Deep learning solution of optimal reinsurance-investment strategies with extra information and multiple risks

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Abstract

This paper investigates an optimal investment-reinsurance problem for an insurer who possesses extra information regarding the future realizations of the claim process and risky asset process. The insurer sells insurance contracts, has access to proportional reinsurance business, and invests in a financial market consisting of three assets: one risk-free asset, one bond and one stock. Here, the nominal interest rate is characterized by the Vasicek model; and the stock price is driven by the Heston's stochastic volatility model. Applying the enlargement of filtration techniques, we establish the optimal control problem in which an insurer maximizes the expected power utility of the terminal wealth. By using the dynamic programming principle, the problem can be changed to four-dimensional Hamilton-Jacobi-Bellman equation. In addition, we adopt a deep neural network method by which the partial differential equation is converted to two backward stochastic differential equations and solved by a stochastic gradient descent-type optimization procedure. Numerical results obtained using TensorFlow in Python and the economic behavior of the approximate optimal strategy and the approximate optimal utility of the insurer are analyzed.

Keywords: Extra information, Hamilton-Jacobi-Bellman equations, Enlarge-

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ment of filtration, Machine learning solution

1. Introduction

Due to the inherent nature of insurance products, insurers often accumulate substantial amounts of cash to be invested in financial markets. This surplus serves as a financial buffer to cover future claims and mitigate the risk of financial ruin. In addition, insurers strategically choose to transfer a portion of their premiums to reinsurance companies, in exchange for protection against adverse claim volatilities. Effectively managing the risks posed by insurance claims and financial market fluctuations necessitates the formulation of reinsurance and investment strategies. Since the groundbreaking work by [1], extensive research has been conducted in this area, with notable contributions from [2–5].

It is evident that insurance companies are typically exposed to risks from both the insurance market and the financial market. In the early days, insurers would attempt to manually collect and analyze relevant information to adapt to the insurance market. For example, an insurer may have exclusive knowledge that the costs associated with treating a specific disease are expected to decline due to advancements in medical technology through information analysis. Consequently, the insurer with this privileged information may outperform competitors who lack such knowledge. Empirical evidence supporting this phenomenon has been documented in the literature, for example, [6] and [7]. As technology and understanding have progressed, insurers have gradually started leveraging emerging technologies such as machine learning and big data to predict accident claims or health risks, which has become quite mature ([8, 9]). For instance, one of the largest insurance companies in the United States, Allstate, developed an automatic driving technology and analytics platform called Arity. By collecting and analyzing vast driving data, they can predict risks, improve driving behavior, and provide more accurate insurance pricing and personalized recommendations for their customers. Oscar Health utilizes a large volume of health data, including customers' personal health records and medical diagnosis data, to predict their risk levels using data analysis and machine learning algorithms. This helps the company develop more accurate insurance strategies and pricing, thereby managing their own insurance risks. Currently, many insurers are actively exploring and applying these technologies to improve their operational strategies and risk management levels. The aforementioned examples imply that some economic agents possess information regarding future claims.

Another type of risk that insurance companies need to approach with caution is the risk associated with the financial market. In the early stages of the 2008 financial crisis, AIG underestimated the potential impact of its derivative trades on the company's liabilities. The company failed to consider the possibility that a downturn in the financial market could lead to a decrease in the value of derivatives on its balance sheet, triggering contractual obligations for higher compensation. As the real estate market collapsed and triggered the subprime crisis, the value of these derivatives rapidly declined. Consequently, AIG was unable to meet the required collateral payments, resulting in a significant increase in its liabilities. Ultimately, the U.S. government intervened by injecting billions of dollars into AIG to prevent its bankruptcy [10]. This case serves as a vital lesson for other insurance companies, highlighting the importance of evaluating not only future financial risks but also the impacts arising from the interaction between liabilities and capital gains, which lead to a phenomenon worth studying that insurers' risk assessment not only helps them adapt to the insurance market but also enables them to improve their investment strategies. This phenomenon aligns with economic intuition, as the evolution of the overall societal environment simultaneously affects various aspects of the economy. Hence, the objective of our study is to investigate the optimal reinsurance and investment problem based on a negative correlation between financial risk and insurance risk where the insurer possess extra information.

Optimal investment for an insider is a classical problem investigated by many scholars, such as [11–14]. In recent years, optimal investment-reinsurance problems with inside information have also gained some attention. [15] examine optimal investment-reinsurance problems under insider information, where the inside information influences insurers' strategy based on an expanded information filtration, leading to a new semimartingale process concerning insurance risk. This framework has been adopted by some studies. For example, [16] focus on an investment-reinsurance game between two insurance companies with differing opinions on extra information based on this model. [17] extends the model proposed by [15], they incorporating jumps and random coefficients in the risk process. [18] consider model uncertainty and investigate an optimal investment-reinsurance problem in the presence of insider information. Our work also builds upon this framework. Specifically, we consider the wealth process of the insurer who obtains some extra information about the future realization of the claim process.

We also consider a negative correlation between claims and risk assets, as supported by relevant literature [10, 19, 20]. This implies that unexpected claim payments may be partially offset by financial market returns for insurers. Furthermore, insurers have access to extra information related to insurance risks, which indirectly affects their understanding of the financial market and subsequently influences their investment strategies. This incorporation of extra information not only impacts the claim process but also introduces mathematical complexities in analyzing the behavior of risky assets. In terms of model setup, we consider the optimal reinsurance and investment for an insider under the joint risks of interest rate, inflation and volatility. As far as we know, when the investment time horizon is long, an insurer can not ignore the interest risk. [21] address the optimal reinsurance and investment problem for insurers exposed to interest rate and inflation risks. They utilize an Ornstein-Uhlenbeck process to model nominal interest rates and employ zero-coupon bonds as a hedging tool against interest rate risks. Building upon their work, they further incorporate ambiguity and volatility risks into the insurer's decision-making framework[22]. As highlighted in [23], stochastic volatility models can accurately capture the characteristics of peaked and heavy-tailed returns seen in stock prices, thereby incorporating a part of the observed volatility smile curve. Thus, we draw upon the framework proposed in [22] and present a model that considers three financial risks: interest rate risk and volatility risk. Specifically, interest rates follow an Ornstein-Uhlenbeck process, while the stock price is driven by Heston's stochastic volatility model.

To solve this problem, we adopt the technique proposed in [21] to transform the original problem into a self-financing problem, which is more theoretically tractable. The stochastic dynamic programming method is used to derive the four-dimensional Hamilton-Jacobi-Bellman (abbr. HJB) equation. Due to the fully nonlinear nature of the four-dimensional parabolic partial differential equation (PDE) we obtained, it is challenging to obtain the solution using conventional methods. Moreover, classical numerical techniques for solving such equations often encounter the "curse of dimensionality". Hence, we will employ recently emerged machine learning methods to solve the PDE [24], [25], [26]. Recently, machine learning and artificial intelligence have changed our community, and offered novel approaches to the insurance industry, such as reducing losses, claim reserve estimation, and policy design [27], [28]. To be specific, firstly, we adopt the analytic techniques of the HJB to derive the semi-analytic form, alleviating the complexity of the computation. Secondly, we will establish a connection between the PDE and second-order backward stochastic differential equations (2BSDEs), which enables us to handle fully nonlinear PDEs and nonlinear expectations [29]. Thirdly, we will utilize the deep 2BSDE method proposed in [30] to solve the PDE.

The contribution of this paper is twofold. First, we formulate the optimal investment-reinsurance problem for an insurer with extra information and multiple risks. In addition, extra information is involved not only with the claims but also with risky assets, which is more aligned with real-world scenarios, and as far as we know, there have been few explorations of this phenomenon to date. Second, we applying a new neural network-based deep learning procedure to overcome the computational difficulty of solving the high-dimensional HJB equation. Hence, we believe that our efforts provide certain inspiration for future work on the optimal control problem in financial markets. The contributions of this paper are twofold. Firstly, we establish the optimal investment reinsurance problem for an insurer with extra information and is exposed to multiple risks. Notably, this extra information encompasses both claims and risky assets, exhibits greater realism to real-world scenarios. Remarkably, there have been limited investigations into this phenomenon thus far. Secondly, we employ a new neural network-based deep learning approach to overcome the computational challenges associated with solving the high-dimensional Hamilton-Jacobi-Bellman (HJB) equation in the optimal investment-reinsurance problem. We believe that our efforts provide valuable inspiration for future research on optimal control problems in financial markets.

The rest of the paper is organized as follows. In section 2, we introduce the extra information and use the enlargement of filtration techniques to obtain the risk model. In section 3, we transform the problem into a self financing problem by introducing an auxiliary problem, and obtain a semi-analytical solution by the stochastic dynamic programming principle. In section 4, we first convert the PDE related to the HJB equation to a second-order backward stochastic differential equation and then obtain the approximate optimal solution of the HJB equation by using a deep learning method called deep 2BSDEs. In section 5, we explain the economic behavior of the insurer through sensitivity analysis based on the approximate optimal solution. In section 6, we provide some conclusions.

2. Reference model

2.1. The Basic Model

Consider the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. All stochastic processes we discuss below are adapted to $\{\mathcal{F}_t\}_{t\geq 0}$, and $[0, T]$ is the time horizon. The surplus process of the insurer is characterized by the classical Lundberg model

$$\begin{cases} dR(t) = cdt - d\left\{\sum_{i=1}^{N_t} Z_i\right\}, \\ R(0) = R_0, \end{cases} \tag{2.1}$$

where Z_i is the size of the i-th claim and c is the premium rate. N_t is a Poisson process subject to the intensity of λ and is independent of Z_i. Claim size Z_i is independent of each other, and its first moment and second moment are μ_1 and μ_2, respectively. The premium rate $c = (1+\eta)\lambda\mu_1$, and η represents the safe load, with $\eta > 0$. Referring to [22], we approximately express the compound Poisson process above to the continuous form

$$\begin{cases} dR(t) = \lambda\mu_1\eta dt + \sqrt{\lambda\mu_2}d\widetilde{W}_0(t), \\ R(0) = R_0, \end{cases} \tag{2.2}$$

where $\left\{\widetilde{W}_0(t), t \in [0,T]\right\}$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$.

To reduce the risks involved in the claims process, the insurer has the option to purchase proportional reinsurance protection. A proportional reinsurance strategy is represented by the function $u(t) : \mathbb{R}^+ \to [0, +\infty)$, where $1 - u(t)$ denotes the proportion of claims covered by the reinsurance company at time t, while the remaining proportion $u(t)$ is covered by the insurer. In the case of proportional reinsurance, a continuous premium is paid at a rate of $(1+\theta_r)(1-u(t))\lambda\mu_1$, where θ_r represents the safety loading of the proportional reinsurance, satisfying $\theta_r \geq \eta$. It is worth noting that the value of $u(t)$ can exceed 1, indicating that the insurance company acquires new business. The surplus process of insurers adopting proportional reinsurance is

$$dX(t) = [\eta - \theta_r + \theta_r u(t)]\lambda\mu_1 dt + u(t)\sqrt{\lambda\mu_2}d\widetilde{W}_0(t). \tag{2.3}$$

In order to secure a relatively stable and sustainable cash flow income during a specific future period, the insurer will allocate investments to both risk-free and risky assets. We consider a market that encompasses interest rate risk. The dynamics of the risk-free assets are denoted by

$$\begin{cases} \dfrac{dS_0(t)}{S_0(t)} = r_n(t)dt, \\ S_0(0) = s_0. \end{cases} \tag{2.4}$$

where $r_n(t)$ is nominal interest and satisfies the following Ornstein-Uhlenbeck process:

$$\begin{cases} dr_n(t) = a(b - r_n(t))dt - \sigma_{r_n} dW_{r_n}(t), \\ r_n(0) = r_0, \end{cases} \tag{2.5}$$

where a is the velocity of mean reversion, b is the mean reversion level, σ_{r_n} is the volatility of interest and is non-zero, and $\{W_{r_n}(t), t \in [0,T]\}$ is a standard Brownian motion.

