Local existence of compressible MHD equations without initial compatibility conditions

Yiying Zhang[1], Zhenhua Guo[2], and Li Fang[1]

[1]Northwest University School of Mathematics
[2]Guangxi University School of Mathematics and Information Science

April 28, 2024

Abstract

In this paper, we study the initial-boundary value problem of three-dimensional viscous, compressible, and heat conductive magnetohydrodynamic equations. Local existence and uniqueness of strong solutions is established with any such initial data that the initial compatibility conditions do not be required. The analysis is based on some suitable prior estimates for the strong coupling term $u\cdot$[?] H and strong nonlinear term curl H x H . Our proof of the existence and uniqueness of solutions is in the Lagrangian coordinates first and then transformed back to the Euler coordinates.

Received: Added at production | Revised: Added at production | Accepted: Added at production

DOI: xxx/xxxx

ARTICLE TYPE

Local existence of compressible MHD equations without initial compatibility conditions

Yiying Zhang[1] | **Zhenhua Guo**[2] | **Li Fang**[1]

[1]School of Mathematics, Northwest University, Xi'an, 710127, China

[2] School of Mathematics and Information Science, Guangxi University, Nanning, 530004, China

Correspondence

Li Fang, School of Mathematics, Northwest University, Xi'an 710127, China.

Email: fangli@nwu.edu.cn

Funding Information

This research was supported by the the National Natural Science Foundation of China, Grant/Award Number: 11931013; the Natural Science Foundation of Guangxi Province , Grant/Award Number: 2022GXNSFDA035078.

Abstract

In this paper, we study the initial-boundary value problem of three-dimensional viscous, compressible, and heat conductive magnetohydrodynamic equations. Local existence and uniqueness of strong solutions is established with any such initial data that the initial compatibility conditions do not be required. The analysis is based on some suitable prior estimates for the strong coupling term $u \cdot \nabla H$ and strong nonlinear term $\mathrm{curl}H \times H$. Our proof of the existence and uniqueness of solutions is in the Lagrangian coordinates first and then transformed back to the Euler coordinates.

KEYWORDS

Full compressible magnetohydrodynamic equations, Existence and uniqueness, Vacuum, Singular-in-time weighted estimates

MSC CLASSIFICATION

35Q55; 35Q30; 76N06; 76N10

1 | INTRODUCTION

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, the compressible and heat conductive magnetohydrodynamic flow viscous can be represented by the following compressible MHD equations [1].:

$$\begin{cases} \rho_t + \mathrm{div}\,(\rho u) = 0, \\ \rho u_t + \rho u \cdot \nabla u - \mu \Delta u - (\lambda + \mu)\nabla \mathrm{div} u + \nabla p = \mathrm{curl} H \times H, \\ c_v \rho\,(\theta_t + u \cdot \nabla \theta) + p\ \mathrm{div}\ u - \kappa \triangle \theta = Q(\nabla u) + \nu \left|\mathrm{curl} H\right|^2, \\ H_t - H \cdot \nabla u + u \cdot \nabla H + H\ \mathrm{div}\ u = \nu \Delta H, \\ \mathrm{div} H = 0, \end{cases} \tag{1.1}$$

where the unknowns $\rho \geq 0$, $u \in \mathbb{R}^3$, $\theta \geq 0$, $H \in \mathbb{R}^3$, respectively, are the density, velocity, absolute temperature, and magnetic field; $p = R\rho\theta$, with positive constant R, is the pressure. And

$$Q(\nabla u) = \frac{\mu}{2}\left|\nabla u + (\nabla u)^T\right|^2 + \lambda\,(\mathrm{div}\ u)^2, \tag{1.2}$$

with $(\nabla u)^T$ being the transpose of ∇u. The constant viscosity coefficients μ and λ satisfy the physical restrictions

$$\mu > 0,\ 2\mu + 3\lambda \geq 0. \tag{1.3}$$

Positive constants c_ν, κ, and ν are the heat capacity, the ratio of the heat conductivity coefficient over the heat capacity, and the magnetic diffusive coefficient, respectively.

Abbreviations: MHD, magnetohydrodynamic; ODE,Oridinary differential equation.

In this paper, we consider the initial-boundary value problem. The initial and boundary conditions read as:

$$(\rho, H, \rho u, \rho\theta)|_{t=0} = (\rho_0, H_0, \rho_0 u_0, \rho_0\theta_0), \tag{1.4}$$

$$u|_{\partial\Omega} = 0,\ \nabla\theta \cdot n\big|_{\partial\Omega} = 0,\ H \cdot n\big|_{\partial\Omega} = \operatorname{curl} H \times n|_{\partial\Omega} = 0, \tag{1.5}$$

where $\rho_0, u_0, H_0, \theta_0$ are given functions.

The Magnetohydrodynamic Equation is used to simulate the motion of conductive fluids under the action of electromagnetic fields, and has wide applications in fields such as astrophysics and plasma[2,3,4]. It is a coupling model of the Navier-Stokes equation of fluid dynamics and the Maxwell electromagnetic equation[5,6,7]. Compared with the compressible Navier-Stokes equations, the study of the well-posedness of the compressible MHD equations is quite complex due to the strong coupling and interaction between fluid motion and magnetic field. Effective methods need to be found to handle strong coupling and strong nonlinearity terms, whose unique characteristics make analytical research a huge challenge but also provide new opportunities[8,9,10]. There has been many works on the mathematical studies the compressible MHD equations in fluid dynamics. Li, Xu, and Zhang studied the global existence of strong solutions for three-dimensional MHD equations under the condition of low total energy of initial data under isentropic conditions[11]. By removing the cricial assumption that the initial total energy is small, Hong et al. proved that as long as the adiabatic index approaches 1 and ν is suitably large, the global classical strong solutions holds[12]. Fan and Yu studied the existence and uniqueness of local strong solutions for three-dimensional MHD equations with vacuum under non isentropic conditions[13]. Liu and Zhong proposed the global existence and uniqueness of strong solutions for the Cauchy problem of a three-dimensional fully compressible MHD system with vacuum[14].

There are already many well-posedness results on compressible Navior-Stokes equations[15,16,17,18,19,20], however, there are few well-posedness results for compressible Navior-Stokes equations without initial compatibility conditions. The following are some research results on the local well-posedness theory of Navier-Stokes equations without initial compatibility conditions. Li established the local well-posedness theory for non-uniform incompressible Navier-Stokes equations without initial compatibility conditions[21]. Gong et al. established the local well-posedness for the isentropic compressible Navier-Stokes equation without any initial compatibility conditions[22]. Lai, Xu, and Zhang established the local well-posedness for full compressible Navier-Stokes equations that satisfies partial compatibility conditions[23]. Li and Zheng obtained the local existence and uniqueness of the compressible Navier-Stokes equation under vacuum and no initial compatibility conditions[24]. The well-posedness theory based on initial compatibility conditions imposes additional limitations on the initial data and cannot fully meet physical requirements. Inspired by the local well-posedness theory of compressible Navier-Stokes equations without any initial compatibility conditions, this paper will further investigate the existence of local strong solutions for a class of three-dimensional compressible MHD equations without initial compatibility conditions.

The strong solutions to be established in this paper are defined as follows.

Definition 1. Given a positive time $T \in (0, \infty)$ and let $q \in (3, 6)$. Assume that θ_0 is no-negative, Lebesgue measurable, and finitely valued a.e. in Ω, and that

$$0 \le \rho_0 \in W^{1,q}(\Omega), \quad u_0 \in H_0^1(\Omega), \quad \sqrt{\rho_0}\theta_0 \in L^2(\Omega), \quad H_0 \in H^2(\Omega).$$

(ρ, u, H, θ) is called a strong solution to system (1.1) in $\Omega \times (0, T)$, subject to (1.4)-(1.5), if it has the regularities

$$\begin{aligned}
&0 \le \rho \in C\left([0,T];L^2\right) \cap L^\infty\left(0,T;W^{1,q}\right), \quad \rho_t \in L^\infty\left(0,T;L^2\right), \quad \sqrt{\rho}u \in C\left([0,T];L^2\right),\\
&u \in L^\infty\left(0,T;H_0^1\right) \cap L^2\left(0,T;H^2\right) \cap L^1\left(0,T;W^{2,q}\right), \quad H \in L^\infty\left(0,T;H^1\right) \cap L^2\left(0,T;H^2\right) \cap L^1\left(0,T;W^{2,q}\right),\\
&\sqrt{\rho}u_t \in L^2\left(0,T;L^2\right), \quad \sqrt{t}u \in L^\infty\left(0,T;H^2\right) \cap L^2\left(0,T;W^{2,q}\right), \quad \sqrt{t}H \in L^\infty\left(0,T;H^2\right) \cap L^2\left(0,T;W^{2,q}\right),\\
&\sqrt{t}u_t \in L^2\left(0,T;H_0^1\right), \quad \sqrt{t}H_t \in L^2\left(0,T;H_0^1\right), \quad \sqrt{\rho}\theta \in C\left([0,T];L^2\right), \quad 0 \le \theta \in L^2\left(0,T;H_0^1\right),\\
&\sqrt{t}\theta \in L^\infty\left(0,T;H_0^1\right) \cap L^2\left(0,T;H^2\right), \quad \sqrt{t}\sqrt{\rho}\theta_t \in L^2\left(0,T;L^2\right), \quad t\theta \in L^2\left(0,T;W^{2,6}\right), \quad t\theta_t \in L^2\left(0,T;H_0^1\right),
\end{aligned}$$

and satisfies equations (1.1) with the initial condition (1.4) a.e. in $\Omega \times (0, T)$.

Our main result can be stated as follows.

Theorem 1. *Let $q \in (3, 6)$. Assume that θ_0 is non-negative, Lebesgue measurable, and finitely valued a.e. in Ω,*

$$0 \le \rho_0 \in W^{1,q}(\Omega), \quad u_0 \in H_0^1(\Omega), \quad \sqrt{\rho_0}\theta_0 \in L^2(\Omega), \quad H_0 \in H^2(\Omega).$$

and that ρ_0 is not identically zero. Then, there exists a positive time T_0 depending only on R, μ, λ, c_v, q, and Φ_0, such that system (1.1), subject to (1.4)-(1.5), has a unique strong solution in $\Omega \times (0, T_0)$, where $\Phi_0 := \|\rho_0\|_\infty + \|\nabla\rho_0\|_q + \left\|\left(\sqrt{\rho_0}\theta_0, \nabla u_0, \nabla H_0\right)\right\|_2^2$.

Remark 1. The arguments presented in this paper with slightly modifications work also for the Cauchy problem and similar result as in Theorem 1 still holds, with the assumptions on u_0 and θ_0 replaced by $(u_0, \sqrt{\rho_0}\theta_0) \in D_0^1 \cap D^2$.

Our method of proof builds up the framework developed by Li and Zheng for compressible Navier-Stokes equations in [24]. Considering the influence of magnetic fields, the key of proving the existence part of Theorem 1 is to carry out some suitable a priori estimates of the following quantity.

$$\begin{aligned}\Phi(t) :=& \int_0^T \left\|\left(\sqrt{\rho}u_t, \nabla^2 u, \nabla^2 H, H_t, \sqrt{s}\nabla u_t, \sqrt{s}\nabla H_t, \nabla\theta, \sqrt{s}\sqrt{\rho}\theta_t, \sqrt{s}\nabla^2\theta\right)\right\|_2^2 \,\mathrm{d}s \\ &\sup_{0\le t\le T}\left(\|\rho\|_\infty + \|\nabla\rho\|_q + \left\|\left(\sqrt{\rho}\theta, \nabla u, \nabla H, \sqrt{t}\nabla\theta, \sqrt{s}\nabla^2 u, \sqrt{s}\nabla^2 H\right)\right\|_2^2\right) + 1. \end{aligned} \tag{1.6}$$

for any approximate solution (ρ, u, H, θ) to system (1.1), subject to (1.4)-(1.5). Compared to [24], one of the main difficulties in our study is to establish the estimates of the item $u \cdot \nabla H$ and $\mathrm{curl}H \times H$.

The structure of the paper is the following. In the next section, we derive some a priori estimates for the strong solutions to system (1.1), subject to (1.4)-(1.5). Section 3 is devoted to proving our main result Theorem 1 in the Lagrangian coordinates first and then transformed back to the Euler coordinates. Finally, the specific proof process of the a preparing existence result Proposition 11 will be provided in Appendix.

2 | A PRIORI ESTIMATES INDEPENDENT OF COMPATIBILITY CONDITIONS

The aim of this section is to derive some a priori estimates for the strong solutions to system (1.1), subject to (1.4)-(1.5). The a priori estimates established in this section do not depend on initial compatibility conditions, which is crucial to finally establish the existence of strong solutions without any compatibility conditions.

The following local existence and uniqueness of strong solutions system (1.1) with (1.4)-(1.5) has been established in [13], where the compatibility conditions are required.

Lemma 1. *Let $q \in (3, 6]$ and assume that $(\rho_0, u_0, H_0, \theta_0)$ satisfies*

$$0 \le \rho_0 \in W^{1,q}(\Omega),\ u_0 \in H_0^1(\Omega) \cap H^2(\Omega),\ 0 \le \theta_0 \in H_0^1(\Omega) \cap H^2(\Omega),\ H_0 \in H^2(\Omega),$$

and the compatibility conditions

$$\begin{aligned}-\mu\Delta u_0 - (\mu + \lambda)\nabla \operatorname{div} u_0 + \nabla\left(R\rho_0\theta_0\right) - \mathrm{curl}H_0 \times H_0 &= \sqrt{\rho_0}g_1, \\ \kappa\Delta\theta_0 + \frac{\mu}{2}\left|\nabla u_0 + (\nabla u_0)^T\right|^2 + \lambda\left(\operatorname{div} u_0\right)^2 - \nu|\mathrm{curl}H_0|^2 &= \sqrt{\rho_0}g_2,\end{aligned}$$

for some $g_1, g_2 \in L^2(\Omega)$. Then, there exists a positive time T_* depending on R, μ, λ, c_v, q, $\|\nabla^2 u_0\|_2$, $\|\nabla^2 H_0\|_2$, $\|\nabla^2\theta_0\|_2$, $\|g_1\|_2$, and $\|g_2\|_2$, such that system (1.1), subject to (1.4)–(1.5), admits a unique strong solution (ρ, u, H, θ) in $\Omega \times (0, T_*)$, satisfying

$$\begin{aligned}&\rho \in C\left([0, T_*]; W^{1,q}\right), \quad \rho_t \in C\left([0, T_*]; L^q\right), (u_t, H_t, \theta_t) \in L^2\left(0, T_*; H_0^1\right), \\ &\left(\sqrt{\rho}u_t, \sqrt{\rho}\theta_t, H_t\right) \in L^\infty\left(0, T_*; L^2\right), (u, H, \theta) \in C\left([0, T_*]; H_0^1 \cap H^2\right) \cap L^2\left(0, T_*; W^{2,q}\right).\end{aligned}$$

It will be shown in this section that the existence time T_* in the above proposition can be chosen depending only on R, μ, λ, c_v, q, and the upper bound of

$$\Phi_0 := \|\rho_0\|_\infty + \|\nabla\rho_0\|_q + \left\|\left(\sqrt{\rho_0}\theta_0, \nabla u_0, \nabla H_0\right)\right\|_2^2.$$

In particular, T_* can be chosen independent of $\|\nabla^2 u_0\|_2$, $\|\nabla^2 H_0\|_2$, $\|\nabla^2\theta_0\|_2$, $\|g_1\|_2$, and $\|g_2\|_2$. Let Φ be the quantity given by (1.6). The main issue of this section is to derive the local in time estimate of Φ independent of $\|\nabla^2 u_0\|_2$, $\|\nabla^2 H_0\|_2$, $\|\nabla^2\theta_0\|_2$, $\|g_1\|_2$, and $\|g_2\|_2$, and therefore independent of the initial compatibility conditions. In the rest of this section, we assume that (ρ, u, H, θ) is a solution to system (1.1), subject to (1.4)–(1.5), in $\Omega \times (0, T)$, for some positive time $T \le 1$, satisfying the regularities in Lemma 1 with T_* there replaced by T. For simplicity, we use the conventions that C denote positive constants depending only on R, μ, λ, c_v, q, and the upper bound of Φ_0.

Proposition 1. *Denote $M_0 := \int_\Omega \rho_0 \mathrm{d}x > 0$, then it holds that*

$$\|\theta\|_2^2 \le C\left(\|\sqrt{\rho}\theta\|_2^2 + \|\rho\|_\infty^2 \|\nabla\theta\|_2^2\right) \tag{2.1}$$

for a positive constant C depending only on M_0 and Ω.

The proof is similar to the proof of Proposition 2.1 in [24], and so we omit it.

Proposition 2. *It holds that*

$$\int_0^T \left(\|\nabla u\|_\infty + \left\|\nabla^2 u\right\|_q\right) \mathrm{d}t \le CT^{\frac{6-q}{4q}} \Phi^{\frac{5}{2}}(T). \tag{2.2}$$

Proof. Applying the elliptic estimates to $(1.1)_2$ yields

$$\left\|\nabla^2 u\right\|_q \le C(\|\rho u_t\|_q + \|\rho\,(u.\nabla)\,u\|_q + \|\nabla P\|_q + \|\mathrm{curl} H \times H\|_q). \tag{2.3}$$

Integrating (2.3) on $[0, T]$, we have

$$\int_0^T \left\|\nabla^2 u\right\|_q \mathrm{d}t \le C\int_0^T \|\rho u_t\|_q \mathrm{d}t + \int_0^T \|\rho\,(u\cdot\nabla)\,u\|_q \mathrm{d}t + \int_0^T \|\nabla P\|_q \mathrm{d}t + \int_0^T \|\mathrm{curl} H \times H\|_q \mathrm{d}t. \tag{2.4}$$

Based on Proposition 1, one finds that

$$\begin{aligned}
\int_0^T \|\rho u_t\|_q \mathrm{d}t \le& C\int_0^T \|\rho\|_\infty^{\frac{5q-6}{4q}} \|\sqrt{\rho}u_t\|_2^{\frac{6-q}{2q}} \|\sqrt{\rho}u_t\|_6^{\frac{3q-6}{2q}} \mathrm{d}t\\
\le& C\Phi^{\frac{5q-6}{4q}}(T)\left(\int_0^T \|\sqrt{\rho}u_t\|_2^2 \mathrm{d}t\right)^{\frac{6-q}{4q}} \left(\int_0^T \|\sqrt{t}\nabla u_t\|_2^2 \mathrm{d}t\right)^{\frac{3q-6}{4q}} T^{\frac{6-q}{4q}}\\
\le& CT^{\frac{6-q}{4q}}\Phi^{\frac{7q-6}{4q}}(T),\\
\int_0^T \|\rho\,(u\cdot\nabla)\,u\|_q \mathrm{d}t \le& C\int_0^T \|\rho\|_\infty \|\nabla u\|_2^{\frac{1}{2}} \left\|\nabla^2 u\right\|_2^{\frac{3}{2}} \mathrm{d}t\\
\le& C\Phi^{\frac{5}{4}}(T)\left(\int_0^T \left\|\nabla^2 u\right\|_2^2 \mathrm{d}t\right)^{\frac{3}{4}} T^{\frac{1}{4}}\\
\le& CT^{\frac{1}{4}}\Phi^2(T),\\
\int_0^T \|\nabla P\|_q \mathrm{d}t \le& C\int_0^T (\|\nabla\rho\|_q + \|\rho\|_\infty)(\|\sqrt{\rho}\theta\|_2 + \|\rho\|_\infty \|\nabla\theta\|_2 + \|\nabla\theta\|_2^{\frac{6-q}{2q}} \left\|\nabla^2\theta\right\|_2^{\frac{3q-6}{2q}})\mathrm{d}t\\
\le& CT\Phi^{\frac{3}{2}}(T) + CT\Phi^2(T)\left(\int_0^T \|\nabla\theta\|_2^2 \mathrm{d}t\right)^{\frac{1}{2}} T^{\frac{1}{2}}\\
&+ CT\Phi(T)\left(\int_0^T \|\nabla\theta\|_2^2 \mathrm{d}t\right)^{\frac{6-q}{4q}} \left(\int_0^T \left\|\sqrt{t}\nabla^2\theta\right\|_2^2 \mathrm{d}t\right)^{\frac{3q-6}{4q}} T^{\frac{1}{2}}\\
\le& CT^{\frac{1}{2}}\Phi^{\frac{2}{5}}(T),\\
\int_0^T \|\mathrm{curl} H \times H\|_q \mathrm{d}t \le& C\int_0^T (\|(H\cdot\nabla)H\|_q + \left\|\nabla|H|^2\right\|_q)\mathrm{d}t\\
\le& C\int_0^T \|\nabla H\|_2^{\frac{1}{2}} \left\|\nabla^2 H\right\|_2^{\frac{3}{2}} \mathrm{d}t\\
\le& CT\Phi^{\frac{1}{4}}(T)\left(\int_0^T \|\nabla H\|_2^2 \mathrm{d}t\right)^{\frac{3}{4}} T^{\frac{1}{4}}\\
\le& CT^{\frac{1}{4}}\Phi(T).
\end{aligned}$$

Inserting the above estimates into (2.4), one gets that

$$\begin{aligned}\int_0^T \left\|\nabla^2 u\right\|_q dt &\leq C(T^{\frac{6-q}{4q}}\Phi^{\frac{7q-6}{4q}}(T) + T^{\frac{1}{4}}\Phi^2(T) + T^{\frac{1}{2}}\Phi^{\frac{2}{5}}(T) + T^{\frac{1}{4}}\Phi(T))\\ &\leq CT^{\frac{6-q}{4q}}\Phi^{\frac{5}{2}}(T),\end{aligned}$$

where $\frac{6-q}{4q} \leq \frac{1}{4}$ for $q \in (3,6)$, $T \leq 1$, and $\Phi(T) \geq 1$ are used. Thus

$$\int_0^T \left(\|\nabla u\|_\infty + \left\|\nabla^2 u\right\|_q\right) \mathrm{d}t \leq CT^{\frac{6-q}{4q}}\Phi^{\frac{5}{2}}(T).$$

□

With the help of Proposition 2, one follows the similar argument in the proof of Proposition 2.3 in [24] to obtain the following Proposition 3 and Corollary 1.

