# Numerical approximation method for hybrid non-linear Caputo fractional differential equations with boundary value conditions

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**Abstract**

We address the problem of existence, uniqueness and approximation of the solution of a large class of fractional integral differential equations with boundary conditions. Based on fixed point techniques with Boyd-Wong type conditions, the existence and uniqueness of the solution is investigated. This, together with the use of certain types of biorthogonal systems, allows us to propose a method to approximate the solution which is tested with several examples.

# Numerical approximation method for hybrid non-linear Caputo fractional differential equations with boundary value conditions

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**Abstract**

We address the problem of existence, uniqueness and approximation of the solution of a large class of fractional integral differential equations with boundary conditions. Based on fixed point techniques with Boyd-Wong type conditions, the existence and uniqueness of the solution is investigated. This, together with the use of certain types of biorthogonal systems, allows us to propose a method to approximate the solution which is tested with several examples.

**Keywords:** Banach algebras Fractional differential equations, Fixed point theory, Biorthogonal systems.

**AMS Classification:** 32A65, 47H10, 47J26, 47G20, 34K37.

## 1 Introduction

Fractional differential equations have emerged as an interesting field of research in recent decades and proof of this is the large number of scientific publications related to this topic such as, for example [1, 3, 5–8, 13–15, 17–19]. These equations involve derivatives of non-integer orders, such as fractional or partial derivatives, and are used to capture complex dynamics and phenomena that cannot be adequately described by traditional integer-order differential equations. In fact, the non-local nature of the fractional derivative has meant that fractional differential equations play a very important role in the mathematical modeling of real-world problems, and very specifically in problems related to control theory, finance, geophysics and biophysics or image and signal processing (see [1, 3, 13, 16, 18]), which has led to the creation of specific computational tools and numerical methods to address the study of these equations. Although the numerical solution of differential equations of integer order has been and is an extremely fruitful topic in computational and numerical mathematics, the state of the art in fractional-order differential equations is much less advanced despite of large number of recent formulations in applied problems. The explicit calculation of the solution of fractional differential equations is feasible only in very simple specific cases and the advancement of numerical techniques aimed at approximating solutions for these equations is widely acknowledged as a crucial and prominent area of study.

In this context, fractional hybrid differential equations with boundary value conditions involving Caputo differential operators of order $1 < \alpha \leq 2$ have caught the attention of numerous authors especially when it comes to establishing results of the existence of solutions for these type of equations. For example, in [19], Sun et al published a study investigating the existing results for the following fractional hybrid differential equations involving Riemann-Liouville differential operators of order $0 < q < 1$,

$$\begin{cases} D^{\alpha}\left(\frac{x(t)}{f(t,x(t))}\right) + g(t,x(t)) = 0, t \in J, \alpha \in (1,2], \\ \\ x(0) = x(1) = 0, \end{cases} \tag{1.1}$$

where $f$ is Lipschitzian and $g$ is Caratheodory. Later, Zakir Ullah et al. [15] have used a Dhage's fixed point result to prove the existence results for

$$\begin{cases} D^{\alpha}\left(\dfrac{x(t) - \mathcal{I}^{\beta}h(t,x(t))}{f(t,x(t))}\right) = g(t,x(t)), t \in J, \alpha \in (1,2], \\ \\ \left(\frac{x(t)-h(t,x(t))}{f(t,x(t))}\right) |_{t=0} = 0 \text{ and } \left(\frac{x(t)-h(t,x(t))}{f(t,x(t))}\right) |_{t=1} = 0, \end{cases} \tag{1.2}$$

where $D^{\alpha}$ is the Caputo derivative of order $\alpha$, $I^{\beta_j}$ is the Riemann-Liouville fractional integral of order $\beta_j > 0$, $j = 1, \ldots, m$, and $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $g, h \in C(J \times \mathbb{R}, \mathbb{R})$. In [8], existence results are obtained for the following thermostat model:

$$\begin{cases} D^{\alpha}\left(\dfrac{x(t)}{f(t,x(t))}\right) = g(t,x(t)), t \in J, \alpha \in (1,2], \\ \\ D\left(\frac{x(t)}{f(t,x(t))}\right) |_{t=0} = 0 \text{ and } \lambda D^{\alpha-1}\left(\frac{x(t)}{f(t,x(t))}\right) |_{t=1} + \left(\frac{x(t)}{f(t,x(t))}\right) |_{t=1} = 0, \end{cases} \tag{1.3}$$

where $f : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$, are given functions. Motivated by the problems considered before and aforementioned above, Ahmad et al. in [4] established an existence result for the nonlinear fractional problem with a nonlocal boundary value:

$$\begin{cases} D^{\alpha}\left(\dfrac{x(t) - \sum_{j=1}^{m} \mathcal{I}^{\beta_j}h_j(t,x(t))}{f(t,x(t))}\right) = g(t,x(t)), t \in J, \\ \\ x(0) = \mu(x) \text{ and } x(1) = A, \end{cases} \tag{1.4}$$

where $A$ is a real constant, $1 < \alpha \leq 2$, and $f : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$, $h_j : J \times \mathbb{R} \to \mathbb{R}$, $j = 1, \ldots, m$, are given functions and $\mu : C(J, \mathbb{R}) \to \mathbb{R}$ satisfying certain conditions. Their results are based on Dhage's hybrid fixed point theorem.

In this work, we established an existence and uniqueness result to the nonlinear hybrid fractional differential equations with boundary conditions (1.4) under more general conditions on $f, g$, and $h_i, i = 1, \dots, m$, and also, we construct an approximated method of the solution, by using a combination method of the Picard operators sequence associated to the modelling problem of (1.4) and the notion of biorthogonal basis in Banach space. Combined techniques of fixed point results and Schauder bases have proven to be effective in approximating the solution of other types of equations, we can cite for example, [9, 10, 12]. This method offers distinct advantages in comparison to other numerical techniques due to its simplicity implementation on a computer and the utilization of approximating functions that consist of sums of piecewise univariate and bivariate polynomials with easily computable coefficients.

The rest of this paper is organized as follows. In Section 2, we establish the existence and uniqueness of solution for the fractional differential equations with boundary conditions (1.4). In Section 3, a numerical method is presented to approximate the obtained solution. In Section 4, we study some numerical examples using our approach. Section 5 concludes the paper.

## 2 Existence and uniqueness of solution

In this section we prove the existence and uniqueness of solutions for (1.4) in the space $C(J)$ of all continuous real-valued functions on $J$ endowed with the supremum norm $\|\cdot\|$. Before stating the study of problem (2.2), we introduce certain notations and some basics notions that will be used later. Let $(X, \|\cdot\|)$ be a Banach space with zero element $\theta$. We denotes by $B_R$ the closed ball of $X$ centered at $\theta$ with radius $R$. For all bounded subset $M$ of $X$, we denote by

$$\|M\| := \sup\{\|x\|, x \in M\}.$$

Let us introduce the class of $\mathcal{D}$-Lipschitzian mappings, which play an efficient tool to solve, generalize and extend many works focussed to the study different integro-differential equations under Lipschitz conditions.