The first risky asset is a zero-coupon nominal bond, denoted as $B(t, T)$. This bond provides a nominal payoff of 1 upon maturity at date T . Considering the absence of arbitrage and the market price of interest rate risk given by λ_{r_n}, the value of $B(t, T)$ can be determined by solving the following partial differential equation, with the boundary condition $B(T, T) = 1$:

$$B_{r_n} a(b - r_n) + \frac{\partial B}{\partial t} + \frac{1}{2} B_{r_n r_n} \sigma_{r_n}^2 = B r_n - B_{r_n} \lambda_{r_n} \sigma_{r_n},$$

where B_{r_n} and $B_{r_n r_n}$ represent the first and second order partial derivatives of $B(t, T)$ with respect to r_n. The solution of $B(t, T)$ is given by the explicit formula:

$$B(t, T) = \exp(q_1(T - t) - q_0(T - t) r_n(t)),$$

with

$$q_1(T - t) = \frac{2(e^{\nu(T-t)} - 1)}{(\nu + a - \lambda_{r_n} \sigma_{r_n})(e^{\nu(T-t)} - 1) + 2\nu},$$

$$q_0(T - t) = \frac{-ab}{\sigma_{r_n}^2} \left[2 \log \frac{(\nu + a - \lambda_{r_n} \sigma_{r_n})(e^{\nu(T-t)} - 1) + 2\nu}{2\nu} - (\nu + a - \lambda_{r_n} \sigma_{r_n})(T - t) \right],$$

where $\nu \equiv \sqrt{(a - \lambda_{r_n} \sigma_{r_n})^2 + 2\sigma_{r_n}^2}$. The returns on the nominal zero-coupon bond could be expressed as the following stochastic differential equation:

$$\frac{dB(t, T)}{B(t, T)} = r_n(t) dt + \sigma_{B_1}(T - t)(\lambda_{r_n} dt + dW_{r_n}(t)), \tag{2.6}$$

$$\sigma_{B_1}(T - t) = \sigma_{r_n} q_1(T - t). \tag{2.7}$$

Due to the presence of bonds with specified maturities in the financial market, the use of a rolling zero coupon bond with a constant time to maturity K is considered as a replacement for the zero coupon bond. The dynamics of this rolling bond could be described as:

$$\frac{dB_K(t)}{B_K(t)} = r_n(t) dt + \sigma_{B_1}(K)(\lambda_{r_n} dt + dW_{r_n}(t)). \tag{2.8}$$

According to the findings reported in Stein (2012), a negative correlation exists between liabilities and capital gains within financial markets. Building upon the research conducted by [19] and [20], under the probability measure $\mathbb{P}$, we assume a negative correlation between the stock price and the claim process, as symbolically denoted by

$$d[cov(\widetilde{W}_S(t), \widetilde{W}_0(t))] = \rho_0 dt.$$

By Cholesky decomposition (see [31]), we can derive

$$d\widetilde{W}_S(t) = \rho_0 d\widetilde{W}_0(t) + \sqrt{1-\rho_0^2}dW_S(t), \tag{2.9}$$

where $\rho_0 \in [-1, 0]$ represents the negative correlation.

Traditionally, many studies have utilized geometric Brownian motion to model stock prices. However, the existence of market anomalies demands more advanced methodologies to incorporate stock returns and volatility, such as stochastic expected returns or stochastic volatility. Moreover, with the introduction of extra information, insurers may identify certain patterns of fluctuation in stochastic volatility. To address this issue, we consider the Heston's stochastic volatility model, providing a comprehensive framework to analyze insurers' investment behavior in stocks supplemented with extra information. The dynamics of the price of stock can be expressed by

$$\begin{cases} \dfrac{dS_1(t)}{S_1(t)} = r_n(t)dt + \sigma_{S_1}(\lambda_{r_n}dt + dW_{r_n}(t)) \\ \qquad\qquad + \nu L(t)dt + \sqrt{L(t)}d\widetilde{W}_S(t), \\ S_1(0) = s_1, \end{cases} \tag{2.10}$$

and the dynamics of volatility are denoted by

$$\begin{cases} dL(t) = \alpha(\beta - L(t))dt + \sigma_L\sqrt{L(t)}d\widetilde{W}_L(t), \\ L(0) = l_0, \end{cases} \tag{2.11}$$

where α is the velocity of mean reversion, β is the reversion level, and σ_L is the risk level. To guarantee that volatility is nonnegative, volatility of the stock needs to satisfy the Feller condition $2\alpha\beta > \sigma_L^2$. In addition, $\widetilde{W}_S(t)$ and $\widetilde{W}_L(t)$ are dependent and we have

$$d[cov(\widetilde{W}_S(t), \widetilde{W}_L(t))] = \rho_S dt, \quad \text{for} \quad \rho_S \in [-1, 1],$$

and we can derive

$$d\widetilde{W}_L(t)) = \rho_S d\widetilde{W}_S(t) + \sqrt{1-\rho_S^2}dW_L(t). \tag{2.12}$$

By substituting (2.9) and (2.12) into (2.11), we have the dynamics of volatility of stock satisfy that

$$\begin{cases} dL(t) = \alpha(\beta - L(t))dt + \sigma_L \sqrt{L(t)}\rho_S \rho_0 d\widetilde{W}_0(t) + \sigma_L \sqrt{L(t)}\rho_S \sqrt{1-\rho_0^2} dW_S(t) \\ \qquad + \sigma_L \sqrt{L(t)} \sqrt{1-\rho_S^2} dW_L(t), \\ L(0) = l_0, \end{cases} \tag{2.13}$$

where $\{W_S(t), t \in [0,T]\}$ and $\{W_L(t), t \in [0,T]\}$ are two standard Brownian motions. In addition, we supposed that $\widetilde{W}_0$, W_{r_n}, W_S and W_L are independent of each other.

2.2. Extra information

In this paper, we will follow the idea that the insurer can obtain some extra information about the future claim. Here, we refer to it as extra information about the future claim. This means that the insurer can leverage new technologies such as big data and machine learning to predict potential events that may have an impact on the insurance market and the timing of their occurrence. Suppose that the extra information at time T_0 is stored in a random variable $\widetilde{W}_0(T_0)$, with the condition $T_0 > T$. We should mention that once the time t is close to T_0, The value of $\delta(t)$ tends to infinity, which means the information is too accurate for the insurer. Hence, the condition that $T_0 > T$ guarantees the regularity of this problem. In addition, the insurer's expectation of the future is affected by the size of the information $\widetilde{W}_0(T_0)$.

To measure the increase in information, we need to consider a new filtration $\mathbb{G} = \{\mathcal{G}_t\}_{t\in[0,T]}$ which is defined as

$$\mathcal{G}_t = \mathcal{F}_t \cup \sigma(\widetilde{W}_0(T_0)),$$

where $\mathcal{F}_t = \sigma\left(\widetilde{W}_0(t), W_{r_n}(t), W_S(t), W_L(t)\right)$ is the natural filtration induced by the standard Brownian motions $\widetilde{W}_0(t), W_{r_n}(t), W_S(t), W_L(t)$, for any $t \in [0,T]$. $\sigma(\widetilde{W}_0(T_0))$ is the $\sigma-$algebra generated by the extra information $\widetilde{W}_0(T_0)$. The new filtration satisfied completeness and right continuity and

$$\mathcal{F}_t \subset \mathcal{G}_t, \forall t \in [0,T].$$

The following lemma will show the decomposition of the process $\widetilde{W}_0$ with respect to $\mathbb{G}$.

Lemma 2.1. *The process $\{\widetilde{W}_0(t), t \geq 0\}$ is a semi-martingale with respect to $\mathbb{G}$ and its semi-martingale decomposition is*

$$\widetilde{W}_0(t) = W_0(t) + \int_0^t \delta(s)ds,$$

where

$$\delta(t) = \frac{\widetilde{W}_0(T_0) - \widetilde{W}_0(t)}{T_0 - t}, 0 \leq t < T_0,$$

and $W_0(t)$ is a $\mathbb{G}-$Brownian motion. Furthermore, set

$$\delta_0 = \frac{\widetilde{W}_0(T_0)}{T_0}, M(t) = \int_0^t \frac{1}{T_0 - s} dW_0(t),$$

where $M(t)$ is the stochastic part of the information drift. Then, we have

$$d\widetilde{W}_0(t) = (\delta_0 - M(t))dt + dW_0(t). \tag{2.14}$$

Proof. Please refer to the proof of Theorem 3.1 and Proposition 3.1 in [15].

Remark 2.2. *For any $t < T_0$, we have*

$$\mathbb{E}\left[\int_0^t \delta(s)^2 ds\right] = \log\left(\frac{T_0}{T_0 - t}\right) < \infty.$$

Remark 2.3. *The lemma 2.1 provides a model framework for understanding how extra information can influence an insurers' strategy. δ_0 represents the influence of the extra information at time T_0 on the surplus process. A positive value for $\widetilde{W}_0(T_0)$ indicates that future events are expected to lead to lower claims than initially anticipated, thereby favoring the insurer's surplus, and vice versa. The term $M(t)$ represents the information drift, acknowledging that the insurer's predictions are subject to potential deviations from actual outcomes. Hence, $\delta_0 - M(t)$ indicates that future forecasts can still be disrupted by unforeseen factors. The factor $1/(T_0 - t)$ signifies that the impact of unexpected factors increases as the forecasted time T_0 approaches. This aligns with reality, as events further in the future have less influence on the present compared to imminent occurrences.*

Consequently, with respect to $\mathbb{G}$, the dynamics of the nominal interest rate, information drift and stochastic volatility are

$$
\begin{aligned}
dr_n(t) &= a(b - r_n(t))dt - \sigma_{r_n} dW_{r_n}(t), r_n(0) = r_0, & (2.15)\\
dM(t) &= \frac{1}{T_0 - t} dW_0(t), M(0) = 0, & (2.16)\\
dL(t) &= \left(\alpha(\beta - L(t)) + \sigma_L \sqrt{L(t)}\rho_S \rho_0(\delta_0 - M(t))\right) dt \\
&+ \sigma_L \sqrt{L(t)} \left(\rho_S \rho_0 dW_0(t) + \rho_S \sqrt{1 - \rho_0^2} dW_S(t)\right) \\
&+ \sigma_L \sqrt{L(t)} \sqrt{1 - \rho_S^2} dW_L(t), L(0) = l_0. & (2.17)
\end{aligned}
$$

2.3. The risk model

In the financial market, the insurer allocates the premiums into the three assets. The wealth of the insurer is influenced by the insurance business and portfolios in the market simultaneously. Denote the investments in cash, zero coupon bond, TIPS and stock at time t by π_0, π_{B_1} and π_S respectively. The surplus process of the insurer is

$$
\begin{cases}
dX^u(t) = (\eta - \theta_r + \theta_r u(t))\lambda\mu_1 dt + u(t)\sqrt{\lambda\mu_2} d\widetilde{W}_0(t) \\
\qquad + \pi_{B_1}(t)\dfrac{dB_K(t)}{B_K(t)} + \pi_0(t)\dfrac{dS_0(t)}{S_0(t)} + \pi_S(t)\dfrac{dS_1(t)}{S_1(t)}, \\
X(0) = x_0.
\end{cases} \tag{2.18}
$$