Proposition 3. *It holds that*

$$\begin{aligned}&\sup_{0\leq t\leq T}\|\rho\|_\infty \leq \|\rho_0\|_\infty \exp\left\{CT^{\frac{6-q}{4q}}\Phi^{\frac{5}{2}}(T)\right\},\\ &\sup_{0\leq t\leq T}\|\rho\|_{W^{1,q}} \leq (1 + T^{\frac{6-q}{4q}}\Phi^{\frac{5}{2}}(T))\exp\left\{CT^{\frac{6-q}{4q}}\Phi^{\frac{5}{2}}(T)\right\}.\end{aligned}$$

Corollary 1. *There is a sufficiently small positive constant $\epsilon_0 \leq 1$ depending only on R, μ, λ, c_v, q, and ϕ_0, such that*

$$\begin{aligned}&\sup_{0\leq t\leq T}\|\rho\|_\infty \leq 2\|\rho_0\|_\infty,\\ &\sup_{0\leq t\leq T}\|\rho\|_{W^{1,q}} \leq C,\end{aligned}$$

as long as

$$T^{\frac{6-q}{4q}}\Phi^{\frac{5}{2}}(T) \leq \epsilon_0. \tag{2.5}$$

Since $\Phi(T) \geq 1$, $T \leq 1$, and by the assumption (2.5) it is easy to check that the following relations hold:

$$T\Phi^3(T) = (T^{\frac{6-q}{4q}}\Phi^{\frac{5}{2}}(T))^{\frac{6}{5}}T^{\frac{13q-18}{10q}} \leq \epsilon_0^{\frac{6}{5}} \leq 1, \tag{2.6}$$

$$T^{\frac{1}{2}}\Phi^2(T) = (T^{\frac{6-q}{4q}}\Phi^{\frac{5}{2}}(T))^{\frac{4}{5}}T^{\frac{7q-12}{10q}} \leq \epsilon_0^{\frac{4}{5}} \leq 1, \tag{2.7}$$

$$T^{\frac{1}{4}}\Phi(T) = (T^{\frac{6-q}{4q}}\Phi^{\frac{5}{2}}(T))^{\frac{2}{5}}T^{\frac{11q-6}{10q}} \leq \epsilon_0^{\frac{2}{5}} \leq 1, \tag{2.8}$$

$$T^{\frac{1}{4}}\Phi^{\frac{3}{2}}(T) = (T^{\frac{6-q}{4q}}\Phi^{\frac{5}{2}}(T))^{\frac{3}{5}}T^{\frac{4q-9}{10q}} \leq \epsilon_0^{\frac{3}{5}} \leq 1. \tag{2.9}$$

Proposition 4. *Let ϵ_0 be the number stated in Corollary 1 and assume that (2.5) holds. Then, the following estimate holds*

$$\sup_{0\leq t\leq T}\|\nabla H\|_2^2 + \int_0^T (\|H_t\|_2^2 + \left\|\nabla^2 H\right\|_2^2)\mathrm{d}t \leq C. \tag{2.10}$$

Proof. Using the Hölder's, Gagliardo-Nirenberg and Young inequalities to $(1.1)_4$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla H\|_2^2 + \|H_t\|_2^2 + \left\|\nabla^2 H\right\|_2^2 &\leq C\int_\Omega |H_t - \nu\Delta H|^2 \mathrm{d}x\\ &= C\int_\Omega \left|H\cdot\nabla u - u\cdot\nabla H - H\mathrm{div}u\right|^2 \mathrm{d}x\\ &\leq C\|\nabla u\|_2^2\|H\|_\infty^2 + C\|u\|_6^2\|\nabla H\|_3^2\\ &\leq C\|\nabla u\|_2^2\|\nabla H\|_2\left\|\nabla^2 H\right\|_2\\ &\leq \frac{1}{2}\|\nabla^2 H\|_2 + C\|\nabla u\|_2^4\|\nabla H\|_2^2.\end{aligned}$$

So,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla H\|_2^2 + \|H_t\|_2^2 + \frac{1}{2}\left\|\nabla^2 H\right\|_2^2 \le C\,\|\nabla u\|_2^4\,\|\nabla H\|_2^2\,. \tag{2.11}$$

Integrating (2.11) over $(0,T)$, one has

$$\sup_{0\le t\le T}\|\nabla H\|_2^2 + \int_0^T (\|H_t\|_2^2 + \left\|\nabla^2 H\right\|_2^2)\mathrm{d}t \le C\int_0^T \|\nabla u\|_2^4\,\|\nabla H\|_2^2\,\mathrm{d}t \le CT\Phi^3(T) \le C,$$

where in the last step (2.6) were used. □

Proposition 5. *Under the assumptions of Proposition 4, it holds that*

$$\sup_{0\le t\le T}\|\sqrt{\rho}\theta\|_2^2 + \int_0^T \|\nabla\theta\|_2^2\,\mathrm{d}t \le C. \tag{2.12}$$

Proof. One multiplies $(1.1)_3$ with θ and integrates it over Ω to get

$$\frac{c_v}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}\theta\|_2^2 + \kappa\|\nabla\theta\|_2^2 = -\int_\Omega \mathrm{div}uP\theta\mathrm{d}x + \int_\Omega \mathcal{Q}(\nabla u)\theta\mathrm{d}x + \int_\Omega \nu\left|\mathrm{cul}H\right|^2\theta\mathrm{d}x. \tag{2.13}$$

By Proposition 1 and Corollary 1, terms on the right-hand side of (2.13) are estimated as

$$\begin{aligned}
\int_\Omega \mathrm{div}\,uP\theta\mathrm{d}x &\le R\int_\Omega \rho|\theta|^2|\nabla u|\mathrm{d}x \le C\|\nabla u\|_\infty\|\sqrt{\rho}\theta\|_2^2,\\
\int_\Omega \mathcal{Q}(\nabla u)\theta\mathrm{d}x &\le C\|\nabla u\|_2^{\frac{3}{2}}\left\|\nabla^2 u\right\|_2^{\frac{1}{2}}(\|\sqrt{\rho}\theta\|_2 + \|\nabla\theta\|_2)\\
&\le \frac{\kappa}{4}\|\nabla\theta\|_2^2 + C\left(\|\sqrt{\rho}\theta\|_2^2 + \|\nabla u\|_2^3\left\|\nabla^2 u\right\|_2\right),\\
\int_\Omega |\mathrm{curl}H|^2\theta\mathrm{d}x &\le \int_\Omega |\nabla H|^2\theta\mathrm{d}x\\
&\le C\|\nabla^2 H\|_2\|\nabla H\|_3\|\theta\|_6\\
&\le C\|\nabla^2 H\|_2^{\frac{3}{2}}\|\nabla H\|_2^{\frac{1}{2}}(\|\sqrt{\rho}\theta\|_2 + \|\nabla\theta\|_2)\\
&\le \frac{\kappa}{4}\|\nabla\theta\|_2^2 + C\left(\|\sqrt{\rho}\theta\|_2^2 + \|\nabla H\|_2^3\left\|\nabla^2 H\right\|_2\right).
\end{aligned}$$

Thus,

$$c_v\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}\theta\|_2^2 + \kappa\|\nabla\theta\|_2^2 \le C(1+\|\nabla u\|_\infty)\|\sqrt{\rho}\theta\|_2^2 + C\|\nabla u\|_2^3\left\|\nabla^2 u\right\|_2 + C\|\nabla H\|_2^3\left\|\nabla^2 H\right\|_2. \tag{2.14}$$

So, it is deduced from Proposition 2 and Corollary 1 that

$$\begin{aligned}
&\sup_{0\le t\le T} c_v\|\sqrt{\rho}\theta\|_2^2 + \kappa\int_0^T \|\nabla\theta\|_2^2\,\mathrm{d}t\\
\le&\left(c_v\,\|\sqrt{\rho_0}\theta_0\|_2^2 + C\int_0^T \|\nabla u\|_2^3\left\|\nabla^2 u\right\|_2 + C\int_0^T \|\nabla H\|_2^3\left\|\nabla^2 H\right\|_2\mathrm{d}t\right)e^{C\int_0^T(1+\|\nabla u\|_\infty)\mathrm{d}t}\\
\le&C\left(1+T^{\frac{1}{2}}\Phi^2(T)\right)e^{C\left(1+T^{\frac{6-q}{4q}}\Phi^{\frac{5}{2}}(T)\right)}\\
\le&C,
\end{aligned}$$

where in the last step (2.5) and (2.7) were used. □

Proposition 6. *Under the assumptions of Proposition 4, it holds that*

$$\sup_{0\le t\le T}\|\nabla u\|_2^2 + \int_0^T \left(\|\sqrt{\rho}u_t\|_2^2 + \left\|\nabla^2 u\right\|_2^2\right)\mathrm{d}t \le C. \tag{2.15}$$

Proof. One deduces from Proposition 1 and Corollary 1 that

$$
\begin{aligned}
\left\|\nabla^2 u\right\|_2^2 &\le C\left(\|\rho u_t\|_2^2+\|\rho(u\cdot\nabla)u\|_2^2+\|\nabla P\|_2^2+\|\mathrm{curl}H\times H\|_2^2\right)\\
&\le C\left(\|\rho\|_\infty\|\sqrt{\rho}u_t\|_2^2+\|\rho\|_\infty^2\|u\|_6^2\|\nabla u\|_3^2+\|\nabla\rho\|_3^2\|\theta\|_6^2+\|\rho\|_\infty^2\|\nabla\theta\|_2^2+\|H\|_6^2\|\nabla H\|_3^2\right)\\
&\le C\left(\|\sqrt{\rho}u_t\|_2^2+\|\nabla u\|_2^3\left\|\nabla^2 u\right\|_2+\|\nabla H\|_2^3\left\|\nabla^2 H\right\|_2+\|\sqrt{\rho}\theta\|_2^2+\|\nabla\theta\|_2^2\right)\\
&\le \frac12\left\|\nabla^2 u\right\|_2^2+C\left(\|\sqrt{\rho}u_t\|_2^2+\|\nabla u\|_2^6+\|\nabla H\|_2^3\left\|\nabla^2 H\right\|_2+\|\sqrt{\rho}\theta\|_2^2+\|\nabla\theta\|_2^2\right).
\end{aligned}
$$

Thus,

$$
\left\|\nabla^2 u\right\|_2^2 \le C\left(\|\sqrt{\rho}u_t\|_2^2+\|\nabla u\|_2^6+\|\nabla H\|_2^3\left\|\nabla^2 H\right\|_2+\|\sqrt{\rho}\theta\|_2^2+\|\nabla\theta\|_2^2\right). \tag{2.16}
$$

Moreover, $(1.1)_3$ implies

$$
P_t=\frac{R}{c_v}\left(\mathcal{Q}(\nabla u)+\nu|\mathrm{curl}H|^2+\kappa\Delta\theta-P\,\mathrm{div}\,u-c_v\,\mathrm{div}(\rho u\theta)\right). \tag{2.17}
$$

So, one integrates (2.17) to get

$$
\begin{aligned}
\int_\Omega \nabla P u_t\,\mathrm{d}x &= -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega P\,\mathrm{div}\,u\mathrm{d}x+\int_\Omega P_t\,\mathrm{div}\,u\mathrm{d}x\\
&= -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega P\,\mathrm{div}\,u\,\mathrm{d}x+\frac{R}{c_v}\int_\Omega \mathrm{div}\,u\left(\mathcal{Q}(\nabla u)+\nu|\mathrm{curl}H|^2+\kappa\Delta\theta-P\,\mathrm{div}\,u-c_v\,\mathrm{div}(\rho u\theta)\right)\mathrm{d}x\\
&= -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega P\,\mathrm{div}\,u\,\mathrm{d}x+\frac{R}{c_v}\int_\Omega \mathrm{div}\,u\mathcal{Q}(\nabla u)\mathrm{d}x+\frac{\nu R}{c_v}\int_\Omega \mathrm{div}\,u|\mathrm{curl}H|^2\mathrm{d}x\\
&\quad-\frac{\kappa R}{c_v}\int_\Omega \nabla\,\mathrm{div}\,u\cdot\nabla\theta\mathrm{d}x-\frac{R}{c_v}\int_\Omega P(\mathrm{div}\,u)^2\,\mathrm{d}x+R\int_\Omega \rho\theta u\cdot\nabla\,\mathrm{div}\,u\,\mathrm{d}x.
\end{aligned}
$$

Multiplying $(1.1)_2$ with u_t, integrating over Ω, and using the above identity, one obtains that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mu}{2}\|\nabla u\|_2^2+\frac{\mu+\lambda}{2}\|\,\mathrm{div}\,u\|_2^2-\int_\Omega P\,\mathrm{div}\,u\,\mathrm{d}x-\int_\Omega \mathrm{curl}H\times H\cdot u\,\mathrm{d}x\right)+\|\sqrt{\rho}u_t\|_2^2\\
&=-\int_\Omega \rho(u\cdot\nabla)u\cdot u_t\,\mathrm{d}x-\frac{R}{c_v}\int_\Omega \mathrm{div}\,u\mathcal{Q}(\nabla u)\mathrm{d}x-\frac{\nu R}{c_v}\int_\Omega \mathrm{div}\,u|\mathrm{curl}H|^2\mathrm{d}x+\frac{\kappa R}{c_v}\int_\Omega \nabla\,\mathrm{div}\,u\cdot\nabla\theta\mathrm{d}x\\
&\quad+\frac{R}{c_v}\int_\Omega P(\mathrm{div}\,u)^2\,\mathrm{d}x-R\int_\Omega \rho\theta u\cdot\nabla\,\mathrm{div}\,u\,\mathrm{d}x-\int_\Omega(\mathrm{curl}H\times H)_t\cdot u\,\mathrm{d}x.
\end{aligned} \tag{2.18}
$$

Each term on the right hand of (2.18) is estimated as follows

$$
\begin{aligned}
\left|\int_\Omega \rho(u\cdot\nabla)u\cdot u_t\,\mathrm{d}x\right| &\le C\|\nabla u\|_2^{\frac32}\left\|\nabla^2 u\right\|_2^{\frac12}\|\sqrt{\rho}u_t\|_2\\
&\le \frac18\|\sqrt{\rho}u_t\|_2^2+\eta\left\|\nabla^2 u\right\|_2^2+C_\eta\|\nabla u\|_2^6,\\
\left|\frac{R}{c_v}\int_\Omega \mathrm{div}\,u\mathcal{Q}(\nabla u)\mathrm{d}x\right| &\le C\|\nabla u\|_2^{\frac32}\left\|\nabla^2 u\right\|_2^{\frac32}\\
&\le \eta\left\|\nabla^2 u\right\|_2^2+C_\eta\|\nabla u\|_2^6,\\
\left|\frac{\nu R}{c_v}\int_\Omega \mathrm{div}\,u|\mathrm{curl}H|^2\,\mathrm{d}x\right| &\le C\int_\Omega |\nabla u||\nabla H|^2\,\mathrm{d}x\\
&\le C\|\nabla u\|_6\|\nabla H\|_3\|\nabla H\|_2\\
&\le \|\nabla^2 u\|_2\|\nabla H\|_2^{\frac32}\|\nabla^2 H\|_2^{\frac12}\\
&\le \eta\left\|\nabla^2 u\right\|_2^2+C_\eta\|\nabla H\|_2^3\|\nabla^2 H\|_2,\\
\left|\frac{\kappa R}{c_v}\int_\Omega \nabla\,\mathrm{div}\,u\cdot\nabla\theta\mathrm{d}x\right| &\le C\left\|\nabla^2 u\right\|_2\|\nabla\theta\|_2
\end{aligned}
$$

$$
\begin{aligned}
&\leq \eta \left\|\nabla^2 u\right\|_2^2 + C_\eta \|\nabla\theta\|_2^2,\\
\left|\frac{R}{c_v}\int_\Omega P(\operatorname{div} u)^2\,\mathrm{d}x\right| &\leq C(\|\sqrt{\rho}\theta\|_2 + \|\nabla\theta\|_2)\|\nabla u\|_2^{\frac{3}{2}}\left\|\nabla^2 u\right\|_2^{\frac{1}{2}}\\
&\leq \eta \left\|\nabla^2 u\right\|_2^2 + C_\eta\left(\|\sqrt{\rho}\theta\|_2^2 + \|\nabla\theta\|_2^2 + \|\nabla u\|_2^6\right),\\
\left|R\int_\Omega \rho\theta u\cdot\nabla \operatorname{div} u\,\mathrm{d}x\right| &\leq C\|\nabla u\|_2^{\frac{1}{2}}\left\|\nabla^2 u\right\|_2^{\frac{3}{2}}\|\sqrt{\rho}\theta\|_2\\
&\leq \eta \left\|\nabla^2 u\right\|_2^2 + C_\eta\|\nabla u\|_2^2\|\sqrt{\rho}\theta\|_2^4,\\
\left|\int_\Omega (\mathrm{curl} H\times H)_t\cdot u\,\mathrm{d}x\right| &\leq C\int_\Omega (H_t\cdot\nabla u\cdot H + H\cdot\nabla u\cdot H_t + H\cdot H_t \mathrm{div} u)\,\mathrm{d}x\\
&\leq C\|\nabla u\|_3\|H_t\|_2\|H\|_6\\
&\leq C\|\nabla u\|_2^{\frac{1}{2}}\left\|\nabla^2 u\right\|_2^{\frac{1}{2}}\|H_t\|_2\|\nabla H\|_2\\
&\leq \eta\left\|\nabla^2 u\right\|_2^2 + C_\eta(\|\nabla u\|_2^2\|\nabla H\|_2^4 + \|H_t\|_2^2),
\end{aligned}
$$

where η is any fixed in $(0,1)$. Based on the above estimates and (2.18), one arrives at

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mu}{2}\|\nabla u\|_2^2 + \frac{\mu+\lambda}{2}\|\operatorname{div} u\|_2^2 - \int_\Omega P\operatorname{div} u\,\mathrm{d}x - \int_\Omega \mathrm{curl} H\times H\cdot u\,\mathrm{d}x\right) + \frac{1}{2}\|\sqrt{\rho}u_t\|_2^2 + \frac{\epsilon_1}{2}\|\nabla^2 u\|_2^2\\
\leq\;& C\left(\|\nabla u\|_2^6 + \|\sqrt{\rho}\theta\|_2^2 + \|\nabla\theta\|_2^2 + \|\nabla u\|_2^2\|\nabla H\|_2^4 + \|H_t\|_2^2 + \|\nabla u\|_2^2\|\sqrt{\rho}\theta\|_2^4\right).
\end{aligned}\tag{2.19}
$$

Integrating (2.19) over $(0,T)$ and using (2.6), Proposition 4 and Proposition 5, one deduces that

$$
\begin{aligned}
&\mu\sup_{0\leq t\leq T}\|\nabla u\|_2^2 + \int_0^T\left(\|\sqrt{\rho}u_t\|_2^2 + \epsilon_1\left\|\nabla^2 u\right\|_2^2\right)\mathrm{d}t\\
\leq&2\sup_{0\leq t\leq T}\left|\int_\Omega P\operatorname{div} u\,\mathrm{d}x\right| + 2\sup_{0\leq t\leq T}\left|\int_\Omega H\cdot\nabla u\cdot H - \frac{1}{2}|H|^2\mathrm{div} u\,\mathrm{d}x\right|\\
&+C\left(\|\nabla u_0\|_2^2 + \left|\int_\Omega P_0\operatorname{div} u_0\,\mathrm{d}x\right| + \left|\int_\Omega H_0\cdot\nabla u_0\cdot H_0 - \frac{1}{2}|H_0|^2\mathrm{div} u_0\,\mathrm{d}x\right|\right)\\
&+C\int_0^T\left(\|\nabla u\|_2^6 + \|\sqrt{\rho}\theta\|_2^2 + \|\nabla\theta\|_2^2 + \|\nabla u\|_2^2\|\nabla H\|_2^4 + \|H_t\|_2^2 + \|\nabla u\|_2^2\|\sqrt{\rho}\theta\|_2^4\right)\\
\leq&\frac{\mu}{2}\sup_{0\leq t\leq T}\|\nabla u\|_2^2 + C[1 + \sup_{0\leq t\leq T}\left(\|\rho\|_\infty\|\sqrt{\rho}\theta\|_2^2\right) + \sup_{0\leq t\leq T}\|\nabla H\|_2^2 + T\Phi^3(T)]\\
\leq&\frac{\mu}{2}\sup_{0\leq t\leq T}\|\nabla u\|_2^2 + C,
\end{aligned}\tag{2.20}
$$

□

Proposition 7. *Under the assumptions of Proposition 4, it holds that*

$$
\sup_{0\leq t\leq T} t\|H_t\|_2^2 + \int_0^T\|\sqrt{t}\nabla H_t\|_2^2\mathrm{d}t \leq C. \tag{2.21}
$$

Proof. Differentiating $(1.1)_4$ with respect to t yields

$$
H_{tt} - H_t\cdot\nabla u - H\cdot\nabla u_t + u_t\cdot\nabla H + u\cdot\nabla H_t + H_t\mathrm{div} u + H\mathrm{div} u_t = \nu\Delta H_t. \tag{2.22}
$$

Multiplying (2.22) with H_t and integrating it over $(0,T)$, one gets that

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|H_t\|_2^2 + \nu\|\nabla H_t\|_2^2 = \int_\Omega (H_t\cdot\nabla u + H\cdot\nabla u_t - u_t\cdot\nabla H - u\cdot\nabla H_t - H_t\mathrm{div} u - H\mathrm{div} u_t)\cdot H_t\,\mathrm{d}x. \tag{2.23}
$$

According to Corollary 1 and Proposition 4, one finds that

$$
\begin{aligned}
&\frac{1}{2}\sup_{0\le t\le T} t\|H_t\|_2^2+\nu\int_0^t \|\sqrt{t}\nabla H_t\|_2^2\mathrm{d}t\\
\le&\int_0^T t\int_\Omega (H_t\cdot\nabla u+H\cdot\nabla u_t-u_t\cdot\nabla H-u\cdot\nabla H_t-H_t\mathrm{div}u-H\mathrm{div}u_t)\cdot H_t\,\mathrm{d}x\mathrm{d}t+\frac{1}{2}\int_0^T\|H_t\|_2^2\mathrm{d}t\\
\le& C+C\int_0^T t\,\|H_t\|_6\,\|\nabla u\|_2\,\|H_t\|_3\,\mathrm{d}t+C\int_0^T t\,\|H\|_6\,\|\nabla u_t\|_2\|H_t\|_3\mathrm{d}t+C\int_0^T t\|u_t\|_6\|\nabla H\|_2\|H_t\|_3\mathrm{d}t\\
&+C\int_0^T t\|u\|_6\|\nabla H_t\|_2\|H_t\|_3\mathrm{d}t+C\int_0^T t\|H_t\|_6\|\nabla u\|_2\|H_t\|_3\mathrm{d}t+C\int_0^T\|H\|_6\|\nabla u_t\|_2\|H_t\|_3\mathrm{d}t\\
\le& CT^{\frac14}\int_0^T\|\sqrt{t}\nabla H_t\|_2^{\frac32}\,\|H_t\|_2^{\frac12}\,\|\nabla u\|_2\mathrm{d}t+CT^{\frac14}\int_0^T\|\nabla H\|_2\|\sqrt{t}\nabla u_t\|_2\|\|H_t\|_2^{\frac12}\|\sqrt{t}\nabla H_t\|_2^{\frac12}\mathrm{d}t\\
&+CT^{\frac14}\int_0^T\|\sqrt{t}\nabla u_t\|_2\|\nabla H\|_2\|H_t\|_2^{\frac12}\|\sqrt{t}\nabla H_t\|_2^{\frac12}\mathrm{d}t+C\\
\le& CT^{\frac14}\Phi^{\frac32}(T)+C\\
\le& C.
\end{aligned}\tag{2.24}
$$

□

Proposition 8. *Under the conditions of Proposition 4, it holds that*

$$
\sup_{0\le t\le T}\left\|\left(\sqrt{t}\nabla\theta,\sqrt{t}\sqrt{\rho}u_t,\sqrt{t}\nabla^2u,\sqrt{t}\nabla^2H\right)\right\|_2^2+\int_0^T\left\|\left(\sqrt{t}\sqrt{\rho}\theta_t,\sqrt{t}\nabla^2\theta,\sqrt{t}\nabla u_t\right)\right\|_2^2\,\mathrm{d}t\le C.\tag{2.25}
$$

Proof. By Corollary 1 and Proposition 1, it follows from$(1.1)_3$ that

$$
\begin{aligned}
\|\nabla^2\theta\|_2^2\le& C\left(\|\rho\theta_t\|_2^2+\|\rho(u\cdot\nabla)\theta\|_2^2+\|\operatorname{div}uP\|_2^2+\|\mathcal{Q}(\nabla u)\|_2^2+\|\nu|\mathrm{curl}H|\|_2\right)\\
\le& C\left(\|\rho\|_\infty\|\sqrt{\rho}\theta_t\|_2^2+\|\rho\|_\infty^2\|u\|_6^2\|\nabla\theta\|_3^2+\|\rho\|_\infty^2\|\theta\|_6^2\|\nabla u\|_3^2+\|\nabla u\|_4^4+\|\nabla H\|_4^4\right)\\
\le& C\left(\|\sqrt{\rho}\theta_t\|_2^2+\|\nabla u\|_2^2\|\nabla\theta\|_2\|\nabla^2\theta\|_2+\|\nabla u\|_2\|\nabla^2u\|_2^3+\|\nabla H\|_2\|\nabla^2H\|_2^3\right)\\
&+C\left(\|\sqrt{\rho}\theta\|_2^2+\|\nabla\theta\|_2^2\right)\|\nabla u\|_2\|\nabla^2u\|_2\\
\le&\frac12\|\nabla^2\theta\|_2^2+C\Big[\|\sqrt{\rho}\theta_t\|_2^2+\|\nabla u\|_2^4\|\nabla\theta\|_2^2+\|\nabla u\|_2\left\|\nabla^2u\right\|_2^3+\|\nabla H\|_2\|\nabla^2H\|_2^3\\
&+\left(\|\sqrt{\rho}\theta\|_2^2+\|\nabla\theta\|_2^2\right)\|\nabla u\|_2\|\nabla^2u\|_2\Big].
\end{aligned}
$$

Thus,

$$
\left\|\nabla^2\theta\right\|_2^2\le C\left(\|\sqrt{\rho}\theta_t\|_2^2+\|\nabla u\|_2^4\|\nabla\theta\|_2^2+\|\nabla u\|_2\|\nabla^2u\|_2^3+\|\nabla H\|_2\|\nabla^2H\|_2^3+(\|\sqrt{\rho}\theta\|_2^2+\|\nabla\theta\|_2^2)\|\nabla u\|_2\|\nabla^2u\|_2\right).\tag{2.26}
$$

Multiplying $(1.1)_3$ with θ_t, we arrive at

$$
\frac{\kappa}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta\|_2^2+c_v\left\|\sqrt{\rho}\theta_t\right\|_2^2\int_\Omega\left[-c_v\rho(u\cdot\nabla)\theta\theta_t-P\operatorname{div}u\theta_t+\mathcal{Q}(\nabla u)\theta_t+\nu|\mathrm{curl}H|^2\theta_t\right]\mathrm{d}x.\tag{2.27}
$$

