2) = 0, \qquad \text{for all } (\varphi_1, \varphi_2) \in X_k. \tag{3.68}$$

We claim that

$$J'_\lambda(u, v)(\varphi_1, \varphi_2) = 0, \qquad \text{for all } (\varphi_1, \varphi_2) \in X. \tag{3.69}$$

In fact, we start observing that for all $\varphi_1 \in W_0^{1,\Phi_1}(\Omega)$, the pair $(\varphi_1, 0) \in X_k$ for all k. Hence, $J'_\lambda(u, v)(\varphi_1, 0) = 0$. On the other hand, for $\varphi_2 \in V_A$, there exists $\chi_n \in V_A^{k(n)}$ such that

$$\lim_{n\to\infty} \chi_n = \varphi_2, \qquad \text{in } V_A.$$

From (3.68),

$$J'_\lambda(u, v)(0, \chi_n) = 0, \qquad \text{for all } n \in \mathbb{N}. \tag{3.70}$$

which implies after passage to the limit as $n \to \infty$ that

$$J'_\lambda(u, v)(0, \varphi_2) = 0, \qquad \text{for all } \varphi_2 \in V_A. \tag{3.71}$$

Thus, (3.69) is proved. Using the fact that $(w_n, y_n) \to (u, v)$ in X and that $J'_\lambda(w_n, y_n) \geq b_n \geq C\sigma^{\ell_2} > 0$, for all $n \in \mathbb{N}$, for some constant $C > 0$ which does not depend on n, we have that $J'_\lambda(u, v) \geq C\sigma^{\ell_2} > 0$, from where it follows that (u, v) is a nontrivial solution for (S_1), and the proof is complete.

□

4. The N-functions Φ_1 and Φ_2 may not verify the Δ_2-condition.

In this section, we study the existence of solutions for the following class of quasilinear systems in Orlicz-Sobolev spaces:

$$(S_2)\qquad \begin{cases} -div(\phi_1(|\nabla u|)\nabla u) = R_u(x,u,v) \text{ in } \Omega \\ -div(\phi_2(|\nabla v|)\nabla v) = -R_v(x,u,v) \text{ in } \Omega \\ u = v = 0 \text{ on } \partial\Omega \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N(N \geq 2)$ with smooth boundary $\partial\Omega$, and $\phi_i(i=1,2)$: $(0,\infty) \to (0,\infty)$ are two functions which satisfy:
(ϕ_1') $\phi_i \in C^1(0,+\infty)$, $t \mapsto t\phi_i(t)$ are stricly increasing and $t \mapsto t^2\phi_i(t)$ is convex in $\mathbb{R}$.
(ϕ_2') $t\phi_i(t) \to 0$ as $t \to 0$ and $t\phi_i(t) \to +\infty$ as $t \to +\infty$
(ϕ_3') $1 < \ell_i \leq \dfrac{t^2\phi_i(t)}{\Phi_i(t)}$, where $\Phi_i(t) = \int_0^{|t|} s\phi_i(s)ds$, $t \in \mathbb{R}$.
(ϕ_4') $\liminf\limits_{t\to+\infty} \dfrac{\Phi_i(t)}{t^{q_i}} > 0$, for some $q_i > N$.
(ϕ_5') $\left|1 - \frac{\Phi_1(t)}{t^2\phi_1(t)}\left(1 + \frac{t\phi_1'(t)}{\phi_1(t)}\right)\right| \leq 1$, $t \in \mathbb{R}$.

The assumption (ϕ_4') implies that the embedding

$$W_0^{1,\Phi_i}(\Omega) \hookrightarrow W^{1,q_i}(\Omega)$$

for some $q_i > N$ is continuous. Hence,

$$W_0^{1,\Phi_i}(\Omega) \hookrightarrow C^{0,\alpha_i}(\overline{\Omega})$$

is continuous for some $\alpha_i \in (0,1)$ and

$$W_0^{1,\Phi_i}(\Omega) \hookrightarrow C(\overline{\Omega}) \tag{4.1}$$

is compact. In what follows, we denote by $\Lambda_i > 0$ the best constant that satisfies

$$\|u\|_{C(\overline{\Omega})} \leq \Lambda_i \|u\|_i, \quad \forall u \in W_0^{1,\Phi_i}(\Omega), \tag{4.2}$$

where $\|\cdot\|_i = \|\nabla\cdot\|_{L^{\Phi_i}(\Omega)}$.

If d is twice the diameter of Ω, then there exists $\delta \geq 0$ such that

$$(\phi_6')\qquad \frac{t^2}{d^2} \leq \Phi_1(t/d), \quad \forall |t| \geq \delta$$

Before continuing this section, we would like to point out that $\Phi_1(t) = (e^{t^2}-1)/2$ and $\Phi_2(t) = |t|^p/p$ with $p > N$ satisfying $(\phi_1') - (\phi_6')$. Moreover, we would like to recall that $(u,v) \in W_0^{1,\Phi_1}(\Omega) \times W_0^{1,\Phi_2}(\Omega)$ is a weak solution of (S_2) whenever

$$\int_\Omega \phi_1(|\nabla u|)\nabla u \nabla\varphi_1 dx - \int_\Omega \phi_2(|\nabla u|)\nabla u \nabla\varphi_2 dx = \int_\Omega R_u(x,u,v)\varphi_1 dx + \int_\Omega R_v(x,u,v)\varphi_2 dx,$$

for all $(\varphi_1,\varphi_2) \in W_0^{1,\Phi_1}(\Omega) \times W_0^{1,\Phi_2}(\Omega)$. Here, let us consider the R function satisfying the following conditions:
(R_1') $R \in C^1(\overline{\Omega} \times \mathbb{R}^2)$ and $R_v(x,u,0) \neq 0$ for all $(x,u) \in \Omega \times \mathbb{R}$.
(R_2') $R(x,u,0) \leq \dfrac{1}{2}\Phi_1(u/d) + \frac{1}{2d^2}|u|^2$, for all $(x,u) \in \Omega \times \mathbb{R}$.
(R_3') $R(x,0,v) \geq -\dfrac{1}{2}\Phi_2(v/d) - Mv$, for all $(x,v) \in \Omega \times \mathbb{R}$, for some constant $M > 0$.

(R_4') There are $\nu > 0$, $\mu > 1$ and $0 < \beta < 1$ such that

$$(i) \qquad \frac{1}{\mu}h(u)R_u(x,u,v)u + \frac{1}{\nu}R_v(x,u,v)v - R(x,u,v) \geq 0, \quad \forall (x,u,v) \in \Omega \times \mathbb{R}^2$$

and

$$(ii) \qquad \beta R(x,u,v) - \frac{1}{\mu}h(u)R_u(x,u,v)u \geq 0, \quad \forall (x,u,v) \in \Omega \times \mathbb{R}^2$$

where $h(u) = \frac{\Phi_1(u)}{u^2\phi_1(u)}$.

The main result of this section is the following.

Theorem 4.1. *Assume that $(\phi_1') - (\phi_6')$ and $(R_1) - (R_4)$ hold. Then, the system (S_2) possesses a nontrivial solution.*

We observe that $R(u,v) = \Phi_1(u)^\sigma \Phi_2(v)^\theta + v^+$ satisfies $(R_1') - (R_4')$ for some $\theta, \sigma > 1$, where $v^+ := \max\{0, v\}$.