Substituting (2.4), (2.8), (2.10), (2.12), (2.13) and (2.14) into (2.18) and using the relationship $X(t) = \pi_{B_1}(t) + \pi_P(t) + \pi_0(t) + \pi_S(t)$, the dynamics of the process $X^{\widetilde{u}}(t)$ can be rewritten as follows:

$$
\begin{cases}
dX^{\widetilde{u}}(t) = (\eta - \theta_r)\lambda\mu_1 dt + \widetilde{u}(t)^\top \sigma(L(t))(\Lambda(M(t), L(t))dt + dW(t)), \\
X(0) = x_0,
\end{cases} \tag{2.19}
$$

where

$$\widetilde{u}(t) \triangleq (u(t), \pi_{B_1}(t), \pi_S(t))^\top, \tag{2.20}$$

$$\Lambda(M(t), L(t)) \triangleq \begin{pmatrix} \dfrac{\lambda\mu_1\theta_r}{\sqrt{\lambda\mu_2}} + (\delta_0 - M(t)) \\ \lambda_{r_n} \\ \left(v\sqrt{L(t)} - \dfrac{\lambda\mu_1\theta_r\rho_0}{\sqrt{\lambda\mu_2}}\right)\dfrac{1}{\sqrt{1-\rho_0^2}} \end{pmatrix}, \tag{2.21}$$

$$\sigma(L(t)) \triangleq \begin{pmatrix} \sqrt{\lambda\mu_2} & 0 & 0 \\ 0 & \sigma_{B_1}(K) & 0 \\ \sqrt{L(t)}\rho_0 & \sigma_{S_1} & \sqrt{L(t)}\sqrt{1-\rho_0^2} \end{pmatrix}, \tag{2.22}$$

$$dW(t) \triangleq \begin{pmatrix} dW_0(t) \\ dW_{r_n}(t) \\ dW_S(t) \end{pmatrix}. \tag{2.23}$$

We denote

$$\mu^d(t, m, l, u, \pi_{B_1}, \pi_S) \triangleq (\eta - \theta_r)\lambda\mu_1 + \overline{u}(t)^\top \sigma(l)\Lambda(m, l),$$

$$\sigma^{d_1}(t, l, u) \triangleq u\left(\sqrt{\lambda\mu_2} + \sqrt{l}\rho_0\right),$$

$$\sigma^{d_2}(t, \pi_{B_1}) \triangleq \pi_B(\sigma_{B_1}(K) + \sigma_{I_1} + \sigma_{S_1}),$$

$$\sigma^{d_3}(t, l, \pi_S) \triangleq \pi_S\left(\sqrt{l}\sqrt{1-\rho_0^2}\right).$$

Then we introduce the admissible set Π of all the admissible strategies as follows:

Definition 2.4. *(Admissible Strategy). A strategy $\widetilde{u}(t) = (u(t), \pi_{B_1}(t), \pi_S(t))_{t\in[0,T]}$ is said to be admissible if*
(1) $u(t) \geq 0, \forall t \in [0, T]$;
(2) $\widetilde{u}(t)$ is progressively measurable with respect to $\{\mathcal{G}_t\}_{t\in[0,T]}$ and
$\mathbb{E}\left(\int_0^T [u(t)^2 + \pi_{B_1}^2(t) + \pi_S^2(t)]dt\right) < \infty$;
(3) $\forall (t, x, m, l) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$, Eq. (2.19) *admits a unique pathwise positive solution $X^{\widetilde{u}}(t) > 0, t \in [0, T]$.*

Assumption 2.5. *(1) A uniformly Lipschitz condition in* Π*: there exists* $K_1 \geq 0$*, such that for all* $x, x', m, m' \in \mathbb{R}$*,* $l, l' \in \mathbb{R}^+$ *and* $(u, \pi_{B_1}, \pi_S) \in \Pi$*,*

$$\begin{aligned}
&|\mu^d(t, x, m, l, u, \pi_{B_1}, \pi_S) - \mu^d(t, x, m', l', u, \pi_{B_1}, \pi_S)| \\
&+ |\sigma^{d_1}(t, l, u) - \sigma^{d_1}(t, l', u)| + |\sigma^{d_3}(t, l, \pi_S) - \sigma^{d_3}(t, l', \pi_S)| \\
&\leq K_1(|x - x'| + |m - m'| + |l - l'|).
\end{aligned} \tag{2.24}$$

(2) L^2*-integrability condition in* Π*: for all* $x, m \in \mathbb{R}$*,* $l \in \mathbb{R}^+$ *and* $(u, \pi_{B_1}, \pi_S) \in \Pi$*,*

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T \Big(|\mu^d(t, m, l, u(t), \pi_{B_1}(t), \pi_S(t))|^2 \\
&+ |\sigma^{d_1}(t, l, u(t)|^2 + |\sigma^{d_2}(t, \pi_{B_1}(t))|^2 + |\sigma^{d_3}(t, l, \pi_S(t))|^2\Big)dt\Big] < \infty.
\end{aligned} \tag{2.25}$$

From the standard stochastic control theory (see, for example [32]), conditions (2.24) and (2.25) ensure that for all $\widetilde{u} = (u, \pi_{B_1}, \pi_S) \in \Pi$, and for any initial condition $(t, x, m, l) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$, the SDE in (2.19) admits a strong unique solution. In this case, we also have

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |X^{\widetilde{u}}(t)|^2\right] < \infty. \tag{2.26}$$

2.4. The optimization problem

Our purpose is to explore how insurers can maximize the utility at the terminal time T through the asset allocation strategy in $[0, T]$. Considering the insurer's wealth, the objective of the insurer is

$$\begin{cases}
\sup_{\widetilde{u}} \mathbb{E}\left(U\left(X^{\widetilde{u}}(T)\right)\right), \\
s.t. \quad X^{\widetilde{u}}(t) \text{ satisfies Eq. 2.18}, \\
\qquad r_n(t) \text{ satisfies Eq. 2.15}, \\
\qquad M(t) \text{ satisfies Eq. 2.16}, \\
\qquad L(t) \text{ satisfies Eq. 2.17}.
\end{cases} \tag{2.27}$$

In this paper, the utility function is specified by the constant relative risk aversion (abbr. CRRA) case

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma}, \gamma > 0, \gamma \neq 1,$$

where the γ is relative risk aversion, which measures the insurers' sensitivity to risk.

3. Solution of the optimization problem

In order to solve Problem (2.27), we first introduce an auxiliary process. The original process (2.19) is not self-financing, i.e., there is a continuous outflow of money for the insurer. The outflow rate of the wealth is $\lambda\mu_1(\eta - \theta_r)$. On the other hand, the value of an asset $D(t, s)$ at time t with payment $\lambda\mu_1(\eta - \theta_r)$ at time s is

$$\begin{cases} \frac{dD(t,s)}{D(t,s)} = r_n(t)dt + \sigma_{B_1}(s-t)[\lambda_{r_n}dt + dW_{r_n}(t)], \\ D(s,s) = \lambda\mu_1(\eta - \theta_r). \end{cases}$$

Next we define the accumulated future outflow of money at time t by

$$F(t,T) = \int_t^T D(t,s)ds = \lambda\mu_1(\eta - \theta_r)\int_t^T B_n(t,s)ds, \ t \in [0,T].$$

A simple calculation shows that $F(t,T)$ satisfies the following BSDE:

$$\begin{cases} dF(t,T) & = -\lambda\mu_1(\eta-\theta_r)dt + F(t,T)\left[r_n(t) + \lambda_{r_n}\sigma_F(t,T)\right]dt \\ & \quad +F(t,T)\sigma_F(t,T)dW_{r_n}(t), \\ F(T,T) & = 0, \end{cases} \tag{3.28}$$

where $\sigma_F(t,T) = \int_t^T \dfrac{\lambda\mu_1(\eta-\theta_r)\sigma_{B_1}(s-t)B_n(t,s)}{F(t,T)}ds.$

3.1. *An auxiliary optimal control problem*

Define an auxiliary process $Y(t) = X(t) + F(t,T)$. We can derive the dynamics of the auxiliary process

$$dY^{\overline{u}}(t) = r_n(t)Y^{\overline{u}}(t)dt + \overline{u}(t)\sigma(L(t))(\Lambda(M(t),L(t))dt + dW(t)), \tag{3.29}$$

where

$$\overline{u}(t) = \widetilde{u}(t) + \left(0, \quad \frac{F(t,T)\sigma_F(t,T)}{\sigma_{B_1}(K)}, \quad 0\right)^\top.$$

Definition 3.1. *A strategy $\overline{u}(t)$ is said to be admissible if*
(1) $u(t) \geq 0, \forall t \in [0,T]$;
(2) $\overline{u}(t)$ is progressively measurable with respect to $\{\mathcal{G}_t\}_{t\in[0,T]}$ and $\mathbb{E}\left(\int_0^T \left[u(t)^2 + \pi_B(t)^2 + \pi_S^2(t)\right]dt\right) < \infty$, where $\pi_B(t) = \pi_{B_1}(t) + \dfrac{F(t,T)\sigma_F(t,T)}{\sigma_{B_1}(K)}$;
(3) $\forall (t,y,m,l) \in [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^+$, Eq. (3.29) admits a unique pathwise positive solution $Y^{\widetilde{u}}(t) > 0, t \in [0,T]$.

We denote

$$\begin{aligned}\mu^y(t,y,r_n,m,l,u,\pi_B,\pi_S) &\triangleq r_n y + \overline{u}(t)^\top \sigma(l)\Lambda(m,l),\\ \sigma^{y_1}(t,l,u) &\triangleq u\left(\sqrt{\lambda\mu_2} + \sqrt{l}\rho_0\right),\\ \sigma^{y_2}(t,\pi_B) &\triangleq \pi_B(\sigma_{B_1}(K) + \sigma_{I_1} + \sigma_{S_1}),\\ \sigma^{y_3}(t,l,\pi_S) &\triangleq \pi_S\left(\sqrt{l}\sqrt{1-\rho_0^2}\right).\end{aligned}$$

Combining with assumption 2.5, we provide another assumption

Assumption 3.2. *(1) A uniformly Lipschitz condition in $A^{\mathbb{G}}$: there exists $K_2 \geq 0$, such that for all $y, y', r_n, r_n', m, m' \in \mathbb{R}$, $l, l' \in \mathbb{R}^+$ and $(u, \pi_B, \pi_S) \in A^{\mathbb{G}}$,*

$$\begin{aligned}&|\mu^y(t,y,r_n,m,l,u,\pi_B,\pi_S) - \mu^y(t,y',r_n',m',l',u,\pi_B,\pi_S)|\\ &+ |\sigma^{y_1}(t,l,u) - \sigma^{y_1}(t,l',u)| + |\sigma^{y_3}(t,l,\pi_S) - \sigma^{y_3}(t,l',\pi_S)|\\ &\leq K_2(|y-y'| + |r_n - r_n'| + |m-m'| + |l-l'|).\end{aligned} \tag{3.30}$$

(2) L^2-integrability condition in $A^{\mathbb{G}}$: for all $y, r_n, m \in \mathbb{R}$, $l \in \mathbb{R}^+$ and $(u, \pi_B, \pi_S) \in A^{\mathbb{G}}$,

$$\begin{aligned}&\mathbb{E}\Big[\int_0^T \Big(|\mu^y(t,y,r_n,m,l,u(t),\pi_B(t),\pi_S(t))|^2\\ &+ |\sigma^{y_1}(t,l,u(t)|^2 + |\sigma^{y_2}(t,\pi_B(t))|^2 + |\sigma^{y_3}(t,l,\pi_S(t))|^2\Big)dt\Big] < \infty.\end{aligned} \tag{3.31}$$

From conditions (2.24) and (2.25), we can ensure that for all $\overline{u} = (u, \pi_B, \pi_S) \in A^{\mathbb{G}}$, and for any initial condition $(t, y, r_n, m, l) \in [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^+$, the SDE in (3.29) admits a strong unique solution. For any initial condition $(t, y, r_n, m, l) \in [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^+$ and all $\overline{u} \in A^{\mathbb{G}}$, (3.29) admits strong unique solutions and

$$\mathbb{E}\left[\sup_{t\leq s\leq T}\left|Y^{\overline{u}}(s)\right|^2 \middle| Y(t) = y, r_n(t) = r_n, M(t) = m, L(t) = l\right] < \infty. \tag{3.32}$$

Now we can transform the original problem (2.27) into the following auxiliary self-financing problem:

$$\begin{cases} \sup\limits_{\overline{u}\in A^{\mathbb{G}}} \mathbb{E}\left(U\left(Y^{\overline{u}}(T)\right)\right),\\ s.t. \quad Y^{\overline{u}}(t) \text{ satisfies Eq. 3.29},\\ \qquad\; r_n(t) \text{ satisfies Eq. 2.15},\\ \qquad\; M(t) \text{ satisfies Eq. 2.16},\\ \qquad\; L(t) \text{ satisfies Eq. 2.17}.\end{cases} \tag{3.33}$$

3.2. Verification Theorem

We apply the stochastic dynamic programming principle to solve the problem (3.33), and we need to have a verification theorem before giving the HJB equation. We focus the problem beginning at time t with the state y, r_n, m, l. We define the value function $V(t, y, r_n, m, l)$ as

$$\begin{aligned}
&V(t, y, r_n, m, l) \\
&= \sup_{\overline{u} \in A^{\mathbb{G}}} \mathbb{E}\left[U\left(Y^{\overline{u}}(T)\right) \Big| Y(t) = y, r_n(t) = r_n, M(t) = m, L(t) = l \right].
\end{aligned}$$

In addition, we define an operator as follows.