Due to Proposition 1, Corollary 1, Proposition 4 and Proposition 5, one gets that

$$
\begin{aligned}
\left|\int_\Omega\rho(u\cdot\nabla)\theta\theta_t\,\mathrm{d}x\right|\le& C\|\rho\|_\infty^{\frac12}\|\nabla u\|_2^{\frac12}\|\nabla^2u\|_2^{\frac12}\|\nabla\theta\|_2\|\sqrt{\rho}\theta_t\|_2\\
\le&\frac{c_v}{8}\|\sqrt{\rho}\theta_t\|_2^2+C\|\nabla u\|_2\|\nabla^2u\|_2\|\nabla\theta\|_2^2,
\end{aligned}\tag{2.28}
$$

$$
\begin{aligned}
\left|\int_\Omega P\operatorname{div}u\theta_t\,\mathrm{d}x\right|\le& C\|\rho\|_\infty^{\frac12}(\|\sqrt{\rho}\theta\|_2+\|\nabla\theta\|_2)\|\nabla u\|_2^{\frac12}\left\|\nabla^2u\right\|_2^{\frac12}\|\sqrt{\rho}\theta_t\|_2\\
\le&\frac{c_v}{8}\|\sqrt{\rho}\theta_t\|_2^2+C\|\nabla u\|_2\|\nabla^2u\|_2\left(\|\sqrt{\rho}\theta\|_2^2+\|\nabla\theta\|_2^2\right),
\end{aligned}\tag{2.29}
$$

$$
\begin{aligned}
\int_\Omega \mathcal{Q}(\nabla u)\theta_t \,\mathrm{d}x &\le \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathcal{Q}(\nabla u)\theta \mathrm{d}x + C\|\nabla u\|_3\|\nabla u_t\|_2\|\theta\|_6 \\
&\le \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathcal{Q}(\nabla u)\theta \mathrm{d}x + C\|\nabla u\|_2^{\frac12}\|\nabla^2 u\|_2^{\frac12}\|\nabla u_t\|_2(\|\sqrt{\rho}\theta\|_2+\|\nabla\theta\|_2),
\end{aligned} \tag{2.30}
$$

$$
\begin{aligned}
\int_\Omega \nu|\mathrm{curl}H|^2\theta_t \,\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \nu|\mathrm{curl}H|^2\theta \mathrm{d}x - \int_\Omega |\mathrm{curl}H|_t^2\theta \mathrm{d}x \\
&\le \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \nu|\mathrm{curl}H|^2\theta \mathrm{d}x + C\|\nabla H\|_3\|\nabla H_t\|_2\|\theta\|_6 \\
&\le \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \nu|\mathrm{curl}H|^2\theta \mathrm{d}x + C\|\nabla H\|_2^{\frac12}\|\nabla^2 H\|_2^{\frac12}\|\nabla H_t\|_2(\|\sqrt{\rho}\theta\|_2+\|\nabla\theta\|_2).
\end{aligned} \tag{2.31}
$$

Inserting (2.28)-(2.31) into (2.27) and adding the resultant with (2.26), multiplying with a small positive number ϵ_2, one obtains that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\kappa}{2}\|\nabla\theta\|_2^2 - \int_\Omega \mathcal{Q}(\nabla u)\theta \mathrm{d}x - \int_\Omega \nu|\mathrm{curl}H|^2\theta \mathrm{d}x\right) + \frac{c_v}{2}\|\sqrt{\rho}\theta_t\|_2^2 + \epsilon_2\left\|\nabla^2\theta\right\|_2^2 \\
\le & C\Big[\|\nabla u\|_2^4\|\nabla\theta\|_2^2 + \left(\|\sqrt{\rho}\theta\|_2^2+\|\nabla\theta\|_2^2\right)\|\nabla u\|_2\left\|\nabla^2 u\right\|_2 + \|\nabla u\|_2\left\|\nabla^2 u\right\|_2^3 + \|\nabla H\|_2\left\|\nabla^2 H\right\|_2^3 \\
&+\left(\|\nabla u\|_2^{\frac12}\|\nabla^2 u\|_2^{\frac12}\|\nabla u_t\|_2 + \|\nabla H\|_2^{\frac12}\|\nabla^2 H\|_2^{\frac12}\|\nabla H_t\|_2\right)(\|\sqrt{\rho}\theta\|_2+\|\nabla\theta\|_2)\Big].
\end{aligned} \tag{2.32}
$$

Multiplying the above inequality with t, yields that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\kappa}{2}\|\sqrt{t}\nabla\theta\|_2^2 - t\int_\Omega \mathcal{Q}(\nabla u)\theta \mathrm{d}x - t\int_\Omega \nu|\mathrm{curl}H|^2\theta \mathrm{d}x\right) + \frac{c_v}{2}\|\sqrt{t}\sqrt{\rho}\theta_t\|_2^2 \\
&\quad + \epsilon_2\|\sqrt{t}\nabla^2\theta\|_2^2 + \int_\Omega \mathcal{Q}(\nabla u)\theta \mathrm{d}x + \int_\Omega \nu|\mathrm{curl}H|^2\theta \mathrm{d}x \\
&\le C[\|\nabla\theta\|_2^2 + \|\nabla u\|_2^4\|\sqrt{t}\nabla\theta\|_2^2 + \left(\|\sqrt{\rho}\theta\|_2^2+\|\sqrt{t}\nabla\theta\|_2^2\right)\|\nabla u\|_2\|\nabla^2 u\|_2 \\
&\quad + (\|\nabla u\|_2^{\frac12}\left\|\nabla^2 u\right\|_2^{\frac12}\|\sqrt{t}\nabla u_t\|_2 + \|\nabla H\|_2^{\frac12}\left\|\nabla^2 H\right\|_2^{\frac12}\|\sqrt{t}\nabla H_t\|_2)(\|\sqrt{\rho}\theta\|_2+\|\sqrt{t}\nabla\theta\|_2) \\
&\quad + \sqrt{t}\|\nabla u\|_2\left\|\nabla^2 u\right\|_2^2\|\sqrt{t}\nabla^2 u\|_2] + \sqrt{t}\|\nabla H\|_2\left\|\nabla^2 H\right\|_2^2\|\sqrt{t}\nabla^2 H\|_2].
\end{aligned} \tag{2.33}
$$

Moreover,

$$
\begin{aligned}
t\int_\Omega \mathcal{Q}(\nabla u)\theta \mathrm{d}x &\le CT^{\frac14}\|\nabla u\|_2^{\frac32}\|\sqrt{t}\nabla^2 u\|_2^{\frac12}(\|\sqrt{\rho}\theta\|_2+\|\sqrt{t}\nabla\theta\|_2) \\
&\le C\left[T\Phi^3(T)\right]^{\frac14}(\|\sqrt{\rho}\theta\|_2+\|\sqrt{t}\nabla\theta\|_2)\|\sqrt{t}\nabla^2 u\|_2^{\frac12} \\
&\le \eta\sup_{0\le t\le T}\|\sqrt{t}\nabla\theta\|_2^2 + C_\eta\left(\sup_{0\le t\le T}\|\sqrt{t}\nabla^2 u\|_2+1\right),
\end{aligned} \tag{2.34}
$$

$$
\begin{aligned}
t\int_\Omega \nu|\mathrm{curl}H|^2\theta \mathrm{d}x &\le Ct\int_\Omega |\nabla H|^2|\theta| \mathrm{d}x \le Ct\|\nabla H\|_2\|\nabla H\|_3\|\theta\|_6 \\
&\le CT^{\frac14}\|\nabla H\|_2^{\frac32}\|\sqrt{t}\nabla^2 H\|_2^{\frac12}(\|\sqrt{\rho}\theta\|_2+\|\sqrt{t}\nabla\theta\|_2) \\
&\le C\left[T\Phi^3(T)\right]^{\frac14}(\|\sqrt{\rho}\theta\|_2+\|\sqrt{t}\nabla\theta\|_2)\|\sqrt{t}\nabla^2 H\|_2^{\frac12} \\
&\le \eta\sup_{0\le t\le T}\|\sqrt{t}\nabla\theta\|_2^2 + C_\eta\left(\sup_{0\le t\le T}\|\sqrt{t}\nabla^2 H\|_2+1\right),
\end{aligned} \tag{2.35}
$$

hold for any fixed $\eta\in(0,1)$. Integrating (2.33) over $(0,T)$, using (2.34) and (2.35), one deduces from Proposition 5 that

$$
\begin{aligned}
&\sup_{0\le t\le T}\|\sqrt{t}\nabla\theta\|_2^2 + \int_0^T (c_v\|\sqrt{t}\sqrt{\rho}\theta_t\|_2^2 + \epsilon_2\|\sqrt{t}\nabla^2\theta\|_2^2)\mathrm{d}t \\
\le & \sup_{0\le t\le T}(t\int_\Omega \mathcal{Q}(\nabla u)\theta \mathrm{d}x) + \sup_{0\le t\le T}(t\int_\Omega \nu|\mathrm{curl}H|^2\theta \mathrm{d}x) + C\int_0^T [\|\nabla\theta\|_2^2 + \|\nabla u\|_2^4\|\sqrt{t}\nabla\theta\|_2^2
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\|\sqrt{\rho}\theta\|_2^2+\|\sqrt{t}\nabla\theta\|_2^2\right)\|\nabla u\|_2\|\nabla^2 u\|_2+\sqrt{t}\|\nabla u\|_2\|\nabla^2 u\|_2^2\|\sqrt{t}\nabla^2 u\|_2\\
&+\sqrt{t}\|\nabla H\|_2\|\nabla^2 H\|_2^2\|\sqrt{t}\nabla^2 H\|_2+\|\nabla u\|_2^{\frac12}\|\nabla^2 u\|_2^{\frac12}\|\sqrt{t}\nabla u_t\|_2(\|\sqrt{\rho}\theta\|_2+\|\sqrt{t}\nabla\theta\|_2)\\
&+\|\nabla H\|_2^{\frac12}\|\nabla^2 H\|_2^{\frac12}\|\sqrt{t}\nabla H_t\|_2(\|\sqrt{\rho}\theta\|_2+\|\sqrt{t}\nabla\theta\|_2)]\mathrm{d}t\\
\le&\eta\sup_{0\le t\le T}\|\sqrt{t}\nabla\theta\|_2^2+C_\eta\left(1+T^{\frac12}\Phi^{\frac32}(T)\right)\sup_{0\le t\le T}\|\sqrt{t}\nabla^2 u\|_2\\
&+C_\eta\left(1+T^{\frac12}\Phi^{\frac32}(T)\right)\sup_{0\le t\le T}\|\sqrt{t}\nabla^2 H\|_2+C_\eta\left(T\Phi^3(T)+T^{\frac12}\Phi^2(T)+T^{\frac14}\Phi^{\frac32}(T)+1\right),
\end{aligned}
$$

for any $\eta>0$. Choosing η sufficiently small, one gets from (2.16)–(2.19) that

$$
\sup_{0\le t\le T}\|\sqrt{t}\nabla\theta\|_2^2+\int_0^T\left(\|\sqrt{t}\sqrt{\rho}\theta_t\|_2^2+\|\sqrt{t}\nabla^2\theta\|_2^2\right)\mathrm{d}t\le C\left(\sup_{0\le t\le T}\|\sqrt{t}\nabla^2 u\|_2+\sup_{0\le t\le T}\|\sqrt{t}\nabla^2 H\|_2+1\right). \tag{2.36}
$$

Differentiating $(1.1)_2$ with respect to t, yields that

$$
\rho\left(u_{tt}+u\cdot\nabla u_t\right)+\rho u_t\cdot\nabla u+\rho_t\left(u_t+u\cdot\nabla u\right)-\mu\Delta u_t-(\mu+\lambda)\nabla\operatorname{div}u_t+\nabla P_t=(\operatorname{curl}H\times H)_t.
$$

It follows from (2.17) that

$$
\int_\Omega\nabla P_t u_t\,\mathrm{d}x=-\int_\Omega P_t\operatorname{div}u_t\,\mathrm{d}x=\frac{R}{c_v}\int_\Omega\operatorname{div}u_t\left(-\mathcal{Q}(\nabla u)-\nu|\operatorname{curl}H|^2-\kappa\Delta\theta+P\operatorname{civ}u+c_v\operatorname{div}(\rho u\theta)\right)\mathrm{d}x. \tag{2.37}
$$

Multiplying (2.37) by u_t and utilizing the above equality, one finds that

$$
\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}u_t\|_2^2+\mu\|\nabla u_t\|_2^2+(\mu+\lambda)\|\operatorname{div}u_t\|_2^2\\
=&\frac{R}{c_v}\int_\Omega\operatorname{div}u_t\left(\mathcal{Q}(\nabla u)+\nu|\operatorname{curl}H|^2+\kappa\Delta\theta-P\operatorname{div}u-c_v\operatorname{div}(\rho u\theta)\right)\mathrm{d}x\\
&+\int_\Omega\operatorname{div}(\rho u)\left(u_t+(u\cdot\nabla)u\right)\cdot u_t\,\mathrm{d}x-\int_\Omega\rho\left(u_t\cdot\nabla\right)u\cdot u_t\,\mathrm{d}x+\int_\Omega(\operatorname{curl}H\times H)_t\cdot u_t\,\mathrm{d}t
\end{aligned} \tag{2.38}
$$

Multiplying the above identity with t and integrating over $(0,T)$, implies that

$$
\begin{aligned}
&\frac12\sup_{0\le t\le T}\|\sqrt{t}\sqrt{\rho}u_t\|_2^2+\mu\int_0^T\|\sqrt{t}\nabla u_t\|_2^2\mathrm{d}t\\
\le&\frac12\int_0^T\|\sqrt{\rho}u_t\|_2^2\mathrm{d}t+C\int_0^T t\int_\Omega\left|\nabla u_t\right||\nabla u|^2\mathrm{d}x\mathrm{d}t+C\int_0^T t\int_\Omega\left|\nabla H_t\right||\nabla H|^2\mathrm{d}x\mathrm{d}t\\
&+\frac{\kappa R}{c_v}\int_0^T t\int_\Omega\operatorname{div}u_t\Delta\theta\mathrm{d}x\mathrm{d}t-\frac{R}{c_v}\int_0^T t\int_\Omega\operatorname{div}u_t P\operatorname{div}u\mathrm{d}x\mathrm{d}t-R\int_0^T t\int_\Omega\operatorname{div}u_t\operatorname{div}(\rho u\theta)\mathrm{d}x\mathrm{d}t\\
&-\int_0^T t\int_\Omega\rho u\cdot\nabla|u_t|^2\mathrm{d}x\mathrm{d}t-\int_0^T t\int_\Omega\rho u\cdot\nabla((u\cdot\nabla)u\cdot u_t)\mathrm{d}x\mathrm{d}t-\int_0^T t\int_\Omega\rho(u_t\cdot\nabla)u\cdot u_t\mathrm{d}x\mathrm{d}t\\
&+\int_0^T t\int_\Omega(H_t\cdot\nabla u_t\cdot H+H\cdot\nabla u_t\cdot H_t+H\cdot H_t\operatorname{div}u_t)\mathrm{d}x\mathrm{d}t=\sum_{i=1}^{10}G_i.
\end{aligned} \tag{2.39}
$$

Each term on the right hand of (2.39) as follows:

$$
\begin{aligned}
&G_1\le C,\\
&G_2\le C\int_0^T\|\sqrt{t}\nabla u_t\|_2\|\nabla u\|_2^{\frac12}\|\sqrt{t}\nabla^2 u\|_2\|\nabla^2 u\|_2^{\frac12}\mathrm{d}t\le CT^{\frac14}\Phi(T)\sup_{0\le t\le T}\|\sqrt{t}\nabla^2 u\|_2,\\
&G_3\le C\int_0^T t\|\nabla H_t\|_2\|\nabla H\|_4^2\mathrm{d}t\le C\int_0^T\|\sqrt{t}\nabla H_t\|_2\|\nabla H\|_2^{\frac12}\|\sqrt{t}\nabla^2 H\|_2\|\nabla^2 H\|_2^{\frac12}\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&\le CT^{\frac{1}{4}}\Phi(T)\sup_{0\le t\le T}\|\sqrt{t}\nabla^2 H\|_2,\\
G_4 &\le \frac{\mu}{2}\int_0^T \|\sqrt{t}\nabla u_t\|_2^2\,\mathrm{d}t + C\int_0^T \|\sqrt{t}\nabla^2\theta\|_2^2\,\mathrm{d}t,\\
G_5 &\le C\int_0^T (\|\sqrt{\rho}\theta\|_2 + \|\sqrt{t}\nabla\theta\|_2)\|\nabla u\|_2^{\frac{1}{2}}\|\nabla^2 u\|_2^{\frac{1}{2}}\|\sqrt{t}\nabla u_t\|_2\mathrm{d}t \le CT^{\frac{1}{4}}\Phi^{\frac{3}{2}}(T),\\
G_6 &\le C\int_0^T t\,\|\nabla u_t\|_2\left(\|\nabla\rho\|_3\|u\|_\infty\|\theta\|_6 + \|\rho\|_\infty\|\nabla u\|_3\|\theta\|_6 + \|\rho\|_\infty\|u\|_6\|\nabla\theta\|_3\right)\mathrm{d}t\\
&\le C\int_0^T \|\sqrt{t}\nabla u_t\|_2\|\nabla u\|_2^{\frac{1}{2}}\left\|\nabla^2 u\right\|_2^{\frac{1}{2}}(\|\sqrt{\rho}\theta\|_2 + \|\sqrt{t}\nabla\theta\|_2)dt\\
&\quad + C\int_0^T \|\sqrt{t}\nabla u_t\|_2\|\nabla u\|_2\|\sqrt{t}\nabla\theta\|_2^{\frac{1}{2}}\|\sqrt{t}\nabla^2\theta\|_2^{\frac{1}{2}}dt \le CT^{\frac{1}{4}}\Phi^{\frac{3}{2}}(T),\\
G_7 &\le C\int_0^T t\|\rho\|_\infty^{\frac{3}{4}}\|\nabla u\|_2\|\sqrt{\rho}u_t\|_2^{\frac{1}{2}}\|\nabla u_t\|_2^{\frac{3}{2}}\,\mathrm{d}t \le CT^{\frac{1}{4}}\Phi^{\frac{3}{2}}(T),\\
G_8 &\le C\int_0^T \|t\|_\infty\left(\|u\|_6\|\nabla u\|_3^2\,\|u_t\|_6 + \|u\|_6^2\left\|\nabla^2 u\right\|_2\|u_t\|_6 + \|u\|_6^2\|\nabla u\|_6\,\|\nabla u_t\|_2\right)\mathrm{d}t\\
&\le C\int_0^T \sqrt{t}\|\nabla u\|_2^2\left\|\nabla^2 u\right\|_2\|\sqrt{t}\nabla u_t\|_2\mathrm{d}t \le CT^{\frac{1}{2}}\Phi^2(T),\\
G_9 &\le C\int_0^T t\|\rho\|_\infty^{\frac{3}{4}}\|\nabla u\|_2\,\|\sqrt{\rho}u_t\|_2^{\frac{1}{2}}\,\|\nabla u_t\|_2^{\frac{3}{2}}\,\mathrm{d}t \le CT^{\frac{1}{4}}\Phi^{\frac{3}{2}}(T),\\
G_{10} &\le C\int_0^2 t\|H_t\|_3\|\nabla u_t\|_2\|H\|_6\mathrm{d}t \le C\int_0^T t\|H_t\|_2^{\frac{1}{2}}\|\nabla H_t\|_2^{\frac{1}{2}}\|\nabla u_t\|_2\|\nabla H\|_2\mathrm{d}t\\
&\le C\int_0^T t^{\frac{1}{4}}\|H_t\|_2^{\frac{1}{2}}\|\sqrt{t}\nabla H_t\|_2^{\frac{1}{2}}\|\sqrt{t}\nabla u_t\|_2\|\nabla H\|_2\mathrm{d}t \le CT^{\frac{1}{4}}\Phi^{\frac{3}{2}}(T).
\end{aligned}
$$

So, it follows from (2.7)–(2.9), and (2.36) that

$$
\begin{aligned}
&\sup_{0\le t\le T}\|\sqrt{t}\sqrt{\rho}u_t\|_2^2 + \mu\int_0^T \|\sqrt{t}\nabla u_t\|_2^2\,\mathrm{d}t\\
\le&CT^{\frac{1}{4}}\Phi(T)\left(\sup_{0\le t\le T}\|\sqrt{t}\nabla^2 u\|_2 + \sup_{0\le t\le T}\|\sqrt{t}\nabla^2 H\|_2\right) + C\int_0^T \|\sqrt{t}\nabla^2\theta\|_2^2dt + C\left(1 + T^{\frac{1}{4}}\Phi^{\frac{3}{2}}(T) + T^{\frac{1}{2}}\Phi^2(T)\right)\\
\le&C\left(\sup_{0\le t\le T}\|\sqrt{t}\nabla^2 u\|_2 + \sup_{0\le t\le T}\|\sqrt{t}\nabla^2 H\|_2 + \int_0^T \|\sqrt{t}\nabla^2\theta\|_2^2dt + 1\right)\\
\le&C\left(\sup_{0\le t\le T}\|\sqrt{t}\nabla^2 u\|_2 + \sup_{0\le t\le T}\|\sqrt{t}\nabla^2 H\|_2 + 1\right).
\end{aligned}
$$

That is,

$$
\sup_{0\le t\le T}\|\sqrt{t}\sqrt{\rho}u_t\|_2^2 + \mu\int_0^T \|\sqrt{t}\nabla u_t\|_2^2\,\mathrm{d}t \le C\left(\sup_{0\le t\le T}\|\sqrt{t}\nabla^2 u\|_2 + \sup_{0\le t\le T}\|\sqrt{t}\nabla^2 H\|_2 + 1\right). \tag{2.40}
$$

Recalling (2.16), one obtains from (2.36), (2.40) and (2.6) that

$$
\begin{aligned}
\sup_{0\le t\le T}\|\sqrt{t}\nabla^2 u\|_2^2 &\le C\sup_{0\le t\le T}\left(\|\sqrt{t}\sqrt{\rho}u_t\|_2^2 + t\|\nabla u\|_2^6 + t\|\sqrt{\rho}\theta\|_2^2 + \|\sqrt{t}\nabla\theta\|_2^2 + t^{\frac{1}{2}}\|\nabla H\|_2^3\|\sqrt{t}\nabla^2 H\|_2\right)\\
&\le C\sup_{0\le t\le T}\left(\|\sqrt{t}\sqrt{\rho}u_t\|_2^2 + \|\sqrt{t}\nabla\theta\|_2^2 + \|\sqrt{t}\nabla^2 H\|_2 + 1\right)\\
&\le C\left(\sup_{0\le t\le T}\|\sqrt{t}\nabla^2 u\|_2 + \|\sqrt{t}\nabla^2 H\|_2 + 1\right).
\end{aligned} \tag{2.41}
$$

Applying the elliptic estimates to $(1.1)_4$, one gets that

$$\begin{aligned}\|\nabla^2 H\|_2^2 &\le C(\|H_t\|_2^2 + \|H\cdot\nabla u\|_2^2 + \|u\cdot\nabla H\|_2^2 + \|H\mathrm{div}u\|_2^2)\\ &\le (\|H_t\|_2^2 + \|H\|_6^2\|\nabla u\|_3^2 + \|u\|_6^2\|\nabla H\|_3^2)\\ &\le C(\|H_t\|_2^2 + \|\nabla H\|_2^2\|\nabla u\|_2\|\nabla^2 u\|_2 + \|\nabla u\|_2^2\|\nabla H\|_2\|\nabla^2 H\|_2)\end{aligned}$$

So,

$$\begin{aligned}\sup_{0\le t\le T}\|\sqrt{t}\nabla^2 H\|_2^2 \le C &\sup_{0\le t\le T}(t\|H_t\|_2^2 + \|\sqrt{t}\nabla^2 u\|_2 + \|\sqrt{t}\nabla^2 H\|_2 + 1)\\ \le C &\sup_{0\le t\le T}(\|\sqrt{t}\nabla^2 u\|_2 + \|\sqrt{t}\nabla^2 H\|_2 + 1).\end{aligned} \tag{2.42}$$

It follows from (2.41) and (2.42) that

$$\begin{aligned}&\sup_{0\le t\le T}\|\sqrt{t}\nabla^2 u\|_2^2 + \sup_{0\le t\le T}\|\sqrt{t}\nabla^2 H\|_2^2\\ \le& C(\sup_{0\le t\le T}\|\sqrt{t}\nabla^2 u\|_2 + \|\sqrt{t}\nabla^2 H\|_2 + 1)\\ \le&\frac{1}{2}(\sup_{0\le t\le T}\|\sqrt{t}\nabla^2 u\|_2^2 + \|\sqrt{t}\nabla^2 H\|_2) + C.\end{aligned}$$

Obviously $\sup_{0\le t\le T}\|\sqrt{t}\nabla^2 u\|_2^2 + \sup_{0\le t\le T}\|\sqrt{t}\nabla^2 H\|_2^2 \le C.$ □

As a direct result of Corollary 1 and Proposition 4-Propositions 8, we have the following corollary.