Under the assumptions $(\phi_1') - (\phi_6')$ it is well known in the literature that the N-functions Φ_1 and Φ_2 might not satisfy the Δ_2-condition, and as a consequence, $W_0^{1,\Phi_1}(\Omega)$ and $W_0^{1,\Phi_2}(\Omega)$ might not be reflexive anymore. Another important fact we can highlight is that under these conditions, it is well known that there are $u \in W_0^{1,\Phi_1}(\Omega)$ and $v \in W_0^{1,\Phi_2}(\Omega)$ such that

$$\int_\Omega \Phi_1(|\nabla u|)dx = \infty \quad \text{and} \quad \int_\Omega \Phi_2(|\nabla v|)dx = \infty$$

In order to avoid this problem, we will work with the space $W_0^1E^{\Phi_1}(\Omega) \times W_0^1E^{\Phi_2}(\Omega)$, because in this space the functional $Q : W_0^1E^{\Phi_1}(\Omega) \times W_0^1E^{\Phi_2}(\Omega) \longrightarrow \mathbb{R}$ given by

$$Q(u,v) = \int_\Omega \Phi_1(|\nabla u|)dx - \int_\Omega \Phi_2(|\nabla v|)dx$$

belongs to $C^1(W_0^1E^{\Phi_1}(\Omega) \times W_0^1E^{\Phi_2}(\Omega), \mathbb{R})$. However, independent of Δ_2-condition, the embedding (4.1) guarantee that the funcional $H : W_0^{1,\Phi_1}(\Omega) \times W_0^{1,\Phi_2}(\Omega) \longrightarrow \mathbb{R}$ given by

$$H(u,v) = \int_\Omega R(x,u,v)dx$$

belongs to $C^1(W_0^{1,\Phi_1}(\Omega) \times W_0^{1,\Phi_2}(\Omega), \mathbb{R})$. In particular, $H|_{W_0^1E^{\Phi_1}(\Omega) \times W_0^1E^{\Phi_2}(\Omega)}$ is also of class C^1. That is, the energy functional $J : W_0^1E^{\Phi_1}(\Omega) \times W_0^1E^{\Phi_2}(\Omega) \longrightarrow \mathbb{R}$ associated to the system (S_2) given by

$$J(u,v) = \int_\Omega \Phi_1(|\nabla u|)dx - \int_\Omega \Phi_2(|\nabla v|)dx - \int_\Omega R(x,u,v)dx$$

belongs to $C^1(W_0^1E^{\Phi_1}(\Omega) \times W_0^1E^{\Phi_2}(\Omega), \mathbb{R})$.

In order to apply the Saddle-point theorem, in the next one we fix some notations. Since $W_0^1E^{\Phi_2}(\Omega)$ is separable, there exists a sequence $(e_n) \subset W_0^1E^{\Phi_2}(\Omega)$ such that

$$W_0^1E^{\Phi_2}(\Omega) = \overline{span\{e_n : n \in \mathbb{N}\}}. \tag{4.3}$$

Hereafter, for each $n \in \mathbb{N}$ we denote by V_n, X_n and X_n' the following spaces

$$V_n = span\{e_j : j = 1, \cdots, n\}, \quad X_n = W_0^1E^{\Phi_1}(\Omega) \times V_n \quad \text{and} \quad X_n' = W_0^{1,\Phi_1}(\Omega) \times V_n.$$

The restriction of J to X_n will be denoted by J_n. Then $J_n : X_n \longrightarrow \mathbb{R}$ is the functional given by

$$J_n(u,v) = \int_\Omega \Phi_1(|\nabla u|)dx - \int_\Omega \Phi_2(|\nabla v|)dx - \int_\Omega R(x,u,v)dx.$$

From the regularity of J, it follows that J_n belongs to $C^1(X_n,\mathbb{R})$ with

$$J_n'(u,v)(w_1,w_2)=\int_\Omega \phi_1(|\nabla u|)\nabla u\nabla w_1 dx-\int_\Omega \phi_2(|\nabla v|)\nabla v\nabla w_2 dx-\int_\Omega R_u(x,u,v)w_1 dx-\int_\Omega R_v(x,u,v)w_2 dx,$$

for all $(w_1,w_2)\in X_n$.

In the following, we prove that J_n satisfies the hypotheses of Theorem 2.4.

Lemma 4.2. *Under the space $Z=W_0^1E^{\Phi_1}(\Omega)\times\{0\}$ the functional J_n is bounded below.*

Proof. By the condition (R_2'),

$$J_n(u,0)\geq \int_\Omega \Phi_1(|\nabla u|)dx-\frac{1}{2}\int_\Omega \Phi_1(|u|/d)dx-\frac{1}{2d^2}\int_\Omega |u|^2dx. \tag{4.4}$$

It is worth remembering here the Poincaré inequality

$$\int_\Omega \Phi_1(|u|/d)dx\leq \int_\Omega \Phi_1(|\nabla u|)dx,\ \forall u\in W_0^1E^{\Phi_1}(\Omega).$$

For more details on this inequality, we infer the reader to [21]. Hence, using the Poincaré inequality together with the hypothesis (ϕ_6') on the inequality (4.4), we obtain

$$J_n(u,0)\geq -\frac{1}{2d^2}\int_{[|u|\leq\delta]}|u|^2\geq -\frac{\delta^2}{2d^2}|\Omega|,\ \ \forall u\in W_0^1E^{\Phi_1}(\Omega).$$

This finishes the proof. □

Lemma 4.3. *If $\|v\|_2\to\infty$, then $J(0,v)\to-\infty$.*

Proof. Let $v\in W_0^1E^{\Phi_1}(\Omega)$ with $\|v\|_1\geq 1$. The assumption (R_3') together with the Poincaré inequality implies that

$$J(0,v)\leq -\frac{1}{2}\int_\Omega \Phi_2(|\nabla v|)dx+M\int_\Omega |v|dx \tag{4.5}$$

From (ϕ_3'),

$$\frac{d}{ds}ln(\Phi_2(rs))=\frac{\psi_2(rs)r^2s}{\Phi_2(rs)}\geq\frac{\ell_2}{s},\ \ \forall s,r>0$$

thus,

$$\int_1^t \frac{d}{ds}ln(\Phi_2(rs))ds\geq \ell_2\int_1^t\frac{1}{s}ds,\quad \forall t\geq 1.$$

Therefore,

$$ln\frac{\Phi_2(rt)}{\Phi_2(r)}\geq ln(t^{\ell_2}),\quad \forall t\geq 1.$$

Because of the monotonicity of the logarithmic function,

$$\frac{\Phi_2(rt)}{\Phi_2(r)}\geq t^{\ell_2},\quad \forall t\geq 1.$$

And as a consequence of this inequality, we have

$$\int_\Omega \Phi_2(|\nabla v|)dx\geq \|v\|_2^{\ell_2}\quad \text{for}\ \ \|v\|_2\geq 1. \tag{4.6}$$

By means of the inequalities (4.5) and (4.6), we conclude that

$$J(0,v)\leq -\|v\|_2^{\ell_2}+M|\Omega|\Lambda_2\|v\|_2.$$

Since $1<\ell_2$, the result follows. □

Corollary 4.4. *If* $\|v\|_2 \to \infty$, *then* $J_n(0,v) \to -\infty$.

Corollary 4.5. *There is* $M > 0$ *such that* $\inf_Z J_n > \max_{\partial \mathcal{M}_n} J_n := b_n$ *where* $\mathcal{M}_n = B_M(0) \cap Y_n$,.

Proof. By the above Corollary $J_n(0,v) \to -\infty$ as $\|v\|_2 \to +\infty$ in Y, then, fix $M > 1$ such that $J_n(0,v) < \inf_Z J_n$ for $\|v\|_2 = M$ and $v \in Y_n$. Since $dim Y_n < \infty$, we can conclude $\inf_Z J_n > \max_{\mathcal{N}_n} J_n$. □

Then, by Lemma 3.2, we can apply the Saddle-point theorem 2.4 to functional J_n using the sets

$$Y_n = \{0\} \times V_n, \quad Z = W_0^1 E^{\Phi_1}(\Omega) \times \{0\}, \quad \text{and} \quad \mathcal{M}_n = B_M(0) \cap Y_n,$$

where $M > 0$ is obtained from Corollary 4.5. Then, there exists a sequence $(u_k, v_k) \subset X_n$ with

$$J_n(u_k, v_k) \longrightarrow c_n \quad \text{and} \quad J_n'(u_k, v_k) \longrightarrow 0 \text{ as } k \to +\infty. \tag{4.7}$$

where

$$c_n = \inf_{\gamma \in \Gamma} \max_{u \in \mathcal{M}_n} J_n(\gamma(u)), \tag{4.8}$$

with

$$\Gamma = \{\gamma \in C(\mathcal{M}_n, X_n) : \ \gamma|_{\mathcal{N}_n} = Id\}.$$

Lemma 4.6. *The sequence* (u_k, v_k) *is bounded in* X_n.