Definition 3.3. *Define the operator by*

$$\begin{aligned}
&\mathscr{L}^{\overline{u}} J(t, y, r_n, m, l) \\
&= (r_n y + \overline{u}^\top \sigma \Lambda) \times J_y(t, y, i, r_n, m, l) + a(b - r_n) \times J_{r_n}(t, y, r_n, m, l) \\
&+ (\alpha(\beta - l) + \sqrt{l} \sigma_L \rho_0 \rho_S (\delta_0 - m)) \times J_l(t, y, r_n, m, l) \\
&+ \frac{1}{2} \overline{u}^\top \sigma \sigma^\top \overline{u} \times J_{yy}(t, y, r_n, m, l) + \frac{1}{2} \sigma_r^\top \sigma_r \times J_{r_n r_n}(t, y, r_n, m, l) \\
&+ \frac{1}{2} \sigma_t^\top \sigma_t \times J_{mm}(t, y, r_n, m, l) \\
&+ \frac{1}{2} \sigma_L^2 l \sigma_{L_1}^\top \sigma_{L_1} \times J_{ll}(t, y, r_n, m, l) + \overline{u}^\top \sigma \sigma_r \times J_{y r_n}(t, y, r_n, m, l) \\
&+ \overline{u}^\top \sigma \sigma_t \times J_{ym}(t, y, r_n, m, l) \\
&+ \sqrt{l} \sigma_L \overline{u}^\top \sigma \sigma_{L_1} \times J_{yl}(t, y, r_n, m, l) \\
&+ \sqrt{l} \sigma_L \sigma_t^\top \sigma_{L_1} \times J_{ml}(t, y, r_n, m, l),
\end{aligned}$$

where

$$\begin{aligned}
&\sigma_r = \begin{pmatrix} 0, & -\sigma_{r_n}, & 0 \end{pmatrix}^\top, \sigma_t = \begin{pmatrix} \frac{1}{T_0 - t}, & 0, & 0 \end{pmatrix}^\top, \\
&\sigma_{L_1} = \begin{pmatrix} \rho_0 \rho_S, 0, & \sqrt{1 - \rho_0^2} \rho_S \end{pmatrix}^\top,
\end{aligned} \tag{3.34}$$

and $J(t, y, r_n, m, l) \in C^{1,2,2,2,2}$ represents the partial derivatives Ψ_y, Ψ_{r_n}, Ψ_l, Ψ_{yy}, $\Psi_{r_n r_n}$, Ψ_{mm}, Ψ_{ll}, $\Psi_{y r_n}$, Ψ_{ym}, Ψ_{yl}, Ψ_{ml} in the space of the value function $\Psi(t, y, r_n, m, l)$ on $[0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$ are continuous.

Using Definition 3.3, we can give a verification theorem as follows:

Theorem 3.4. *(Verification Theorem) Define* $\Omega \triangleq [0,T) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$ *and* $\overline{\Omega} \triangleq [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$. *Let* J *be a function in* $C^{1,2,2,2,2}(\Omega) \cap C^0(\overline{\Omega})$*, which satisfies a quadratic growth condition, i.e. there exists* $K_3 > 0$ *such that*

$$|J(t,y,r_n,m,l)| \leq K_3\left(1 + |y|^2 + |m|^2 + |l|^2\right), \tag{3.35}$$

for all $(t,y,r_n,m,l) \in \overline{\Omega}$.

Define $\overline{u}^* = \left(u^*, \pi_B^*, \pi_S^*\right)$*, we have*

$$\overline{u}^*(t) = \widetilde{u}^*(t) + \left(0, \quad \frac{F(t,T)\sigma_F(t,T)}{\sigma_{B_1}(K)}, \quad 0\right), \tag{3.36}$$

and

$$\begin{aligned} \overline{u}^*(t,y,r_n,m,l) = argmax_{\overline{u}}\Bigg\{ & \frac{\partial J}{\partial y}\overline{u}^\top \sigma\Lambda + \frac{\partial^2 J}{\partial y \partial r_n}\overline{u}^\top \sigma\sigma_r \\ + \frac{\partial^2 J}{\partial y \partial m}\overline{u}^\top\sigma\sigma_t + \frac{\partial^2 J}{\partial y \partial l} & \sqrt{l}\sigma_L \overline{u}^\top \sigma\sigma_{L_1} + \frac{1}{2}\frac{\partial^2 J}{\partial y^2}\overline{u}^\top\sigma\sigma^\top\overline{u}\Bigg\}. \end{aligned} \tag{3.37}$$

(i) Suppose that

$$-\frac{\partial J(t,y,r_n,m,l)}{\partial t} - \sup_{\overline{u}\in A^{\mathbb{G}}}\{\mathscr{L}^{\overline{u}} J(t,y,r_n,m,l)\} \quad \geq \quad 0, \tag{3.38}$$

$$J(T,y,r_n,m,l) \quad \geq \quad U(y), \tag{3.39}$$

for all $(t,y,r_n,m,l) \in \Omega$*, we have* $J \geq V$ *on* Ω.

(ii) The process $\overline{u}^* = (u^*, \pi_B^*, \pi_S^*)$ *lies in* $A^{\mathbb{G}}$*, and it follows that*

$$-\frac{\partial J(t,y,r_n,m,l)}{\partial t} - \mathscr{L}^{\overline{u}^*} J(t,y,r_n,m,l) = 0. \tag{3.40}$$

Then

$$J(t,y,r_n,m,l) = V(t,y,r_n,m,l) \quad on \quad \Omega. \tag{3.41}$$

Proof. See Appendix 8.1. □

Under the assumptions of Theorem 3.4, we can easily solve the value function

$$V(t,y,r_n,m,l) = \sup_{\overline{u}\in A^{\mathbb{G}}} \mathbb{E}\left[U\left(Y^{\overline{u}}(T)\right)\right].$$

In fact, it suffices to solve $J(t, y, r_n, m, l)$, which satisfies the Hamilton-Jacobi-Bellman equation (3.40). Moreover, Theorem 3.4 states that the optimal strategy can be expressed as (3.36) and (3.37) when $J(\cdot) = V(\cdot)$.

Furthermore, let the function $h(t, r_n, m, l)$ satisfy

$$V(t, y, r_n, m, l) = \frac{1}{1-\gamma} y^{1-\gamma} h(t, r_n, m, l), \quad \text{with} \quad h(T, r_n, m, l) = 1. \tag{3.42}$$

The optimal strategies for auxiliary problems will be presented in the following proposition.

Proposition 3.5. *The optimal reinsurance and investment strategy is*

$$\overline{u}^*(t) \;=\; (X^*(t) + F(t,T))\left(\frac{1}{\gamma}\begin{pmatrix} P_1(t) \\ P_2(t) \\ P_3(t) \end{pmatrix}\right), \tag{3.43}$$

where

$$\begin{aligned}
P_1(t) &= \frac{v\rho_0\sqrt{L(t)}}{\sqrt{\lambda\mu_2}(\rho_0^2-1)} + \frac{\delta_0 - M(t)}{\sqrt{\lambda\mu_2}} + \frac{\mu_1\theta_r}{\mu_2(1-\rho_0^2)} + \frac{1}{T_0 - t}\frac{1}{\sqrt{\lambda\mu_2}}\frac{\tilde{h}_m(t)}{\tilde{h}(t)}, \\
P_2(t) &= \frac{1}{\sigma_{B_1}}\left(\lambda_{r_n} + \frac{\sigma_{S_1} v}{\rho_0^2 - 1} - \frac{\rho_0\mu_1\sigma_{S_1}\theta_r\sqrt{\lambda}}{\sqrt{L(t)\mu_2}(\rho_0^2-1)} - \rho_s\sigma_L\sigma_{S_1}\frac{\tilde{h}_l(t)}{\tilde{h}(t)} - \sigma_{r_n}\frac{\tilde{h}_{r_n}(t)}{\tilde{h}(t)}\right), \\
P_3(t) &= -\frac{v}{\rho_0^2-1} + \frac{\mu_1\theta_r\rho_0\sqrt{\lambda}}{\sqrt{L(t)\mu_2}(\rho_0^2-1)} + \rho_s\sigma_L\frac{\tilde{h}_l(t)}{\tilde{h}(t)}.
\end{aligned}$$

Here, $\tilde{h}(t)$ is short for $h(t, r_n(t), M(t), L(t))$, which satisfies

$$\begin{aligned}
&\frac{\tilde{h}_t}{\tilde{h}} + \frac{\tilde{h}_{r_n}}{\tilde{h}}\left(a(b - r_n) - \left(1 - \frac{1}{\gamma}\right)\Lambda^\top\sigma_r\right) + \frac{\tilde{h}_m}{\tilde{h}}\frac{1-\gamma}{\gamma}\Lambda^\top\sigma_t \\
&+ \frac{\tilde{h}_l}{\tilde{h}}\left(\alpha(\beta - l) + \rho_0\rho_s\sigma_L\sqrt{l}(\delta_0 - m) + \frac{1-\gamma}{\gamma}\sigma_L\sqrt{l}\sigma_{L_1}^\top\Lambda\right) + \frac{1}{2}\frac{\tilde{h}_{r_n r_n}}{\tilde{h}}\sigma_r^\top\sigma_r \\
&+ \frac{1}{2}\frac{\tilde{h}_{mm}}{\tilde{h}}\sigma_t^\top\sigma_t + \frac{1}{2}\frac{\tilde{h}_{ll}}{\tilde{h}}\sigma_L^2 l + \frac{\tilde{h}_m\tilde{h}_l}{\tilde{h}^2}\frac{1-\gamma}{\gamma}\sigma_L\sqrt{l}\sigma_{L_1}^\top\sigma_t - \frac{\tilde{h}_{r_n}^2}{\tilde{h}^2}\frac{\gamma-1}{2\gamma}\sigma_r^\top\sigma_r \\
&- \frac{\tilde{h}_m^2}{\tilde{h}^2}\frac{\gamma-1}{2\gamma}\sigma_t^\top\sigma_t - \frac{\tilde{h}_l^2}{\tilde{h}^2}\frac{\gamma-1}{2\gamma}\sigma_L^2 l\sigma_{L_1}^\top\sigma_{L_1} + \frac{1-\gamma}{2\gamma}\Lambda^\top\Lambda \\
&+ \frac{\tilde{h}_{ml}}{\tilde{h}}\sigma_L\sqrt{l}\sigma_t^\top\sigma_{L_1} + (1-\gamma)r_n = 0,
\end{aligned} \tag{3.44}$$

in which

$$\tilde{h}_t = \frac{\partial \tilde{h}}{\partial t}, \tilde{h}_{r_n} = \frac{\partial \tilde{h}}{\partial r_n}, h_m = \frac{\partial \tilde{h}}{\partial m}, \tilde{h}_l = \frac{\partial \tilde{h}}{\partial l},$$

$$\tilde{h}_{r_n r_n} = \frac{\partial^2 \tilde{h}}{\partial r_n^2}, \tilde{h}_{mm} = \frac{\partial^2 \tilde{h}}{\partial m^2}, \tilde{h}_{ll} = \frac{\partial^2 \tilde{h}}{\partial l^2}, \tilde{h}_{ml} = \frac{\partial^2 \tilde{h}}{\partial m \partial l}.$$