**Definition 2.1** *Let $T : X \to X$. We say that $T$ is $\mathcal{D}$-Lipschitzian if there exists a nondecreasing continuous function $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ with $\phi(0) = 0$ such that*

$$\|Tx - Ty\| \leq \phi(\|x - y\|) \text{ for all } x, y \in X.$$

*The function $\phi$ is called the $\mathcal{D}$-function of $T$. When $\phi$ is not necessary nondecreasing and $\phi(r) < r, r > 0$, $T$ defines a nonlinear contraction operator.*

**Remark 2.1** *According to Lemma 2.1 in [2], if $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a $\mathcal{D}$-function with $\phi(r) < r, r > 0$, then $\lim_{n\to\infty} \phi^n(t) = 0$ for any $t > 0$.*

The following technical lemma ensures that every $\mathcal{D}$-Lipschitzian operator maps bounded subsets into bounded one.

**Lemma 2.1** *Let $T : X \longrightarrow X$ be a $\mathcal{D}$-Lipschitzian mapping with $\mathcal{D}$-function $\phi$. Then $T(M)$ is bounded for all bounded subset $M$ of $X$, with bound $\phi(\|M\|) + \|T(\theta)\|$.*

*Proof.* Let $x \in M$. By using the triangulary inequality, it follows that

$$\|T(x)\| \leq \|T(x) - T(\theta)\| + \|T(\theta)\|.$$

According to the fact that $T$ is $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-function $\phi$ we get

$$\|T(x)\| \leq \phi(\|x\|) + \|T(\theta)\|.$$

Passing to the supremum over $x \in M$, since $\phi$ is nondecreasing we obtain that

$$\|T(M)\| \leq \phi(\|M\|) + \|T(\theta)\|.$$

□

As a generalisation of the celebrated Banach's fixed point theorem, Boyd and Wong have established in [11] the following fixed point result.

**Theorem 2.1** *Let $T : X \to X$. Assume that there exists a continuous function $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(r) < r$ if $r > 0$, and*

$$\|T(x) - T(y)\| \leq \phi(\|x - y\|), \quad \forall x, y \in X.$$

*Then $T$ has a unique fixed point $\tilde{x} \in X$. Moreover, for any $x_0 \in X$, the sequence $\{T^n(x_0)\}_{n\in\mathbb{N}}$ converges to $\tilde{x}$.*

Also we recall fundamental concepts of fractional Caputo derivative and Riemann-Liouville integral, as introduced in [13, 18].

**Definition 2.2** *Let $f : [0, \infty) \to \mathbb{R}$ be an $(n-1)$-times absolutely continuous mapping. The Caputo derivative of fractional order $q$ of the function $f$ is given by*

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(n)}(s) ds, n-1 < q < n, n = [q] + 1,$$

*where $[q]$ denotes the integer part of the real number $q$.*

**Remark 2.2** *The Caputo derivative of order $q$ for a function $f : [0, \infty) \to \mathbb{R}$ can be written as*

$$D^q f(t) = D^q \left( f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), t > 0, n-1 < q < n.$$

**Definition 2.3** *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function. The Riemann-Liouville fractional integral of order $q > 0$ is given by*

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}} ds, t \in \mathbb{R}_+$$

*provided the right-hand side is point-wise defined on $(0, \infty)$. Here the mapping $\Gamma$ is defined by $\Gamma(n) = \int_0^{\infty} t^{n-1} e^{-t} dt,\ n > 0$.*

In [4], the authors prove that $x$ is a solution of the hybrid fractional integro-differential problem

$$\begin{cases} D^{\alpha}\left(\dfrac{x(t) - \sum_{j=1}^{m} I^{\beta_j} h_j(t, x(t))}{f(t, x(t))}\right) = y(t), t \in J, \\ x(0) = a \text{ and } x(1) = b, \end{cases} \tag{2.1}$$

with $y \in C(J)$, if and only if

$$x(t) = \sum_{j=1}^{m} I^{\beta_j} h_j(t, x(t)) + f(t, x(t)) \left[ \int_0^1 G(t,s) y(s) ds + \frac{(1-t)a}{f(0,a)} + \frac{t}{f(1,b)} \left( b - \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\beta_i - 1}}{\Gamma(\beta_i)} h_i(s, x(s)) ds \right) \right],$$

where $G(t,s) = \dfrac{((t-s)^+)^{\alpha-1} - t(1-s)^{\alpha-1}}{\Gamma(\alpha)}$ with $x^+ = \max(x, 0)$ for all $x \in \mathbb{R}$.

Based on this result, we notice that the problem (1.4) can be transformed as fixed point problem of the hybrid operator equation of the form:

$$x = A(x) \cdot B(x) + C(x), \tag{2.2}$$

with $A, B, C : C(J) \to C(J)$ are defined by

$$\begin{cases} (Ax)(t) &= f(t, x(t)), \\ (Bx)(t) &= \dfrac{(1-t)a}{f(0,a)} + \dfrac{tb}{f(1,b)} + \displaystyle\int_0^1 \left( G(t,s) g(s, x(s)) - \frac{t}{f(1,b)} \sum_{i=1}^{m} \frac{(1-s)^{\beta_i - 1}}{\Gamma(\beta_i)} h_i(s, x(s)) \right) ds, \\ (Cx)(t) &= \displaystyle\sum_{j=1}^{m} \int_0^t \frac{1}{\Gamma(\beta_j)(t-s)^{1-\beta_j}} h_j(s, x(s)) ds, \end{cases}$$

where $x \in C(J), t \in J$.

Our intention is to apply the Boyd-Wong theorem and throughout this work we suppose that following assumptions *(i)-(iv)* hold true:

($i$) For every $x \in \mathbb{R}$, the partial mapping $t \mapsto f(t, x)$ is continuous on $J$.

($ii$) There exist $R > 0$, a nondecreasing, continuous function $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and continuous functions $\eta, \gamma_i, i = \{1, \ldots, m\}$, such that

($a$) $\left| f(t, x) - f(t, y) \right| \leq \alpha(t) \phi(|x - y|), t \in J$ and $x, y \in \mathbb{R}$ with $|x|, |y| \leq R$,

(b) $\left|g(t,x)-g(t,y)\right| \leq \eta(t)\psi(|x-y|), t \in J$ and $x, y \in \mathbb{R}$ with $|x|, |y| \leq R$,

(c) $\left|h_i(t,x)-h_i(t,y)\right| \leq \gamma_i(t)\varphi_i(|x-y|), t \in J$ and $x, y \in \mathbb{R}$ with $|x|, |y| \leq R$.

(iii) For every $x \in \mathbb{R}$, the partial mappings $t \mapsto g(t,x)$ and $t \mapsto h_i(t,x), i = 1, \ldots, m$, are continuous on $J$.