Corollary 2. *Under the assumptions of Proposition 4, it holds that*

$$\Phi(T) + \sup_{0\le t\le T}\left\|\left(\sqrt{t}\nabla^2 u, \sqrt{t}\nabla^2 H, \sqrt{t}\sqrt{\rho}u_t\right)\right\|_2^2 \le C,$$

that is

$$\begin{aligned}&\sup_{0\le t\le T}\left(\|\rho\|_\infty + \|\nabla\rho\|_q + \left\|\left(\sqrt{\rho}\theta, \nabla u, \nabla H, \sqrt{t}\nabla\theta, \sqrt{t}\nabla^2 u, \sqrt{t}\nabla^2 H, \sqrt{t}\sqrt{\rho}u_t\right)\right\|_2^2\right)\\ &+\int_0^T \left\|\left(\sqrt{\rho}u_t, \nabla^2 u, \nabla^2 H, H_t, \sqrt{t}\nabla u_t, \sqrt{t}\nabla H_t, \nabla\theta, \sqrt{t}\sqrt{\rho}\theta_t, \sqrt{t}\nabla^2\theta\right)\right\|_2^2 \,\mathrm{d}t \le C.\end{aligned} \tag{2.43}$$

Proposition 9. *Under the assumptions of Proposition 4, it holds that*

$$\sup_{0\le t\le T}\left\|\left(\rho_t, t\nabla^2\theta, t\sqrt{\rho}\theta_t\right)\right\|_2^2 + \int_0^T\left(\|t\nabla\theta_t\|_2^2 + \|t\nabla^2\theta\|_6^2 + \|\sqrt{t}\nabla^2 u\|_q^2 + \|\sqrt{t}\nabla^2 H\|_q^2\right)\mathrm{d}t \le C. \tag{2.44}$$

Proof. It follows from $(1.1)_1$ and Corollary 2 that

$$\|\rho_t\|_2 \le \|u\|_6\|\nabla\rho\|_3 + \|\nabla u\|_2\|\rho\|_\infty \le C\left(\|\nabla\rho\|_3 + \|\rho\|_\infty\right)\|\nabla u\|_2 \le C. \tag{2.45}$$

Differentiating $(1.1)_3$ with respect to t, yields that

$$c_v\rho\left(\theta_{tt} + u\cdot\nabla\theta_t\right) + c_v\rho_t\left(\theta_t + u\cdot\nabla\theta\right) + c_v\rho u_t\cdot\nabla\theta - \kappa\Delta\theta_t + P_t\,\mathrm{div}\,u + P\,\mathrm{div}\,u_t = 4\mu Du : Du_t + 2\lambda\,\mathrm{div}\,u\,\mathrm{div}\,u_t + \nu|\,\mathrm{curl}\,H|_t^2.$$

Multiplying the above equality with θ_t and integrating over Ω, one gets that

$$\begin{aligned}&\frac{c_v}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}\theta_t\|_2^2 + \kappa\|\nabla\theta_t\|_2^2\\ =&-c_v\int_\Omega\rho_t\left|\theta_t\right|^2\mathrm{d}x - c_v\int_\Omega\rho_t u\cdot\nabla\theta\theta_t\mathrm{d}x - c_v\int_\Omega\rho u_t\cdot\nabla\theta\theta_t\mathrm{d}x - \int_\Omega P_t\,\mathrm{div}\,u\theta_t\mathrm{d}x\\ &-\int_\Omega P\,\mathrm{div}\,u_t\theta_t\,\mathrm{d}x + \int_\Omega\left(4\mu Du : Du_t + 2\lambda\,\mathrm{div}\,u\,\mathrm{div}\,u_t\right)\theta_t\,\mathrm{d}x + \int_\Omega\nu|\,\mathrm{curl}\,H|_t^2\theta_t\end{aligned} \tag{2.46}$$

Now, we estimates each terms on the right-hand side of (2.46) as follows

$$
\begin{aligned}
\left|c_v\int_\Omega \rho_t\left|\theta_t\right|^2\mathrm{d}x\right| &\le C\|\rho\|_\infty^{\frac{3}{4}}\|\nabla u\|_2\left(\|\sqrt{\rho}\theta_t\|_2\|\nabla\theta_t\|_2+\|\sqrt{\rho}\theta_t\|_2^{\frac12}\|\nabla\theta_t\|_2^{\frac32}\right)\\
&\le \frac{\kappa}{8}\|\nabla\theta_t\|_2^2+C\|\sqrt{\rho}\theta_t\|_2^2\\
\left|c_v\int_\Omega \rho_t u\cdot\nabla\theta\theta_t\mathrm{d}x\right| &\le C\|\rho\|_\infty(\|u\|_\infty\|\nabla u\|_3\|\nabla\theta\|_2\|\theta_t\|_6+\|u\|_6^2\left\|\nabla^2\theta\right\|_2\|\theta_t\|_6\\
&\quad+\|u\|_6^2\|\nabla\theta\|_6\|\nabla\theta_t\|_2)\\
&\le \frac{\kappa}{8}\|\nabla\theta_t\|_2^2+C\left(\|\sqrt{\rho}\theta_t\|_2^2+\left\|\nabla^2u\right\|_2^2\|\nabla\theta\|_2^2+\|\nabla\theta\|_2^2+\|\nabla^2\theta\|_2^2\right),\\
\left|c_v\int_\Omega \rho u_t\cdot\nabla\theta\theta_t\mathrm{d}x\right| &\le C\|\rho\|_\infty^{\frac34}\|\sqrt{\rho}u_t\|_2^{\frac12}\|\nabla u_t\|_2^{\frac12}\|\nabla\theta\|_2\left(\|\sqrt{\rho}\theta_t\|_2+\|\nabla\theta_t\|_2\right)\\
&\le \frac{\kappa}{8}\|\nabla\theta_t\|_2^2+C(\|\sqrt{\rho}\theta_t\|_2^2+\|\sqrt{\rho}u_t\|_2\|\nabla u_t\|_2\|\nabla\theta\|_2),\\
\left|\int_\Omega P_t\operatorname{div}u\theta_t\mathrm{d}x\right| &\le C\left[(\|\sqrt{\rho}\theta\|_2+\|\nabla\theta\|_2)\left\|\nabla^2u\right\|_2+\|\sqrt{\rho}\theta_t\|_2+\|\sqrt{\rho}\theta_t\|_2^{\frac12}\|\nabla\theta_t\|_2^{\frac12}\right]\left(\|\sqrt{\rho}\theta_t\|_2+\|\nabla\theta_t\|_2\right)\\
&\le \frac{\kappa}{8}\|\nabla\theta_t\|_2^2+C\left[\|\sqrt{\rho}\theta_t\|_2^2+\left(1+\|\nabla\theta\|_2^2\right)\|\nabla^2u\|_2^2\right],\\
\left|\int_\Omega P\operatorname{div}u_t\theta_t\,\mathrm{d}x\right| &\le C\|\rho\|_\infty^{\frac34}(\|\sqrt{\rho}\theta\|_2+\|\nabla\theta\|_2)\|\nabla u_t\|_2\left(\|\sqrt{\rho}\theta_t\|_2+\|\nabla\theta_t\|_2\right)\\
&\le \frac{\kappa}{8}\|\nabla\theta_t\|_2^2+C\left[\|\sqrt{\rho}\theta_t\|_2^2+\left(1+\|\nabla\theta\|_2^2\right)\|\nabla u_t\|_2^2\right],\\
\left|\int_\Omega(4\mu Du:Du_t+2\lambda\operatorname{div}u\operatorname{div}u_t\theta_t)\,\mathrm{d}x\right| &\le C\|\nabla u\|_2^{\frac12}\left\|\nabla^2u\right\|_2^{\frac12}\|\nabla u_t\|_2\left(\|\sqrt{\rho}\theta_t\|_2+\|\nabla\theta_t\|_2\right)\\
&\le \frac{\kappa}{8}\|\nabla\theta_t\|_2^2+C\left(\|\sqrt{\rho}\theta_t\|_2^2+\left\|\nabla^2u\right\|_2\|\nabla u_t\|_2^2\right),\\
\left|\int_\Omega\nu|\operatorname{curl}H|_t^2\theta_t\right| &\le C\int_\Omega\left|\nabla H\right|\left|\nabla H_t\right|\left|\theta_t\right|\mathrm{d}x\le C\|\nabla H\|_3\|\nabla H_t\|_2\|\theta_t\|_6\\
&\le C\|\nabla H\|_2^{\frac12}\left\|\nabla^2H\right\|_2^{\frac12}\|\nabla H_t\|_2\left(\|\sqrt{\rho}\theta_t\|_2+\|\nabla\theta_t\|_2\right)\\
&\le \frac{\kappa}{8}\|\nabla\theta_t\|_2^2+C\left(\|\sqrt{\rho}\theta_t\|_2^2+\left\|\nabla^2H\right\|_2\|\nabla H_t\|_2^2\right),
\end{aligned}
$$

So, one deduces from (2.46) that

$$
\begin{aligned}
\frac{c_v}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}\theta_t\|_2^2+\frac{\kappa}{8}\|\nabla\theta_t\|_2^2\le& C\left[\|\sqrt{\rho}\theta_t\|_2^2+\left(1+\|\nabla^2u\|_2^2\right)\left(1+\|\nabla\theta\|_2^2\right)+\|\nabla^2\theta\|_2^2+\|\sqrt{\rho}u_t\|_2\|\nabla u_t\|_2\|\nabla\theta\|_2^2\right]\\
&+C\left(1+\|\nabla\theta\|_2^2+\left\|\nabla^2u\right\|_2\right)\|\nabla u_t\|_2^2+C\left\|\nabla^2H\right\|_2\|\nabla H_t\|_2^2.
\end{aligned}
$$

Multiplying the above inequality by t^2 and using Corollary 2, one obtains that

$$
\frac{c_v}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|t\sqrt{\rho}\theta_t\|_2^2+\frac{\kappa}{8}\|t\nabla\theta_t\|_2^2\le C\left(\|\sqrt{t}\sqrt{\rho}\theta_t\|_2^2+\|\sqrt{t}\nabla^2\theta\|_2^2+\|\sqrt{t}\nabla u_t\|_2^2+\|\sqrt{t}\nabla H_t\|_2^2+1\right). \tag{2.47}
$$

Integrating (2.47) over $(0,T)$ and using Corollary 2, one gets that

$$
\sup_{0\le t\le T}\|t\sqrt{\rho}\theta_t\|_2^2+\int_0^T\|t\nabla\theta_t\|_2^2\,\mathrm{d}t\le C. \tag{2.48}
$$

Recalling (2.26), one deduces from (2.48) and Corollary 2 that

$$
\begin{aligned}
\sup_{0\le t\le T}\left\|t\nabla^2\theta\right\|_2^2\le& C\sup_{0\le t\le T}\left(\|t\sqrt{\rho}\theta_t\|_2^2+t\|\nabla u\|_2^4\|\sqrt{t}\nabla\theta\|_2^2+\sqrt{t}\|\nabla u\|_2\|\sqrt{t}\nabla^2u\|_2^3+\sqrt{t}\|\nabla H\|_2\|\sqrt{t}\nabla^2H\|_2^3\right)\\
&+C\sup_{0\le t\le T}\left(t^{\frac32}\|\sqrt{\rho}\theta\|_2^2+\sqrt{t}\|\sqrt{t}\nabla\theta\|_2^2\right)\|\nabla u\|_2\|\sqrt{t}\nabla^2u\|_2\\
\le& C.
\end{aligned} \tag{2.49}
$$

Applying the elliptic estimates to $(1.1)_2$, one obtains from Proposition 1 and Corollary 2 that

$$\begin{aligned}\left\|\nabla^2 u\right\|_q &\le C\left(\|\rho u_t\|_q + \|\rho(u\cdot\nabla)u\|_q + \|\nabla P\|_q + \|\mathrm{curl} H\times H\|_q\right)\\ &\le C\left(\|\rho\|_\infty\|u_t\|_6 + \|\rho\|_\infty\|u\|_\infty\|\nabla u\|_6 + \|\nabla\rho\|_q\|\theta\|_\infty + \|\rho\|_\infty\|\nabla\theta\|_6 + \|H\|_\infty\|\nabla H\|_6\right)\\ &\le C\left(1+\|\nabla u_t\|_2 + \left\|\nabla^2 u\right\|_2^2 + \|\nabla\theta\|_2 + \left\|\nabla^2\theta\right\|_2 + \left\|\nabla^2 H\right\|_2\right).\end{aligned}$$

Thus,

$$\begin{aligned}\int_0^T \|\sqrt{t}\nabla^2 u\|_q^2\,\mathrm{d}t \le& C\int_0^T\left(1+\|\sqrt{t}\nabla u_t\|_2^2 + \|\sqrt{t}\nabla^2 u\|_2^2\|\nabla^2 u\|_2^2 + \|\sqrt{t}\nabla^2 u\|_2^2\|\nabla^2 H\|_2^2\right)\mathrm{d}t\\ &+C\int_0^T\left(\|\nabla\theta\|_2^2 + \|\sqrt{t}\nabla^2\theta\|_2^2\right)\mathrm{d}t\\ \le& C.\end{aligned}\tag{2.50}$$

Applying the elliptic estimates to $(1.1)_4$ and using Corollary 2, one obtains that

$$\begin{aligned}\|\nabla^2 H\|_q &\le C(\|H_t\|_q + \|H\cdot\nabla u\|_q + \|u\cdot\nabla H\|_q + \|H\mathrm{div} u\|_q)\\ &\le C(\|H_t\|_6 + \|H\|_\infty\|\nabla u\|_6 + \|u\|_\infty\|\nabla H\|_6 + \|H\|_\infty\|\nabla u\|_6)\\ &\le C(\|\nabla H_t\|_2 + \|\nabla^2 H\|_2\|\nabla^2 u\|_2).\end{aligned}$$

So,

$$\begin{aligned}\int_0^T \|\sqrt{t}\nabla^2 H\|_q^2\mathrm{d}t &\le C\int_0^t(\|\sqrt{t}\nabla H_t\|_2^2 + \|\nabla^2 H\|_2\|\sqrt{t}\nabla^2 u\|_2^2)\mathrm{d}t\\ &\le C.\end{aligned}\tag{2.51}$$

Finally, one applies the elliptic estimates to $(1.1)_3$ and deduces from Corollary 2 that

$$\begin{aligned}\left\|\nabla^2\theta\right\|_6^2 \le& C\left(\|\rho\theta_t\|_6^2 + \|\rho(u\cdot\nabla)\theta\|_6^2 + \|\rho\theta\,\mathrm{div}\,u\|_6^2 + \|\mathcal{Q}(\nabla u)\|_6^2 + \|\nu|\mathrm{curl}H|^2\|_6^2\right)\\ \le& C(\|\rho\|_\infty^2\|\theta_t\|_6^2 + \|\rho\|_\infty^2\|u\|_\infty^2\|\nabla\theta\|_6^2 + \|\rho\|_\infty^2\|\theta\|_6^2\|\nabla u\|_\infty^2 + \|\nabla u\|_\infty^2\|\nabla u\|_6^2 + \|\nabla H\|_\infty^2\|\nabla H\|_6^2)\\ \le& C\left[\|\sqrt{\rho}\theta_t\|_2^2 + \|\nabla\theta_t\|_2^2 + \left\|\nabla^2 u\right\|_2\left(\|\nabla\theta\|_2^2 + \|\nabla^2\theta\|_2^2\right) + \left(1+\|\nabla\theta\|_2^2\right)\|\nabla^2 u\|_q^2\right]\\ &+C\left[\|\nabla^2 u\|_2^2\|\nabla^2 u\|_q^2 + \|\nabla^2 H\|_2^2\|\nabla^2 H\|_q^2\right].\end{aligned}\tag{2.52}$$

Hence, it follows from (2.48), (2.50), (2.51) and Corollary 2 that

$$\begin{aligned}\int_0^T \left\|t\nabla^2\theta\right\|_6^2\,\mathrm{d}t \le& C\int_0^T\left[\|\sqrt{t}\sqrt{\rho}\theta_t\|_2^2 + \|t\nabla\theta_t\|_2^2 + \left(1+\|\sqrt{t}\nabla\theta\|_2^2\right)\|\sqrt{t}\nabla^2 u\|_q^2\right]\mathrm{d}t\\ &+C\int_0^T\left[\|\sqrt{t}\nabla^2 u\|_2\left(\|\sqrt{t}\nabla\theta\|_2^2 + \|\sqrt{t}\nabla^2\theta\|_2^2\right) + \|\sqrt{t}\nabla^2 u\|_2^2\|\sqrt{t}\nabla^2 u\|_q^2\right]\mathrm{d}t\\ &+C\int_0^T\|\sqrt{t}\nabla^2 H\|_2^2\|\sqrt{t}\nabla^2 H\|_q^2\mathrm{d}t\\ \le& C.\end{aligned}\tag{2.53}$$

Combining (2.45) with (2.48)-(2.53), the desire result (2.44) is obtained. □

Next we prove that the existence time T_0 depends only on R, μ, λ, c_v, q, and the upper bound of Φ_0, but is independent of the quantities $\|\nabla^2 u_0\|_2$, $\|\nabla^2 H_0\|_2$, $\|\nabla^2\theta_0\|_2$, $\|g_1\|_2$, and $\|g_2\|_2$ stated in Lemma 1.

Proposition 10. *Let $q\in(3,6)$ and assume that $(\rho_0, u_0, H_0, \theta_0)$ satisfies*

$$\underline{\rho}\le\rho_0\in W^{1,q}(\Omega),\ u_0\in H_0^1(\Omega)\cap H^2(\Omega),\ 0\le\theta_0\in H_0^1(\Omega)\cap H^2(\Omega),\ H_0\in H^2(\Omega)$$

for some positive number $\underline{\rho}$. Then, there exist two positive constants T_0 and C depending only on R, μ, λ, c_v, q, and the upper bound of Φ_0, such that system (1.1), subject to (1.4)-(1.5), admits a unique solution (ρ, u, H, θ), in $\Omega \times (0, T_0)$, satisfying

$$\sup_{0\le t\le T_0}\left(\|\rho\|_\infty + \|\rho\|_{W^{1,q}} + \left\|\left(\rho_t, \sqrt{\rho}\theta, \nabla u, \nabla H\right)\right\|_2^2\right) + \int_0^{T_0} \left\|\left(\nabla\theta, \sqrt{\rho}u_t, \nabla^2 u, \nabla^2 H, H_t\right)\right\|_2^2 \,\mathrm{d}t \le C,$$

and

$$\sup_{0\le t\le T_0}\|(\sqrt{t}\nabla\theta, t\sqrt{\rho}\theta_t, t\nabla^2\theta, \sqrt{t}\sqrt{\rho}u_t, \sqrt{t}\nabla^2 u, \sqrt{t}\nabla^2 H)\|_2^2 + \int_0^{T_0}(\|\sqrt{t}\nabla^2 u\|_q^2 + \|t\nabla^2\theta\|_6^2)dt + \int_0^{T_0}\|(\sqrt{t}\sqrt{\rho}\theta_t, \sqrt{t}\nabla^2\theta, t\nabla\theta_t, \sqrt{t}\nabla u_t, \sqrt{t}\nabla H_t)\|_2^2 \,\mathrm{d}t \le C.$$

Proof. By Lemma 1, there is a unique local strong solution (ρ, u, H, θ) on $\Omega \times (0, T_*)$ satisfying the regularities stated in Lemma 1. By Lemma 1, one extends the local solution uniquely to the maximal time of existence $T_{\max}$. Then, the following holds

$$\sup_{T_*\le t<T_{\max}}\left(\left\|\frac{1}{\rho}\right\|_\infty + \|\rho\|_{W^{1,q}} + \|u\|_{H^2} + \|\theta\|_{H^2} + \|H\|_{H^2}\right) = \infty. \tag{2.54}$$

(ρ, u, H, θ) satisfies the regularities in Lemma 1 with T_* replaced by T for any $T \in (0, T_{\max})$. Let ϵ_0 be the constant stated in Corollary 1, Φ be the function given by (1.8), and set

$$T_0 = \sup\left\{T \in (0, T_{\max}) \mid T^{\frac{6-q}{4q}}\Phi^{\frac{5}{2}}(T) \le \epsilon_0\right\}.$$

Claim: $T_0 < T_{\max}$.

Assume that $T_0 = T_{\max}$. Then, by definition

$$T^{\frac{6-q}{4q}}\Phi^{\frac{5}{2}}(T) \le \epsilon_0 \quad (\forall T \in (0, T_{\max})). \tag{2.55}$$

Since $\Phi(T) \ge 1$, it follows from (2.55) that $T_{\max} \le \epsilon_0^{\frac{4q}{6-q}}$. Thanks to (2.55), one deduces from Corollary 2 and Proposition 9 that

$$\Phi(T) + \sup_{0\le t\le T}\|(\sqrt{t}\nabla^2 u, \sqrt{t}\nabla^2 H, t\nabla^2\theta)\|_2^2 \le C \quad (\forall T \in (0, T_{\max}))$$

holds for some positive constant C, where C independent of $T \in (0, T_{\max})$. Following the arguments in Proposition 3, one deduces from Proposition 2 and (2.55) that

$$\inf_{\Omega\times(0,T)}\rho \ge \underline{\rho}e^{-\int_0^T\|\operatorname{div} u\|_\infty dt} \ge \underline{\rho}e^{-C\epsilon_0} \quad (\forall T \in (0, T_{\max}))$$

for some positive constant C independent of $T \in (0, T_{\max})$. So,

$$\sup_{T_*\le t<T_{\max}}\left(\left\|\frac{1}{\rho}\right\|_\infty + \|\rho\|_{W^{1,q}} + \|u\|_{H^2} + \|H\|_{H^2} + \|\theta\|_{H^2}\right) \le C,$$

which contradicts to (2.54). This contradiction proves the claim. Since $T_0 < T_{\max}$ and noticing that $\Phi(T)$ is continuous on $[0, T_{\max})$, one gets from the definition of T_0 that

$$T_0^{\frac{6-q}{4q}}\Phi^{\frac{5}{2}}(T_0) = \epsilon_0. \tag{2.56}$$

Recalling that $\Phi(T) \ge 1$, it follows from Corollary 2 that $T_0 \le \epsilon_0^{\frac{4q}{6-q}}$ and $\Phi(T_0) \le C_0$ for a positive constant C_0 depending only on R, μ, λ, c_v, q, and the upper bound of Φ_0. Therefore, it follows from (2.56) that $T_0 \ge \left(\frac{\epsilon_0}{C_0^{\frac{5}{2}}}\right)^{\frac{4q}{6-q}}$. The corresponding estimates follow from Corollary 2 and Proposition 9. $\square$

3 | PROOF OF THEOREM 1.1

In order to prove Theorem 1, we take similar argument in [24]. At first, a preparing existence result (see Proposition 11) is given. The proof of Proposition 11 is proved by similar argument of Proposition 3.1 in [24], and the specific proof process is postponed in the “Appendix”.

Proposition 11. *Assume that all the conditions of Theorem 1 hold. Denote*

$$\Phi_0 := \|\rho_0\|_\infty + \|\nabla\rho_0\|_q + \left\|\left(\sqrt{\rho_0}\theta_0, \nabla u_0, \nabla H_0\right)\right\|_2^2.$$

(i) Then, there exists a positive time T_0 depending only on R, μ, λ, c_v, q, and Φ_0, such that system (1.1), subject to (1.4)-(1.5), in $\Omega\times(0,T_0)$, admits a solution (ρ,u,H,θ), which satisfies all the properties stated in Definition 1 except that the regularities $\sqrt{\rho}u, \sqrt{\rho}\theta \in C\left([0,T_0];L^2\right)$ are replaced by $\rho u\in C\left([0,T_0];L^2\right)$ and $\rho\theta\in C_w\left([0,T_0];L^2\right)$, where C_w represents the weak continuity.

(ii) Moreover, for any $t\in(0,T_0)$, it holds that

$$\|\sqrt{\rho}u\|_2^2(t)\le\|\sqrt{\rho_0}u_0\|_2^2+Ct,\quad \|\sqrt{\rho}\theta\|_2^2(t)\le\|\sqrt{\rho_0}\theta_0\|_2^2+C\sqrt{t}.$$

Based on this result, the well-posedness of the strong solutions is first proven in Lagrangian coordinates, and finally transformed back into Euler coordinates.