Proof. Define the function

$$\eta(t) = \begin{cases} \dfrac{\Phi_1(t)}{t\phi_1(t)} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$$

and consider the sequence

$$g_k(x) = \eta(u_k(x)), \ \ x \in \Omega.$$

A direct computation leads to

$$\nabla g_k = \left[1 - \frac{\Phi_1(u_k)}{u_k^2 \phi_1(u_k)}\Big(1 + \frac{u_k \phi_1'(u_k)}{\phi_1(u_k)}\Big)\right] \nabla u_k.$$

Furthermore, considering the hypothesis (ϕ_4'), it is shown without difficulty that $g_k \in W_0^1 E^{\Phi_1}(\Omega)$ and $\|g_k\|_1 \leq \|u_k\|_1$ for each $k \in \mathbb{N}$. Being (u_k, v_k) a sequence $(PS)_{c_n}$, then by $(R_4')(i)$ and (ϕ_3')

(4.9)

$$\begin{aligned} c_n + 1 + o_k(1)\|(u_k, v_k)\| &\geq J_n(u_k, v_k) - J_n'(u_k, v_k)\big(\frac{1}{\mu} g_k, \frac{1}{\nu} v_k\big) \\ &= \int_\Omega \Phi(|\nabla u_k|)dx - \frac{1}{\mu}\int_\Omega \phi(|\nabla u_k|)|\nabla u_k|^2 S(u_k)dx - \int_\Omega \Psi(|\nabla v_k|)dx + \frac{1}{\nu}\int_\Omega \psi(|\nabla v_k|)|\nabla v_k|^2 dx \\ &\quad + \frac{1}{\mu}\int_\Omega R_u(x, u_k, v_k) u_k h(u_k)dx + \frac{1}{\nu}\int_\Omega R_v(x, u_k, v_k) v_k dx - \int_\Omega R(x, u_k, v_k)dx \\ &\geq \int_\Omega \Phi_1(|\nabla u_k|)dx - \frac{1}{\mu}\int_\Omega \phi_1(|\nabla u_k|)|\nabla u_k|^2 S(u_k)dx + \left(\frac{\ell_2}{\nu} - 1\right)\int_\Omega \Phi_2(|\nabla v_k|)dx. \end{aligned}$$

where $h(t)=\frac{\Phi_1(t)}{t^2\phi_1(t)}$ and $S(t)=1-\frac{\Phi_1(t)}{t^2\phi_1(t)}\Big(1+\frac{t\phi_1'(t)}{\phi_1(t)}\Big)$.(The functions S and h were introduced by Alves et al in [3]) On the other hand, it follows from $(R_4')(ii)$ that

$$\begin{aligned}
c_n+1+o_k(1)\|g_k\|_1\geq&-\beta J_n(u_k,v_k)+J_n'(u_k,v_k)\big(\frac{1}{\mu}g_k,0\big)\\
=&-\beta\int_\Omega \Phi_1(|\nabla u_k|)dx+\frac{1}{\mu}\int_\Omega \phi_1(|\nabla u_k|)|\nabla u_k|^2S(u_k)dx+\beta\int_\Omega \Phi_2(|\nabla v_k|)dx\\
&-\frac{1}{\mu}\int_\Omega R_u(x,u_k,v_k)u_kh(u_k)dx+\beta\int_\Omega R(x,u_k,v_k)dx,\\
\geq&-\beta\int_\Omega \Phi_1(|\nabla u_k|)dx+\frac{1}{\mu}\int_\Omega \phi_1(|\nabla u_k|)|\nabla u_k|^2S(u_k)dx,
\end{aligned}$$

i.e,

$$-\frac{1}{\mu}\int_\Omega \phi_1(|\nabla u_k|)|\nabla u_k|^2S(u_k)dx\geq -c_n-1-o_k(1)\|u_k\|_1-\beta\int_\Omega \Phi_1(|\nabla u_k|)dx. \tag{4.10}$$

From (4.9) and (4.10),

$$2(c_n+1)+o_k(1)\|(u_k,v_k)\|\geq(1-\beta)\int_\Omega \Phi_1(|\nabla u_k|)dx+\left(\frac{\ell_2}{\nu}-1\right)\int_\Omega \Phi_2(|\nabla v_k|)dx.$$

Suppose for contradiction that, up to a subsequence, $\|(u_k,v_k)\|\to+\infty$ as $k\to+\infty$. This way, we need to study the following situations:
(i) $\|u_k\|_1\to+\infty$ and $\|v_k\|_2\to\infty$
(ii) $\|u_k\|_1\to+\infty$ and $\|v_k\|_2$ is bounded
(iii) $\|v_k\|_2\to\infty$ and $\|u_k\|_1$ is bounded

In the first case, there is $k_0\in\mathbb{N}$ such that

$$\int_\Omega \Phi_1(|\nabla u_k|)dx\geq\|u_k\|_1 \quad \text{and} \quad \int_\Omega \Phi_2(|\nabla v_k|)dx\geq\|v_k\|_2,\ \forall k\geq k_0.$$

Hence, the inequality (3.15) implies that

$$2c_n^2+o_k(1)\|(u_k,v_k)\|^2\geq(1-\beta)^2\|u_k\|_1^2+\left(\frac{\ell_2}{\nu}-1\right)^2\|v_k\|_2^2,\ \forall k\geq k_0.$$

Which is absurd.

In case (ii), there is $k_0\in\mathbb{N}$ such that

$$\int_\Omega \Phi_1(|\nabla u_k|)dx\geq\|u_k\|_1,\ \forall k\geq k_0.$$

Thus, the inequality (3.15) is reduced to

$$2c_n^2+C_1+o_k(1)\|u_k\|_1\geq(1-\beta)^2\|u_k\|_1^2,\ \forall k\geq k_0.$$

which is absurd. The last case is similar to the case (ii). The above analysis shows that (u_k,v_k) is now a bounded sequence in X_n.

□

From Lemmas 4.6 and 4.12, we may assume that there exists a subsequence of (u_k,v_k), still denoted by itself, and $(w_n,y_n)\in X_n'$ such that

$$u_k\overset{*}{\rightharpoonup} w_n \text{ weakly in } W_0^{1,\Phi_1}(\Omega) \quad\text{and}\quad v_k\overset{*}{\rightharpoonup} y_n \text{ weakly in } V_n,\quad \text{as}\quad k\to\infty. \tag{4.11}$$

Here, we highlight that the pair (w_n,y_n) may not belong to the space X_n, because whenever Φ_1 does not satisfy the Δ_2-condition the space X_n is not a weak* closed subspace of X_n'.

The results below will be used to ensure that the sequence (w_n, y_n) is bounded in $W_0^{1,\Phi_1}(\Omega) \times W_0^{1,\Phi_2}(\Omega)$, moreover, we will do some results that will be fundamental.

Lemma 4.7. *The sequence (u_k, v_k) obtained in (4.7) satisfies*

$$\int_\Omega \phi_1(|\nabla u_k|)\nabla u_k \nabla \varphi dx = \int_\Omega R_u(x, u_k, v_k)\varphi dx + o_k(1), \ \ \forall k \in \mathbb{N} \ \ \text{and} \ \ \varphi \in W_0^{1,\Phi_1}(\Omega).$$

Proof. From 4.7,

$$J_n'(u_k, v_k)(\varphi, 0) = o_k(1)\|\varphi\|_1, \ \ \forall \varphi \in W_0^1 E^{\Phi_1}(\Omega). \tag{4.12}$$

By definition, the space $W_0^{1,\Phi_1}(\Omega)$ is the weak* closure of $C_0^\infty(\Omega)$ in $W^{1,\Phi_1}(\Omega)$, thus, given $\varphi \in W_0^{1,\Phi_1}(\Omega)$ there will be a sequence (φ_m) in $C_0^\infty(\Omega)$ such that

$$\varphi_m \xrightarrow{\ *\ } \varphi \ \text{ in } W_0^{1,\Phi_1}(\Omega). \tag{4.13}$$

It is clear that $(\|\varphi_m\|_1)$ is bounded in $\mathbb{R}$, so by (4.12),

$$o_k(1) = \int_\Omega \phi_1(|\nabla u_k|)\nabla u_k \nabla \varphi_m dx - \int_\Omega R_u(x, u_k, v_k)\varphi_m dx, \ \ \forall k \in \mathbb{N}$$

Using the fact that $\phi_1(|\nabla u_k|)\frac{\partial u_k}{\partial x_i} \in E^{\tilde{\Phi}_1}(\Omega)$ along with the limit (4.13), we will get

$$\lim_{m\to\infty} \int_\Omega \phi_1(|\nabla u_k|)\nabla u_k \nabla \varphi_m dx = \int_\Omega \phi_1(|\nabla u_k|)\nabla u_k \nabla \varphi dx$$