In addition, $\tilde{h}_{r_n}(t)$, $\tilde{h}_m(t)$ *and* $\tilde{h}_l(t)$ *are short for* $h_{r_n}(t, r_n(t), M(t), L(t))$, $h_m(t, r_n(t), M(t), L(t))$ *and* $h_l(t, r_n(t), M(t), L(t))$*, respectively.*

Proof. See Appendix 8.2. □

According to Proposition 3.5, in order to determine the optimal strategy for the auxiliary problem, it is crucial to employ a suitable approach to solve for (3.44), which will be the main focus of our investigation. The solution to the original problem can be obtained by subtracting the additional zero coupon bonds, as described below:

$$\widetilde{u}^*(t) = \overline{u}^*(t) - \left(0, \quad \frac{F(t,T)\sigma_F(t,T)}{\sigma_{B_1}(K)}, \quad 0\right)^\top.$$

4. Problem solving by deep learning

Due to the difficulty in obtaining the analytical solution directly, a numerical method is necessary to solve the problem represented by (3.44). However, this equation is a four-dimensional nonlinear parabolic partial differential equation (PDE), which is prone to the curse of dimensionality when solved using traditional grid-based methods. Additionally, the presence of nonlinear terms further complicates the problem of finding a solution.

To circumvent these challenges, we will employ a deep learning method to solve this PDE. Specifically, we will utilize a deep learning technique called deep 2BSDE. This approach establishes a unified formulation that combines the PDE and the two-dimensional backward stochastic differential equation (2BSDE) based on their underlying connection. Subsequently, we approximate the solution through temporal forward discretization and spatial approximation using deep neural networks. According to [29], under certain smoothness and regularity conditions, the 2BSDE has at most one solution. This provides a stochastic representation for solutions of fully nonlinear parabolic PDEs. For a more comprehensive understanding of the theoretical foundations of this method, we refer readers to Theorem 1 and Theorem 2 in [33].

4.1. 2BSDE system

In this section, we will represent the nonlinear parabolic PDE into second-order forward backward stochastic differential equations.

The deep 2BSDE method establishes a connection between fully nonlinear second-order partial differential equations (PDEs) and second-order backward stochastic differential equations (2BSDEs). The relationship is demonstrated in Theorem 4.10 of [29] and Lemma 3.1 of [30].

Lemma 4.1. *Let $d \in \mathbb{N}$, $\mathcal{T} \in (0, \infty)$, let $\mathbf{u} = (\mathbf{u}(t, \mathbf{x}))_{t\in[0,\mathcal{T}],\mathbf{x}\in\mathbb{R}^d} \in C^{1,2}([0, \mathcal{T}] \times \mathbb{R}^d, \mathbb{R})$, $\mu_e \in C(\mathbb{R}^d \times \mathbb{R}^d)$, $\sigma_e \in C(\mathbb{R}^d, \mathbb{R}^{d\times d})$, $\mathbf{f} : [0, \mathcal{T}] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R}$, and $\mathbf{g} : \mathbb{R}^d \to \mathbb{R}$ be functions which satisfy for all $t \in [0, \mathcal{T})$, $\mathbf{x} \in \mathbb{R}^d$ that $\nabla_{\mathbf{x}}\mathbf{u} \in C^{1,2}([0, \mathcal{T}] \times \mathbb{R}^d, \mathbb{R}^d)$, $\mathbf{u}(\mathcal{T}, \mathbf{x}) = \mathbf{g}(x)$, and*

$$\frac{\partial \mathbf{u}}{\partial t}(t, \mathbf{x}) = \mathbf{f}(t, \mathbf{x}, \mathbf{u}(t, \mathbf{x}), (\nabla_{\mathbf{x}}\mathbf{u})(t, \mathbf{x}), (\Delta_{\mathbf{x}}\mathbf{u})(t, \mathbf{x})),$$

where $\nabla_{\mathbf{x}}$ and $\Delta_{\mathbf{x}}$ are operators of the first-order partial derivation and second-order partial derivation of $\mathbf{x}$, respectively. Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space, let$\mathbf{W} = (W^{(1)}, ..., W^{(d)}) : [0, \mathcal{T}] \times \Omega \to \mathbb{R}^d$ be a standard Brownian motion on $(\Omega, \mathbb{F}, \mathbb{P})$, let $\{\mathbb{F}_t\}_{t\in[0,\mathcal{T}]}$ be the normal filtration on $(\Omega, \mathbb{F}, \mathbb{P})$ generated by $\mathbf{W}$, let $\xi : \Omega \to \mathbb{R}^d$ be an $\mathcal{F}_0/\mathcal{B}(\mathbb{R}^d)$-measurable function, let $\mathbf{X} = (\mathbf{X}^{(1)}, ..., \mathbf{X}^{(d)}) : [0, \mathcal{T}] \times \Omega \to \mathbb{R}^d$ be an $\mathbb{F}$-adapted stochastic processes with continuous sample paths which satisfies that for all $t \in [0, \mathcal{T}]$ it holds $\mathbb{P}$-a.s. that

$$\mathbf{X}_t = \xi + \int_0^t \mu_e(\mathbf{X}_s)ds + \int_0^t \sigma_e(\mathbf{X}_s)d\mathbf{W}_s,$$

for every $\psi \in C^{1,2}([0, \mathcal{T}] \times \mathbb{R}^d, \mathbb{R})$, let $\mathcal{A}\psi : [0, \mathcal{T}] \times \mathbb{R}^d \to \mathbb{R}$ be the function which satisfies for all $(t, \mathbf{x}) \in [0, \mathcal{T}] \times \mathbb{R}^d$ that

$$(\mathcal{A}\psi)(t, \mathbf{x}) = \left(\frac{\partial \psi}{\partial t}\right)(t, \mathbf{x}) + \frac{1}{2} Trace(\sigma_e \sigma_e^\top (\Delta_{\mathbf{x}}\psi)(t, \mathbf{x})),$$

and let $\mathbf{Y} : [0, \mathcal{T}] \times \Omega \to \mathbb{R}$, $\mathbf{Z} = (Z^{(1)}, ..., Z^{(d)}) : [0, \mathcal{T}] \times \Omega \to \mathbb{R}^d$, $\Gamma = (\Gamma(i, j))_{(i,j)\in\{1,...,d\}^2} : [0, \mathcal{T}] \times \Omega \to \mathbb{R}^{d\times d}$, and $\mathbf{A} = (A^{(1)}, ..., A^{(d)}) : [0, \mathcal{T}] \times \Omega \to \mathbb{R}^d$ be the stochastic processes which satisfy for all $t \in [0, \mathcal{T}], i \in \{1, 2, ..., d\}$ that

$$\begin{aligned} \mathbf{Y}_t &= \mathbf{u}(t, \mathbf{X}_t), & \mathbf{Z}_t &= (\nabla_{\mathbf{x}}\mathbf{u})(t, \mathbf{X}_t), \\ \Gamma_t &= (\Delta_{\mathbf{x}}\mathbf{u})(t, \mathbf{X}_t), & \mathbf{A}_t^{(i)} &= \left(\mathcal{L}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}_i}\right)\right)(t, \mathbf{X}_t). \end{aligned}$$

Then, it holds that $\mathbf{Y}$*,* $\mathbf{Z}$*,* Γ*, and* $\mathbf{A}$ *are* $\mathbb{F}$*-adapted stochastic processes with continuous sample paths which satisfy that for all* $t \in [0, \mathcal{T}]$ *it holds* $\mathbb{P}$*-a.s. that*

$$\begin{aligned}\mathbf{Y}_t = \mathbf{g}(\mathbf{X}_{\mathcal{T}}) - \int_t^{\mathcal{T}} (\mathbf{f}(s, \mathbf{X}_s, \mathbf{Y}_s, \mathbf{Z}_s, \Gamma_s) \\ + \frac{1}{2} Trace(\sigma(\mathbf{X}_s)\sigma(\mathbf{X}_s)^\top \Gamma_s))ds - \int_t^{\mathcal{T}} \langle \mathbf{Z}_s, d\mathbf{X}_s \rangle_{\mathbb{R}^d}\end{aligned}$$

and

$$\mathbf{Z}_t = \mathbf{Z}_0 + \int_0^t \mathbf{A}_s ds + \int_0^t \Gamma_s d\mathbf{X}_s.$$

Proof. Please see the proof of Lemma 3.1 in [30] for details. □

Let $\mathbf{X} = (r_n, m, l)$. $\nabla_{\mathbf{X}}$ and $\Delta_{\mathbf{X}}$ are operators of the first-order partial derivation and second-order partial derivation of $\mathbf{X}$, respectively. Define $H(t, \mathbf{X}) = \ln[h(t, \mathbf{X})]$. Then, equation (3.44) can be rewritten as

$$H_t(t, \mathbf{X}) = G(t, \mathbf{X}, H(t, \mathbf{X}), \nabla_{\mathbf{X}} H(t, \mathbf{X}), \Delta_{\mathbf{X}} H(t, \mathbf{X})), \tag{4.45}$$

where

$$\begin{aligned}&G(t, \mathbf{X}, H(t, \mathbf{X}), \nabla_{\mathbf{X}} H(t, \mathbf{X}), \Delta_{\mathbf{X}} H(t, \mathbf{X})) = \\ &- H_{r_n}\left(a(b - r_n) - \left(1 - \frac{1}{\gamma}\right)\Lambda^\top \sigma_r\right) - H_m \frac{1-\gamma}{\gamma}\Lambda^\top \sigma_t \\ &- H_l\left(\alpha(\beta - l) + \rho_0\rho_S\sigma_L\sqrt{l}(\delta_0 - m) + \frac{1-\gamma}{\gamma}\sigma_L\sqrt{l}\sigma_{L_1}^\top \Lambda\right) \\ &- \frac{1}{2}(H_{r_n r_n} + H_{r_n}^2)\sigma_r^\top \sigma_r - \frac{1}{2}(H_{mm} + H_m^2)\sigma_t^\top \sigma_t - \frac{1}{2}(H_{ll} + H_l^2)\sigma_L^2 l \sigma_{L_1}^\top \sigma_{L_1} \\ &- H_m H_l \frac{1-\gamma}{\gamma}\sigma_L\sqrt{l}\sigma_{L_1}^\top \sigma_t + H_{r_n}^2 \frac{\gamma - 1}{2\gamma}\sigma_r^\top \sigma_r + H_m^2 \frac{\gamma-1}{2\gamma}\sigma_t^\top \sigma_t \\ &+ H_l^2 \frac{\gamma - 1}{2\gamma}\sigma_L^2 l \sigma_{L_1}^\top \sigma_{L_1} - (H_m H_l + H_{ml})\sigma_L \sqrt{l}\sigma_{L_1}^\top \sigma_t - \frac{1-\gamma}{2\gamma}\Lambda^\top \Lambda \\ &- (1-\gamma) r_n.\end{aligned} \tag{4.46}$$

The terminal condition is $H(T, \mathbf{X}) = 0$.