3.1 | Lagrangian Coordinates and some lemmas

Given a velocity field $u\in L^1\left(0,T_0;C^1(\bar{\Omega})\right)$ satisfying $u|_{\partial\Omega}=0$. Let $x=\varphi(y,t)$ be the corresponding coordinates transform, governed by the velocity field u, between the Euler coordinates (x,t) and the Lagrangian coordinates (y,t), that is,

$$\begin{cases}\partial_t\varphi(y,t)=u(\varphi(y,t),t), & \forall t\in[0,T_0],\\ \varphi(y,0)=y.\end{cases}\tag{3.1}$$

By the classical theory for ODEs, φ is well-defined and $\varphi:\Omega\times[0,T_0]\to\Omega$. Moreover, by the unique solvability of ODEs, for each $t\in[0,T_0]$, $\varphi(\cdot,t):\Omega\to\Omega$ is bijective. Denote by $y=\psi(x,t)$ the inverse mapping of $x=\varphi(y,t)$ with respect to y, which satisfies

$$\begin{cases}\partial_t\psi(x,t)+(u(x,t)\cdot\nabla)\psi(x,t)=0,\\ \psi(x,t)\big|_{t=0}=x.\end{cases}\tag{3.2}$$

Set

$$A(y,t)=\left(a_{ij}(y,t)\right)_{3\times3}, a_{ij}(y,t)=\partial_i\psi_j(x,t)|_{x=\varphi(y,t)},\tag{3.3}$$

$$J(y,t)=\det A(y,t)=\det\nabla\psi(x,t)|_{x=\varphi(y,t)},\tag{3.4}$$

$$B(y,t)=\left(b_{ij}(y,t)\right)_{3\times3},\quad b_{ij}(y,t)=\partial_i\varphi_j(y,t).\tag{3.5}$$

Then, one can check that

$$\begin{cases}\partial_tA(y,t)=-\nabla u(\varphi(y,t),t)A(y,t),\\ A(y,t)\big|_{t=0}=I,\end{cases}\tag{3.6}$$

and

$$\begin{cases}\partial_tB(y,t)=B(y,t)\nabla u(\varphi(y,t),t),\\ B(y,t)\big|_{t=0}=I,\end{cases}\tag{3.7}$$

here $\nabla u=\left(\partial_iu_j\right)_{3\times3}$. Recalling the definition of J, one derives from (3.6) that

$$\begin{cases}\partial_tJ(y,t)=-\operatorname{div}u(\varphi(y,t),t)J(y,t),\\ J(y,t)\big|_{t=0}=1.\end{cases}\tag{3.8}$$

Lemma 2. *Li and Zheng.*[24] *Given $u \in L^\infty\left(0,T_0;H_0^1\right) \cap L^1\left(0,T_0;W^{2,q}\right)$, with $q \in (3,6)$, and let φ, ψ, A, B, and J be defined as before. Then, $J > 0$ on $\Omega \times (0,T_0)$ and the following hold:*

$$\sup_{0\le t\le T_0}\left(\left\|\left(\frac{1}{J},J,A,B\right)\right\|_\infty + \|(J_t,A_t,B_t)\|_2 + \|(\nabla J,\nabla A,\nabla B)\|_q\right) \le C, \tag{3.9}$$

$$\|\nabla[g(\varphi(\cdot,t))]\|_{W^{1,\alpha}} \simeq \|\nabla g\|_{W^{1,\alpha}} \simeq \|\nabla[g(\psi(\cdot,t))]\|_{W^{1,\alpha}}, \quad \forall \alpha \in [1,q], \tag{3.10}$$

$$\|\nabla[g(\varphi(\cdot,t))]\|_\alpha \simeq \|\nabla g\|_\alpha \simeq \|\nabla[g(\psi(\cdot,t))]\|_\alpha, \quad \forall \alpha \in [1,\infty], \tag{3.11}$$

$$\|g(\varphi(\cdot,t))\|_\alpha \simeq \|g\|_\alpha \simeq \|g(\psi(\cdot,t))\|_\alpha, \quad \forall \alpha \in [1,\infty], \tag{3.12}$$

for any function g such that all the relevant quantities are finite, here we denote $Q_1 \simeq Q_2$ means $\frac{Q_1}{\bar C} \le Q_2 \le \bar C Q_1$ for a positive constant $\bar C$ depending only on Ω, α, q, T_0, and $\|u\|_{L^\infty(0,T_0;H^1(\Omega))\cap L^1(0,T_0;W^{2,q}(\Omega))}$.

Lemma 3. *Li and Zheng.*[24] *Under the assumptions as in Proposition 11, the following hold:*

(i) $h(\varphi(\cdot,t),t) \in C\left([0,T_0];L^2\right)$ *if* $h \in C\left([0,T_0];L^2\right)$;

(ii) $h(\varphi(\cdot,t),t) \in C_w\left([0,T_0];L^2\right)$ *if* $h \in C_w\left([0,T_0];L^2\right)$.

3.2 | Regularities and Reduced System in the Lagrangian Coordinates

Let initial data $(\rho_0,u_0,H_0,\theta_0)$ satisfy the assumptions in Theorem 1. Let (ρ,u,H,θ) be the solution established in Proposition 11 and φ be the corresponding mapping defined by (3.1). Set

$$\begin{cases} \varrho(y,t) = \rho(\varphi(y,t),t), \\ v(y,t) = u(\varphi(y,t),t), \\ \vartheta(y,t) = \theta(\varphi(y,t),t), \\ h(y,t) = H(\varphi(y,t),t). \end{cases} \tag{3.13}$$

Recalling the regularities of (ρ,u,H,θ) in Proposition 11, one gets from Lemma 2 and Lemma 3 that

$$\begin{cases} \varrho \in L^\infty\left(0,T_0;W^{1,q}\right), \sqrt{\varrho}\vartheta \in L^\infty\left(0,T_0;L^2\right), \vartheta \in L^2\left(0,T_0;H^1\right), \\ v \in L^\infty\left(0,T_0;H^1\right) \cap L^2\left(0,T_0;H^2\right) \cap L^1\left(0,T_0;W^{2,q}\right), \\ h \in L^\infty\left(0,T_0;H^1\right) \cap L^2\left(0,T_0;H^2\right) \cap L^1\left(0,T_0;W^{2,q}\right), \\ \sqrt{t}v \in L^\infty\left(0,T_0;H^2\right) \cap L^2\left(0,T_0;W^{2,q}\right), \\ \sqrt{t}h \in L^\infty\left(0,T_0;H^2\right) \cap L^2\left(0,T_0;W^{2,q}\right), \\ \sqrt{t}\vartheta \in L^\infty\left(0,T_0;H^1\right) \cap L^2\left(0,T_0;H^2\right), \\ \varrho v \in C\left([0,T_0];L^2\right), \varrho\vartheta \in C_w\left([0,T_0];L^2\right). \end{cases} \tag{3.14}$$

Obviously,

$$\begin{aligned} \partial_t\varrho(y,t) &= (\partial_t\rho + u\cdot\nabla\rho)(\varphi(y,t),t), \\ \partial_t v(y,t) &= [\partial_t u + (u\cdot\nabla)u](\varphi(y,t),t), \\ \partial_t\vartheta(y,t) &= (\partial_t\theta + u\cdot\nabla\theta)(\varphi(y,t),t), \\ \partial_t h(y,t) &= (\partial_t H + u\cdot\nabla H)(\varphi(y,t),t). \end{aligned}$$

Recalling the regularities of (ρ,u,H,θ) stated in Proposition 11, one deduces from Lemma 2 that

$$\begin{aligned} \sup_{0\le t\le T_0}\|\varrho_t\|_2 &\le C\sup_{0\le t\le T_0}\left(\|\rho_t\|_2 + \|u\|_6\|\nabla\rho\|_3\right) \\ &\le C\sup_{0\le t\le T_0}\left(\|\rho_t\|_2 + \|\nabla u\|_2\|\nabla\rho\|_q\right) \le C, \\ \int_0^{T_0}\|\sqrt{\varrho}v_t\|_2^2\,\mathrm{d}t &\le C\int_0^{T_0}\left(\|\sqrt{\rho}u_t\|_2^2 + \|u\|_6^2\|\nabla u\|_3^2\right)\mathrm{d}t \\ &\le C\int_0^{T_0}\left(\|\sqrt{\rho}u_t\|_2^2 + \|\nabla u\|_2^3\|\nabla^2 u\|_2\right)\mathrm{d}t \le C, \end{aligned}$$

$$\begin{aligned}\int_0^{T_0} t\left\|\nabla v_t\right\|_2^2 \mathrm{~d}t &\leq C \int_0^{T_0} t\left\|\nabla\left(\partial_t u+(u \cdot \nabla) u\right)\right\|_2^2 \mathrm{~d}t \\ &\leq C \int_0^{T_0} t\left(\left\|\nabla u_t\right\|_2^2+\|u\|_{\infty}^2\left\|\nabla^2 u\right\|_2^2+\|\nabla u\|_4^4\right) \mathrm{d}t \\ &\leq C \int_0^{T_0} t\left(\left\|\nabla u_t\right\|_2^2+\|\nabla u\|_2\left\|\nabla^2 u\right\|_2^3\right) \mathrm{d}t \leq C .\end{aligned}$$

Similarly,

$$\begin{aligned}\int_0^{T_0} t\left\|\sqrt{\varrho} \vartheta_t\right\|_2^2 \mathrm{~d}t &\leq C \int_0^{T_0} t\left\|\sqrt{\rho} \theta_t+\sqrt{\rho} u \cdot \nabla \theta\right\|_2^2 \mathrm{~d}t \\ &\leq C \int_0^{T_0} t\left(\left\|\sqrt{\rho} \theta_t\right\|_2^2+\|\rho\|_{\infty}\|u\|_6^2\|\nabla \theta\|_3^2\right) \mathrm{d}t \\ &\leq C \int_0^{T_0} t\left(\left\|\sqrt{\rho} \theta_t\right\|_2^2+\|\nabla u\|_2^2\|\nabla \theta\|_2\left\|\nabla^2 \theta\right\|_2\right) \mathrm{d}t \leq C, \\ \int_0^{T_0} t^2\left\|\nabla \vartheta_t\right\|_2^2 \mathrm{~d}t &\leq C \int_0^{T_0} t^2 \int_{\Omega}\left(\left|\nabla \theta_t\right|^2+|u|^2\left|\nabla^2 \theta\right|^2+|\nabla u|^2|\nabla \theta|^2\right) \mathrm{d}x \mathrm{~d}t, \\ &\leq C \int_0^{T_0} t^2\left(\left\|\nabla \theta_t\right\|_2^2+\|\nabla u\|_2\left\|\nabla^2 u\right\|_2\left\|\nabla^2 \theta\right\|_2^2\right) \mathrm{d}t \\ &\quad+C \int_0^{T_0} t^2\left\|\nabla^2 u\right\|_2^2\left(\|\nabla \theta\|_2^2+\|\nabla \theta\|_2\left\|\nabla^2 \theta\right\|_2\right) \mathrm{d}t \leq C, \\ \int_0^{T_0}\left\|h_t\right\|_2^2 \mathrm{~d}t &= \int_0^{T_0}\left\|\left[\left(\partial_t H+(u \cdot \nabla) H\right)\right](\varphi(y, t), t)\right\|_2^2 \mathrm{~d}t \\ &\leq C \int_0^{T_0}\left\|\left(\partial_t H+(u \cdot \nabla) H\right)\right\|_2^2 \mathrm{~d}t \\ &\leq C \int_0^{T_0}\left(\left\|H_t\right\|_2^2+\|u\|_6^2\|\nabla H\|_3^2\right) \mathrm{d}t \\ &\leq C \int_0^{T_0}\left(\left\|H_t\right\|_2^2+\|\nabla u\|_2^3\left\|\nabla^2 H\right\|_2\right) \mathrm{d}t \leq C . \\ \int_0^{T_0} t\left\|\nabla h_t\right\|_2^2 \mathrm{~d}t &= \int_0^{T_0} t\left\|\nabla\left[\left(\partial_t H+(u \cdot \nabla) H\right)\right](\varphi(y, t), t)\right\|_2^2 \mathrm{~d}t \\ &\leq C \int_0^{T_0} t\left\|\nabla\left(\partial_t H+(u \cdot \nabla) H\right)\right\|_2^2 \mathrm{~d}t \\ &\leq C \int_0^{T_0} t\left(\left\|\nabla H_t\right\|_2^2+\|\nabla u\|_4^2\|\nabla H\|_4^2+\|u\|_{\infty}^2\left\|\nabla^2 H\right\|_2^2\right) \mathrm{d}t \\ &\leq C \int_0^{T_0}\left(\left\|\sqrt{t} \nabla H_t\right\|_2^2+t\left(\|\nabla u\|_2^{\frac{1}{2}}\|\nabla H\|_2^{\frac{1}{2}}\|\nabla u\|_2^{\frac{3}{2}}\|\nabla H\|_2^{\frac{3}{2}}\right.\right. \\ &\quad\left.\left.+\|\nabla u\|_2\left\|\nabla^2 u\right\|_2\left\|\nabla^2 H\right\|_2\right)\right) \mathrm{d}t \leq C,\end{aligned}$$

Therefore,

$$\left.\begin{array}{l}\varrho_t \in L^{\infty}\left(0, T_0 ; L^2\right), \quad \sqrt{\varrho} v_t \in L^2\left(0, T_0 ; L^2\right), \quad \sqrt{t} v_t \in L^2\left(0, T_0 ; H^1\right), \\ \sqrt{t} \sqrt{\varrho} \vartheta_t \in L^2\left(0, T_0 ; L^2\right), \quad t \vartheta_t \in L^2\left(0, T_0 ; H^1\right), \\ h_t \in L^{\infty}\left(0, T_0 ; L^2\right), \quad \sqrt{t} h_t \in L^2\left(0, T_0 ; H^1\right).\end{array}\right\} \tag{3.15}$$

Based on the above discussion, we get the following Proposition 12 and Proposition 13.

Proposition 12. *Given initial data $(\rho_0, u_0, H_0, \theta_0)$ satisfying the assumptions in Theorem 1. Let (ρ, u, H, θ) be the solution established in Proposition 11 and φ be the corresponding mapping defined by (3.1). Then, $(\varrho, v, h, \vartheta)$ defined by (3.13) satisfies (3.14) and (3.15).*

Let A be defined as before in the previous subsection and denote

$$\nabla_A f:=A \nabla f, \quad \operatorname{div}_A v:=A:(\nabla v)^T, \quad \nabla v=\left(\partial_i v_j\right)_{3 \times 3} \quad \operatorname{cul}_A h=(A \nabla) \times h .$$

Then we can derive from $(1.1)_1$–$(1.1)_5$ and (3.6)–(3.8) that

$$\varrho_t + \mathrm{div}_A\, v\varrho = 0, \tag{3.16}$$

$$J\rho_0 v_t - \mu\, \mathrm{div}_A\, (\nabla_A v) - (\mu+\lambda)\nabla_A\, (\mathrm{div}_A\, v) + R\nabla_A\, (J\rho_0\vartheta) = h\cdot\nabla_A h - \frac{1}{2}\nabla_A |h|^2, \tag{3.17}$$

$$c_v J\rho_0\vartheta_t + RJ\rho_0\vartheta\, \mathrm{div}_A\, v - \kappa\, \mathrm{div}_A\, (\nabla_A\vartheta) = \frac{\mu}{2}|\nabla_A v + (\nabla_A v)^T|^2 + \lambda\, (\mathrm{div}_A\, v)^2 + \nu\, (\mathrm{curl}_A\, h)^2\,, \tag{3.18}$$

$$h_t - h\cdot\nabla_A v + v\cdot\nabla_A h + h\mathrm{div}_A u = \nu \mathrm{div}_A(\nabla_A h), \tag{3.19}$$

$$\mathrm{div}_A h = 0, \tag{3.20}$$

$$A_t + \nabla_A v A = 0, \tag{3.21}$$

$$J_t + \mathrm{div}_A\, vJ = 0. \tag{3.22}$$

System (3.16)-(3.22) are satisfied a.e. in $\Omega\times(0,T_0)$. Here in (3.17) and (3.18) we have used the fact that

$$\frac{\varrho}{J} = \frac{\rho_0}{J_0} = \rho_0 \tag{3.23}$$

to replace ϱ with $J\rho_0$, as $\partial_t\left(\frac{\varrho}{J}\right) = 0$ guaranteed by (3.16) and (3.22).

The initial-boundary conditions read as

$$(\varrho v, \varrho\vartheta, h, A, J)\big|_{t=0} = (\rho_0 u_0, \rho_0\theta_0, H_0, I, 1)\,, \tag{3.24}$$

$$v|_{\partial\Omega} = 0, \quad \nabla\vartheta\cdot n\big|_{\partial\Omega} = 0, \quad h\cdot n\big|_{\partial\Omega} = 0. \tag{3.25}$$

Since $\varrho v \in C\left([0,T_0]\,;L^2\right)$ and $\varrho\vartheta \in C_w\left([0,T_0]\,;L^2\right)$, $h\in C\left([0,T_0]\,;L^2\right)$, one deduces from (3.14), and $A, J \in C\left([0,T_0]\,;L^2\right)$, one deduces from Lemma 2, the initial condition (3.24) is well-defined. Finally, we state the continuities of $\sqrt{\rho_0}v$ and $\sqrt{\rho_0}\vartheta$, whose proof is similar to the proof of proposition 4.4 in [24]. For simplify, we omit the proof of Proposition 13.

Proposition 13. *Under the same assumptions in Proposition 12, it holds that*

$$\sqrt{\rho_0}v, \sqrt{\rho_0}\vartheta \in C\left([0,T_0]\,;L^2\right),$$

$$\left(\sqrt{\rho_0}v, \sqrt{\rho_0}\vartheta\right) \to \left(\sqrt{\rho_0}u_0, \sqrt{\rho_0}\theta_0\right) \quad \text{in } L^2 \quad \text{as } t\to 0.$$

3.3 | Proof of Theorem 1

Now, it is ready to give the proof of Theorem 1.

Proof of Theorem 1. (i) Existence. By virtue of Proposition 11, it remains to show that $\sqrt{\rho}u, \sqrt{\rho}\theta \in C\left([0,T_0]\,;L^2\right)$. Let $(\varrho, v, h, \vartheta)$ be given by (3.13). Then, it follows from Proposition 13 that $\sqrt{\rho_0}v, \sqrt{\rho_0}\vartheta \in C\left([0,T_0]\,;L^2\right)$. Recalling that $\varrho = J\rho_0$ and noticing that $J\in C\left([0,T_0]\,;C(\bar{\Omega})\right)$ guaranteed by Lemma 2, one gets $\sqrt{\varrho}v, \sqrt{\varrho}\vartheta \in C\left([0,T_0]\,;L^2\right)$. Then, similarly to the proof of (ii) of Lemma 3 (since ψ has the same properties as those of φ), one can then show that $\sqrt{\rho}u, \sqrt{\rho}\theta \in C\left([0,T_0]\,;L^2\right)$.

(ii) **Uniqueness**. Let $(\hat{v}, \hat{\vartheta}, \hat{h}, \hat{A}, \hat{J})$ in $\Omega\times(0,T_0)$, with the same initial data $(\rho_0, u_0, \theta_0, H_0)$. Let $(\hat{\varphi}, \hat{\psi}, \hat{\varrho}, \hat{v}, \hat{\vartheta}, \hat{h}, \hat{A}, \hat{J})$ and $(\check{\varphi}, \check{\psi}, \check{\varrho}, \check{v}, \check{\vartheta}, \check{h}, \check{A}, \check{J})$ be the corresponding quantities defined as before and denote

$$(v, \vartheta, A, J) = (\hat{v}, \hat{\vartheta}, \hat{A}, \hat{J}) - (\check{v}, \check{\vartheta}, \check{A}, \check{J}).$$

Then, $(\hat{v}, \hat{\vartheta}, \hat{h}, \hat{A}, \hat{J})$ and $(\check{v}, \check{\vartheta}, \check{A}, \check{J})$ have the regularities (3.14) and (3.15), satisfy system (3.17)–(3.22) a.e. in $\Omega\times(0,T_0)$, and fulfill the initial-boundary conditions (3.23)–(3.24). One can check by direct calculations that (v, ϑ, h, A, J) satisfies

$$\begin{aligned}
\rho_0\hat{J}v_t - \mu\,\mathrm{div}_{\hat{A}}\left(\nabla_{\hat{A}}v\right) - (\mu+\lambda)\nabla_{\hat{A}}\left(\mathrm{div}_{\hat{A}}\,v\right) &= -\rho_0 J\check{v}_t + \mu\,\mathrm{div}_{\hat{A}}\left(\nabla_A\check{v}\right) + \mu\,\mathrm{div}_A\left(\nabla_{\check{A}}\check{v}\right) \\
&+ (\mu+\lambda)\nabla_{\hat{A}}\left(\mathrm{div}_A\,\check{v}\right) + (\mu+\lambda)\nabla_A\left(\mathrm{div}_{\check{A}}\,\check{v}\right) - R\nabla_{\hat{A}}\left(\rho_0\hat{J}\vartheta + \rho_0 J\check{\vartheta}\right) - R\nabla_A\left(\rho_0\check{J}\check{\vartheta}\right) \\
&+ \hat{h}\cdot\nabla_{\hat{A}}v + \hat{h}\cdot\nabla_A\check{v} + h\cdot\nabla_{\check{A}}\check{v} - \frac{1}{2}\nabla_{\hat{A}}|\hat{h}||h| - \frac{1}{2}\nabla_{\hat{A}}|h||\check{h}| - \frac{1}{2}\nabla_A|\check{h}|^2,
\end{aligned} \tag{3.26}$$

$$
\begin{aligned}
c_v\rho_0\hat{J}\vartheta_t-\kappa\,\mathrm{div}_{\hat{A}}\left(\nabla_{\hat{A}}\vartheta\right) &= -c_v\rho_0 J\check{\vartheta}_t+\kappa\,\mathrm{div}_{\hat{A}}\left(\nabla_A\check{\vartheta}\right)+\kappa\,\mathrm{div}_A\left(\nabla_{\check{A}}\check{\vartheta}\right)\\
&\quad -R\rho_0\left(\hat{J}\hat{\vartheta}\,\mathrm{div}_{\hat{A}}\,v+\hat{J}\vartheta\,\mathrm{div}_A\,\check{v}+\hat{J}\vartheta\,\mathrm{div}_{\check{A}}\,\check{v}+J\check{\vartheta}\,\mathrm{div}_{\check{A}}\,\check{v}\right)\\
&\quad +\frac{\mu}{2}\left(\nabla^i_{\hat{A}}\hat{v}_j+\nabla^j_{\hat{A}}\hat{v}_i+\nabla^i_{\check{A}}\check{v}_j+\nabla^j_{\check{A}}\check{v}_i\right)\left(\nabla^i_{\hat{A}}v_j+\nabla^j_{\hat{A}}v_i+\nabla^i_A\check{v}_j+\nabla^j_A\check{v}_i\right)\\
&\quad +\lambda\left(\mathrm{div}_{\hat{A}}\,\hat{v}+\mathrm{div}_{\check{A}}\,\check{v}\right)\left(\mathrm{div}_{\hat{A}}\,v+\mathrm{div}_A\,\check{v}\right)\\
&\quad +\nu\left(|\,\mathrm{curl}_{\hat{A}}\,\hat{h}|+|\,\mathrm{curl}_{\check{A}}\,\check{h}|\right)\left(\mathrm{curl}_{\hat{A}}\,h+\mathrm{curl}_A\,\check{h}\right), \qquad (3.27)\\
h_t-\nu\,\mathrm{div}_{\hat{A}}\left(\nabla_{\hat{A}}h\right) &= \hat{h}\cdot\nabla_{\hat{A}}v+\hat{h}\cdot\nabla_A\check{v}+h\cdot\nabla_{\check{A}}\check{v}-(\hat{v}\cdot\nabla_{\hat{A}}h+\hat{v}\cdot\nabla_A\check{h}+v\cdot\nabla_{\check{A}}\check{h})\\
&\quad -(\hat{h}\cdot\mathrm{div}_{\hat{A}}\,v+\hat{h}\cdot\mathrm{div}_A\,\check{v}+h\cdot\mathrm{div}_{\check{A}}\,\check{v})+\nu\,\mathrm{div}_{\hat{A}}\left(\nabla_A\check{h}\right)+\nu\,\mathrm{div}_A\left(\nabla_{\check{A}}\check{h}\right), \qquad (3.28)\\
\mathrm{div}_{\hat{A}}\,h+\mathrm{div}_A\,\check{h} &= 0, \qquad (3.29)\\
A_t+\nabla_{\hat{A}}\hat{v}A+\nabla_{\hat{A}}v\check{A}+\nabla_A\check{v}\check{A} &= 0, \qquad (3.30)\\
J_t+\mathrm{div}_{\hat{A}}\,\hat{v}J+\mathrm{div}_{\hat{A}}\,v\check{J}+\mathrm{div}_A\,\check{v}\check{J} &= 0. \qquad (3.31)
\end{aligned}
$$

For any vector field W and function f such that either $W\big|_{\partial\Omega}=0$ or $f\big|_{\partial\Omega}=0$, one has that

$$
\begin{aligned}
&\int_\Omega \frac{1}{\hat{J}}\nabla_{\hat{A}}f\cdot W dy=\int_\Omega \frac{1}{\hat{J}}\hat{a}_{il}\partial_l f W_i dy\\
&=-\int_\Omega\left(\partial_l\left(\frac{\hat{a}_{il}}{\hat{J}}\right)W_i+\frac{1}{\hat{J}}\hat{a}_{il}\partial_l W_i\right) f dy=-\int_\Omega \frac{1}{\hat{J}}\,\mathrm{div}_{\hat{A}}\,W f dy. \qquad (3.32)
\end{aligned}
$$