(iv) There exist positive numbers $c_1$ and $c_2$ such that

$$\left|\frac{a}{f(0,a)}\right| \leq c_1 \text{ and } \left|\frac{b}{f(1,b)}\right| \leq c_2.$$

We need to prove some auxiliary Lemmas:

**Lemma 2.2** *The operators $A, B$ and $C$ map $B_R$ into $C(J)$.*

*Proof.* The claim regarding $A$ follows immediately from assumptions $(i)$. Then we have only to prove Lemma 2.2 for the operators $B$ and $C$. To do this, let $x \in B_R$ and $t, t' \in J$. We have

$$\begin{aligned}|(Bx)(t)-(Bx)(t')| \leq & \left|\frac{(1-t)a}{f(0,a)}+\frac{tb}{f(1,b)}-\frac{(1-t')a}{f(0,a)}-\frac{t'b}{f(1,b)}\right| \\ &+\int_0^1 |G(t,s)-G(t',s)| \left|g(s,x(s))\right| ds + \frac{|t-t'|}{\left|f(1,b)\right|}\int_0^1 \sum_{i=1}^{m} \frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)} |h_i(s,x(s))|\, ds.\end{aligned}$$

From assumptions $(ii)-(b)$ and $(ii)-(c)$, it follows that the partial functions $x \mapsto g(t,x)$, $x \mapsto h_i(t,x)$, $i = 1, \ldots, m$, are $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-function $\eta(t)\,\psi(\cdot)$ and $\gamma_i(t)\varphi_i(\cdot)$, respectively. Using Lemma 2.1 one obtains

$$\int_0^1 |G(t,s)-G(t',s)| \left|g(s,x(s))\right| ds \leq \int_0^1 |G(t,s)-G(t',s)|\ \eta(s)\,\psi(|x(s)|)ds + \int_0^1 |G(t,s)-G(t',s)| \left|g(s,0)\right| ds,$$

and

$$\sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_0^1 (1-s)^{\beta_i-1} |h_i(s,x(s))|\, ds \leq \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i+1)} \left(\int_0^1 (1-s)^{\beta_i-1}\gamma_i(s)\varphi_i(|x(s)|)ds + \int_0^1 (1-s)^{\beta_i-1} |h_i(s,0)|\, ds\right).$$

Considering the continuity of the functions $\eta$ and $\gamma_i$ for $i = 1, \ldots, m$, and the mapping $s \mapsto g(s,0)$ over the compact interval $J$, we can deduce that

$$\begin{aligned}|B(x)(t)-B(x)(t')| \leq & \Delta_{\eta,\psi,g}\left[\frac{|t-t'|}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha)}\left(\int_0^{t'} \left|(t-s)^{\alpha-1}-(t'-s)^{\alpha-1}\right| ds + \int_{t'}^{t} (t-s)^{\alpha-1} ds\right)\right] \\ &+\frac{|(t-t')a|}{|f(0,a)|}+\frac{|(t-t')b|}{|f(1,b)|}+\frac{|t-t'|}{|f(1,b)|}\sum_{i=1}^{m}\frac{\Delta_{\gamma_i,\varphi_i,h_i}}{\Gamma(\beta_i+1)},\end{aligned}$$

where

$$\Delta_{\eta,\psi,g} := \psi(R)\left\|\eta(\cdot)\right\| + \left\|g(\cdot,0)\right\| \text{ and } \Delta_{\gamma_i,\varphi_i,h_i} = \left\|\gamma_i(\cdot)\right\|\ \varphi_i(R) + \|h_i(\cdot,0)\|.$$

As $t \longrightarrow t'$, the expression on the right side of the above inequality tends to zero, leading to the conclusion that $B(x) \in C(J)$.

Similarly, we have

$$|C(x)(t)-C(x)(t')| \leq \sum_{j=1}^{m}\int_{t'}^{t}\frac{(t-s)^{\beta_j-1}}{\Gamma(\beta_j)}\left|h_j(s,x(s))\right|ds+\sum_{j=1}^{m}\frac{1}{\Gamma(\beta_j)}\int_0^{t'}\left|(t-s)^{\beta_j-1}-(t'-s)^{\beta_j-1}\right|\left|h_j(s,x(s))\right|ds.$$

By using Lemma 2.1 we get

$$|C(x)(t)-C(x)(t')| \leq \sum_{j=1}^{m}\frac{1}{\Gamma(\beta_j)}\Delta_{1,\varphi_j,h_j}\left(\frac{|t-t'|^{\beta_j}}{\beta_j}+\int_0^{t'}\left|(t-s)^{\beta_j-1}-(t'-s)^{\beta_j-1}\right|ds\right).$$

Therefore, as $t \longrightarrow t'$, As $t \longrightarrow t'$, the expression on the right side of the above inequality converges to zero, and consequently we get $C(x) \in C(J)$. □

**Lemma 2.3** *The operators $A, B$ and $C$ are $\mathcal{D}$-Lipschitzian.*

*Proof.* Using assumption (*ii*) together with the continuity of $\alpha(\cdot)$ on $J$, we can see that $A$ is $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-function $\Phi$ given by

$$\Phi(t) = \|\alpha\|_\infty \phi(t), t \in J.$$

Now, let us prove that $B$ is $\mathcal{D}$-Lipschitzian. To see this, let $x, y \in B_R$, and $t \in J$. Keeping in mind the assumption (*ii*), we obtain

$$\begin{aligned}
\left|B(x)(t)-B(y)(t)\right| &\leq \int_0^1\left|G(t,s)(g(s,x(s))-g(s,y(s)))\right|ds \\
&\quad +\frac{t}{|f(1,b)|}\sum_{i=1}^{m}\frac{1}{\Gamma(\beta_i)}\int_0^1(1-s)^{\beta_i-1}\left|h_i(s,x(s))-h_i(s,y(s)))\right|ds \\
&\leq \int_0^1|G(t,s)|\eta(s)\psi(|x(s)-y(s)|)ds+\frac{t}{|f(1,b)|}\sum_{i=1}^{m}\frac{1}{\Gamma(\beta_i)}\int_0^1(1-s)^{\beta_i-1}\gamma_i(s)\varphi_i(|x(s)-y(s)|ds \\
&\leq \frac{2}{\Gamma(\alpha+1)}\|\eta(\cdot)\|\psi(\|x-y\|)+\sum_{i=1}^{m}\frac{1}{\Gamma(\beta_i+1)|f(1,b)|}\|\gamma_i\|\varphi_i(\|x-y\|).
\end{aligned}$$

Taking the supremum over $t$, we obtain that $B$ is $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-function $\Psi$ such that

$$\Psi(t) = \frac{2}{\Gamma(\alpha+1)}\left\|\eta(\cdot)\right\|\psi(t)+\sum_{i=1}^{m}\frac{1}{\Gamma(\beta_i+1)|f(1,b)|}\left\|\gamma_i\right\|\varphi_i(t), t \geq 0.$$

Proceeding as above, assumption (*ii*) ensures that

$$\begin{aligned}
\left|C(x)(t)-C(y)(t)\right| &\leq \sum_{j=1}^{m}\frac{1}{\Gamma(\beta_j)}\int_0^t\frac{1}{(t-s)^{1-\beta_j}}\left|h_j(s,x(s))-h_j(s,y(s))\right|ds \\
&\leq \sum_{j=1}^{m}\frac{1}{\Gamma(\beta_j+1)}\left\|\gamma_j\right\|\varphi_j(\|x-y\|),
\end{aligned}$$

from where, it follows that

$$\left\|C(x)-C(y)\right\| \leq \sum_{j=1}^{m}\frac{1}{\Gamma(\beta_j+1)}\left\|\gamma_j\right\|\varphi_j(\|x-y\|).$$

Consequently, $C$ defines a $\mathcal{D}$-Lipschitzian mapping on $C(J)$ with $\mathcal{D}$-function $\Xi$ defined by

$$\Xi(t) = \sum_{j=1}^{m} \frac{1}{\Gamma(\beta_j + 1)} \left\|\gamma_j\right\| \varphi_j(t), t \geq 0.$$

□

**Remark 2.3** *Notice that the sets $A(B_R)$, $B(B_R)$ and $C(B_R)$ are bounded with bounds $\|A(B_R)\| := \sup_{x\in B_R} \|A(x)\|$, $\|B(B_R)\| := \sup_{x\in B_R} \|B(x)\|$ and $\|C(B_R)\| := \sup_{x\in B_R} \|C(x)\|$. From Lemma 2.1, it follows that $\|A(B_R)\| \leq \Phi(R) + \|A(0)\|$, $\|B(B_R)\| \leq \Psi(R) + \|B(0)\|$, and $\|C(B_R)\| \leq \Xi(R) + \|C(0)\|$.*

Now let us presented the main result of this section.