Therefore, since the spaces $W_0^{1,\Phi_1}(\Omega)$, $W_0^{1,\Phi_2}(\Omega)$ are embedded in $C(\overline{\Omega})$, we can conclude that

$$o_k(1) = \int_\Omega \phi_1(|\nabla u_k|)\nabla u_k \nabla \varphi dx - \int_\Omega R_u(x, u_k, v_k)\varphi dx, \ \ \forall k \in \mathbb{N}.$$

□

Before proceeding with the results, we need to make the following definitions:

- We will denote by $D(J_{\Phi_i}) \subset W_0^{1,\Phi_i}(\Omega)$, the following set:

$$D(J_{\Phi_i}) = \left\{ u \in W_0^{1,\Phi_i}(\Omega) : \int_\Omega \Phi_i(|\nabla u|)dx < +\infty \right\}$$

- We will denote by $dom(\phi(t)t) \subset W_0^{1,\Phi}(\Omega)$, the following set:

$$dom(\phi_i(t)t) = \left\{ u \in W_0^{1,\Phi_i}(\Omega) : \int_\Omega \tilde{\Phi}_i\big(\phi_i(|\nabla u|)|\nabla u|\big)dx < +\infty \right\}$$

Lemma 4.8. *Let (w_n) the sequence obtained in (4.11). Then $(w_n) \subset D(J_{\Phi_1}) \cap dom(\phi_1(t)t)$, furthermore,*

$$c_n = \lim_{k\to\infty} J_n(u_k, v_k) = J_n(w_n, y_n)$$

and

$$\begin{aligned} &\int_\Omega \Phi_1(|\nabla \varphi_1|)dx - \int_\Omega \Phi_1(|\nabla w_n|)dx - \int_\Omega \phi_2(|\nabla y_n|)\nabla y_n \nabla(\varphi_2 - y_n)dx \\ &\qquad \geq \int_\Omega R_u(x, w_n, y_n)(\varphi_1 - w_n)dx + \int_\Omega R_v(x, w_n, y_n)(\varphi_2 - y_n)dx, \end{aligned} \tag{4.14}$$

for all $(\varphi_1, \varphi_2) \in W_0^{1,\Phi_1}(\Omega) \times V_n$.

Proof. Using the fact that $J'_n(u_k, v_k) \to 0$ as $k \to \infty$ together with Lemma 4.7, we can conclude that

$$\int_\Omega \phi_1(|\nabla u_k|)\nabla u_k \nabla \varphi_1 dx - \int_\Omega \phi_2(|\nabla v_k|)\nabla v_k \nabla \varphi_2 dx = \int_\Omega R_u(x, u_k, v_k)\varphi_1 dx + \int_\Omega R_v(x, u_k, v_k)\varphi_2 dx + o_k(1), \tag{4.15}$$

for each $(\varphi_1, \varphi_2) \in W_0^{1,\Phi_1}(\Omega) \times V_n$ and $k \in \mathbb{N}$. Since Φ_1 is convex, we have

$$\int_\Omega \Phi_1(|\nabla \eta_1|)dx - \int_\Omega \Phi_1(|\nabla u_k|)dx \geq \int_\Omega \phi_1(|\nabla u_k|)\nabla u_k \nabla(\eta_1 - u_k)dx,$$

for all $\eta_1 \in W_0^{1,\Phi_1}(\Omega)$. Hence, considering $\varphi_1 = \eta_1 - u_k$ in (4.15) and using the inequality above, we get

$$\int_\Omega \Phi_1(|\nabla \eta_1|)dx - \int_\Omega \Phi_1(|\nabla u_k|)dx - \int_\Omega \phi_2(|\nabla v_k|)\nabla v_k \nabla \varphi_2 dx \geq \int_\Omega R_u(x, u_k, v_k)(\eta_1 - u_k)dx + \int_\Omega R_v(x, u_k, v_k)\varphi_2 dx + o_k(1), \tag{4.16}$$

for every $(\eta_1, \varphi_2) \in W_0^{1,\Phi_1}(\Omega) \times V_n$ and $k \in \mathbb{N}$. Since $u_k \xrightarrow{*} w_n$ in $W_0^{1,\Phi_1}(\Omega)$ follows from [18, Theorem 2.1, Chapter 8] that

$$\int_\Omega \Phi_1(|\nabla w_n|)dx \leq \varliminf_{k\to\infty} \int_\Omega \Phi_1(|\nabla u_k|)dx, \tag{4.17}$$

Remember that $dim V_n = n$, so $v_k \to y_n$ in V_n. Hence,

$$\int_\Omega \Phi_1(|\nabla \eta_1|)dx - \int_\Omega \Phi_1(|\nabla w_n|)dx - \int_\Omega \phi_2(|\nabla y_n|)\nabla y_n \nabla \varphi_2 dx \geq \int_\Omega R_u(x, w_n, y_n)(\eta_1 - w_n)dx + \int_\Omega R_v(x, w_n, y_n)\varphi_2 dx, \tag{4.18}$$

for each $(\eta_1, \varphi_2) \in W_0^{1,\Phi_1}(\Omega) \times V_n$. Justifying the inequality (4.14).

Considering $(\eta_1, \varphi_2) = (w_n, 0)$ in the inequality (4.16), we get

$$\int_\Omega \Phi_1(|\nabla w_n|)dx - \int_\Omega \Phi_1(|\nabla u_k|)dx \geq \int_\Omega R_u(x, u_k, v_k)(w_n - u_k)dx + o_k(1).$$

Thus,

$$\int_\Omega \Phi_1(|\nabla w_n|)dx \geq \varlimsup_{k\to\infty} \int_\Omega \Phi_1(|\nabla u_k|)dx. \tag{4.19}$$

Combining (4.17) and (4.19),

$$\lim_{k\to\infty} \int_\Omega \Phi_1(|\nabla u_k|)dx = \int_\Omega \Phi_1(|\nabla w_n|)dx.$$

Therefore, we can conclude that

$$c_n = \lim_{k\to\infty} J_n(u_k, v_k) = J_n(w_n, y_n).$$

Finally, we will show that $w_n \in dom(\phi_1(t)t)$. By the inequality (4.16),

$$\int_\Omega \Phi_1(|\nabla u_k - \frac{1}{k}\nabla u_k|)dx - \int_\Omega \Phi_1(|\nabla u_k|)dx \geq -\frac{1}{k}\int_\Omega R_u(x, u_k, v_k)u_k dx + o_k(1),$$

i.e,

$$\int_\Omega \frac{\big(\Phi_1(|\nabla u_k - \frac{1}{k}\nabla u_k|) - \Phi_1(|\nabla u_k|)\big)}{-\frac{1}{k}}dx \le \int_\Omega R_u(x,u_k,v_k)u_k dx + o_k(1).$$

As (u_k) and (v_k) are bounded in $W_0^{1,\Phi_1}(\Omega)$ and $W_0^{1,\Phi_2}(\Omega)$, respectively, there will be $M>0$ such that

$$\int_\Omega \Phi_1(|\nabla u_k - \frac{1}{k}\nabla u_k|)dx - \int_\Omega \Phi_1(|\nabla u_k|)dx \le M, \quad \forall k\in\mathbb{N}.$$

Since Φ_1 is in C^1 class, there exists $\theta_k(x)\in[0,1]$ such that

$$\frac{\Phi_1(|\nabla u_k - \frac{1}{k}\nabla u_k|) - \Phi_1(|\nabla u_k|)}{-\frac{1}{k}} = \phi_1(|\big(1-\frac{\theta_k}{k}(x)\big)\nabla u_k|)\big(1-\frac{\theta_k(x)}{k}\big)|\nabla u_k|^2.$$

Recalling that $0 < 1-\frac{\theta_k(x)}{k} \le 1$, we know that $1-\frac{\theta_k(x)}{k} \ge \big(1-\frac{\theta_k(x)}{k}\big)^2$ which leads to

$$\int_\Omega \phi_1(|\big(1-\frac{\theta_k}{k}(x)\big)\nabla u_k|)\big(1-\frac{\theta_k(x)}{k}\big)^2|\nabla u_k|^2 dx \le M, \quad \forall k\in\mathbb{N}.$$