Step 1. Energy inequalities. Multiplying (3.26) with $\frac{v}{\hat{J}}$ and using (3.30), one gets that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho_0}v\|_2^2+\mu\left\|\frac{\nabla_{\hat{A}}v}{\sqrt{\hat{J}}}\right\|_2^2+(\mu+\lambda)\left\|\frac{\mathrm{div}_{\hat{A}}\,v}{\sqrt{\hat{J}}}\right\|_2^2\\
=&-\int_\Omega \rho_0 J\check{v}_t\frac{v}{\hat{J}}\,\mathrm{d}y-\int_\Omega\frac{1}{\hat{J}}\left[\mu\nabla_A\check{v}:\nabla_{\hat{A}}v+(\mu+\lambda)\,\mathrm{div}_A\,\check{v}\,\mathrm{div}_{\hat{A}}\,v\right]\mathrm{d}y\\
&+\int_\Omega\frac{1}{\hat{J}}\left[\mu\,\mathrm{div}_A\left(\nabla_{\check{A}}\check{v}\right)\cdot v+(\mu+\lambda)\nabla_A\left(\mathrm{div}_{\check{A}}\,\check{v}\right)\cdot v\right]\mathrm{d}y-R\int_\Omega\nabla_A\left(\rho_0\check{J}\check{\vartheta}\right)\frac{v}{\hat{J}}\,\mathrm{d}y\\
&+R\int_\Omega\left(\rho_0\hat{J}\vartheta+\rho_0 J\check{\vartheta}\right)\frac{\mathrm{div}_{\hat{A}}\,v}{\hat{J}}\,\mathrm{d}y+\int_\Omega\frac{1}{\hat{J}}\left[\hat{h}\cdot\nabla_{\hat{A}}v+\hat{h}\cdot\nabla_A\check{v}+h\cdot\nabla_{\check{A}}\check{v}\right]\cdot v\,\mathrm{d}y\\
&-\frac{1}{2}\int_\Omega\left[\nabla_{\hat{A}}|\hat{h}||h|+\nabla_{\hat{A}}|h||\check{h}|+\nabla_A|\check{h}|^2\right]\frac{v}{\hat{J}}\,\mathrm{d}y=:\sum_{i=1}^7 N_i. \qquad (3.33)
\end{aligned}
$$

By Lemma 2, it follows

$$
\sup_{0\le t\le T_0}\left(\left\|\frac{1}{\hat{J}}\right\|_\infty+\|(\hat{J},\check{J},\hat{A})\|_\infty+\|(\nabla\check{A},\nabla\check{J})\|_3\right)\le C.
$$

Then terms $N_i, i=1,2,\cdots,7$, are estimated as follows:

$$
\begin{aligned}
N_1 &\le C\|J\|_2\,\|\rho_0\check{v}_t\|_3\,\|v\|_6\\
&\le \frac{\mu}{8}\left\|\frac{\nabla_{\hat{A}}v}{\sqrt{\hat{J}}}\right\|_2^2+C\,\|\sqrt{\rho_0}\check{v}_t\|_2\,\|\nabla\check{v}_t\|_2\,\|J\|_2^2,\\
N_2 &\le \frac{\mu}{8}\left\|\frac{\nabla_{\hat{A}}v}{\sqrt{\hat{J}}}\right\|_2^2+C\|\nabla\check{v}\|_\infty^2\|A\|_2^2\\
&\le \frac{\mu}{8}\left\|\frac{\nabla_{\hat{A}}v}{\sqrt{\hat{J}}}\right\|_2^2+C\|\nabla^2\check{v}\|_q^2\|A\|_2^2,\\
N_3 &\le C\int_\Omega |A|\left(|\check{A}|\left|\nabla^2\check{v}\right|+|\nabla\check{A}||\nabla\check{v}|\right)|v|dy
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{\mu}{8}\left\|\frac{\nabla_{\hat A} v}{\sqrt{\hat J}}\right\|_2^2 + C\|\nabla^2\check v\|_q^2\|A\|_2^2,\\
N_4 &\le C\|A\|_2\left(\|\nabla\rho_0\|_3\|\check\vartheta\|_\infty + \|\nabla\check J\|_3\|\check\vartheta\|_\infty + \|\nabla\check\vartheta\|_3\right)\|v\|_6\\
&\le \frac{\mu}{8}\left\|\frac{\nabla_{\hat A} v}{\sqrt{\hat J}}\right\|_2^2 + C(1+\|\nabla\check\vartheta\|_2)\|\nabla^2\check\vartheta\|_2\|A\|_2^2,\\
N_5 &\le \frac{\mu+\lambda}{4}\left\|\frac{\mathrm{div}_{\hat A}\, v}{\sqrt{\hat J}}\right\|_2^2 + C\left(\|\sqrt{\rho_0}\vartheta\|_2^2 + \|\check\vartheta\|_\infty^2\|J\|_2^2\right)\\
&\le \frac{\mu+\lambda}{4}\left\|\frac{\mathrm{div}_{\hat A}\, v}{\sqrt{\hat J}}\right\|_2^2 + C\left(\|\sqrt{\rho_0}\vartheta\|_2^2 + (1+\|\nabla\check\vartheta\|_2)\left\|\nabla^2\check\vartheta\right\|_2\|J\|_2^2\right),\\
N_6 &= \int_\Omega \frac{1}{\hat J}\left[\hat h\cdot\nabla_{\hat A} v + \hat h\cdot\nabla_A\check v + h\cdot\nabla_{\check A}\check v\right]\cdot v\,\mathrm{d}y\\
&\le C\int_\Omega\left(|\hat h||\hat A||\nabla h| + |\hat h||A||\nabla\check h| + |h||\check A||\nabla\check h|\right)|v|\mathrm{d}y\\
&\le C\left(\|\hat h\|_6\|\nabla h\|_2 + \|\hat h\|_6\|A\|_2\|\nabla\check h\|_6 + \|h\|_6\|\|\nabla\check h\|_2\right)\|v\|_6\mathrm{d}y\\
&\le \frac{\mu}{8}\left\|\frac{\nabla_{\hat A} v}{\sqrt{\hat J}}\right\|_2^2 + \|\nabla(\hat h,\check h)\|_2^2\left\|\frac{\nabla_{\hat A} h}{\sqrt{\hat J}}\right\|_2^2 + \|\nabla\hat h\|_2^2\|\nabla^2\check h\|_2^2\|A\|_2^2,\\
N_7 &\le \int_\Omega\left[|\hat A||\nabla\hat h||h| + |\hat A||\hat h||\nabla h| + |\hat A||\nabla h||\check h| + |\hat A||h||\nabla\check h| + |A||\nabla\check h||\check h|\right]|v|\mathrm{d}y\\
&\le C\left(\|\nabla(\hat h,\check h)\|_2\|h\|_6 + \|\hat h,\check h\|_6\|\nabla h\|_2 + \|A\|_2\|\nabla\check h\|_6\|\check h\|_6\right)\\
&\le \frac{\mu}{8}\left\|\frac{\nabla_{\hat A} v}{\sqrt{\hat J}}\right\|_2^2 + \|\nabla(\hat h,\check h)\|_2^2\left\|\frac{\nabla_{\hat A} h}{\sqrt{\hat J}}\right\|_2^2 + \|\nabla\check h\|_2^2\|\nabla^2\check h\|_2^2\|A\|_2^2.
\end{aligned}
$$

Substituting these estimates into (3.33) yields

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho_0}v\|_2^2 + \mu\left\|\frac{\nabla_{\hat A} v}{\sqrt{\hat J}}\right\|_2^2 \le & C\|\sqrt{\rho_0}\vartheta\|_2^2 + C\left(\|\sqrt{\rho_0}\check v_t\|_2\|\nabla\check v_t\|_2 + (1+\|\nabla\check\vartheta\|_2)\|\nabla^2\check\vartheta\|_2 + \|\nabla^2\check v\|_q^2\right)\|(A,J)\|_2^2\\
&+ C\|\nabla(\hat h,\check h)\|_2^2\left\|\frac{\nabla_{\hat A} h}{\sqrt{\hat J}}\right\|_2^2 + \|\nabla(\hat h,\check h)\|_2^2\|\nabla^2\check h\|_2^2\|A\|_2^2.
\end{aligned}
\tag{3.34}
$$

Multiplying (3.27) with $\frac{\vartheta}{\hat J}$ and using (3.32), one gets that

$$
\begin{aligned}
&\frac{c_v}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho_0}\vartheta\|_2^2 + \kappa\left\|\frac{\nabla_{\hat A}\vartheta}{\sqrt{\hat J}}\right\|_2^2\\
=&-c_v\int_\Omega \rho_0 J\check\vartheta_t\frac{\vartheta}{\hat J}\,\mathrm{d}y - \kappa\int_\Omega \nabla_A\check\vartheta\cdot\frac{\nabla_{\hat A}\vartheta}{\hat J}\,\mathrm{d}y + \kappa\int_\Omega \mathrm{div}_A\left(\nabla_{\check A}\check\vartheta\right)\frac{\vartheta}{\hat J}\,\mathrm{d}y\\
&-R\int_\Omega \rho_0\left(\hat J\hat\vartheta\,\mathrm{div}_{\hat A}\, v + \hat J\hat\vartheta\,\mathrm{div}_A\,\check v + \hat J\vartheta\,\mathrm{div}_{\check A}\,\check v + J\check\vartheta\,\mathrm{div}_{\check A}\,\check v\right)\frac{\vartheta}{\hat J}\,\mathrm{d}y\\
&+\frac{\mu}{2}\int_\Omega\left(\nabla^i_{\hat A}\hat v_j + \nabla^j_{\hat A}\hat v_i + \nabla^i_{\check A}\check v_j + \nabla^j_{\check A}\check v_i\right)\left(\nabla^i_{\hat A}v_j + \nabla^j_{\hat A}v_i + \nabla^i_A\check v_j + \nabla^j_A\check v_i\right)\frac{\vartheta}{\hat J}\,\mathrm{d}y\\
&+\lambda\int_\Omega\left(\mathrm{div}_{\hat A}\,\hat v + \mathrm{div}_{\check A}\,\check v\right)\left(\mathrm{div}_{\hat A}\, v + \mathrm{div}_A\,\check v\right)\frac{\vartheta}{\hat J}\,\mathrm{d}y\\
&+\nu\int_\Omega\left(|\,\mathrm{curl}_{\hat A}\,\hat h| + |\,\mathrm{curl}_{\check A}\,\check h|\right)\left(\mathrm{curl}_{\hat A}\, h + \mathrm{curl}_A\,\check h\right)\frac{\vartheta}{\hat J}\,\mathrm{d}y =: \sum_{i=1}^{7} O_i.
\end{aligned}
\tag{3.35}
$$

Similar to N_1, N_2 , and N_3, the terms O_1, O_2 and O_3 are estimates as follows

$$O_1 \le \frac{\kappa}{8}\left\|\frac{\nabla_{\hat{A}}\vartheta}{\sqrt{\hat{J}}}\right\|_2^2 + C\left(\left\|\sqrt{\rho_0}\check{\vartheta}_t\right\|_2^2 + \left\|\sqrt{\rho_0}\check{\vartheta}_t\right\|_2 \left\|\nabla\check{\vartheta}_t\right\|_2\right)\|J\|_2^2,$$

$$O_2 \le \frac{\kappa}{8}\left\|\frac{\nabla_{\hat{A}\vartheta}\vartheta}{\sqrt{\hat{J}}}\right\|_2^2 + C\|\nabla\check{\vartheta}\|_\infty^2\|A\|_2^2,$$

$$O_3 \le \frac{\kappa}{8}\left\|\frac{\nabla_{\hat{A}}\vartheta}{\sqrt{\hat{J}}}\right\|_2^2 + C\left(\left\|\nabla^2\check{\vartheta}\right\|_3^2 + \|\nabla\check{\vartheta}\|_\infty^2\right)\|A\|_2^2.$$

Moreover,

$$\begin{aligned}
O_4 \le& C\int_\Omega \rho_0|\vartheta|\left(|\hat{\vartheta}||\operatorname{div}_{\hat{A}} v| + |\hat{\vartheta}||A||\nabla\check{v}| + |\vartheta||\nabla\check{v}| + |J||\check{\vartheta}||\nabla\check{v}|\right)\mathrm{d}y\\
\le& C\left(\|\hat{\vartheta}\|_\infty + \|\check{\vartheta}\|_\infty\right)\|\nabla\check{v}\|_\infty\left\|\sqrt{\rho_0}\vartheta\right\|_2(\|A\|_2 + \|J\|_2)\\
&+ C\|\hat{\vartheta}\|_\infty\left\|\frac{\nabla_{\hat{A}}v}{\sqrt{\hat{J}}}\right\|_2\left\|\sqrt{\rho_0}\vartheta\right\|_2 + C\|\nabla\check{v}\|_\infty\left\|\sqrt{\rho_0}\vartheta\right\|_2^2,\\
O_5 + O_6 \le& C\int_\Omega(|\nabla\hat{v}| + |\nabla\check{v}|)\left(|\nabla_{\hat{A}}v| + |A||\nabla\check{v}|\right)|\vartheta|\mathrm{d}y\\
\le& \frac{\kappa}{8}\left\|\frac{\nabla_{\hat{A}}\vartheta}{\sqrt{\hat{J}}}\right\|_2^2 + C\left(\|\sqrt{\rho_0}\vartheta\|_2^2 + \|\nabla(\hat{v},\check{v})\|_3^2\left\|\frac{\nabla_{\hat{A}}v}{\sqrt{\hat{J}}}\right\|_2^2 + \left\|\nabla^2(\hat{v},\check{v})\right\|_2^4\|A\|_2^2\right).\\
O_7 \le& C\int_\Omega(|\nabla\hat{h}| + |\nabla\check{h}|)\left(|\nabla_{\hat{A}}h| + |A||\nabla\check{h}|\right)|\vartheta|\,\mathrm{d}y\\
\le& C\|\nabla(\hat{h},\check{h})\|_3\left\|\frac{\nabla_{\hat{A}}h}{\sqrt{\hat{J}}}\right\|_2\|\vartheta\|_6 + C\|\nabla(\hat{h},\check{h})\|_6^2\|A\|_2\|\vartheta\|_6\\
\le& \frac{\kappa}{8}\left\|\frac{\nabla_{\hat{A}}\vartheta}{\sqrt{\hat{J}}}\right\|_2^2 + C\left(\|\sqrt{\rho_0}\vartheta\|_2^2 + \|\nabla(\hat{h},\check{h})\|_3^2\left\|\frac{\nabla_{\hat{A}}h}{\sqrt{\hat{J}}}\right\|_2^2 + \|\nabla^2(\hat{h},\check{h})\|_2^4\|A\|_2^2\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&c_v\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho_0}\vartheta\|_2^2 + \kappa\left\|\frac{\nabla_{\hat{A}}\vartheta}{\sqrt{\hat{J}}}\right\|_2^2\\
\le& C\left(\left\|\sqrt{\rho_0}\check{\vartheta}_t\right\|_2^2 + \|\sqrt{\rho_0}\check{\vartheta}_t\|_2\|\nabla\check{\vartheta}_t\|_2\right)\|J\|_2^2 + C\|\nabla(\hat{v},\check{v})\|_3^2\left\|\frac{\nabla_{\hat{A}}v}{\sqrt{\hat{J}}}\right\|_2^2 + C\|\nabla(\hat{h},\check{h})\|_3^2\left\|\frac{\nabla_{\hat{A}}h}{\sqrt{\hat{J}}}\right\|_2^2\\
&+ C\left(\|\nabla\check{\vartheta}\|_\infty^2 + \|\nabla^2\check{\vartheta}\|_3^2 + \|\nabla^2(\hat{v},\check{v})\|_2^4 + \|\nabla^2(\hat{h},\check{h})\|_2^4\right)\|A\|_2^2 + C\|\hat{\vartheta}\|_\infty\left\|\frac{\nabla_{\hat{A}}v}{\sqrt{\hat{J}}}\right\|_2\|\sqrt{\rho_0}\vartheta\|_2\\
&+ C\left(1 + \|\nabla\check{v}\|_\infty\right)\|\sqrt{\rho_0}\vartheta\|_2^2 + C\|\nabla\check{v}\|_\infty\left(\|\hat{\vartheta}\|_\infty + \|\check{\vartheta}\|_\infty\right)\left\|\sqrt{\rho_0}\vartheta\right\|_2(\|A\|_2 + \|J\|_2).
\end{aligned} \tag{3.36}$$

Next multplying (3.28) with $\frac{h}{\hat{J}}$ and using (3.32), one gets that

$$\begin{aligned}
&\frac{1}{\hat{J}}\frac{\mathrm{d}}{\mathrm{d}t}\|h\|_2^2 + \nu\left\|\frac{\nabla_{\hat{A}}h}{\sqrt{\hat{J}}}\right\|_2^2\\
=& \int_\Omega\frac{1}{\hat{J}}(\hat{h}\cdot\nabla_{\hat{A}}v + h\cdot\nabla_{\check{A}}\check{v} - \hat{v}\cdot\nabla_{\hat{A}}h - v\cdot\nabla_{\check{A}}\check{h} - \hat{h}\cdot\operatorname{div}_{\hat{A}}v - h\cdot\operatorname{div}_{\check{A}}\check{v})\cdot h\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&+\int_\Omega \frac{1}{\hat{J}}(\hat{h}\cdot\nabla_A\check{v}\cdot h-\hat{v}\cdot\nabla_A\check{h}\cdot h-\hat{h}\cdot \mathrm{div}_A\,\check{v}\cdot h)\mathrm{d}y-\nu\int_\Omega\frac{1}{\hat{J}}(\nabla_{\hat{A}}h:\nabla_A\check{h})\mathrm{d}y\\
&+\nu\int_\Omega\frac{1}{\hat{J}}\,\mathrm{div}_A\left(\nabla_{\check{A}}\check{h}\right)\cdot h\mathrm{d}y\\
\le& C(\|\hat{h}\|_\infty\|\nabla v\|_2\|h\|_2+\|v\|_6\|\nabla\check{h}\|_6\|h\|_2+\|h\|_2\|\nabla\check{v}\|_\infty\|h\|_2+\|\hat{v}\|_\infty\|\nabla h\|_2\|h\|_2\\
&+\|\hat{h}\|_6\|A\|_2\|\nabla\check{v}\|_6\|h\|_6+\|\hat{v}\|_6\|A\|_2\|\nabla\check{h}\|_6\|h\|_6+\|A\|_2\|\nabla\check{h}\|_\infty\|\hat{A}\|_\infty\|\nabla h\|_2\\
&+\|\check{A}\|_\infty\|\nabla\check{h}\|_\infty\|A\|_2\|\nabla h\|_2)\\
\le&\eta\left\|\frac{\nabla_{\hat{A}}v}{\sqrt{\hat{J}}}\right\|_2^2+C(\|\hat{h}\|_\infty^2+\|\nabla^2\check{h}\|_2+\|\nabla\check{v}\|_\infty)\|h\|_2^2+\frac{\nu}{2}\left\|\frac{\nabla_{\hat{A}}h}{\sqrt{\hat{J}}}\right\|_2^2+C(\|\hat{v}\|_\infty\|h\|_2^2\\
&+\|\nabla\hat{h}\|_2^2\|\nabla^2\check{v}\|_2^2\|A\|_2^2+\|\nabla\hat{v}\|_2^2\|\nabla^2\hat{h}\|_2^2\|A\|_2^2+\|\nabla\check{h}\|_\infty^2\|A\|_2^2).
\end{aligned}
$$

So,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|h\|_2^2+\nu\left\|\frac{\nabla_{\hat{A}}h}{\sqrt{\hat{J}}}\right\|_2^2\le&\eta\left\|\frac{\nabla_{\hat{A}}v}{\sqrt{\hat{J}}}\right\|_2^2+C(\|\hat{h}\|_\infty^2+\|\nabla^2\check{h}\|_2+\|\nabla\check{v}\|_\infty+\|\hat{v}\|_\infty)\|h\|_2^2\\
&+C(\|\nabla\hat{h}\|_2^2\|\nabla^2\check{v}\|_2^2+\|\nabla\hat{v}\|_2^2\|\nabla^2\hat{h}\|_2^2+\|\nabla\check{h}\|_\infty^2)\|A\|_2^2
\end{aligned}
\tag{3.37}
$$

Step 2. Growth estimates. We proceed to consider the growth estimates of A and J. Multiplying (3.30) and (3.31), respectively, with A and J, and summing the resultants up, one obtains that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\|A\|_2^2+\|J\|_2^2\right)\le C\|\nabla(\hat{v},\check{v})\|_\infty\left(\|A\|_2^2+\|J\|_2^2\right)+C\|\nabla v\|_2(\|A\|_2+\|J\|_2).\tag{3.38}
$$

Thus,

$$
\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\|A\|_2^2+\|J\|_2^2}\le C\|\nabla(\hat{v},\check{v})\|_\infty\sqrt{\|A\|_2^2+\|J\|_2^2}+C\|\nabla v\|_2.\tag{3.39}
$$

Since $\hat{v},\check{v}\in L^1\left(0,T_0;W^{2,q}\right)\cap L^\infty\left(0,T_0;H^1\right)$ and $W^{1,q}\hookrightarrow L^\infty$ for $q\in(3,6)$, one has $\|\nabla(\hat{v},\check{v})\|_\infty\in L^1((0,T_0))$ and $\|\nabla v\|_2\in L^\infty((0,T_0))$. It follows from (3.35) that

$$
\sqrt{\|A\|_2^2+\|J\|_2^2}\le Ce^{C\int_0^t\|\nabla(\hat{v},\check{v})\|_\infty\mathrm{d}s}\int_0^t\|\nabla v\|_2\mathrm{d}s\le Ct\quad(\forall t\in(0,T_0)).\tag{3.40}
$$

Recalling

$$
\begin{aligned}
&\sqrt{\rho_0}\check{v}_t\in L^2\left(0,T_0;L^2\right),\quad \sqrt{t}\check{v}_t\in L^2\left(0,T_0;H^1\right),\quad \check{\vartheta}\in L^2\left(0,T_0;H^1\right),\\
&\sqrt{t}\check{\vartheta}\in L^2\left(0,T_0;H^2\right),\quad \sqrt{t}\check{v}\in L^2\left(0,T_0;W^{2,q}\right),\quad \hat{h},\check{h}\in L^\infty\left(0,T_0;H^1\right)\cap L^2\left(0,T_0;H^2\right),
\end{aligned}
$$

one gets that

$$
\omega_1(t)=t\left(\|\sqrt{\rho_0}\check{v}_t\|_2\|\nabla\check{v}_t\|_2+(1+\|\nabla\check{\vartheta}\|_2)\|\nabla^2\check{\vartheta}\|_2+\|\nabla^2\check{v}\|_q^2+\|\nabla(\hat{h},\check{h})\|_2^2\|\nabla^2\check{h}\|_2^2\right)\in L^1((0,T_0)).
$$

Since $\sqrt{\rho_0}\vartheta,\sqrt{\rho_0}v\in C\left([0,T_0];L^2\right)$ (guaranteed by Proposition 13), $\sqrt{\rho_0}v\big|_{t=0}=0$ and $\|\nabla h\|_2\in L^\infty((0,T_0))$, one integrates (3.34) with respect to t and uses (3.40) arrive at

$$
\|\sqrt{\rho_0}v\|_2^2(t)+\mu\int_0^t\left\|\frac{\nabla_{\hat{A}}v}{\sqrt{\hat{J}}}\right\|_2^2\mathrm{d}s\le Ct+Ct\int_0^t\omega_1\,\mathrm{d}s+C\int_0^t\|\nabla h\|_2^2\,\mathrm{d}s\le Ct,\quad(\forall t\in(0,T_0)).
$$

Combining this with (3.40) leads to

$$
\sup_{0\le t\le T_0}\left(\|A\|_2+\|J\|_2+\|\sqrt{\rho_0}v\|_2^2\right)+\int_0^t\|\nabla v\|_2^2\mathrm{d}s\le Ct,\quad(\forall t\in(0,T_0)).\tag{3.41}
$$