**Theorem 2.2** *Suppose that*

$$\begin{cases} \|A(B_R)\|\,\Psi(t) + \|B(B_R)\|\,\Phi(t) + \Xi(t) < t, \ t > 0 \\ \|A(B_R)\|\ \|B(B_R)\| + \|C(B_R)\| \leq R. \end{cases} \tag{2.3}$$

*Then the nonlinear problem* (1.4) *has an unique continuous solution $\tilde{x}$ in $B_R$, and*

$$\tilde{x} = \lim_{n\longrightarrow\infty} (A \cdot B + C)^n (x_0) \text{ for all } x_0 \in B_R. \tag{2.4}$$

*Proof.* In order to applied Theorem 2.1, we need only to prove that $A \cdot B + C$ defines a nonlinear contraction from $B_R$ into itself. To see this, let $x, y \in B_R$. Using Lemma 2.2, it follows that $A(x), B(x), C(x) \in C(J)$, and

$$\begin{aligned} \|A(x) \cdot B(x) + C(x) - A(y) \cdot B(y) - C(y)\| &\leq \|A(x) \cdot (B(x) - B(y))\| + \|(A(x) - A(y)) \cdot B(y)\| + \|C(x) - C(y)\| \\ &\leq \|A(B_R)\|\ \Psi(\|x - y\|) + \|B\,(B_R)\|\ \Phi(\|x - y\|) + \Xi(\|x - y\|). \end{aligned}$$

Hence, from the first estimate of (2.3) and Remark 2.3 we deduce that $A \cdot B + C$ is a nonlinear contraction on $B_R$ with $\mathcal{D}$-function

$$\Theta(r) = \|A(B_R)\|\,\psi(r) + \|B(B_R)\|\,\Phi(r) + \Xi(r), \ r > 0.$$

On the other hand, It can be seen from the second estimate of (2.3) that $A{\cdot}B{+}C$ maps $B_R$ into itself. By applying Theorem 2.1, we conclude establish the existence of an unique point $\tilde{x} \in B_R$ such that $A(\tilde{x}) \cdot B(\tilde{x}) + C(\tilde{x}) = \tilde{x}$. Additionally, it is confirmed that the sequence $\{(A \cdot B + C)^n(x_0), n \in \mathbb{N}\}$ converges to $\tilde{x}$ for all $x_0 \in B_R$. □

**Remark 2.4** *It's worth noting that we can establish the following inequality through an inductive argument:*

$$\|(A \cdot B + C)^n(x) - (A \cdot B + C)^n(y)\| \leq \Theta^n(\|x - y\|), \text{ for all } n = 1, 2, \ldots. \tag{2.5}$$

## 3 Method to approximate the solution and algorithm

The previous section enables us to represent the solution of (1.4) as the limit of the function sequence $\{(A \cdot B + C)^n(x_0), n \in \mathbb{N}\}$, where $x_0 \in B_R$. It's evident that if we could explicitly compute $(A \cdot B + C)^n(x_0)$ for each iteration, we would obtain an approximation of the fixed point $\tilde{x}$ for each value of $n$. However, in practice, explicit calculations of this nature are only viable in very simple cases. Consequently, we we need to construct an alternative method for approximating the fixed point, one that is more computationally accessible.

In the context of our problem, given that $B$ and $C$ are represented by integral operators, we choice to approximate only the power terms of these operators, which are generally difficult to compute. In contrast, operator $A$ is comparatively straightforward to calculate, necessitating no power term approximations. Thus, we initiate the process with an initial function $x_0 \in B_R$ and construct two sequences of operators, denoted as $\left\{\widetilde{B}_n, n \in \mathbb{N}\right\}$ and $\left\{\widetilde{C}_n, n \in \mathbb{N}\right\}$, in order to obtain successive approximations $N_n \circ \ldots \circ N_1(x_0)$ of $(A \cdot B + C)^n(x_0)$ and as a result, approximations of the fixed point $\tilde{x}$. Specifically, the main idea to construct the approximated scheme is the following: take $x_0 \in B_R$, in the first term, we approximate $B(x_0)$ by $\widetilde{B}_1(x_0)$, and $C(x_0)$ by $\widetilde{C}_1(x_0)$, then we obtain $N_1(x_0) := A(x_0) \cdot \widetilde{B}_1(x_0) + \widetilde{C}_1(x_0)$ as an approximation of the first term of the Picard iterate, $A(x_0) \cdot B(x_0) + C(x_0)$. In the second term of our scheme, we approximate the second term of the Picard iterate $(A \cdot B + C)^2(x_0)$, by combining the first term $N_1(x_0)$, with an approximation of the operators $B, C$ which will be denoted by $\widetilde{B}_2$ and $\widetilde{C}_2$ and consequently we obtain a second term of our scheme $N_2 \circ N_1(x_0) = (A \cdot \widetilde{B}_2 + \widetilde{C}_2)(N_1(x_0))$ which approximate $(A \cdot B + C)^2(x_0)$.

In order to construct an approximation schema for the solution $\tilde{x}$, we have to prove the following technical Lemma.

**Lemma 3.1** *Let $\tilde{x}$ be the unique solution of the nonlinear problem (1.4). Let $x_0 \in B_R$, $\varepsilon > 0$, and $k \in \mathbb{N}$ such that $\left\|(A \cdot B + C)^k(x_0) - \tilde{x}\right\| \leq \varepsilon/2$. Additionally, assume that $N_0, N_1, \ldots, N_k$ are operators on $C(J)$ with $N_0$ the identity operator on $C(J)$ and $\varepsilon_1, \ldots, \varepsilon_k > 0$, such that*

$$\sum_{p=1}^{k} \Theta^{k-p}\left(\varepsilon_p\right) < \varepsilon/2. \tag{3.1}$$

*If*

$$\left\|(A \cdot B + C) \circ N_{p-1} \circ \ldots \circ N_1(x_0) - N_p \circ \ldots \circ N_1(x_0)\right\| \leq \varepsilon_p, \text{ for all } p = 1, \ldots, k, \tag{3.2}$$

*then,*

$$\|\tilde{x} - N_k \circ \ldots \circ N_1(x_0)\| \leq \varepsilon.$$

*Proof.* Let $x_0 \in B_R$. We have that

$$\begin{aligned}
\left\|(A \cdot B + C)^k(x_0) - N_k \circ \ldots \circ N_1(x_0)\right\| \leq\; & \left\|(A \cdot B + C)^k(x_0) - (A \cdot B + C)^{k-1} \circ N_1(x_0)\right\| \\
& + \left\|(A \cdot B + C)^{k-1} \circ N_1(x_0) - (A \cdot B + C)^{k-2} \circ N_2 \circ N_1(x_0)\right\| \\
& + \\
& \vdots \\
& + \left\|(A \cdot B + C)^2 \circ N_{k-2} \circ \ldots \circ N_1(x_0) - (A \cdot B + C) \circ N_{k-1} \circ \ldots \circ N_1(x_0)\right\| \\
& + \|(A \cdot B + C) \circ N_{k-1} \circ \ldots \circ N_1(x_0) - N_k \circ \ldots \circ N_1(x_0)\|.
\end{aligned}$$