As $\nabla u_k \xrightarrow{*} \nabla w_n$ in $\big(L^{\Phi_1}(\Omega)\big)^{N-1}$, we also have $\big(1-\frac{\theta_k(x)}{k}\big)\nabla u_k \xrightarrow{*} \nabla w_n$ in $\big(L^{\Phi_1}(\Omega)\big)^{N-1}$ as $k\to\infty$. Then, by using the fact that $\phi_1(t)t^2$ is convex, we can apply [18, Theorem 2.1, Chapter 8] to get

$$\liminf_{k\to\infty}\int_\Omega \phi_1(|\big(1-\frac{\theta_k}{k}(x)\big)\nabla u_k|)\big(1-\frac{\theta_k(x)}{k}\big)^2|\nabla u_k|^2 \ge \int_\Omega \phi_1(|\nabla w_n|)|w_n|^2 dx$$

and so,

$$\int_\Omega \phi_1(|\nabla w_n|)|w_n|^2 dx \le M.$$

Recalling that

$$\phi_1(t)t^2 = \Phi_1(t) + \tilde{\Phi}_1(\phi_1(t)t), \quad \forall t\in\mathbb{R}$$

we have

$$\phi_1(|\nabla w_n|)|\nabla w_n|^2 = \Phi_1(|\nabla w_n|) + \tilde{\Phi}_1(\phi_1(|\nabla w_n|)|\nabla w_n|)$$

which leads to

$$\int_\Omega \phi_1(|\nabla w_n|)|\nabla w_n|^2 dx = \int_\Omega \Phi_1(|\nabla w_n|)dx + \int_\Omega \tilde{\Phi}_1(\phi_1(|\nabla w_n|)|\nabla w_n|^2)dx.$$

Since $\int_\Omega \phi_1(|\nabla w_n|)|\nabla w_n|^2 dx$ is finite, we see that $\int_\Omega \Phi_1(|\nabla w_n|)dx$ and $\int_\Omega \tilde{\Phi}_1(\phi_1(|\nabla w_n|)|\nabla w_n|^2)dx$ are also finite, showing that $w_n \in D(J_{\Phi_1})\cap dom(\phi_1(t)t)$. This finishes the proof.

□

Lemma 4.9. *For each $(\varphi_1,\varphi_2)\in W_0^{1,\Phi_1}(\Omega)\times V_n$, the following equality holds*

$$\int_\Omega \phi_1(|\nabla w_n|)\nabla w_n \nabla\varphi_1 dx - \int_\Omega \phi_2(|\nabla y_n|)\nabla y_n\nabla\varphi_2 dx = \int_\Omega R_u(x,w_n,y_n)\varphi_1 dx + \int_\Omega R_u(x,w_n,y_n)\varphi_2 dx.$$

Proof. Given $\varepsilon\in(0,1/2)$ and $\varphi_1\in C_0^\infty(\Omega)$, we set the function

$$v_\varepsilon = \frac{1}{1-\frac{\varepsilon}{2}}((1-\varepsilon)w_n + \varepsilon\varphi_1).$$

Consider $\varphi_2 \in V_n$ and apply $(v_\varepsilon, \varepsilon\varphi_2 + y_n)$ on the inequality (4.14), hence

$$\int_\Omega \Phi_1(|\nabla v_\varepsilon|)dx - \int_\Omega \Phi_1(|\nabla w_n|)dx - \varepsilon \int_\Omega \phi_2(|\nabla y_n|)\nabla y_n \nabla \varphi_2 dx \geq \int_\Omega R_u(x, w_n, y_n)(v_\varepsilon - w_n)dx + \varepsilon \int_\Omega R_v(x, w_n, y_n)\varphi_2 dx,$$

and so,

$$\frac{\int_\Omega \Phi_1(|\nabla v_\varepsilon|)dx - \int_\Omega \Phi_1(|\nabla w_n|)dx}{\varepsilon} - \int_\Omega \phi_2(|\nabla y_n|)\nabla y_n \nabla \varphi_2 dx \geq \int_\Omega R_u(x, w_n, y_n)\Big(\frac{v_\varepsilon - w_n}{\varepsilon}\Big)dx + \int_\Omega R_v(x, w_n, y_n)\varphi_2 dx.$$

Note that

$$\frac{\varepsilon\varphi_1}{1-\frac{\varepsilon}{2}} = 2\left(1 - \frac{1-\varepsilon}{1-\frac{\varepsilon}{2}}\right)\varphi_1,$$

hence, by the convexity of Φ_1,

$$\Phi_1\Big(\frac{1}{1-\frac{\varepsilon}{2}}\big((1-\varepsilon)\nabla w_n + \varepsilon\nabla\varphi_1\big)\Big) \leq \frac{1-\varepsilon}{1-\frac{\varepsilon}{2}}\Phi_1(|\nabla w_n|) + \big(1 - \frac{1-\varepsilon}{1-\frac{\varepsilon}{2}}\big)\Phi_1(2|\nabla\varphi_1|).$$

Hence, by Lebesgue dominated convergence theorem, we get

$$\int_\Omega \phi_1(|\nabla w_n|)\nabla w_n(\nabla w_n - \frac{\nabla w_n}{2})dx - \int_\Omega \phi_2(|\nabla y_n|)\nabla y_n \nabla\varphi_2 dx \geq \int_\Omega R_u(x, w_n, y_n)\Big(\varphi_1 - \frac{w_n}{2}\Big)dx + \int_\Omega R_v(x, w_n, y_n)\varphi_2 dx. \tag{4.20}$$

Therefore

$$\int_\Omega \phi_1(|\nabla w_n|)\nabla w_n \nabla\varphi_1 - \int_\Omega R_u(x, w_n, y_n)\varphi_1 dx \geq A, \quad \forall \varphi_1 \in C_0^\infty(\Omega), \tag{4.21}$$

where

$$A = \frac{1}{2}\int_\Omega \phi_1(|\nabla w_n|)|\nabla w_n|^2 dx - \frac{1}{2}\int_\Omega R_u(x, w_n, y_n)w_n dx.$$

As $C_0^\infty(\Omega)$ is a vector space, the last inequality gives

$$\int_\Omega \phi_1(|\nabla w_n|)\nabla w_n \nabla\varphi_1 - \int_\Omega R_u(x, w_n, y_n)\varphi_1 dx = 0, \quad \forall \varphi_1 \in C_0^\infty(\Omega). \tag{4.22}$$

We know that $W_0^{1,\Phi_1}(\Omega)$ is the weak* closure of $C_0^\infty(\Omega)$ in $W^{1,\Phi_1}(\Omega)$, then using the fact that $\phi_1(|\nabla w_n|)|\nabla w_n| \in L^{\tilde{\Phi}_1}(\Omega)$ we can conclude that

$$\int_\Omega \phi_1(|\nabla w_n|)\nabla w_n \nabla\varphi_1 dx - \int_\Omega R_u(x, w_n, y_n)\varphi_1 dx = 0, \ \ \forall \varphi_1 \in W_0^{1,\Phi_1}(\Omega). \tag{4.23}$$

Still by (4.20), we have

$$-\int_\Omega \phi_2(|\nabla y_n|)\nabla y_n \nabla\varphi_2 \geq \int_\Omega R_v(x, w_n, y_n)\varphi_2 dx, \ \ \forall \varphi_2 \in V_n.$$

Since V_n is a vector space, the above inequality gives

$$\int_\Omega \phi_2(|\nabla y_n|)\nabla y_n \nabla\varphi_2 = -\int_\Omega R_v(x, w_n, y_n)\varphi_2 dx, \ \ \forall \varphi_2 \in V_n. \tag{4.24}$$

From (4.23) and (4.24)

$$\int_\Omega \phi_2(|\nabla w_n|)\nabla w_n \nabla\varphi_1 dx - \int_\Omega \phi_2(|\nabla y_n|)\nabla y_n\nabla\varphi_2 = \int_\Omega R_u(x,w_n,y_n)\varphi_1 dx + \int_\Omega R_v(x,w_n,y_n)\varphi_2 dx,$$

for any $(\varphi_1,\varphi_2)\in W_0^{1,\Phi_2}(\Omega)\times V_n$.