Step 3. Singular t-weighted energy inequalities. Multiplying (3.38) by $t^{-\frac{3}{2}}$, yields that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\|A\|_2^2}{t^{\frac{3}{2}}}+\frac{\|J\|_2^2}{t^{\frac{3}{2}}}\right)+\frac{3}{2}\left(\frac{\|A\|_2^2}{t^{\frac{5}{2}}}+\frac{\|J\|_2^2}{t^{\frac{5}{2}}}\right)\\
\le& C\|\nabla(\hat{v},\check{v})\|_\infty\left(\frac{\|A\|_2^2}{t^{\frac{3}{2}}}+\frac{\|J\|_2^2}{t^{\frac{3}{2}}}\right)+C\frac{\|\nabla v\|_2}{t^{\frac{1}{4}}}\left(\frac{\|A\|_2}{t^{\frac{5}{4}}}+\frac{\|J\|_2}{t^{\frac{5}{4}}}\right)\\
\le& \frac{1}{2}\left(\frac{\|A\|_2^2}{t^{\frac{5}{2}}}+\frac{\|J\|_2^2}{t^{\frac{5}{2}}}\right)+\frac{C}{\sqrt{t}}\left\|\frac{\nabla_{\hat{A}}v}{\sqrt{\hat{J}}}\right\|_2^2+C\|\nabla(\hat{v},\check{v})\|_\infty\left(\frac{\|A\|_2^2}{t^{\frac{3}{2}}}+\frac{\|J\|_2^2}{t^{\frac{3}{2}}}\right).
\end{aligned}
$$

So,

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\|A\|_2^2}{t^{\frac{3}{2}}}+\frac{\|J\|_2^2}{t^{\frac{3}{2}}}\right)\le C\|\nabla(\hat{v},\check{v})\|_\infty\left(\frac{\|A\|_2^2}{t^{\frac{3}{2}}}+\frac{\|J\|_2^2}{t^{\frac{3}{2}}}\right)+\frac{C}{\sqrt{t}}\left\|\frac{\nabla_{\hat{A}}v}{\sqrt{\hat{J}}}\right\|_2^2. \tag{3.42}
$$

Multiplying (4.34) with $\frac{1}{\sqrt{t}}$ and recalling the definition of $\omega_1(t)$ one arrives at

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\|\sqrt{\rho_0}v\|_2^2}{\sqrt{t}}\right)+\frac{\|\sqrt{\rho_0}v\|_2^2}{2t^{\frac{3}{2}}}+\frac{\mu}{\sqrt{t}}\left\|\frac{\nabla_{\hat{A}}v}{\sqrt{\hat{J}}}\right\|_2^2\le C\left(\frac{1}{\sqrt{t}}+\omega_1(t)\right)\left(\|\sqrt{\rho_0}\vartheta\|_2^2+\frac{\|A\|_2^2}{t^{\frac{3}{2}}}+\frac{\|J\|_2^2}{t^{\frac{3}{2}}}\right)+\frac{C}{\sqrt{t}}\left\|\frac{\nabla_{\hat{A}}h}{\sqrt{\hat{J}}}\right\|_2^2. \tag{3.43}
$$

Denote

$$
\begin{aligned}
&\omega_{21}(t)=t^{\frac{3}{2}}\left(\|\sqrt{\rho_0}\check{\vartheta}_t\|_2^2+\|\sqrt{\rho_0}\check{\vartheta}_t\|_2\|\nabla\check{\vartheta}_t\|_2\right),\quad \omega_{22}(t)=\sqrt{t}\|\nabla(\hat{v},\check{v})\|_3^2,\\
&\omega_{23}(t)=\sqrt{t}\|\nabla(\hat{h},\check{h})\|_3^2,\quad \omega_{24}(t)=t^{\frac{3}{2}}\left(\|\nabla\check{\vartheta}\|_\infty^2+\|\nabla^2\check{\vartheta}\|_3^2+\|\nabla^2(\hat{v},\check{v})\|_2^4\right),\\
&\omega_{25}(t)=\sqrt{t}\|\hat{\vartheta}\|_\infty^2,\quad \omega_{26}(t)=1+\|\nabla\check{v}\|_\infty,\quad \omega_{27}(t)=t^{\frac{3}{4}}\|\nabla\check{v}\|_\infty\|(\hat{\vartheta},\check{\vartheta})\|_\infty.
\end{aligned}
$$

Recalling the regularities of $(\hat{v},\hat{\vartheta},\hat{h})$ and $(\check{v},\check{\vartheta},\check{h})$, we have

$$
\begin{aligned}
&\omega_{21}(t)\le\left(\|\sqrt{t}\sqrt{\rho_0}\check{\vartheta}_t\|_2^2+\|\sqrt{t}\sqrt{\rho_0}\check{\vartheta}_t\|_2\|t\nabla\check{\vartheta}_t\|_2\right)\in L^1\left((0,T_0)\right),\\
&\omega_{22}(t)\le C\|\nabla(\hat{v},\check{v})\|_2\|\sqrt{t}\nabla^2(\hat{v},\check{v})\|_2\in L^\infty\left((0,T_0)\right),\\
&\omega_{23}(t)\le C\|\nabla(\hat{h},\check{h})\|_2\|\sqrt{t}\nabla^2(\hat{h},\check{h})\|_2\in L^\infty\left((0,T_0)\right),\\
&\omega_{25}(t)\le C(1+\|\nabla\hat{\vartheta}\|_2)\|\sqrt{t}\nabla^2\hat{\vartheta}\|_2\in L^1\left((0,T_0)\right),\quad \omega_{26}(t)=1+\|\nabla\check{v}\|_\infty\in L^1\left((0,T_0)\right),\\
&\omega_{27}(t)\le C\|\sqrt{t}\nabla^2\check{v}\|_q(1+\|\nabla(\hat{\vartheta},\check{\vartheta})\|_2^{\frac{1}{2}})\|\sqrt{t}\nabla^2(\hat{\vartheta},\check{\vartheta})\|_2^{\frac{1}{2}}\in L^1\left((0,T_0)\right).
\end{aligned}
$$

For ω_{24}, one gets from Lemma 2 and Gagliardo-Nirenberg inequality that

$$
\begin{aligned}
\omega_{24}(t)\le& Ct^{\frac{3}{2}}\left(\|\nabla\check{\vartheta}\|_\infty^2+\|\nabla^2\check{\vartheta}\|_3^2\right)+C\sqrt{t}\|\nabla^2(\hat{v},\check{v})\|_2^2\|\sqrt{t}\nabla^2(\hat{v},\check{v})\|_2^2\\
&+C\sqrt{t}\|\nabla^2(\hat{h},\check{h})\|_2^2\|\sqrt{t}\nabla^2(\hat{h},\check{h})\|_2^2\\
\le& C\left(\|\sqrt{t}\nabla\check{\vartheta}\|_2+\|\sqrt{t}\nabla^2\check{\vartheta}\|_2\right)\left\|t\nabla^2\check{\vartheta}\right\|_6+C\sqrt{t}\|\nabla^2(\hat{v},\hat{h},\check{v},\check{h})\|_2^2\|\sqrt{t}\nabla^2(\hat{v},\hat{h},\check{v},\check{h})\|_2^2\\
\in& L^1\left((0,T_0)\right).
\end{aligned}
$$

For terms ω_{2i}, $i=1,2,\cdots,7$, one gets from (3.37) that

$$
\begin{aligned}
c_v\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho_0}\vartheta\|_2^2+\kappa\left\|\frac{\nabla_{\hat{A}}\vartheta}{\sqrt{\hat{J}}}\right\|_2^2\le& C\left(\omega_{21}(t)\frac{\|J\|_2^2}{t^{\frac{3}{2}}}+\frac{\omega_{22}(t)}{\sqrt{t}}\left\|\frac{\nabla_{\hat{A}}v}{\sqrt{\hat{J}}}\right\|_2^2+\frac{\omega_{23}(t)}{\sqrt{t}}\left\|\frac{\nabla_{\hat{A}}h}{\sqrt{\hat{J}}}\right\|_2^2+\omega_{24}(t)\frac{\|A\|_2^2}{t^{\frac{3}{2}}}\right)\\
&+C\sqrt{\omega_{25}(t)}\left(\frac{1}{\sqrt{t}}\left\|\frac{\nabla_{\hat{A}}v}{\sqrt{\hat{J}}}\right\|_2^2\right)^{\frac{1}{2}}\|\sqrt{\rho_0}\vartheta\|_2+C\omega_{26}(t)\|\sqrt{\rho_0}\vartheta\|_2^2\\
&+C\omega_{27}(t)\left(\|\sqrt{\rho_0}\vartheta\|_2^2+\frac{\|A\|_2^2}{t^{\frac{3}{2}}}+\frac{\|J\|_2^2}{t^{\frac{3}{2}}}\right).
\end{aligned}
$$

Noting that $\omega_{23} \in L^\infty((0,T_0))$, one gets

$$
\begin{aligned}
& c_v \frac{\mathrm{d}}{\mathrm{d}t} \|\sqrt{\rho_0}\vartheta\|_2^2 + \kappa \left\| \frac{\nabla_{\hat{A}}\vartheta}{\sqrt{\hat{J}}} \right\|_2^2 \\
\leq & C\left(\omega_{23}(t)+1\right) \frac{1}{\sqrt{t}} \left\| \frac{\nabla_{\hat{A}} v}{\sqrt{\hat{J}}} \right\|_2^2 + C\omega_{27}(t) \frac{1}{\sqrt{t}} \left\| \frac{\nabla_{\hat{A}} h}{\sqrt{\hat{J}}} \right\|_2^2 + C\omega_2(t) \left(\|\sqrt{\rho_0}\vartheta\|_2^2 + \frac{\|A\|_2^2}{t^{\frac{3}{2}}} + \frac{\|J\|_2^2}{t^{\frac{3}{2}}} \right) \\
\leq & \frac{C}{\sqrt{t}} \left(\left\| \frac{\nabla_{\hat{A}} v}{\sqrt{\hat{J}}} \right\|_2^2 + \left\| \frac{\nabla_{\hat{A}} h}{\sqrt{\hat{J}}} \right\|_2^2 \right) + C\omega_2(t) \left(\|\sqrt{\rho_0}\vartheta\|_2^2 + \frac{\|A\|_2^2}{t^{\frac{3}{2}}} + \frac{\|J\|_2^2}{t^{\frac{3}{2}}} \right),
\end{aligned} \tag{3.44}
$$

where

$$
\omega_2(t) := \omega_{21}(t) + \omega_{24}(t) + \omega_{25}(t) + \omega_{26}(t) + \omega_{27}(t) \in L^1((0,T_0)).
$$

Denote

$$
\begin{aligned}
\omega_{31}(t) &= (\|\hat{h}\|_\infty^2 + \|\nabla^2 \check{h}\|_2 + \|\nabla \check{v}\|_\infty + \|\hat{v}\|_\infty), \\
\omega_{32}(t) &= t(\|\nabla \hat{h}\|_2^2 \|\nabla^2 \check{v}\|_2^2 + \|\nabla \hat{v}\|_2^2 \|\nabla^2 \hat{h}\|_2^2 + \|\nabla \check{h}\|_\infty^2).
\end{aligned}
$$

Recalling the regularities of $(\hat{v},\hat{h})$ and $(\check{v},\check{h})$, one gets that

$$
\begin{aligned}
\omega_{31}(t) &\leq (\|\nabla \hat{h}\|_2 \|\nabla^2 \hat{h}\|_2 + \|\nabla^2 \check{h}\|_2 + \|\nabla \check{v}\|_\infty + \|\hat{v}\|_\infty) \in L^1((0,T_0)), \\
\omega_{32}(t) &\leq (\|\nabla \hat{h}\|_2^2 \|\sqrt{t}\nabla^2 \check{v}\|_2^2 + \|\nabla \hat{v}\|_2^2 \|\sqrt{t}\nabla^2 \hat{h}\|_2^2 + \|\sqrt{t}\nabla \check{h}\|_q^2) \in L^1((0,T_0)).
\end{aligned}
$$

Multiplying (3.37) with $\frac{1}{\sqrt{t}}$ arrives at

$$
\frac{\mathrm{d}}{\mathrm{d}t} \frac{\|h\|_2^2}{\sqrt{t}} + \frac{\|h\|_2^2}{2t^{\frac{2}{3}}} + \frac{\nu}{\sqrt{t}} \left\| \frac{\nabla_{\hat{A}} h}{\sqrt{\hat{J}}} \right\|_2^2 \leq \frac{\eta}{\sqrt{t}} \left\| \frac{\nabla_{\hat{A}} v}{\sqrt{\hat{J}}} \right\|_2^2 + C\omega_{31}(t) \frac{\|h\|_2^2}{\sqrt{t}} + C\omega_{32}(t) \frac{\|A\|_2^2}{t^{\frac{3}{2}}}. \tag{3.45}
$$

Multiplying (3.43) with a small positive number ζ, adding the resultants with (3.45) and taking $\eta = \frac{\mu\zeta}{2}$, one obtains that

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\|h\|_2^2}{\sqrt{t}} + \zeta \frac{\|\sqrt{\rho_0} v\|_2^2}{\sqrt{t}} \right] + \frac{\nu}{2\sqrt{t}} \left\| \frac{\nabla_{\hat{A}} h}{\sqrt{\hat{J}}} \right\|_2^2 + \frac{\mu\zeta}{2\sqrt{t}} \left\| \frac{\nabla_{\hat{A}} v}{\sqrt{\hat{J}}} \right\|_2^2 \\
\leq & C \left(\frac{1}{\sqrt{t}} + \omega_1(t) + \omega_{31}(t) + \omega_{32}(t) \right) \left(\|\sqrt{\rho_0}\vartheta\|_2^2 + \frac{\|A\|_2^2}{t^{\frac{3}{2}}} + \frac{\|J\|_2^2}{t^{\frac{3}{2}}} + \frac{\|h\|_2^2}{\sqrt{t}} \right).
\end{aligned} \tag{3.46}
$$

Multiplying (3.42) and (3.44) with a small positive number ζ_1 and adding the resultants with (3.46), one gets that

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\|h\|_2^2}{\sqrt{t}} + \zeta \frac{\|\sqrt{\rho_0} v\|_2^2}{\sqrt{t}} + \zeta_1 \left(\frac{\|A\|_2^2}{t^{\frac{3}{2}}} + \frac{\|J\|_2^2}{t^{\frac{3}{2}}} + c_v \|\sqrt{\rho_0}\vartheta\|_2^2 \right) \right] + \frac{\nu}{4\sqrt{t}} \left\| \frac{\nabla_{\hat{A}} h}{\sqrt{\hat{J}}} \right\|_2^2 + \frac{\mu\zeta}{4\sqrt{t}} \left\| \frac{\nabla_{\hat{A}} v}{\sqrt{\hat{J}}} \right\|_2^2 + \kappa\zeta \left\| \frac{\nabla_{\hat{A}}\vartheta}{\sqrt{\hat{J}}} \right\|_2^2 \\
\leq & C \left(\frac{1}{\sqrt{t}} + \omega_1(t) + \omega_2(t) + \omega_{31}(t) + \omega_{32}(t) + \|\nabla(\hat{v},\check{v})\|_\infty \right) \left(\|\sqrt{\rho_0}\vartheta\|_2^2 + \frac{\|A\|_2^2}{t^{\frac{3}{2}}} + \frac{\|J\|_2^2}{t^{\frac{3}{2}}} + \frac{\|h\|_2^2}{\sqrt{t}} \right).
\end{aligned}
$$

By Proposition 13 and recalling (3.41), it follows that

$$
\lim_{t\to 0} \left[\frac{\|h\|_2^2}{\sqrt{t}} + \zeta \frac{\|\sqrt{\rho_0} v\|_2^2}{\sqrt{t}} + \zeta_1 \left(\frac{\|A\|_2^2}{t^{\frac{3}{2}}} + \frac{\|J\|_2^2}{t^{\frac{3}{2}}} + c_v \|\sqrt{\rho_0}\vartheta\|_2^2 \right) \right](t) = 0.
$$

By the Grönwall inequality, one gets that

$$
\left(\frac{\|\sqrt{\rho_0} v\|_2^2}{\sqrt{t}} + \frac{\|A\|_2^2}{t^{\frac{3}{2}}} + \frac{\|J\|_2^2}{t^{\frac{3}{2}}} + \|\sqrt{\rho_0}\vartheta\|_2^2 + \frac{\|h\|_2^2}{\sqrt{t}} \right)(t) + \int_0^t \left(\frac{1}{\sqrt{t}} \left\| \frac{\nabla_{\hat{A}} v}{\sqrt{\hat{J}}} \right\|_2^2 + \frac{1}{\sqrt{t}} \left\| \frac{\nabla_{\hat{A}} h}{\sqrt{\hat{J}}} \right\|_2^2 + \left\| \frac{\nabla_{\hat{A}}\vartheta}{\sqrt{\hat{J}}} \right\|_2^2 \right) \mathrm{d}\tau = 0,
$$

which implies $A = J = v = \vartheta = h = 0$. $\check{v}(y,t)$, one has $\hat{\varphi} = \check{\varphi}$ and further that $\hat{\psi} = \check{\psi}$. Then, it follows

$$\hat{u}(x,t) = \hat{v}(\hat{\psi}(x,t),t) = \check{v}(\check{\psi}(x,t),t) = \check{u}(x,t),$$

that is $\hat{u} \equiv \check{u}$. Similarly, one has $\hat{H} \equiv \check{H}$, $\hat{\theta} \equiv \check{\theta}$ and $\hat{\rho} \equiv \check{\rho}$. This proves the uniqueness. $\square$

REFERENCES

1. Li T, Qin T. Physics and Partial Differential Equations, vol. 1. Society for Industrial and Applied Mathematics (SIAM). Philadelphia, PA; 2012.
2. Ducomet B, Feireisl E. The equations of magnetohydrodynamics: on the interaction between matter and radiation in the evolution of gaseous stars. *Communications in mathematical Physics.* 2006;266:595–629.
3. Huang X, Li J. Serrin-type blowup criterion for viscous, compressible, and heat conducting Navier-Stokes and magnetohydrodynamic flows. *Communications in Mathematical Physics.* 2013;324:147–171.
4. Zhong X. On formation of singularity of the full compressible magnetohydrodynamic equations with zero heat conduction. *Indiana University Mathematics Journal.* 2019;68(5):1379–1407.
5. Hu X, Wang D. Global solutions to the three-dimensional full compressible magnetohydrodynamic flows. *Communications in mathematical physics.* 2008;283:255–284.
6. Wen H, Zhu C. Global solutions to the three-dimensional full compressible Navier–Stokes equations with vacuum at infinity in some classes of large data. *SIAM Journal on Mathematical Analysis.* 2011;49(1):162–221.
7. Choe HJ, Kim H. Strong solutions of the Navier–Stokes equations for isentropic compressible fluids. *Journal of Differential Equations.* 2003;190(2):504–523.
8. Hu X, Wang D. Global existence and large-time behavior of solutions to the three-dimensional equations of compressible magnetohydrodynamic flows. *Archive for Rational Mechanics and Analysis.* 2010;197(1):203–238.
9. Zhong X. Strong solutions to the Cauchy problem of two-dimensional non-barotropic non-resistive magnetohydrodynamic equations with zero heat conduction. *Journal of Differential Equations.* 2020;268(9):4921–4944.
10. Li Y, Sun Y. Global weak solutions to a two-dimensional compressible MHD equations of viscous non-resistive fluids. *Journal of Differential Equations.* 2019;267(6):3827–3851.
11. Li HL, Xu X, Zhang J. Global classical solutions to 3D compressible magnetohydrodynamic equations with large oscillations and vacuum. *SIAM Journal on Mathematical Analysis.* 2013;45(3):1356–1387.
12. Hong G, Hou X, Peng H, Zhu C. Global existence for a class of large solutions to three-dimensional compressible magnetohydrodynamic equations with vacuum. *SIAM Journal on Mathematical Analysis.* 2017;49(4):2409–2441.
13. Fan J, Yu W. Strong solution to the compressible magnetohydrodynamic equations with vacuum. *Nonlinear Analysis: Real World Applications.* 2009;10(1):392–409.
14. Liu Y, Zhong X. Global well-posedness to three-dimensional full compressible magnetohydrodynamic equations with vacuum. *Zeitschrift für angewandte Mathematik und Physik.* 2012;71:1–25.
15. Hoff D, Smoller J. Non-formation of vacuum states for compressible Navier–Stokes equations. *Communications in Mathematical Physics.* 2001;216:255–276.
16. Li J, Xin Z. Global well-posedness and large time asymptotic behavior of classical solutions to the compressible Navier–Stokes equations with vacuum. *Annals of PDE.* 2019;5:1–37.
17. Feireisl E. On the motion of a viscous, compressible, and heat conducting fluid. *Indiana University mathematics journal.* 2004;53(6):1705–1738.
18. Cho Y, Choe HJ, Kim H. Unique solvability of the initial boundary value problems for compressible viscous fluids. *Journal de mathématiques pures et appliquées.* 2004;83(2):243–275.
19. Cho Y, Kim H. Existence results for viscous polytropic fluids with vacuum. *Journal of Differential Equations.* 2006;228(2):377–411.
20. Li J. Global small solutions of heat conductive compressible Navier–Stokes equations with vacuum: smallness on scaling invariant quantity. *Archive for Rational Mechanics and Analysis.* 2020;237(2):899–919.
21. Li J. Local existence and uniqueness of strong solutions to the Navier–Stokes equations with nonnegative density. *Journal of Differential Equations.* 2017;263(10):6512–6536.
22. Gong H, Li J, Zhang X. Local well-posedness of isentropic compressible Navier-Stokes equations with vacuum. *Commun Math Sci.* 2020;18(7):1891–1909.
23. Lai S, Xu H, Zhang J. Well-posedness and exponential decay for the Navier–Stokes equations of viscous compressible heat-conductive fluids with vacuum. *Mathematical Models and Methods in Applied Sciences.* 2022;32(09):1725–1784.
24. Li J, Zheng Y. Local Existence and Uniqueness of Heat Conductive Compressible Navier–Stokes Equations in the Presence of Vacuum Without Initial Compatibility Conditions. *Journal of Mathematical Fluid Mechanics.* 2023;25(1):14.

How to cite this article: Yiying Zhang, Zhenhua Guo and Li Fang. Local existence of compressible MHD equations without initial compatibility conditions. *Math Meth Appl Sci.* 2023.

APPENDIX

In this "Appendix", we give the prove of Proposition 11.

Proof of Proposition 11. (i) The proof of statement (i) is divided into the following steps.