Taking into account (2.5), we get

$$\left\|(A\cdot B+C)^k(x_0)-N_k\circ\ldots\circ N_1(x_0)\right\| \le \sum_{p=1}^{k-1}\Theta^{k-p}\left(\left\|(A\cdot B+C)\circ N_{p-1}\circ\ldots\circ N_1(x_0)-N_p\circ\ldots\circ N_1(x_0)\right\|\right)$$
$$+\left\|(A\cdot B+C)\circ N_{k-1}\circ\ldots\circ N_1(x_0)-N_k\circ\ldots\circ N_1(x_0)\right\|.$$

By using (3.1) and (3.2), this implies that

$$\left\|(A\cdot B+C)^k(x_0)-N_m\circ\ldots\circ N_1(x_0)\right\|\le\sum_{p=1}^{k-1}\Theta^{k-p}\left(\varepsilon_p\right)<\varepsilon/2,$$

and we conclude that

$$\|\tilde{x}-N_k\circ\ldots\circ N_1(x_0)\|\le\left\|\tilde{x}-(A\cdot B+C)^k(x_0)\right\|+\left\|(A\cdot B+C)^k(x_0)-N_k\circ\ldots\circ N_1(x_0)\right\|<\varepsilon.$$

□

At this time we are in a position to make a proposal for the construction of the approximation terms $N_1, \ldots, N_k$. To this end, let us consider a Schauder basis $\{\tau_n\}_{n\ge1}$ in $C(J)$ and the sequence of associated projections $\{\xi_n\}_{n\ge1}$. Note that, in view of the Baire category theorem, for all $n\ge1$, $\xi_n$ is continuous. This yields, in particular, that

$$\lim_{n\to\infty}\|\xi_n(x)-x\|=0. \tag{3.3}$$

For each $p\in\mathbb{N}$, we consider $n_p\in\mathbb{N}$ and we define $\widetilde{B}_p$ by

$$\begin{cases}\widetilde{B}_p: C(J)\longrightarrow C(J)\\ x\longrightarrow \widetilde{B}_p(x)(t)=\dfrac{(1-t)a}{f(0,a)}+\dfrac{tb}{f(1,b)}+\displaystyle\int_0^1 G(t,s)\xi_{n_p}\left(U_0(x)\right)(s)ds-\dfrac{t}{f(1,b)}\sum_{i=1}^m\int_0^1\dfrac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)}\xi_{n_p}\left(V_0^i(x)\right)(s)ds,\end{cases}$$

where $U_0: C(J)\longrightarrow C(J)$ is defined by

$$U_0(x)(s)=g(s,x(s))$$

and $V_0: C(J)\longrightarrow C(J)$ is defined by

$$V_0^i(x)(s)=h_i(s,x(s)),$$

and $\widetilde{C}_p$ by

$$\begin{cases}\widetilde{C}_p: C(J)\longrightarrow C(J)\\ x\longrightarrow \widetilde{C}_p(x)(t)=\displaystyle\sum_{j=1}^m\frac{1}{\Gamma(\beta_j)}\int_0^t\frac{1}{(t-s)^{1-\beta_j}}\xi_{n_p}\left(V_0^i(x)\right)(s)ds,\end{cases}$$

Let us defined the operator $N_p: C(J)\longrightarrow C(J)$ by

$$N_p(x)(t)=A(x)(t)\widetilde{B}_p(x)(t)+\widetilde{C}_p(x)(t), t\in J,$$

where $A: C(J)\longrightarrow C(J)$ is given by

$$A(x)(t)=f(t,x(t)).$$

**Lemma 3.2** *For all fixed $p \geq 1$, the operator $N_p \circ \ldots \circ N_1$ maps $B_R$ into itself.*

*Proof.* Let $p \in \mathbb{N} \backslash \{0\}$ be fixed. For all $l = 1, \ldots, p$, and all $x \in B_R$, we have

$$\begin{aligned} |N_l(x)(t)| \;\leq\; & \left|f(t,x(t))\right| \left( \left| \frac{(1-t)a}{f(0,a)} + \frac{tb}{f(1,b)} \right| + \int_0^1 |G(t,s)| \left| \xi_{n_l}\left(U_0(x)\right)(s) \right| ds \right. \\ & \left. + \frac{t}{|f(1,b)|} \sum_{i=1}^m \int_0^1 \frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)} \left| \xi_{n_l}\left(V_0^i(x)\right)(s)\right| ds \right) + \sum_{j=1}^m \frac{1}{\Gamma(\beta_j)} \int_0^t \frac{1}{(t-s)^{1-\beta_j}} \left| \xi_{n_l}\left(V_0^i(x)\right)(s) \right| ds, \end{aligned}$$

Proceeding essentially as in the above section and using the fact that $\xi_{n_l}$ is a bounded linear operator on $C(J)$, we get

$$\sup_{t \in J} |N_l(x)(t)| \leq \|A(B_R)\| \, \|B(B_R)\| + \|C(B_R)\| \leq R.$$

Consequently, $N_l$ maps $B_R$ into itself, from which we see that $N_p \circ \ldots \circ N_1$ is also maps $B_R$ into itself. □

The aim of the following result is to provide a rationale for the selection of $n_1, \ldots, n_k$, so that operators $N_1, \ldots, N_k$ can be constructed to approximate the unique solution to (1.4).

**Theorem 3.1** *Let $\tilde{x}$ be the unique solution of the nonlinear problem (1.4). Let $x_0 \in B_R$ and $\varepsilon > 0$, then there exists $k \in \mathbb{N}$ and $n_1, n_2, \ldots, n_k$ such that*

$$\|\tilde{x} - N_k \circ \ldots \circ N_1(x_0)\| \leq \varepsilon.$$

*Proof.* In view of (2.4), there exists $k \in \mathbb{N}$ such that $\left\|(A \cdot B + C)^k(x_0) - \tilde{x}\right\| \leq \varepsilon/2$. Let $\varepsilon_1, \ldots, \varepsilon_k > 0$ satisfying (3.1). Using Lemma 3.2 we have $N_p \circ N_{p-1} \circ \ldots \circ N_1(x_0) \in B_R$ and $N_{p-1} \circ \ldots \circ N_1(x_0) \in B_R$, for all $p = 1, 2, \ldots, k$, and from the triangular inequality, one obtains

$$\left|(A \cdot B + C) \circ N_{p-1} \circ \ldots \circ N_1(x_0)(t) - N_p \circ N_{p-1} \circ \ldots \circ N_1(x_0)(t)\right| \leq H_{p,1}(x_0)(t) + H_{p,2}(x_0)(t),$$

where

$$H_{p,1}(x_0)(t) = \left|(A \cdot B) \circ N_{p-1} \circ \ldots \circ N_1(x_0)(t) - (A \cdot \widetilde{B}_p) \circ N_{p-1} \circ \ldots \circ N_1^{(n_1)}(x_0)(t)\right|$$

and

$$H_{p,2}(x_0)(t) = \left|C \circ N_{p-1} \circ \ldots \circ N_1(x_0)(t) - \widetilde{C}_p \circ N_{p-1} \circ \ldots \circ N_1(x_0)(t)\right|.$$