□

Lemma 4.10. *The sequence (w_n, y_n) is bounded in X.*

Proof. Consider the sequence

$$g_n(x) = \eta(w_n(x)), \;\; x\in\Omega.$$

where η is given in Lemma 4.6. A direct computation leads to

$$\nabla g_n = \left[1 - \frac{\Phi_1(w_n)}{w_n^2\phi_1(w_n)}\Big(1+\frac{w_n\phi_1'(w_n)}{\phi_1(w_n)}\Big)\right]\nabla w_n.$$

The last identity together with (ϕ_4') implies that

$$|\nabla g_n| \le |\nabla w_n|, \;\; \forall n\in\mathbb{N} \tag{4.25}$$

On the other hand, (ϕ_3') also gives

$$|g_n(x)| \le \frac{1}{\ell_1}|\nabla w_n(x)|, \;\; \forall x\in\Omega. \tag{4.26}$$

From (4.25) and (4.26), $g_n\in D(J_{\Phi_1})$ with

$$\|g_n\|_1 \le \|w_n\|_1, \;\forall n\in\mathbb{N}.$$

By the Lemmas 4.8 and 4.9,

$$\begin{aligned} c_n = &\int_\Omega \Phi_1(|\nabla w_n|)dx - \frac{1}{\mu}\int_\Omega \phi_1(|\nabla w_n|)|\nabla w_n|^2 S(w_n)dx - \int_\Omega \Phi_2(|\nabla y_n|)dx + \frac{1}{\nu}\int_\Omega \phi_2(|\nabla y_n|)|\nabla y_n|^2 dx \\ &+\frac{1}{\mu}\int_\Omega R_u(x,w_n,v_k)w_n h(w_n)dx + \frac{1}{\nu}\int_\Omega R_v(x,w_n,y_n)y_n dx - \int_\Omega R(x,w_n,y_n)dx, \end{aligned}$$

where $h(t)=\frac{\Phi_1(t)}{t^2\phi_1(t)}$ and $S(t) = 1-\frac{\Phi(t)}{t^2\phi_1(t)}\Big[1+\frac{t\phi_1'(t)}{\phi_1(t)}\Big]$. By $(R_4')(i)$ together with (ϕ_3'),

$$c_n \ge \int_\Omega \Phi_1(|\nabla w_n|)dx - \frac{1}{\mu}\int_\Omega \phi_1(|\nabla w_n|)|\nabla w_n|^2 S(w_n)dx + \left(\frac{\ell_2}{\nu}-1\right)\int_\Omega \Phi_2(|\nabla y_n|)dx. \tag{4.27}$$

On the other hand, the Lemmas 4.8 and 4.9 together with $(R_4')(ii)$ imply that

$$\begin{aligned} -\beta c_n = &-\beta\int_\Omega \Phi_1(|\nabla w_n|)dx + \frac{1}{\mu}\int_\Omega \phi_1(|\nabla w_n|)|\nabla w_n|^2 S(w_n)dx + \beta\int_\Omega \Phi_2(|\nabla y_n|)dx \\ &-\frac{1}{\mu}\int_\Omega R_u(x,w_n,y_n)w_n h(w_n)dx + \beta\int_\Omega R(x,w_n,y_n)dx, \\ \ge &-\beta\int_\Omega \Phi_1(|\nabla w_n|)dx + \frac{1}{\mu}\int_\Omega \phi_1(|\nabla w_n|)|\nabla w_n|^2 S(w_n)dx, \end{aligned}$$

i.e,

$$-\frac{1}{\mu}\int_\Omega \phi_1(|\nabla w_n|)|\nabla w_n|^2 S(w_n)dx \ge \alpha c_n - \alpha\int_\Omega \Phi_1(|\nabla w_n|)dx. \tag{4.28}$$

From (4.27) and (4.28),

$$(1-\beta)c_n \ge (1-\beta)\int_\Omega \Phi_1(|\nabla w_n|)dx + \left(\frac{\ell_2}{\nu}-1\right)\int_\Omega \Phi_2(|\nabla y_n|)dx.$$

As a consequence of Lemma 4.3, we have that (c_n) is bounded. Therefore the sequences $\big(\int_\Omega \Phi_1(|\nabla w_n|)dx\big)$ and $\big(\int_\Omega \Phi_2(|\nabla y_n|)dx\big)$ are bounded and consequently (w_n, y_n) is bounded at $W_0^{1,\Phi_1}(\Omega) \times W_0^{1,\Phi_2}(\Omega)$.

□

Since (w_n, y_n) is bounded, there is no loss of generality in assuming that

$$(w_n, y_n) \xrightarrow{\;*\;} (u, v) \quad \text{in } W_0^{1,\Phi_1}(\Omega) \times W_0^{1,\Phi_2}(\Omega) \ \text{ as } \ n \to \infty. \tag{4.29}$$

By [18, Theorem 2.1, Chapter 8], we can conclude that it is worth

$$\int_{\mathbb{R}^N} \Phi_1(|\nabla u|)dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} \Phi_1(|\nabla w_n|)dx \tag{4.30}$$

and

$$\int_{\mathbb{R}^N} \Phi_2(|\nabla v|)dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} \Phi_2(|\nabla y_n|)dx. \tag{4.31}$$

Proposition 4.11. *The pair (u, v) is a nontrivial solution of (S_2).*

Proof. Fixing $k, n \in \mathbb{N}$ with $n \ge k$, we have $X_k' \subset X_n'$. Thus, for $(\varphi_1, \varphi_2) \in X_k'$, it follows from Lemma 4.9 that

$$\begin{aligned}\int_\Omega \phi_1(|\nabla w_n|)\nabla w_n \nabla \varphi_1 dx - \int_\Omega \phi_2(|\nabla y_n|)\nabla y_n \nabla \varphi_2 dx = &\int_\Omega R_u(x, w_n, y_n)\varphi_1 dx \\ &+ \int_\Omega R_v(x, w_n, y_n)\varphi_2 dx, \quad \forall n \ge k.\end{aligned} \tag{4.32}$$

By the above equality together with the convexity of Φ_1, we will obtain

$$\int_\Omega \Phi_1(|\nabla \varphi_1|)dx - \int_\Omega \Phi_1(|\nabla w_n|)dx \ge \int_\Omega R_u(x, w_n, y_n)(\varphi_1 - w_n)dx, \quad \forall \varphi_1 \in W_0^{1,\Phi_1}(\Omega). \tag{4.33}$$

From this inequality, we can conclude that

$$\int_\Omega \Phi_1(|\nabla w_n - \frac{1}{n}\nabla w_n|)dx - \int_\Omega \Phi_1(|\nabla w_n|)dx \ge -\frac{1}{n}\int_\Omega R_u(x, w_n, y_n)w_n dx,$$

i.e,

$$\int_\Omega \frac{\big(\Phi_1(|\nabla w_n - \frac{1}{n}\nabla w_n|) - \Phi_1(|\nabla w_n|)\big)}{-\frac{1}{n}}dx \le \int_\Omega R_u(x, w_n, y_n)w_n dx.$$

As (w_n) and (y_n) are bounded in $W_0^{1,\Phi_1}(\Omega)$ and $W_0^{1,\Phi_2}(\Omega)$, respectively, there will be $M > 0$ such that

$$\int_\Omega \frac{\Phi_1(|\nabla w_n - \frac{1}{n}\nabla w_n|) - \Phi_1(|\nabla w_n|)}{-\frac{1}{n}}dx \le M, \quad \forall n \in \mathbb{N}.$$

Since Φ_1 is in C^1 class, there exists $\theta_n(x) \in [0, 1]$ such that

$$\frac{\Phi_1(|\nabla w_n - \frac{1}{n}\nabla w_n|) - \Phi_1(|\nabla w_n|)}{-\frac{1}{n}} = \phi_1(|\big(1 - \frac{\theta_n(x)}{n}\big)\nabla w_n|)\big(1 - \frac{\theta_n(x)}{n}\big)|\nabla w_n|^2.$$

Recalling that $0 < 1 - \frac{\theta_n(x)}{n} \le 1$, we know that $1 - \frac{\theta_n(x)}{n} \ge \big(1 - \frac{\theta_n(x)}{n}\big)^2$ which leads to

$$\int_\Omega \phi_1(|\big(1 - \frac{\theta_n(x)}{n}\big)\nabla w_n|)\big(1 - \frac{\theta_n(x)}{n}\big)^2|\nabla w_n|^2 dx \le M, \quad \forall n \in \mathbb{N}.$$