Assumption 4.2. *In this paper, we assume that* $H, \nabla_{\mathbf{X}} H$ *belongs to* $C^{1,2,2,2}([0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+)$.

For all $t \in [0, T]$, let the stochastic processes $W_t^D = (W_{r_n}(t), W_0(t), W_S(t), W_L(t))$ on $(\Omega, \mathbb{G}, \mathbb{P})$, and $\mathbf{X}_t = (r_n(t), M(t), L(t))$ satisfying that

$$\mathbf{X}_t = \xi + \int_0^t \mu^D(\mathbf{X}_s)ds + \int_0^t \sigma^D(\mathbf{X}_s)dW_s^D,$$

where $\mathbf{X}_0 = \xi$,

$$\mu^D(\mathbf{X}_s) \triangleq \begin{pmatrix} a(b - r_n(s)) \\ 0 \\ \alpha(\beta - L(s)) + \sigma_L\rho_0\rho_S \sqrt{L(s)}(\delta_0 - M(s)) \end{pmatrix}, \tag{4.47}$$

and

$$\sigma^D(\mathbf{X}_s) \triangleq \begin{pmatrix} -\sigma_{r_n} & 0 & 0 & 0 \\ 0 & \dfrac{1}{T_0 - t} & 0 & 0 \\ 0 & \rho_S\rho_0\sigma_L \sqrt{L(s)} & \rho_S\sigma_L \sqrt{(1-\rho_0^2)L(s)} & \sigma_L \sqrt{(1-\rho_S^2)L(s)} \end{pmatrix} \tag{4.48}$$

Let $e_1^{(3)} = (1, 0, 0)$, $e_2^{(3)} = (0, 1, 0)$, $e_3^{(3)} = (0, 0, 1) \in \mathbb{R}^3$ be the standard basis vectors of $\mathbb{R}^3$, for every $\phi \in C^{1,2}([0, T] \times \mathbb{R}^3, \mathbb{R}^3)$ let $\mathcal{A}\phi : [0, T] \times \mathbb{R}^3 \to \mathbb{R}^3$ be the function which satisfies for all $(t, \mathbf{x}) \in [0, T] \times \mathbb{R}^3$ that

$$(\mathcal{A}\phi)(t, \mathbf{x}) = \frac{\partial\phi}{\partial t} + \frac{1}{2}\sum_{i=1}^{3}\left(\frac{\partial^2\phi}{\partial\mathbf{x}^2}\right)(t, \mathbf{x})\left(\sigma^D e_i^{(3)}, \sigma^D e_i^{(3)}\right),$$

and let $\mathscr{Y} : [0, T] \times \Omega \to \mathbb{R}$, $\mathscr{Z} : [0, T] \times \Omega \to \mathbb{R}^3$, $\Gamma : [0, T] \times \Omega \to \mathbb{R}^{3\times 3}$, $\mathscr{A} : [0, T] \times \Omega \to \mathbb{R}^3$ be the stochastic processes which satisfy for all $t \in [0, T]$ that

$$\mathscr{Y}_t \triangleq H(t, \mathbf{X}_t), \quad \mathscr{Z}_t \triangleq (\nabla_{\mathbf{X}} H)(t, \mathbf{X}_t), \tag{4.49}$$

$$\mathscr{G}_t \triangleq (\Delta_{\mathbf{X}} H)(t, \mathbf{X}_t), \tag{4.50}$$

$$\mathscr{A}_t \triangleq (\mathcal{A}(\nabla_{\mathbf{x}} H))(t, \mathbf{X}_t). \tag{4.51}$$

Lemma 4.1 implies that for all $\tau_1, \tau_2 \in [0, T]$ with $\tau_1 < \tau_2$, it holds $\mathbb{P}$-a.s. that

$$\mathbf{X}_{\tau_2} = \mathbf{X}_{\tau_1} + \int_{\tau_1}^{\tau_2} \mu^D(\mathbf{X}_s)ds + \int_{\tau_1}^{\tau_2} \sigma^D(\mathbf{X}_s)dW_s^D, \tag{4.52}$$

$$\mathscr{Y}_{\tau_2} = \mathscr{Y}_{\tau_1} + \int_{\tau_1}^{\tau_2} \left(G(s, \mathbf{X}_s, \mathscr{Y}_s, \mathscr{Z}_s, \mathscr{G}_s) + \frac{1}{2} Trace(\sigma^D(\mathbf{X}_s)^\top \sigma^D(\mathbf{X}_s) \mathscr{G}_s) \right) ds \\ + \int_{\tau_1}^{\tau_2} \langle \mathscr{Z}_s, d\mathbf{X}_s \rangle, \tag{4.53}$$

and

$$\mathscr{Z}_{\tau_2} = \mathscr{Z}_{\tau_1} + \int_{\tau_1}^{\tau_2} \mathscr{A}_s ds + \int_{\tau_1}^{\tau_2} \mathscr{G}_s dW_s^D. \tag{4.54}$$

The PDE (4.46)is related to the 2BSDE system (4.52)-(4.54).

We define the forward discretization of the 2BSDE system. Let $t_0, t_1, ..., t_N \in [0, T]$ be real numbers with

$$0 = t_0 < t_1 < ... < t_N = T$$

which makes the mesh size is sufficiently small. Here, the paths of $\mathbf{X}$, $\mathscr{Y}$, $\mathscr{Z}$ for all $n \in \{0, 1, ..., N-1\}$ are as follows:

$$\mathbf{X}_{t_0} = \mathbf{X}_0 = \xi, \quad \mathscr{Y}_{t_0} = \mathscr{Y}_0 = H(0, \xi), \quad \mathscr{Z}_{t_0} = \mathscr{Z}_0 = (\nabla_{\mathbf{X}} H)(0, \xi),$$

$$\mathbf{X}_{t_{n+1}} \approx \mathbf{X}_{t_n} + \mu^D(\mathbf{X}_{t_n})(t_{n+1} - t_n) + \sigma^D(\mathbf{X}_{t_n})(W_{t_{n+1}}^D - W_{t_n}^D),$$

$$\mathscr{Y}_{n+1} \approx \mathscr{Y}_{t_n} + \left(G(t_n, \mathbf{X}_{t_n}, \mathscr{Y}_{t_n}, \mathscr{Z}_{t_n}, \mathscr{G}_{t_n}) + \frac{1}{2} Trace(\sigma^D(\mathbf{X}_{t_n})^\top \sigma^D(\mathbf{X}_{t_n}) \mathscr{G}_{t_n}) \right) (t_{n+1} - t_n) \\ + \langle \mathscr{Z}_{t_n}, \mathbf{X}_{t_{n+1}} - \mathbf{X}_{t_n} \rangle,$$

and

$$\mathscr{Z}_{t_{n+1}} \approx \mathscr{Z}_{t_n} + \mathscr{A}_{t_n}(t_{n+1} - t_n) + \mathscr{G}_{t_n}(\mathbf{X}_{t_{n+1}} - \mathbf{X}_{t_n}).$$

4.2. Deep Learning Approximations

Now, we will use a deep neural network to approximate the unknown function of the 2BSDE system. We consider

$$\mathfrak{G}_n^\theta(\mathbf{x}) \approx \mathscr{G}(t_n, \mathbf{x})$$

and

$$\mathfrak{A}_n^\theta(\mathbf{x}) \approx \mathscr{A}(t_n, \mathbf{x})$$

as an approximation of $\mathscr{G}$ and $\mathscr{A}$ for all suitable θ, $\mathbf{x}$ and all $n \in \{0, 1, ..., N-1\}$, where θ are the parameters of the deep neural network. We denote the approximation 2BSDE system as

$$\begin{aligned}\mathfrak{Y}_{n+1}^{\theta} &= \mathfrak{Y}_{n}^{\theta} + \left(G(t_n, \mathbf{X}_n, \mathfrak{Y}_n^{\theta}, \mathfrak{Z}_n^{\theta}, \mathfrak{G}_n^{\theta}(\mathbf{X}_n)) + \frac{1}{2} Trace(\sigma^{D}(\mathbf{X}_n)^{\top} \sigma^{D}(\mathbf{X}_n) \mathfrak{G}_n^{\theta}(\mathbf{X}_n)) \right) (t_{n+1} - t_n) \\ &\quad + \langle \mathfrak{Z}_n^{\theta}, \mathbf{X}_{n+1} - \mathbf{X}_n \rangle,\end{aligned} \tag{4.55}$$

and

$$\mathfrak{Z}_{n+1}^{\theta} = \mathfrak{Z}_n^{\theta} + \mathfrak{A}_n^{\theta}(\mathbf{X}_n)(t_{n+1} - t_n) + \mathfrak{G}_{t_n}^{\theta}(\mathbf{X}_n)(\mathbf{X}_{n+1} - \mathbf{X}_n). \tag{4.56}$$

Meanwhile, we aim to ensure that the initial state ξ in the 2BSDE system can attain $H(T, \mathbf{X}_T) = 0$ at the terminal time T. To achieve this, the loss function is defined as

$$\mathbb{E}\left[|\mathfrak{Y}_N^{\theta} - H(t_N, \mathbf{X}_{t_N})|^2 \right] = \mathbb{E}\left[|\mathfrak{Y}_N^{\theta}|^2 \right]. \tag{4.57}$$

By combining the Monte Carlo and stochastic gradient descent algorithms, we can iteratively update the parameter θ until convergence. This allows us to obtain the approximate relationships between $\mathscr{G}_t$ and $\mathbf{X}_t$, as well as between $\mathscr{A}_t$ and $\mathbf{X}_t$. Subsequently, we can calculate the approximate solution of (4.46) on the interval $[0, T]$ using (4.55), with the initial condition $\mathbf{X}_0 = \xi$ and the terminal condition $H(T, \mathbf{X}_T) = 0$.

After training the neural network, we can use the input $\mathbf{X}_t = (r_n(t), M(t), L(t))$ and the network $\mathfrak{Z}^{\theta}$ to obtain the partial derivative values of the function H with respect to $\mathbf{x}$ at time t, denoted as $\nabla_{\mathbf{x}} H(t, \mathbf{x}_t)$. Then, according to Proposition 3.5, we can obtain the optimal reinsurance and investment strategy at time t.

The procedure of deep 2BSDE is illustrated in Figure 1. For brevity, we omit unnecessary details in this description and refer the readers to [30, 34, 35] for more comprehensive explanations. In the figure, the solid line represents the forward calculation process. Initially, we initialize $\mathfrak{Z}_0^{\theta}$, $\mathfrak{Y}_0^{\theta}$, $\mathfrak{A}^{\theta}$, and $\mathfrak{G}^{\theta}$. By applying equations (4.55) and (4.56), we calculate $\mathfrak{Y}^{\theta}$ from $\mathbf{X}_0$ until $\mathfrak{Y}_N^{\theta}$, as well as the corresponding loss based on (4.57). In the figure, the long dotted line represents the parameter updating process of the neural network. After obtaining the loss function, we update the neural networks $\mathfrak{G}^{\theta}$ and $\mathfrak{A}^{\theta}$ using the stochastic gradient descent method. Simultaneously, we update the initial values $\mathfrak{Z}_0^{\theta}$ and $\mathfrak{Y}_0^{\theta}$. The short dotted line indicates the recursive calculation process of $\mathbf{X}^\theta$, $\mathfrak{Y}^\theta$, and $\mathfrak{Z}^\theta$, which we omit in this description.