Taking the supremum over $t \in J$, we get

$$\begin{aligned} \left\|H_{p,1}(x_0)\right\| &\leq \left\|A_{p-1}(x_0)\right\| \left( \int_0^1 |G(t,s)| ds \left\|\xi_{n_p}(U_{p-1})(x_0) - U_{p-1}(x_0)\right\| + \frac{t}{|f(1,b)|} \sum_{i=1}^m \frac{1}{\Gamma(\beta_i+1)} \left\| \xi_{n_p}\left(V_{p-1}\right)(x_0) - V_{p-1}(x_0) \right\| \right) \\ &\leq \left\|A_{p-1}(x_0)\right\| \left( \|G(t,\cdot)\|_{L^1} \left\|\xi_{n_p}\left(U_{p-1}\right)(x_0) - U_{p-1}(x_0)\right\| + \frac{1}{|f(1,b)|} \sum_{i=1}^m \frac{1}{\Gamma(\beta_i+1)} \left\| \xi_{n_p}\left(V_{p-1}\right)(x_0) - V_{p-1}(x_0) \right\| \right). \end{aligned}$$

Likewise, we can demonstrate that

$$\begin{aligned}\left\|H_{p,2}(x_0)\right\| &\leq \sum_{j=1}^{m} \frac{1}{\Gamma(\beta_j)} \int_0^t \frac{1}{(t-s)^{1-\beta_j}} ds \left\|\xi_{n_p}\left(V_{p-1}\right)(x_0) - V_{p-1}(x_0)\right\| \\ &\leq \sum_{j=1}^{m} \frac{1}{\Gamma(\beta_j+1)} \left\|\xi_{n_p}\left(V_{p-1}\right)(x_0) - V_{p-1}(x_0)\right\|,\end{aligned}$$

from where it follows that

$$\begin{aligned}&\left\|(A\cdot B+C)\circ N_{p-1}\circ\ldots\circ N_1(x_0) - N_p\circ\ldots\circ N_1(x_0)\right\| \\ \leq\ &\left\|A_{p-1}(x_0)\right\| \left(\|G(t,\cdot)\|_{L^1} \left\|\xi_{n_p}\left(U_{p-1}\right)(x_0) - U_{p-1}(x_0)\right\| + \frac{1}{|f(1,b)|}\sum_{i=1}^{p} \frac{1}{\Gamma(\beta_i+1)} \left\|\xi_{n_p}\left(V_{p-1}\right)(x_0) - V_{p-1}(x_0)\right\|\right) \\ &+\sum_{j=1}^{p} \frac{1}{\Gamma(\beta_j+1)} \left\|\xi_{n_p}\left(V_{p-1}\right)(x_0) - V_{p-1}(x_0)\right\|.\end{aligned}$$

and therefore, by the convergence property (3.3) we can find $n_p \geq 1$ and consequently $N_p$, such that

$$\begin{aligned}&\left\|(A\cdot B+C)\circ N_{p-1}\circ\ldots\circ N_1(x_0) - N_p\circ\ldots\circ N_1(x_0)\right\| \\ \leq\ &\|A(B_R)\| \left(\sup_{s\in J}\|G(s,\cdot)\|_{L^1} \left\|\xi_{n_p}\left(U_{p-1}\right)(x_0) - U_{p-1}(x_0)\right\| + \frac{1}{|f(1,b)|}\sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i+1)} \left\|\xi_{n_p}\left(V_{p-1}\right)(x_0) - V_{p-1}(x_0)\right\|\right) \\ &+\sum_{j=1}^{m} \frac{1}{\Gamma(\beta_j+1)} \left\|\xi_{n_p}\left(V_{p-1}\right)(x_0) - V_{p-1}(x_0)\right\| \leq \varepsilon_p.\end{aligned}$$

Using these obtained estimates, together with Lemma 3.1, we conclude that $\|\tilde{x} - N_k\circ\ldots\circ N_1(x_0)\| < \varepsilon$. □

Before presenting on the approximation algorithm we need the following auxiliary result. In what follows, we will use the following notation:

$$x_k^{(n_1,\ldots,n_k)} = N_k\circ\ldots\circ N_1(x_0).$$

**Lemma 3.3** *Let $x_0 \in B_R$, $\varepsilon > 0$ and let $\overline{k} \in \mathbb{N}$ such that $\Theta^{\overline{k}-1}(2R) \leq \varepsilon/4$. Then, there are $n_1, n_2, \ldots, n_{\overline{k}} \in \mathbb{N}$ such that $\left\|x_{\overline{k}}^{(n_1,n_2,\ldots,n_{\overline{k}})} - (A\cdot B+C)\left(x_{\overline{k}}^{(n_1,n_2,\ldots,n_{\overline{k}})}\right)\right\| \leq \varepsilon$.*

*Proof.* Let $x_0 \in X$, and $\varepsilon > 0$. Since $A\cdot B+C$ is $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-function $\Theta$, by using the triangular inequality we get

$$\begin{aligned}\left\|x_{\overline{k}}^{(n_1,n_2,\ldots,n_{\overline{k}})} - (A\cdot B+C)\left(x_{\overline{k}}^{(n_1,n_2,\ldots,n_{\overline{k}})}\right)\right\| &\leq \left\|x_{\overline{k}}^{(n_1,n_2,\ldots,n_{\overline{k}})} - (A\cdot B+C)\left(x_{\overline{k}-1}^{(n_1,n_2,\ldots,n_{\overline{k}-1})}\right)\right\| \\ &\quad+\left\|(A\cdot B+C)\left(x_{\overline{k}-1}^{(n_1,n_2,\ldots,n_{\overline{k}-1})}\right) - (A\cdot B+C)\left(x_{\overline{k}}^{(n_1,n_2,\ldots,n_{\overline{k}})}\right)\right\| \\ &\leq \left\|\left(A\cdot B_{n_{\overline{k}}} + C_{n_{\overline{k}}}\right)\left(x_{\overline{k}-1}^{(n_1,n_2,\ldots,n_{\overline{k}-1})}\right) - (A\cdot B+C)\left(x_{\overline{k}-1}^{(n_1,n_2,\ldots,n_{\overline{k}-1})}\right)\right\| \\ &\quad+\Theta\left(\left\|x_{\overline{k}-1}^{(n_1,n_2,\ldots,n_{\overline{k}-1})} - x_{\overline{k}}^{(n_1,n_2,\ldots,n_{\overline{k}})}\right\|\right).\end{aligned}$$

From Equality (2.5) and taking into account that $\Theta^{\bar{k}-1}(2R) \leq \varepsilon/4$, it follows that

$$\begin{aligned}
\left\|x_{\bar{k}-1}^{(n_1,n_2,\dots,n_{\bar{k}-1})} - x_{\bar{k}}^{(n_1,n_2,\dots,n_{\bar{k}})}\right\| &\leq \left\|x_{\bar{k}-1}^{(n_1,n_2,\dots,n_{\bar{k}-1})} - (A\cdot B + C)^{\bar{k}-1}(x_0)\right\| + \left\|(A\cdot B + C)^{\bar{k}-1}(x_0) - (A\cdot B + C)^{\bar{k}}(x_0)\right\| \\
&\quad + \left\|(A\cdot B + C)^{\bar{k}}(x_0) - x_{\bar{k}}^{(n_1,n_2,\dots,n_{\bar{k}})}\right\| \\
&\leq \left\|x_{\bar{k}-1}^{(n_1,n_2,\dots,n_{\bar{k}-1})} - (A\cdot B + C)^{\bar{k}-1}(x_0)\right\| + \Theta^{\bar{k}-1}\left(\|x_0 - (A\cdot B + C)(x_0)\|\right) \\
&\quad + \left\|(A\cdot B + C)^{\bar{k}}(x_0) - x_{\bar{k}}^{(n_1,n_2,\dots,n_{\bar{k}})}\right\| \\
&\leq \left\|x_{\bar{k}-1}^{(n_1,n_2,\dots,n_{\bar{k}-1})} - (A\cdot B + C)^{\bar{k}-1}(x_0)\right\| + \left\|(A\cdot B + C)^{\bar{k}}(x_0) - x_{\bar{k}}^{(n_1,n_2,\dots,n_{\bar{k}})}\right\| + \varepsilon/4.
\end{aligned}$$