As $\nabla w_n \xrightarrow{*} \nabla u$ in $\big(L^{\Phi_1}(\Omega)\big)^{N-1}$, we also have $\big(1-\frac{\theta_n(x)}{n}\big)\nabla w_n \xrightarrow{*} \nabla u$ in $\big(L^{\Phi_1}(\Omega)\big)^{N-1}$ as $n\to\infty$. Then, by using the fact that $\phi_1(t)t^2$ is convex, we can apply [18, Theorem 2.1, Chapter 8] to get

$$\liminf_{n\to\infty}\int_\Omega \phi_1(|\big(1-\frac{\theta_n(x)}{n}\big)\nabla w_n|)\big(1-\frac{\theta_n(x)}{n}\big)^2|\nabla w_n|^2 \geq \int_\Omega \phi_1(|\nabla u|)|\nabla u|^2 dx$$

and so,

$$\int_\Omega \phi_1(|\nabla w_n|)|w_n|^2 dx \leq M.$$

Recalling that

$$\phi_1(t)t^2 = \Phi_1(t) + \tilde{\Phi}_1(\phi_1(t)t), \quad \forall t \in \mathbb{R}$$

we have

$$\phi_1(|\nabla w_n|)|\nabla w_n|^2 = \Phi_1(|\nabla w_n|) + \tilde{\Phi}_1(\phi_1(|\nabla w_n|)|\nabla w_n|)$$

which leads to

$$\int_\Omega \phi_1(|\nabla w_n|)|\nabla w_n|^2 dx = \int_\Omega \Phi_1(|\nabla w_n|)dx + \int_\Omega \tilde{\Phi}_1(\phi_1(|\nabla w_n|)|\nabla w_n|^2)dx.$$

Since $\int_\Omega \phi_1(|\nabla u|)|\nabla u|^2 dx$ is finite, we see that $\int_\Omega \Phi_1(|\nabla u|)dx$ and $\int_\Omega \tilde{\Phi}_1(\phi_1(|\nabla u|)|\nabla u|^2)dx$ are also finite, showing that $u \in D(J_{\Phi_1})$ and $u \in dom(\phi_1(t)t)$. Furthermore, it follows from (4.30) and (4.33) that

$$\int_\Omega \Phi_1(|\nabla \varphi_1|)dx - \int_\Omega \Phi_1(|\nabla u|)dx \geq \int_\Omega R_u(x,u,v)(\varphi_1 - u)dx, \quad \forall \varphi_1 \in W_0^{1,\Phi_1}(\Omega). \tag{4.34}$$

On the other hand, it follows from the equality (4.32) that

$$\int_\Omega \Phi_2(|\nabla \varphi_2|)dx - \int_\Omega \Phi_2(|\nabla y_n|)dx \geq -\int_\Omega R_v(x,w_n,y_n)(\varphi_2 - y_n)dx, \quad \forall \varphi_2 \in V_k. \tag{4.35}$$

From this inequality, we can conclude that

$$\int_\Omega \Phi_2(|\nabla y_n - \frac{1}{n}\nabla w_n|)dx - \int_\Omega \Phi_2(|\nabla y_n|)dx \geq \frac{1}{n}\int_\Omega R_v(x,w_n,y_n)y_n dx,$$

i.e,

$$\int_\Omega \frac{\Phi_2(|\nabla y_n - \frac{1}{n}\nabla y_n|) - \Phi_2(|\nabla y_n|)}{-\frac{1}{n}}dx \leq -\int_\Omega R_v(x,w_n,y_n)y_n dx.$$

As (w_n) and (y_n) are bounded in $W_0^{1,\Phi_1}(\Omega)$ and $W_0^{1,\Phi_2}(\Omega)$, respectively, there will be $M>0$ such that

$$\int_\Omega \frac{\Phi_2(|\nabla y_n - \frac{1}{n}\nabla y_n|) - \Phi_2(|\nabla y_n|)}{-\frac{1}{n}}dx \leq M, \quad \forall n \in \mathbb{N}.$$

Since Φ_2 is in C^1 class, there exists $\theta_n(x) \in [0,1]$ such that

$$\frac{\Phi_2(|\nabla y_n - \frac{1}{n}\nabla y_n|) - \Phi_2(|\nabla y_n|)}{-\frac{1}{n}} = \phi_2(|\big(1-\frac{\theta_n(x)}{n}\big)\nabla y_n|)\big(1-\frac{\theta_n(x)}{n}\big)|\nabla y_n|^2.$$

Recalling that $0 < 1-\frac{\theta_n(x)}{n} \leq 1$, we know that $1-\frac{\theta_n(x)}{n} \geq \big(1-\frac{\theta_n(x)}{n}\big)^2$ which leads to

$$\int_\Omega \phi_2(|\big(1-\frac{\theta_n(x)}{n}\big)\nabla y_n|)\big(1-\frac{\theta_n(x)}{n}\big)^2|\nabla y_n|^2 dx \leq M, \quad \forall n \in \mathbb{N}.$$

As $\nabla y_n \overset{*}{\rightharpoonup} \nabla v$ in $\left(L^{\Phi_2}(\Omega)\right)^{N-1}$, we also have $\left(1-\frac{\theta_n(x)}{n}\right)\nabla y_n \overset{*}{\rightharpoonup} \nabla v$ in $\left(L^{\Phi_2}(\Omega)\right)^{N-1}$ as $n\to\infty$. Then, by using the fact that $\phi_2(t)t^2$ is convex, we can apply [18, Theorem 2.1, Chapter 8] to get

$$\liminf_{n\to\infty}\int_\Omega \phi_2(|\big(1-\frac{\theta_n(x)}{n}\big)\nabla y_n|)\big(1-\frac{\theta_n(x)}{n}\big)^2|\nabla y_n|^2 \geq \int_\Omega \phi_2(|\nabla v|)|\nabla v|^2dx$$

and so,

$$\int_\Omega \phi_2(|\nabla v|)|\nabla v|^2dx \leq M.$$

Recalling that

$$\phi_2(t)t^2 = \Phi_2(t) + \tilde{\Phi}_2(\phi_2(t)t), \ \ \forall t\in\mathbb{R}$$

we have which leads to

$$\int_\Omega \phi_2(|\nabla v|)|\nabla v|^2dx = \int_\Omega \Phi_2(|\nabla v|)dx + \int_\Omega \tilde{\Phi}_2(\phi_2(|\nabla v|)|\nabla v|^2)dx.$$

Since $\int_\Omega \phi_2(|\nabla v|)|\nabla v|^2dx$ is finite, we see that $\int_\Omega \Phi_2(|\nabla v|)dx$ and $\int_\Omega \tilde{\Phi}_2(\phi_2(|\nabla u|)|\nabla v|^2)dx$ are also finite, showing that $v\in D(J_{\Phi_2})$ and $v\in dom(\phi_2(t)t)$.

Now, for $\varphi\in W_0^1E^{\Phi_2}(\Omega)$, there exists $\chi_m\in V_m$ such that

$$\lim_{m\to\infty}\chi_m = \varphi \quad \text{in } W_0^1E^{\Phi_2}(\Omega). \tag{4.36}$$

From (4.32),

$$-\int_\Omega \phi_2(|\nabla y_n|)\nabla y_n\nabla(\chi_m - y_n)dx = \int_\Omega R_v(x,w_n,y_n)(\chi_m-y_n)dx, \ \ \forall n\geq m.$$

The convexity of Φ_2 implies that

$$\int_\Omega \Phi_2(|\nabla\chi_m|)dx - \int_\Omega \Phi_2(|\nabla y_n|)dx \geq -\int_\Omega R_v(x,w_n,y_n)(\chi_m-y_n)dx, \ \ \forall n\geq m. \tag{4.37}$$

Thus, by the limit (4.31) we have

$$\int_\Omega \Phi_2(|\nabla\chi_m|)dx - \int_\Omega \Phi_2(|\nabla v|)dx \geq -\int_\Omega R_v(x,u,v)(\chi_m-v)dx. \tag{4.38}$$

Now we use (4.36) in the above inequality to get

$$\int_\Omega \Phi_2(|\nabla\varphi|)dx - \int_\Omega \Phi_2(|\nabla v|)dx \geq -\int_\Omega R_v(x,u,v)(\varphi-v)dx. \tag{4.39}$$