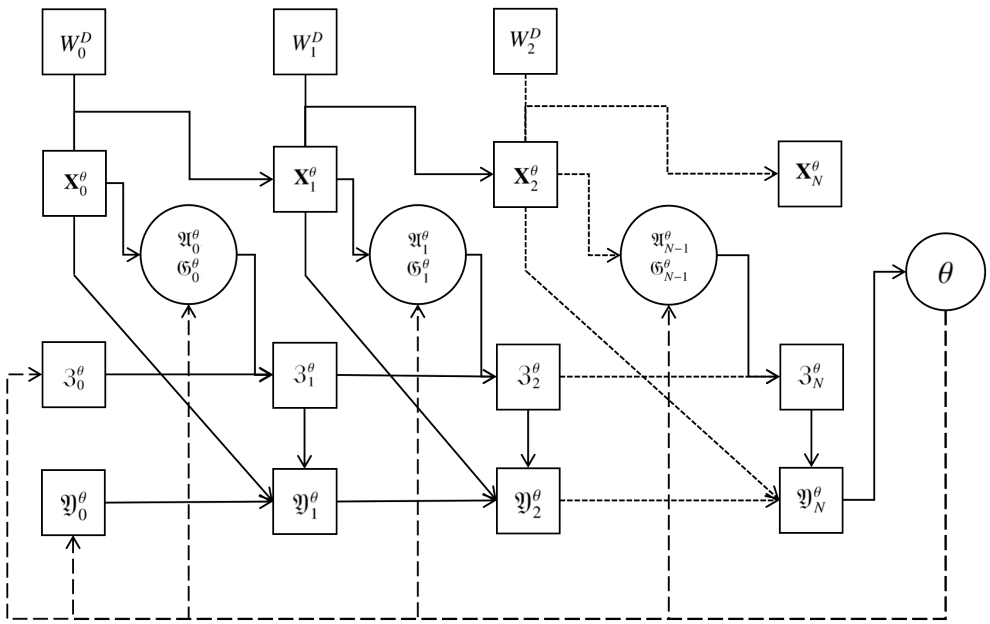


Figure 1: The algorithm procedure of deep 2BSDE

We utilize a 4-layer neural network architecture with two hidden layers for our analysis, where the number of nodes in each layer corresponds to the dimension of (4.46). Specifically, the output layer, hidden layer I, hidden layer II, and output layer comprise 4, 4, and 16 nodes, respectively. All layers, except for the output layer, employ the Rectified Linear Unit (ReLU) activation function, while the output layer does not utilize any activation function. Our training process adopts a batch size of 64 and employs the Adam optimizer for optimization. We conduct 2000 training iterations. Furthermore, we implement the code using Python3 programming language and the TensorFlow package. The computations are performed on an Intel Core i7-8700K CPU with a clock speed of 3.70GHz and 16GB RAM.

5. Sensitivity analysis

In this section, we further investigate the impact of different factors on the approximate optimal strategy and the approximate optimal value function through

Table 1: Notation summary.

T	r_0	l_0	λ	η	θ	a	b	σ_{r_n}	μ_1
5	0.03	0.4	3	0.05	0.1	0.1	0.02	0.02	0.08
μ_2	K	σ_{S_1}	λ_{r_n}	γ	δ_0	T_0	α	β	σ_L
0.05	10	0.1	0.2	2	0.1	20	0.1	0.2	0.2
v	ρ_0	ρ_S							
0.5	-0.2	0.5							

sensitivity analysis. The values of the model parameters are provided in Table 1, unless stated otherwise. The training process and the absolute error are depicted in Figure 2 and Figure 3, respectively, under the specified parameter values. Figure 2 represents the absolute error obtained by recalculating the termination condition based on our trained neural network. On the other hand, Figure 3 illustrates the L^2 error of termination condition, which corresponds to the loss function (4.57) computed using the trained neural network.

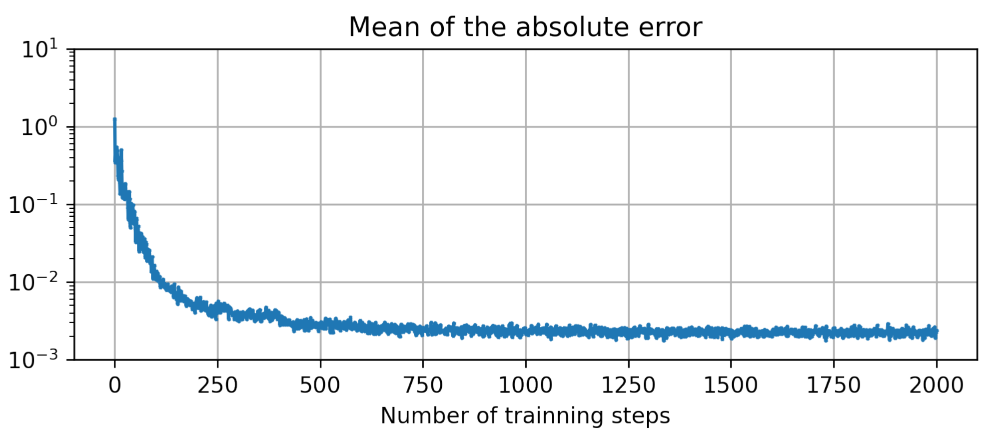


Figure 2: The absolute error of $H(T, \mathbf{X}(T))$

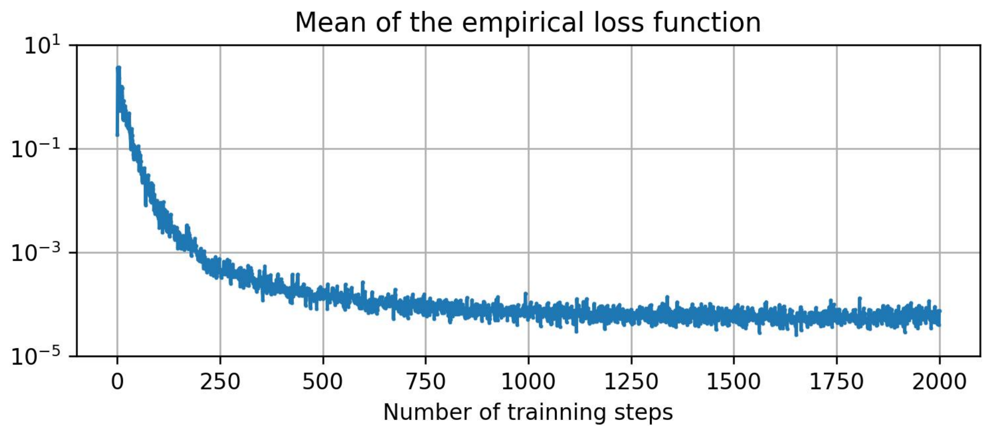


Figure 3: The empirical loss

5.1. Impact on reinsurance and investment strategy

The impact of the risk aversion coefficient γ on reinsurance and investment strategies is illustrated in Figure 4 and Figure 5. When the risk aversion coefficient increases, insurers are inclined to retain fewer insurance policies and purchase more reinsurance. This is because the risk aversion coefficient measures an individual's aversion to facing risks. Higher risk aversion coefficient signifies a greater concern for potential losses, leading insurer to purchase more reinsurance to mitigate the potential losses associated with insurance risk. In line with the increasing risk aversion coefficient, insurers tend to reduce their investments in zero-coupon bonds. In our opinion, this phenomenon can be attributed to insurers' heightened sensitivity to long-term interest rate fluctuations, leading to increased concerns regarding market uncertainty and volatility. Consequently, there exists an inverse relationship between investments in zero-coupon bonds and the risk aversion coefficient. Stock investments are generally perceived as high-risk, high-reward investments. As a result, insurers tend to have a heightened level of interest and concern regarding the risks and volatility associated with stocks. Consequently, the volatility in stock prices is more likely to trigger insurers' concerns, leading to a tendency to decrease the allocation of stocks in their investment portfolios to reduce the overall risk level, especially when the risk aversion coefficient is high. These finding is consistent with the results in proposition 3.5.

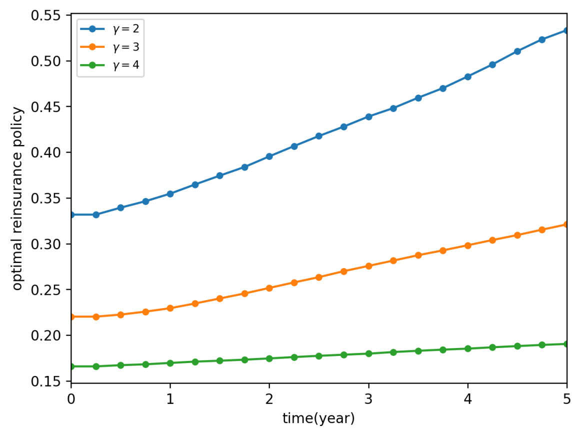
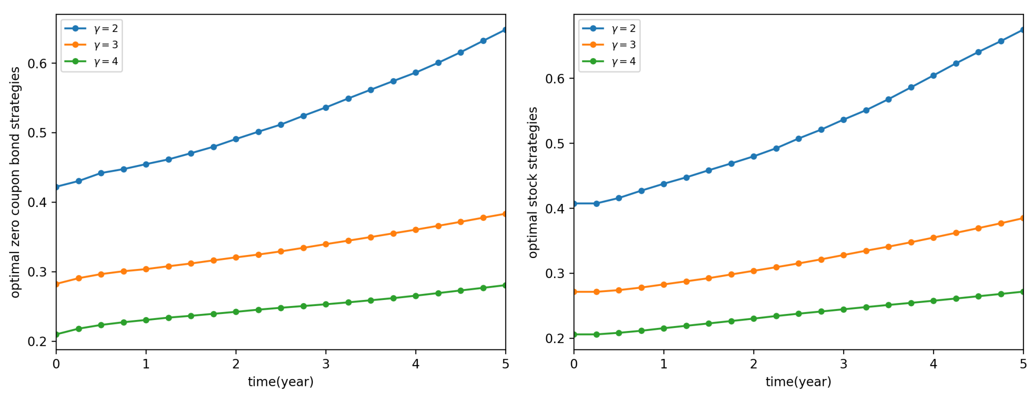


Figure 4: Effect of γ on reinsurance policy



(a) Effect of γ on zero-coupon bond strategies

(b) Effect of γ on stock strategies

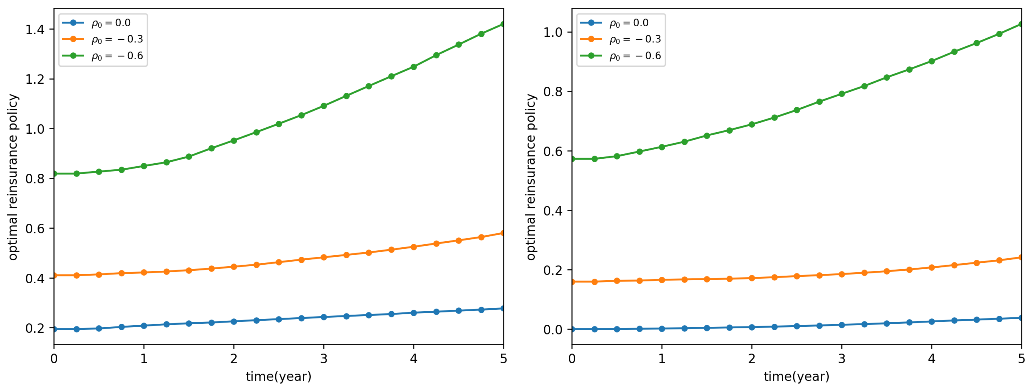
Figure 5: Effect of γ on investment strategies

Figure 6 - Figure 8 depict the influence of the negative correlation coefficient ρ_0 on reinsurance and investment strategies. With the increase of negative correlation between insurance risk and stock risk, insurers tend to hold more insurance policy. Simultaneously, a reduced negative correlation is also linked to an increase in stock investments by insurers. We offer the following explanation: a larger negative correlation coefficient denotes a more pronounced inverse association between insurance risk and stock risk. Furthermore, insurers can mitigate insurance risk through the acquisition of reinsurance, which allows the transfer of risks to reinsurers. The cumulative effect of these factors leads to a further

reduction in insurance risk. As a consequence, this manifests in a lower reinsurance policy and a higher allocation of investments towards stocks. Figure 6b and Figure 8b depict the influence of the negative correlation coefficient ρ_0 on reinsurance and investment strategies when extra information is negative ($\delta_0 = -0.1$). These findings indicate that when there is higher-than-expected insurance risk in the future, an increase in the negative correlation leads to an increased demand for reinsurance by insurers. However, the magnitude of the increase in stock investment is smaller than expected. In our opinion, this can be attributed to the transfer of insurance risk through the reinsurance policy, for which potentially diverts the profits insurers could have gained from stocks when there is a negative correlation between insurance risk and stock risk. As a result, the increase in stock investment is not significant.