In other words, through the application of the triangle inequality and inequality (2.5), we can derive

$$\begin{aligned}
\left\|(A\cdot B + C)^{\bar{k}}(x_0) - x_{\bar{k}}^{(n_1,n_2,\dots,n_{\bar{k}})}\right\| &\leq \left\|(A\cdot B + C)^{\bar{k}}(x_0) - (A\cdot B + C)\left(x_{\bar{k}-1}^{((n_1,n_2,\dots,n_{\bar{k}-1}))}\right)\right\| + \\
&\quad \left\|(A\cdot B + C)\left(x_{\bar{k}-1}^{(n_1,n_2,\dots,n_{\bar{k}-1})}\right) - x_{\bar{k}}^{(n_1,n_2,\dots,n_{\bar{k}})}\right\| \\
&\leq \Theta\left(\left\|(A\cdot B + C)^{\bar{k}-1}(x_0) - x_{\bar{k}-1}^{(n_1,n_2,\dots,n_{\bar{k}-1})}\right\|\right) + \left\|(A\cdot B + C)\left(x_{\bar{k}-1}^{(n_1,n_2,\dots,n_{\bar{k}-1})}\right) - x_{\bar{k}}^{(n_1,n_2,\dots,n_{\bar{k}})}\right\|.
\end{aligned}$$

Considering the convergence property of the projection operators $\xi_{n_1},\dots,\xi_{\bar{k}-1}$, we can infer the existence of $n_1, n_2, \dots, n_{\bar{k}-1} \in \mathbb{N}$ such that

$$\left\|(A\cdot B + C)^{\bar{k}-1}(x_0) - x_{\bar{k}-1}^{(n_1,n_2,\dots,n_{\bar{k}-1})}\right\| \leq \varepsilon/16,$$

and consequently

$$\Theta\left(\left\|(A\cdot B + C)^{\bar{k}-1}(x_0) - x_{\bar{k}-1}^{(n_1,n_2,\dots,n_{\bar{k}-1})}\right\|\right) \leq \varepsilon/16.$$

On the other hand, by (3.3), we can find $n_{\bar{k}}$ such that

$$\left\|\left(A\cdot \widetilde{B}_{n_{\bar{k}}} + \widetilde{C}_{n_{\bar{k}}}\right)\left(x_{\bar{k}-1}^{(n_1,n_2,\dots,n_{\bar{k}-1})}\right) - (A\cdot B + C)\left(x_{\bar{k}-1}^{(n_1,n_2,\dots,n_{\bar{k}-1})}\right)\right\| \leq \varepsilon/8,$$

and

$$\left\|(A\cdot B + C)\left(X_{\bar{k}-1}^{(n_1,n_2,\dots,n_{\bar{k}-1})}\right) - x_{\bar{k}}^{(n_1,n_2,\dots,n_{\bar{k}})}\right\| \leq \varepsilon/8.$$

This implies that

$$\left\|x_{\bar{k}-1}^{(n_1,n_2,\dots,n_{\bar{k}-1})} - x_{\bar{k}}^{(n_1,n_2,\dots,n_{\bar{k}})}\right\| \leq \varepsilon/2,$$

and consequently we conclude that

$$\left\|x_{\bar{k}}^{(n_1,n_2,\dots,n_{\bar{k}})} - (A\cdot B + C)\left(x_{\bar{k}}^{(n_1,n_2,\dots,n_{\bar{k}})}\right)\right\| \leq \varepsilon/8 + \Theta(\varepsilon/2) \leq \varepsilon.$$

□

We are in position to give the following algorithm to obtain a sequence of approximations:

$$x_0 \to x_1 = x_1^{(n_1)} \to x_2 = x_2^{(n_1,n_2)} \to \cdots \to x_i = x_i^{(n_1,n_2,\dots,n_i)} \to \cdots$$

**Algorithm**

**Input:** $x_0 \in B_R$, $\varepsilon > 0$, $\bar{k}$ such that $\Theta^{\bar{k}-1}(2R) \le \varepsilon/8$ and $\varepsilon_1, \dots, \varepsilon_{\bar{k}} > 0$ satisfying (3.1).

Choose $n_1 \in \mathbb{N}$ large enough.

**for** $j = n_1, n_1 + 1, n_1 + 2 \dots$

Calculate $N_1(x_0)$

Set $x_1^{(n_1)} \leftarrow N_1(x_0)$

Calculate $\|x_1^{(n_1)} - (AB + C)(x_1^{(n_1)})\|$

**if** $\|x_1^{(n_1)} - (AB + C)(x_1^{(n_1)})\| \ge \varepsilon_1$

Set $n_1 \leftarrow n_1 + 1$

**if else** Set $x_1 \leftarrow x_1^{(n_1)}$

**end (for)**

**for** $i = 2, \dots, \bar{k}$

Choose $n_i \in \mathbb{N}$ large enough.

**for** $j = n_i, n_i + 1, n_i + 2 \dots$

Calculate $N_i(x_{i-1})$

Set $x_i^{(n_1, n_2, \dots, n_i)} \leftarrow N_i(x_{i-1})$

Calculate $\|x_i^{(n_1, n_2, \dots, n_i)} - (AB + C)(x_i^{(n_1, n_2, \dots, n_i)}\|$

**if** $\|x_i^{(n_1, n_2, \dots, n_i)} - (AB + C)(x_i^{(n_1, n_2, \dots, n_i)})\| \ge \varepsilon_i$

Set $n_i \leftarrow n_i + 1$

**if else** Set $x_i \leftarrow x_i^{(n_1, n_2, \dots, n_i)}$

**end (for)**

**end (for)**

# 4 Numerical examples

In this section we show the numerical results obtained in several examples using the Faber-Schauder system in $C(J)$ which is the usual Schauder basis in this Banach space.

**Example 1:** Consider the following problem with boundary conditions:

$$\begin{cases} D^{5/3}\left(x(t) - I^{\frac{3}{2}} h(t, x(t))\right) = g(t, x(t)), t \in J, \\ \\ x(0) = 0 \text{ and } x(1) = b. \end{cases} \tag{4.1}$$

Notice that this problem is a particular case of (1.4), where $a = 0$, $m = 1$, $\alpha = 5/3, \beta = \frac{3}{2}$, $f \equiv 1$, $g(t, x) =$

$d4^{-2/3}\sqrt{|x|}$, $h(t,x) = r$, with $d \in \mathbb{R}$ and $r > 0$. It is clear that, $f, g, h$ satisfy $(i)$-$(iv)$ with $\alpha(\cdot), \phi(\cdot) \equiv 0$, $\tau(\cdot) = d4^{-2/3}, \gamma(\cdot) \equiv 0$, $\varphi \equiv 0$ and $\psi(t) = \sqrt{t}, t \geq 0$. If $b = 0.5$, $d = -0.490673$ and $r = 0.5$, then the conditions (2.3) are satisfied. An application of Theorem 2.2 achieves that the nonlinear fractional integro-differential equation (4.1) has a unique solution in $B_R$, with $R = 1/2$. The solution is $\tilde{x}(t) = bt$.