Repeating the arguments used in Lemma 4.9, the inequalities (4.34) and (4.39) imply

$$\int_\Omega \phi_1(|\nabla u|)\nabla u\nabla\varphi_1dx = \int_\Omega R_u(x,u,v)\varphi_1dx, \ \ \forall\varphi_2\in W_0^{1,\Phi_1}(\Omega),$$

$$\int_\Omega \phi_2(|\nabla v|)\nabla v\nabla\varphi_2dx = -\int_\Omega R_v(x,u,v)\varphi_2dx, \ \ \forall\varphi_2\in W_0^1E^{\Phi_2}(\Omega).$$

Finally, the fact that $\phi_2(|\nabla v|)|\nabla v|\in L^{\tilde{\Phi}_2}(\Omega)$ together with the density weak* of $C_0^\infty(\Omega)$ in $W_0^{1,\Phi_2}(\Omega)$ give

$$\int_\Omega \phi_1(|\nabla u|)\nabla u\nabla\varphi_1dx - \int_\Omega \phi_2(|\nabla v|)\nabla v\nabla\varphi_2dx = \int_\Omega R_u(x,u,v)\varphi_1dx + \int_\Omega R_v(x,u,v)\varphi_2dx,$$

for every $(\varphi_1,\varphi_2)\in W_0^{1,\Phi_1}(\Omega)\times W_0^{1,\Phi_2}(\Omega)$. To conclude, the hypothesis (R_1') guarantees that (u,v) is a nontrivial solution for (S_2), and the proof is complete.

□

Appendix

Basics On Orlicz-Sobolev Spaces. In this section we recall some properties of Orlicz and Orlicz-Sobolev spaces, which can be found in [1,11,29]. First of all, we recall that a continuous function $\Phi : \mathbb{R} \to [0, +\infty)$ is a N-function if:

(i) Φ is convex;
(ii) $\Phi(t) = 0 \Leftrightarrow t = 0$;
(iii) Φ is even;
(iv) $\lim_{t\to 0} \frac{\Phi(t)}{t} = 0$ and $\lim_{t\to +\infty} \frac{\Phi(t)}{t} = +\infty$.

We say that a N-function Φ verifies the Δ_2-condition, and we denote by $\Phi \in (\Delta_2)$, if there are constants $K > 0,\ t_0 > 0$ such that

$$\Phi(2t) \leq K\Phi(t), \quad \forall t \geq t_0.$$

In the case of $|\Omega| = +\infty$, we will consider that $\Phi \in (\Delta_2)$ if $t_0 = 0$. For instance, it can be shown that $\Phi(t) = |t|^p/p$ for $p > 1$ satisfies the Δ_2-condition, while $\Phi(t) = (e^{t^2} - 1)/2$ does not verify it.

If Ω is an open set of $\mathbb{R}^N$, where N can be a natural number such that $N \geq 1$, and Φ a N-function then define the Orlicz space associated with Φ as

$$L^\Phi(\Omega) = \left\{ u \in L^1_{\text{loc}}(\Omega) : \int_\Omega \Phi\left(\frac{|u|}{\lambda}\right) dx < +\infty \text{ for some } \lambda > 0 \right\}.$$

The space $L^\Phi(\Omega)$ is a Banach space endowed with the Luxemburg norm given by

$$\|u\|_{L^\Phi(\Omega)} = \inf\left\{ \lambda > 0 : \int_\Omega \Phi\left(\frac{|u|}{\lambda}\right) dx \leq 1 \right\}.$$

In the case that Φ verifies Δ_2-condition we have

$$L^\Phi(\Omega) = \left\{ u \in L^1_{\text{loc}}(\Omega) : \int_\Omega \Phi(|u|) dx < +\infty \right\}.$$

The complementary function $\tilde{\Phi}$ associated with Φ is given by the Legendre transformation, that is,

$$\tilde{\Phi}(s) = \max_{t\geq 0}\{st - \Phi(t)\}, \quad \forall\, t \geq 0.$$

The functions Φ and $\tilde{\Phi}$ are complementary to each other and satisfy the inequality below

$$\tilde{\Phi}(\Phi'(t)) \leq \Phi(2t), \quad \forall\, t > 0.$$

Moreover, we also have a Young type inequality given by

$$st \leq \Phi(t) + \tilde{\Phi}(s), \quad \forall s, t \geq 0.$$

Using the above inequality, it is possible to establish the following Holder type inequality:

$$\left| \int_\Omega uv dx \right| \leq 2\|u\|_{L^\Phi(\Omega)}\|v\|_{L^{\tilde{\Phi}}(\Omega)}, \quad \text{for all } \ u \in L^\Phi(\Omega) \ \text{ and } \ v \in L^{\tilde{\Phi}}(\Omega).$$

The corresponding Orlicz-Sobolev space is defined by

$$W^{1,\Phi}(\Omega) = \left\{ u \in L^\Phi(\Omega) : \frac{\partial u}{\partial x_i} \in L^\Phi(\Omega), i = 1, ..., N \right\},$$

with the norm

$$\|u\|_{1,\Phi} = \|\nabla u\|_\Phi + \|u\|_\Phi.$$

The space $W_0^{1,\Phi}(\Omega)$ is defined as the weak* clousure of $C_0^\infty(\Omega)$ in $W^{1,\Phi}(\Omega)$. Moreover, by the Modular Poincaré's inequality

$$\int_\Omega \Phi(|u|/d)dx \leq \int_\Omega \Phi(|\nabla u|)dx, \ \forall u \in W_0^{1,\Phi}(\Omega), \tag{4.40}$$

where $d = 2diam(\Omega)$, and it follows that

$$\|u\|_\Phi \leq 2d\|\nabla u\|_\Phi, \ \forall u \in W_0^{1,\Phi}(\Omega).$$

The last inequality yields that the functional $\|\cdot\| := \|\nabla \cdot\|_\Phi$ defines an equivalent norm in $W_0^{1,\Phi}(\Omega)$. The spaces $L^\Phi(\Omega)$, $W^{1,\Phi}(\Omega)$ and $W_0^{1,\Phi}(\Omega)$ are separable and reflexive, when Φ and $\tilde{\Phi}$ satisfy Δ_2-condition .

If $|\Omega| < \infty$, the space $E^\Phi(\Omega)$ denotes the closing of $L^\infty(\Omega)$ in $L^\Phi(\Omega)$ with respect to the norm $\|\cdot\|_\Phi$. When $|\Omega| = \infty$, the space $E^\Phi(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $L^\Phi(\Omega)$ with respect to norm $\|\cdot\|_\Phi$. In any of these cases, $L^\Phi(\Omega)$ is the dual space of $E^{\tilde{\Phi}}(\Omega)$, while $L^{\tilde{\Phi}}(\Omega)$ is the dual space of $E^\Phi(\Omega)$. Moreover, $E^\Phi(\Omega)$ and $E^{\tilde{\Phi}}(\Omega)$ are separable and all continuous functional $M : E^\Phi(\Omega) \longrightarrow \mathbb{R}$ are of the form

$$M(v) = \int_\Omega v(x)g(x)dx, \quad \text{for some function } g \in L^{\tilde{\Phi}}(\Omega).$$

We recall that if Φ verifies the Δ_2-condition, we then have $E^\Phi(\Omega) = L^\Phi(\Omega)$.

The next result is crucial in the approach explored in Sections 3 and 4, and its proof follows directly from the Banach-Alaoglu-Bourbaki theorem [16].

Lemma 4.12. *Assume that Φ is a N-function. If $(u_n) \subset W_0^{1,\Phi}(\Omega)$ is a bounded sequence, then are a subsequence of (u_n), still denoted by itself, and $u \in W_0^{1,\Phi}(\Omega)$ such that*

$$u_n \xrightarrow{*} u \quad \text{in } L^\Phi(\Omega) \quad \text{and} \quad \frac{\partial u_n}{\partial x_i} \xrightarrow{*} \frac{\partial u}{\partial x_i} \quad \text{in } L^\Phi(\Omega) \tag{4.41}$$

and

$$\int_\Omega u_n v dx \to \int_\Omega uv dx, \quad \int_\Omega \frac{\partial u_n}{\partial x_i} w dx \to \int_\Omega \frac{\partial u}{\partial x_i} w dx, \ \forall v, w \in E^{\tilde{\Phi}}(\Omega).$$

We denote the limit (4.41) by $u_n \xrightarrow{*} u$ in $W_0^{1,\Phi}(\Omega)$. As an immediate consequence of the last lemma, we have the following corollary.

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