Step 1. Construction of the initial data. Choose $\{u_{0n}\}_{n=1}^{\infty} \subseteq H_0^1 \cap H^2$ such that $u_{0n} \to u_0$ in H^1 as $n \to \infty$, $\{H_{0n}\}_{n=1}^{\infty} \subseteq H^2$ such that $H_{0n} \to H_0$ in H^1 as $n \to \infty$. Set $\rho_{0n} = \rho_0 + \frac{1}{n^2}$. Then, it is clear that

$$\|\rho_{0n}\|_{\infty} + \|\nabla \rho_{0n}\|_q + \|\nabla u_{0n}\|_2^2 \leq \|\rho_0\|_{\infty} + \|\nabla \rho_0\|_q + \|\nabla u_0\|_2^2 + \frac{1}{2} \tag{4.1}$$

for large n. Put

$$\bar{\theta}_{0n}(x) = \begin{cases} 0, & x \in \left\{x \in \Omega \mid \rho_0(x) < \frac{1}{n}\right\} \\ \theta_0, & x \in \left\{x \in \Omega \mid \rho_0(x) \geq \frac{1}{n}\right\} \end{cases}$$

and take $\theta_{0n} \geq 0$ such that

$$\|\theta_{0n} - \bar{\theta}_{0n}\|_2 \leq \frac{1}{n}. \tag{4.2}$$

Note that such θ_{0n} exists. For example, one can take $\theta_{0n} = j_{\varepsilon_n} * \widetilde{\theta}_{0n}$ for sufficiently small positive ε_n, where j_{ε_n} is the standard mollifier and $\widetilde{\theta}_{0n}$ is the zero extension of $\bar{\theta}_{0n}$ on R^3 , that is, $\widetilde{\theta}_{0n} = \bar{\theta}_{0n}$ on Ω and $\widetilde{\theta}_{0n} = 0$ on $\mathbb{R}^3 \backslash \Omega$. We want to show

$$\|\rho_{0n}\|_{\infty} + \|\nabla \rho_{0n}\|_q + \|\nabla u_{0n}\|_2^2 + \|\nabla H_{0n}\|_2^2 + \|\sqrt{\rho_{0n}}\theta_{0n}\|_2^2 \leq \Phi_0 + 1, \tag{4.3}$$

for large n, and

$$\int_{\Omega} \rho_{0n}\theta_{0n}\chi dx \to \int_{\Omega} \rho_0\theta_0\chi dx \quad \text{as } n \to \infty, \quad \forall \chi \in L^2(\Omega). \tag{4.4}$$

we can get

$$\sqrt{\rho_{0n}}\theta_{0n} = \sqrt{\rho_0 + \frac{1}{n^2}}\left(\theta_{0n} - \bar{\theta}_{0n}\right) + \sqrt{\rho_0 + \frac{1}{n^2}}\bar{\theta}_{0n} \leq \left(\sqrt{\rho_0} + \frac{1}{n}\right)\left|\theta_{0n} - \bar{\theta}_{0n}\right| + \sqrt{\rho_0}\theta_0 + \frac{\bar{\theta}_{0n}}{n}.$$

Recalling (4.2), one gets

$$\|\sqrt{\rho_{0n}}\theta_{0n}\| \leq \left(\|\rho_0\|_{\infty}^{\frac{1}{2}} + \frac{1}{n}\right)\frac{1}{n} + \|\sqrt{\rho_0}\theta_0\|_2 + \frac{\left\|\bar{\theta}_{0n}\right\|_2}{n}.$$

With the aid of the above and noticing that

$$\begin{aligned} \|\bar{\theta}_{0n}\|_2 &= \left(\int_{\Omega \cap \left\{x \mid \rho_0(x) \geq \frac{1}{n}\right\}} \theta_0^2 dx\right)^{\frac{1}{2}} \leq \sqrt{n}\left(\int_{\Omega \cap \left\{x \mid \rho_0 \geq \frac{1}{n}\right\}} \rho_0\theta_0^2 dx\right)^{\frac{1}{2}} \\ &\leq \sqrt{n}\left(\int_{\Omega} \rho_0\theta_0^2 dx\right)^{\frac{1}{2}} = \sqrt{n}\,\|\sqrt{\rho_0}\theta_0\|_2, \end{aligned} \tag{4.5}$$

one obtains

$$\|\sqrt{\rho_{0n}}\theta_{0n}\|_2 \leq \left(\|\rho_0\|_{\infty}^{\frac{1}{2}} + \frac{1}{n}\right)\frac{1}{n} + \left(1 + \frac{1}{\sqrt{n}}\right)\|\sqrt{\rho_0}\theta_0\|_2. \tag{4.6}$$

This implies

$$\|\sqrt{\rho_{0n}}\theta_{0n}\|_2^2 \leq \|\sqrt{\rho_0}\theta_0\|_2^2 + \frac{1}{2} \quad \text{for large } n. \tag{4.7}$$

Combining (4.1) with (4.7) leads to (4.3). It follows (4.2) and (4.5) that

$$\begin{aligned} \left|\int_{\Omega} \rho_{0n}\theta_{0n}\chi dx - \int_{\Omega} \rho_0\theta_0\chi dx\right| &= \left|\int_{\Omega} \left[\rho_{0n}\left(\theta_{0n} - \bar{\theta}_{0n}\right) + (\rho_{0n} - \rho_0)\,\bar{\theta}_{0n} + \rho_0\left(\bar{\theta}_{0n} - \theta_0\right)\right]\chi dx\right| \\ &\leq \|\rho_{0n}\|_{\infty}\left\|\theta_{0n} - \bar{\theta}_{0n}\right\|_2\|\chi\|_2 + \frac{1}{n^2}\left\|\bar{\theta}_{0n}\right\|_2\|\chi\|_2 + \int_{\Omega \cap \left\{x \mid \rho_0(x) < \frac{1}{n}\right\}} \rho_0\theta_0|\chi|dx \\ &\leq \frac{\|\chi\|_2}{n}\left(\|\rho_0\|_{\infty} + \frac{1}{n^2}\right) + \frac{\|\chi\|_2}{n^{\frac{3}{2}}}\|\sqrt{\rho_0}\theta_0\|_2 + \frac{\|\chi\|_2}{\sqrt{n}}\|\sqrt{\rho_0}\theta_0\|_2, \end{aligned}$$

for any $\chi \in L^2(\Omega)$ holds, which implies (4.4).

Step 2. Approximate solutions and convergence. Thanks to (4.3) and Proposition 10, there are two positive constants T_0 and C independent of n such that system (1.1), subject to (1.4)-(1.5), admits a unique solution $(\rho_n, u_n, \theta_n, H_n)$, in $\Omega\times(0,T_0)$, and the following a priori estimates hold

$$\left.\begin{array}{l}\int_0^{T_0}\left\|\left(\nabla\theta_n,\sqrt{\rho_n}\partial_t u_n,\nabla^2 u_n,\partial_t H_n,\nabla^2 H_n\right)\right\|_2^2 dt\le C,\\ \sup\limits_{0\le t\le T}\left(\|\rho_n\|_\infty+\|\rho_n\|_{W^{1,q}}+\left\|\left(\partial_t\rho_n,\sqrt{\rho_n}\theta_n,\nabla u_n,\nabla H_n\right)\right\|_2^2\right)\le C,\\ \int_0^{T_0}\left(\|(\sqrt{t}\sqrt{\rho_n}\partial_t\theta_n,\sqrt{t}\nabla^2\theta_n,\sqrt{t}\nabla\partial_t u_n,\sqrt{t}\nabla\partial_t H_n)\|_2^2+\|\sqrt{t}\nabla^2 u_n\|_q^2\right)\mathrm{d}t\le C,\\ \int_0^{T_0}t^2\left(\|\nabla\partial_t\theta_n\|_2^2+\left\|\nabla^2\theta_n\right\|_6^2\right)\mathrm{d}t\le C,\\ \sup\limits_{0\le t\le T}\left\|\left(\sqrt{t}\nabla\theta_n,t\sqrt{\rho_n}\partial_t\theta_n,t\nabla^2\theta_n,\sqrt{t}\sqrt{\rho_n}\partial_t u_n,\sqrt{t}\nabla^2 u_n,\sqrt{t}\sqrt{\rho_n}\partial_t u_n,\sqrt{t}\nabla^2 H_n\right)\right\|_2^2\le C,\end{array}\right\}\tag{4.8}$$

for large n. By the Banach-Alaoglu theorem and using the Cantor's diagonal arguments, there is a subsequence, still denoted by $(\rho_n, u_n, H_n, \theta_n)$, and (ρ, u, H, θ) satisfying

$$\rho\in L^\infty\left(0,T_0;W^{1,q}\right),\quad \rho_t\in L^\infty\left(0,T_0;L^2\right),\ \theta\in L^2\left(0,T_0;H_0^1\right),\tag{4.9}$$

$$u\in L^\infty\left(0,T_0;H_0^1\right)\cap L^2\left(0,T_0;H^2\right),\ H\in L^\infty\left(0,T_0;H^1\right)\cap L^2\left(0,T_0;H^2\right)\tag{4.10}$$

$$\sqrt{t}\nabla\theta,\sqrt{t}\nabla^2 u,\sqrt{t}\nabla^2 H\in L^\infty\left(0,T_0;L^2\right),\quad t\nabla^2\theta\in L^\infty\left(0,T_0;L^2\right)\cap L^2\left(0,T_0;L^6\right),\tag{4.11}$$

$$\sqrt{t}\nabla^2\theta\in L^2\left(0,T_0;L^2\right),\quad t\nabla\theta_t\in L^2\left(0,T_0;L^2\right),\tag{4.12}$$

$$\sqrt{t}\nabla u_t,\ \sqrt{t}\nabla H_t\in L^2\left(0,T_0;L^2\right),\quad \sqrt{t}\nabla^2 u,\ \sqrt{t}\nabla^2 H\in L^2\left(0,T_0;L^q\right),\tag{4.13}$$

such that

$$\rho_n\overset{*}{\rightharpoonup}\rho,\ \text{in } L^\infty\left(0,T_0;W^{1,q}\right),\tag{4.14}$$

$$\partial_t\rho_n\overset{*}{\rightharpoonup}\rho_t,\ \text{in } L^\infty\left(0,T_0;L^2\right),\tag{4.15}$$

$$u_n\overset{*}{\rightharpoonup}u,\ \text{in } L^\infty\left(0,T_0;H_0^1\right),\tag{4.16}$$

$$u_n\rightharpoonup u,\quad \text{in } L^2\left(0,T_0;H^2\right),\tag{4.17}$$

$$\partial_t u_n\rightharpoonup u_t,\ \text{in } L^2\left(\delta,T_0;H_0^1\right),\tag{4.18}$$

$$\theta_n\overset{*}{\rightharpoonup}\theta,\ \text{in } L^\infty\left(\delta,T_0;H_0^1\right),\tag{4.19}$$

$$\theta_n\rightharpoonup\theta,\ \text{in } L^2\left(\delta,T_0;W^{2,6}\right),\tag{4.20}$$

$$\partial_t\theta_n\rightharpoonup\theta_t,\ \text{in } L^2\left(\delta,T_0;H_0^1\right),\tag{4.21}$$

$$H_n\overset{*}{\rightharpoonup}H,\ \text{in } L^\infty\left(0,T_0;H^1\right),\tag{4.22}$$

$$H_n\rightharpoonup u,\quad \text{in } L^2\left(0,T_0;H^2\right),\tag{4.23}$$

$$\partial_t H_n\rightharpoonup H_t,\ \text{in } L^2\left(\delta,T_0;H^1\right),\tag{4.24}$$

for any $\delta\in(0,T_0)$. Moreover, since $W^{1,q}$ for $q\in(3,6)$, and $H^2\hookrightarrow\hookrightarrow H^1\hookrightarrow\hookrightarrow L^2$, it follows from the Aubin-Lions lemma and (4.14)-(4.24) that

$$\rho_n\to\rho,\quad \text{in } C\left([0,T_0];C(\bar\Omega)\right),\tag{4.25}$$

$$u_n\to u,\quad \text{in } C\left([\delta,T_0];L^2(\Omega)\right)\cap L^2\left(\delta,T_0;H_0^1(\Omega)\right),\tag{4.26}$$

$$\theta_n\to\theta,\quad \text{in } C\left([\delta,T_0];L^2(\Omega)\right)\cap L^2\left(\delta,T_0;H_0^1(\Omega)\right).\tag{4.27}$$

$$H_n\to H,\quad \text{in } C\left([\delta,T_0];L^2(\Omega)\right)\cap L^2\left(\delta,T_0;H^1(\Omega)\right),\tag{4.28}$$

Due to the convergence (4.18)(4.21)and (4.25)-(4.28), one has the following convergence of the nonlinear terms

$$\left(\rho_n u_n,\sqrt{\rho_n}u_n,\rho_n\theta_n,\sqrt{\rho_n}\theta_n\right)\to(\rho u,\sqrt{\rho}u,\rho\theta,\sqrt{\rho}\theta)\quad \text{in } C\left([\delta,T_0];L^2\right),\tag{4.29}$$

$$\left(\rho_n\partial_t u_n,\sqrt{\rho_n}\partial_t u_n,\rho_n\partial_t\theta_n,\sqrt{\rho_n}\partial_t\theta_n\right)\to\left(\rho u_t,\sqrt{\rho}u_t,\rho\theta_t,\sqrt{\rho}\theta_t\right)\quad \text{in } L^2\left(\Omega\times(\delta,T_0)\right),\tag{4.30}$$

$$\rho_n\left(u_n\cdot\nabla\right)u_n\to\rho(u\cdot\nabla)u,\quad \rho_n\left(u_n\cdot\nabla\right)\theta_n\to\rho(u\cdot\nabla)\theta,\quad \text{in } L^1\left(\Omega\times(\delta,T_0)\right),\tag{4.31}$$

$$\rho_n\theta_n \operatorname{div} u_n \to \rho\theta \operatorname{div} u, \quad \mathcal{Q}(\nabla u_n) \to \mathcal{Q}(\nabla u), \quad \text{in } L^1(\Omega\times(\delta,T_0)), \tag{4.32}$$

$$\operatorname{curl} H_n \times H_n \to \operatorname{curl} H \times H, \quad |\operatorname{curl} H_n|^2 \to |\operatorname{curl} H|^2, \quad \text{in } L^1(\Omega\times(\delta,T_0)), \tag{4.33}$$

$$H_n\cdot\nabla u_n \to H\cdot\nabla u, u_n\cdot\nabla H_n \to u\cdot\nabla H, H_n\operatorname{div} u_n \to H\operatorname{div} u, \text{ in } L^1(\Omega\times(\delta,T_0)), \tag{4.34}$$

for any $\delta\in(0,T_0)$. By the weakly lower semi-continuity of norms, it follows from (4.8), (4.29) and (4.30) that

$$\int_\delta^{T_0}\left(\|\sqrt{\rho}u_t\|_2^2+\|\sqrt{t}\sqrt{\rho}\theta_t\|_2^2\right)dt \le \varliminf_{n\to\infty}\int_\delta^{T_0}\left(\|\sqrt{\rho_n}\partial_t u_n\|_2^2+\|\sqrt{t}\sqrt{\rho_n}\partial_t\theta_n\|_2^2\right)dt \le C,$$
$$\|\sqrt{\rho}\theta\|_2(t)=\lim_{n\to\infty}\|\sqrt{\rho_n}\theta_n\|_2\le C$$

for any $\delta, t\in(0,T_0)$ and some positive constant C independent of δ and t. Therefore,

$$\sqrt{\rho}\theta\in L^\infty\left(0,T_0;L^2\right), \quad \sqrt{\rho}u_t, \sqrt{t}\sqrt{\rho}\theta_t\in L^2\left(0,T_0;L^2\right), \tag{4.35}$$

The regularity $u\in L^1\left(0,T_0;W^{2,q}\right)$ can be proved in the same as in Proposition 3.

Step 3. The existence. Thanks to the convergence (4.14)-(4.34), one can take the limit as $n\to\infty$ to the equations of $(\rho_n,u_n,H_n,\theta_n)$ to show that (ρ,u,H,θ) satisfies equations (1.1) in the sense of distribution. Due to the regularities (4.9)-(4.13), one can further show that (ρ,u,H,θ) satisfies (1.1), a.e. in $\Omega\times(0,T_0)$. The initial condition $\rho\big|_{t=0}=\rho_0$ is guaranteed by (4.29) by recalling that $\rho_n\big|_{t=0}=\rho_0+\frac{1}{n^2}$. To complete the proof of (i), one still needs to show the regularities $\rho u\in C\left([0,T_0];L^2\right)$ and $\rho\theta\in C_w\left([0,T_0];L^2\right)$, as well as the initial condition $(\rho u,\rho\theta)\big|_{t=0}=(\rho_0u_0,\rho_0\theta_0)$. To this end, noticing that $\rho u,\rho\theta\in C\left((0,T_0];L^2\right)$ guaranteed by (4.35), it suffices to show

$$\rho u\to\rho_0u_0 \quad \text{in } L^2, \quad \text{as } t\to 0 \tag{4.36}$$

$$\rho\theta\to\rho_0\theta_0 \quad \text{in } L^2 \quad \text{as } t\to 0. \tag{4.37}$$

We first verify (4.36). By (4.8), one gets that

$$\begin{aligned}\int_0^{T_0}\|\partial_t(\rho_nu_n)\|_2^2\,\mathrm{d}t &\le 2\int_0^{T_0}\left(\|\partial_t\rho_nu_n\|_2^2+\|\rho_n\partial_tu_n\|_2^2\right)\,\mathrm{d}t\\ &\le C\int_0^{T_0}\left(\|\partial_t\rho_n\|_2^2\|u_n\|_\infty^2+\|\rho_n\|_\infty\|\sqrt{\rho_n}\partial_tu_n\|_2^2\right)\,\mathrm{d}t\\ &\le C\int_0^{T_0}\|\nabla u_n\|_2\|\nabla^2u_n\|_2\mathrm{d}t+C\le C\end{aligned} \tag{4.38}$$

for large n. Thanks to this, it follows from the Newton-Leibnitz formula, the Minkowski and Hölder inequalities that

$$\begin{aligned}&\|\rho u(\cdot,t)-\rho_0u_0\|_2\\ &\le\|\rho u-\rho_nu_n\|_2(t)+\|\rho_nu_n-\rho_{0n}u_{0n}\|_2(t)+\|\rho_{0n}u_{0n}-\rho_{0n}u_0\|_2+\|\rho_{0n}u_0-\rho_0u_0\|_2\\ &\le\|\rho u-\rho_nu_n\|_2(t)+\int_0^t\|\partial_t(\rho_nu_n)\|_2\mathrm{d}\tau+\|\rho_{0n}(u_{0n}-u_0)\|_2+\frac{C}{n^2}\|u_0\|_2\\ &\le\|\rho u-\rho_nu_n\|_2(t)+C\sqrt{t}+\|\rho_{0n}\|_\infty\|u_{0n}-u_0\|_2+\frac{C}{n^2}\|u_0\|_2\end{aligned} \tag{4.39}$$

for large n, from which, recalling (4.29) and $u_{0n}\to u_0$ in H^1 as $n\to\infty$, one gets taking $n\to\infty$ that $\|\rho u-\rho_0u_0\|_2(t)\le C\sqrt{t}$, proving (4.36). Then, we verify (4.37). Since $\rho\theta\in L^\infty\left(0,T_0;L^2\right)$ and $C_c^\infty(\Omega)$ is dense in L^2, it suffices to verify

$$\left(\int_\Omega\rho\theta\phi\mathrm{d}x\right)(t)\to\int_\Omega\rho_0\theta_0\phi\mathrm{d}x \quad as \quad t\to 0, \quad \forall\phi\in C_c^\infty(\Omega) \tag{4.40}$$

Rewrite the equation for θ_n as

$$c_v\left[\partial_t(\rho_n\theta_n)+\operatorname{div}(\rho_n\theta_nu_n)\right]+R\rho_n\theta_n\operatorname{div}u_n-\kappa\Delta\theta_n=\mathcal{Q}(\nabla u_n)+\nu|\operatorname{cul}H_n|^2.$$

Multiplying the above equation with $\phi \in C_c^\infty(\Omega)$ and integrating over $\Omega \times (0,t)$ yield

$$\begin{aligned}
&c_v\left[\left(\int_\Omega \rho_n\theta_n\phi dx\right)(t) - \int_\Omega \rho_{0n}\theta_{0n}\phi \mathrm{d}x\right]\\
=&c_v\int_0^t\int_\Omega \rho_n\theta_n u_n\cdot\nabla\phi \mathrm{d}x d\tau + \kappa\int_0^t\int_\Omega \Delta\theta_n\phi \mathrm{d}x d\tau - R\int_0^t\int_\Omega \rho_n\theta_n \operatorname{div} u_n\phi \mathrm{d}x d\tau\\
&+\int_0^t\int_\Omega \mathcal{Q}(\nabla u)\phi \mathrm{d}x d\tau + \int_0^t\int_\Omega \nu|\mathrm{cul}H_n|^2\phi \mathrm{d}x d\tau =: \sum_{i=1}^5 M_i.
\end{aligned}$$

Terms on the right-hand side are estimated as follows:

$$\begin{aligned}
&\left|M_1\right| \le c_v\int_0^t \|\rho_n\|_\infty^{\frac12}\|\sqrt{\rho_n}\theta_n\|_2\|u_n\|_6\|\nabla\phi\|_3 d\tau \le Ct,\\
&\left|M_2\right| \le \kappa\int_0^t\int_\Omega |\nabla\theta_n||\nabla\phi| \mathrm{d}x d\tau \le C\left(\int_0^t \|\nabla\theta_n\|_2^2 d\tau\right)^{\frac12}\sqrt{t} \le C\sqrt{t},\\
&\left|M_3\right| \le C\int_0^t \|\rho_n\|_\infty^{\frac12}\|\sqrt{\rho}\theta_n\|_2\|\nabla u_n\|_2\|\phi\|_\infty d\tau \le Ct,\\
&\left|M_4\right| \le C\int_0^t\int_\Omega \left|\nabla u_n\right|^2|\phi| \mathrm{d}x d\tau \le C\int_0^t \|\nabla u_n\|_2^2\|\phi\|_\infty d\tau \le Ct,\\
&\left|M_5\right| \le C\int_0^t\int_\Omega \left|\nabla H_n\right|^2|\phi| \mathrm{d}x d\tau \le C\int_0^t \|\nabla H_n\|_2^2\|\phi\|_\infty d\tau \le Ct,
\end{aligned}$$

for large n. Therefore, one gets for large n that

$$\left|\left(\int_\Omega \rho_n\theta_n\phi \mathrm{d}x\right)(t) - \int_\Omega \rho_{0n}\theta_{0n}\phi \mathrm{d}x\right| < C\sqrt{t} \quad (\forall t \in (0,T_0))$$

for any $\phi \in C_c^\infty(\Omega)$ and for a positive constant C independent of n. Thanks to this and recalling (4.4) and (4.29), one gets by taking $n \to \infty$ that

$$\left|\left(\int_\Omega \rho\theta\phi \mathrm{d}x\right)(t) - \int_\Omega \rho_0\theta_0\phi \mathrm{d}x\right| \le C\sqrt{t}, \quad \forall t \in (0,T_0), \quad \forall \phi \in C_c^\infty(\Omega),$$

verifying (4.37).

(ii) Now we prove the statement (ii). Multiplying equation $(1.1)_2$ for $(\rho_n, u_n, H_n, \theta_n)$ with u_n and integrating over Ω, one gets that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\rho_n}{2}\left|u_n\right|^2 \mathrm{d}x + \mu\int_\Omega \left|\nabla u_n\right|^2 \mathrm{d}x + (\mu+\lambda)\int_\Omega \left|\operatorname{div} u_n\right|^2 \mathrm{d}x - \int_\Omega P_n \operatorname{div} u_n \mathrm{d}x&\\
= \int_\Omega \mathrm{curl}H_n \times H_n \cdot u_n \mathrm{d}x.&
\end{aligned} \tag{4.41}$$

Integrating the above with respect to t, one deduces for large n that

$$\begin{aligned}
\|\sqrt{\rho_n}u_n\|_2^2(t) &\le \|\sqrt{\rho_{0n}}u_{0n}\|_2^2 + 2\int_0^t\int_\Omega P_n \operatorname{div} u_n \mathrm{d}x \mathrm{d}\tau + 2\int_0^t\int_\Omega \mathrm{curl}H_n \times H_n \cdot u_n \mathrm{d}x\\
&\le \|\sqrt{\rho_{0n}}u_{0n}\|_2^2 + 2R\int_0^t \|\rho_n\|_\infty^{\frac12}\|\sqrt{\rho_n}\theta_n\|_2\|\nabla u_n\|_2 \mathrm{d}\tau + 2\int_0^t \|\nabla H_n\|_2\|\nabla u_n\|_2 \mathrm{d}\tau\\
&\le \|\sqrt{\rho_{0n}}u_{0n}\|_2^2 + Ct \quad (\forall t \in (0,T_0))
\end{aligned}$$

for a positive constant C independent of n. Thanks to the above, recalling (4.29) and noticing that $\sqrt{\rho_{0n}}u_{0n} \to \sqrt{\rho_0}u_0$ in L^2 as $n \to \infty$, one takes $n \to \infty$ to get that

$$\|\sqrt{\rho}u\|_2^2(t) \le \|\sqrt{\rho_0}u_0\|_2^2 + Ct, \quad \forall t \in (0,T_0) \tag{4.41}$$

Multiplying equation $(1.1)_3$ for $(\rho_n, u_n, H_n, \theta_n)$ with θ_n and integrating over Ω, one gets from (4.8) that

$$\begin{aligned}
&\frac{c_v}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho_n}\theta_n\|_2^2+\kappa\|\nabla\theta_n\|_2^2\\
=&-\int_\Omega \operatorname{div} u_n P_n\theta_n\,\mathrm{d}x+\int_\Omega \mathcal{Q}(\nabla u_n)\,\theta_n\,\mathrm{d}x+\int_\Omega \mathcal{Q}(\nabla u_n)\,\theta_n\,\mathrm{d}x+\nu\int_\Omega |\mathrm{curl}H_n|^2\theta_n\,\mathrm{d}x\\
\leq& C\left(\|\rho_n\|_\infty^{\frac{1}{2}}\|\sqrt{\rho_n}\theta_n\|_2\|\nabla u_n\|_3+\|\nabla u_n\|_2\|\nabla u_n\|_3+\|\nabla H_n\|_2\|\nabla H_n\|_3\right)\|\theta_n\|_6\\
\leq&\frac{\kappa}{2}\|\nabla\theta_n\|_2^2+C\left(1+\left\|\nabla^2 u_n\right\|_2+\left\|\nabla^2 H_n\right\|_2\right),
\end{aligned}$$

for large n. Integrating the above inequality with respect to t and using (4.8), one obtains that

$$\begin{aligned}
\|\sqrt{\rho_n}\theta_n\|_2^2(t)&\leq\|\sqrt{\rho_{0n}}\theta_{0n}\|_2^2+C\int_0^t\left(1+\left\|\nabla^2 u_n\right\|_2+\left\|\nabla^2 H_n\right\|_2\right)\mathrm{d}\tau\\
&\leq\|\sqrt{\rho_{0n}}\theta_{0n}\|_2^2+C\sqrt{t}\quad(\forall t\in(0,T_0)).
\end{aligned}$$

Thanks to this and recalling (4.6) and (4.29), one can take $n\to\infty$ to arrive at

$$\|\sqrt{\rho}\theta\|_2^2(t)\leq\|\sqrt{\rho_0}\theta_0\|_2^2+C\sqrt{t},\quad\forall t\in(0,T_0)$$

Combining this with (4.41), the conclusion follows. □