Table 1 presented the numerical results for (4.1) with the values $a = 0.05, b = 1/4$, and the initial function $x_0(t) = \frac{1}{2}$ using the proposed method.

Table 1: Numerical results for problem (4.1) with initial $x_0(t) = 0.5$.

| $t$ | $\lvert x_2^{(9,9)}(t) - \tilde{x}(t)\rvert$ | $\lvert x_4^{(9,9,9,9)}(t) - \tilde{x}(t)\rvert$ |
|---|---|---|
| 0.1 | 0.0000861323 | 0.00497238 |
| 0.2 | 0.00275949 | 0.0106329 |
| 0.3 | 0.00516259 | 0.0136675 |
| 0.4 | 0.00678153 | 0.0145598 |
| 0.5 | 0.00769718 | 0.014127 |
| 0.6 | 0.00782594 | 0.0126382 |
| 0.7 | 0.00769718 | 0.0103659 |
| 0.8 | 0.00565374 | 0.0074308 |
| 0.9 | 0.00327359 | 0.00394672 |
| 1 | 0 | 0 |
| $\max \lvert x^*(t) - \tilde{x}(t)\rvert$ | 0.00782594 | 0.01455980 |

**Example 2:** Now we consider the problem

$$\begin{cases} D^{5/3}\left(\dfrac{x(t)}{f(t,x(t))}\right) = g(t,x(t)), t \in J, \\ \\ x(0) = \delta \text{ and } x(1) = 2\delta. \end{cases} \tag{4.2}$$

which can be regarded as a special case of (1.4), with the following parameter values: $a = \delta$, $b = 2\delta$, $m = 2$, $\alpha = 5/3, \beta_j = \frac{2j-1}{2}$ for $j = 1,2$, $f(t,x) = \dfrac{\delta(t+1)}{\delta(t+1) + k1(t)}, g(t,x) = dx^2$, and $h_j(t,x) = 0, j = 1,2$, with $d \in \mathbb{R}$ and $r > 0$. It is easy to see the $f, g$ and $h$ satisfy $(i)$-$(iv)$ with $\alpha(\cdot), \phi(\cdot) \equiv 0$, $\eta(\cdot) = 2dR$ and $\psi(t) = t, t \geq 0$. If $\delta = 0.1$, $d = 0.5$, then the conditions (2.3) are satisfied. An application of Theorem 2.2 achieves that the nonlinear fractional integro-differential equation (4.2) has a unique solution in $B_{1/2}$. The solution is $\tilde{x}(t) = \delta t + \delta$. The obtained numerical results are include in Table 2.

Table 2: Numerical results for problem (4.2) with initial $x_0(t) = \frac{1}{10}t$.

| $t$ | $\lvert x_1^{(9)}(t) - \tilde{x}(t)\rvert$ | $\lvert x_2^{(9,9)}(t) - \tilde{x}(t)\rvert$ |
|---|---|---|
| 0.1 | 0.0000182256 | 0.0000197691 |
| 0.2 | 0.0000154635 | 0.0000383688 |
| 0.3 | 0.0000250424 | 0.0000828754 |
| 0.4 | 0.000101898 | 0.000215839 |
| 0.5 | 0.000201094 | 0.000521411 |
| 0.6 | 0.000293377 | 0.0010852 |
| 0.7 | 0.000326333 | 0.00200598 |
| 0.8 | 0.000219447 | 0.0034218 |
| 0.9 | 0.000147197 | 0.00550835 |
| 1 | 0.000979496 | 0.00852049 |
| $\max\lvert x^*(t) - \tilde{x}(t)\rvert$ | 0.000979496 | 0.00852049 |

**Example 3:** Considere the following problem

$$\begin{cases} D^{5/3}\left(\dfrac{x(t) - \sum_{j=1}^{2} I^{\frac{2j-1}{2}} h_j(t, x(t))}{f(t, x(t))}\right) = g(t, x(t)), t \in J, \\ \\ x(0) = \delta \text{ and } x(1) = 2\delta. \end{cases} \tag{4.3}$$

where $a = \delta$, $b = 2\delta$, $m = 2$, $\alpha = 5/3$, $\beta_j = \frac{2j-1}{2}$, $j = 1, 2$, $g(t, x) = dx^2$, and $h_j(t, x) = rx^2$, $j = 1, 2$, with $d \in \mathbb{R}$ and $r > 0$ and $f(t, x) = \dfrac{105\sqrt{\pi}b(t+1) - \left(2\delta^2 r\sqrt{t}\,(105 + 2t\,(105 + 8t(7+t)))\right)}{105\sqrt{\pi}\,(\delta(t+1) + k(t))}$, with
$k(t) = \frac{1}{2}\delta^2 \int_0^t \left(G(t,s) - \frac{t}{\sqrt{\pi}}\left((1-s)^{-1/2} + 2(1-s)^{1/2}\right)\right)(s+1)^2 ds.$

It is clear that the mappings $f, g$ and $h$ satisfy $(i)$-$(iv)$ with $\alpha(t) = 0$, $\Phi(t) = t$, $\eta(t) = 2Rd$ and $\gamma_i(t) = 2Rr$. If $\delta = 0.1$, $d = 0.5$ and $r = 0.5$, then the conditions (2.3) are satisfied. An application of Theorem 2.2 achieves that the nonlinear fractional integro-differential equation (4.2) has a unique solution in $B_R$, with $R = 1/2$. The solution is $\tilde{x}(t) = \delta(t+1)$ and the numerical results are presented in Table 3.

Table 3: Numerical results for problem (4.3) with initial $x_0(t) = 0.4(t+1)$.

| $t$ | $\lvert x_1^{(9)}(t) - \tilde{x}(t)\rvert$ | $\lvert x_2^{(9,9)}(t) - \tilde{x}(t)\rvert$ |
|---|---|---|
| 0.1 | 0.00207136 | 0.00208625 |
| 0.2 | 0.00316195 | 0.00325773 |
| 0.3 | 0.00402196 | 0.00425228 |
| 0.4 | 0.00472513 | 0.00508871 |
| 0.5 | 0.00529221 | 0.00574753 |
| 0.6 | 0.00567915 | 0.00619244 |
| 0.7 | 0.00574453 | 0.00636505 |
| 0.8 | 0.0052411 | 0.00615544 |
| 0.9 | 0.00378861 | 0.00526269 |
| 1 | $1.31621\ 10^{-10}$ | $7.87242\ 10^{-12}$ |
| $\max\lvert x^*(t) - \tilde{x}(t)\rvert$ | 0.00576502 | 0.00636505 |

## 5 Conclusions

In this paper we have analyzed a problem of fractional order differential equations ($1 < \alpha \leq 2$). The existence of solutions was obtained via the Boyd-Wong fixed point theorem. The approximation method was constructed via a combination of the Picard operators terms and the basis Schauder notion. The illustrative examples proved the applicability of the approximated method.


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## Conflict of interest statement

The authors declare no potential conflict of interests.

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