From Figure 7a and Figure 7b, we can observe that the investment in zero-coupon bonds decreases as the negative correlation coefficient increases. A stronger negative correlation indicates that insurers can balance stock risk and insurance risk more effectively through portfolio and reinsurance policy, leading to more aggressive strategies. Conversely, if insurance risk and stock risk cannot be easily diversified, insurers tend to choose conservative investments in zero-coupon bonds.

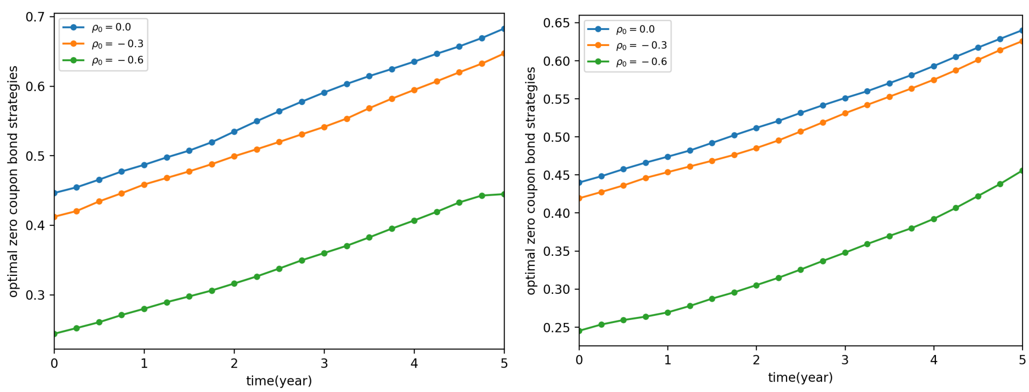
Based on these findings, it is observed that if insurers can ascertain certain extra information regarding the future, they tend to opt for more aggressive reinsurance and investment strategies, regardless of whether the extra information is positive or negative. This also partly elucidates the adoption of aggressive strategies by AIG insurance company prior to the 2008 financial crisis. However, it is only when insurers have the ability to ascertain the authenticity of extra information regarding future claim processes that they can benefit from aggressive reinsurance and investment strategies.



(a) Effect of ρ_0 on reinsurance policy ($\delta_0 = 0.1$)

(b) Effect of ρ_0 on reinsurance policy ($\delta_0 = -0.1$)

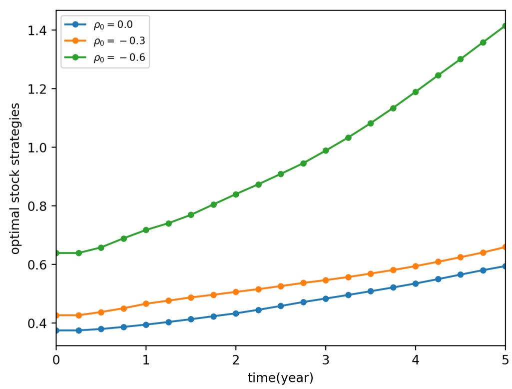
Figure 6: Effect of ρ_0 on reinsurance policy



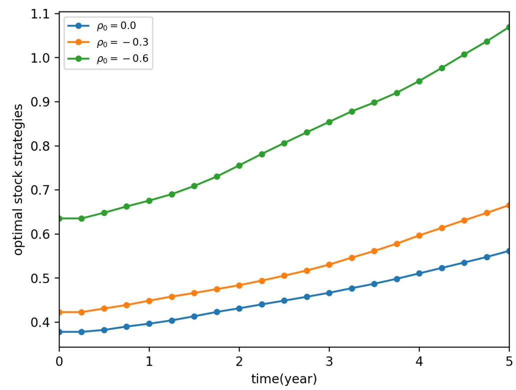
(a) Effect of ρ_0 on zero-coupon bond strategies ($\delta_0 = 0.1$)

(b) Effect of ρ_0 on zero-coupon bond strategies ($\delta_0 = -0.1$)

Figure 7: Effect of ρ_0 on zero-coupon bond strategies



(a) Effect of ρ_0 on stock strategies ($\delta_0 = 0.1$)



(b) Effect of ρ_0 on stock strategies ($\delta_0 = -0.1$)

Figure 8: Effect of ρ_0 on stock strategies

Figure 9 and Figure 10 demonstrate the impact of extra information on reinsurance policies and stock strategies. The demand for reinsurance decreases as the magnitude of extra information increases. This is because higher levels of extra information indicate lower expected claims, prompting insurers to retain more policies when extra information is higher. This result aligns with our expectations. Conversely, stock investments only increase when the absolute value of extra information is higher. We believe this may be due to the negative correlation between insurance risk and stock risk. Insurers can balance insurance risk and stock risk through reinsurance policies. As a result, stock strategies are only influenced by the magnitude of extra information, regardless of its direction.

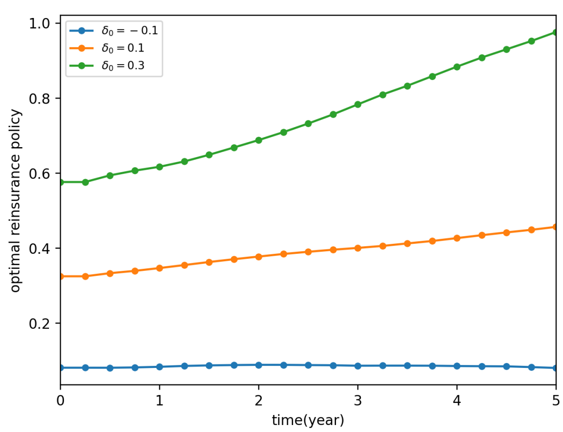


Figure 9: Effect of δ_0 on reinsurance policy

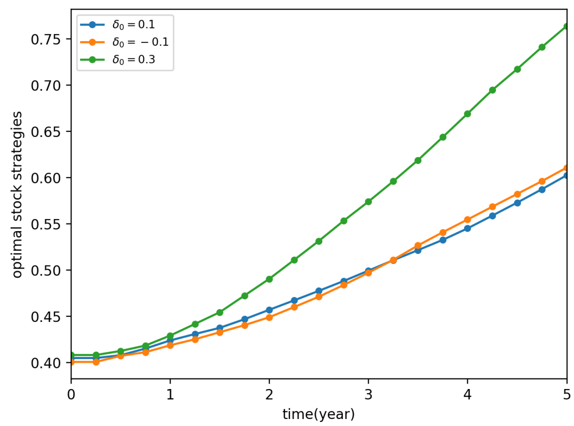


Figure 10: Effect of δ_0 on stock strategies

Figure 11-Figure 12 exhibit the influence of the volatility coefficient v on reinsurance and investment strategies. It can be observed that when the volatility coefficient is higher, insurers tend to retain more policies while increasing investments in stocks. This can be attributed to the presence of negative correlation and positive extra information, which enable insurers to have more accurate predictions regarding the insurance process and future stock dynamics. The availability of extra information allows insurers to anticipate risk in advance. The negative correlation, on the other hand, leads the anticipated extra information to propagate into both stock dynamics and volatility dynamics, resulting in a deterministic drift in stock dynamics that is positively correlated with the volatility coefficient. In this scenario, reinsurance policies can be used by insurers to diversify the risks associated with both stock and volatility, thereby partially mitigating such risks. Consequently, a positive relationship between the volatility coefficient v and insurers' inclination towards more aggressive reinsurance and investment strategies is observed. Conversely, risk-averse investments such as zero-coupon bonds tend to decrease as the volatility coefficient v increases.

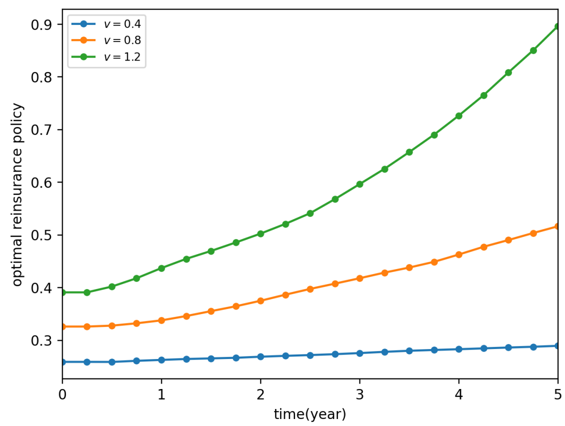
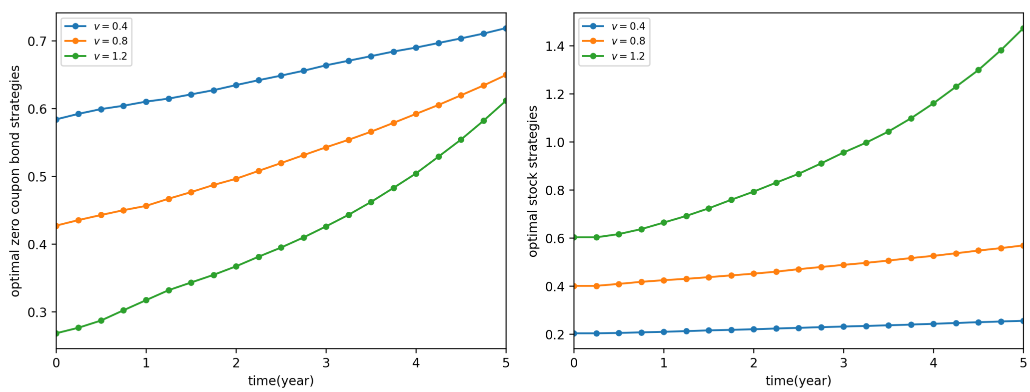


Figure 11: Effect of v on reinsurance policy



(a) Effect of v on zero-coupon bond strategies

(b) Effect of v on stock strategies

Figure 12: Effect of v on investment strategies

5.2. Impact on initial utility

Figure 13 illustrates the relationship between extra information and initial utility for insurers. It can be observed that the initial utility of insurers increases with the presence of extra information. It is important to note that both negative and positive extra information lead to an increase in initial utility for insurers, however, the magnitude of the increase is significantly lower for negative extra information as compared to positive extra information. In other words, the absolute value of extra information is a direct factor influencing insurers' utility, regardless of whether it is negative or positive. Here is our explanation: Negative extra information signifies higher future claims than expected, leading to an increase in the insurers' demand for reinsurance. However, insurers still need to pay a certain reinsurance premium to diversify the anticipated risks. On the other hand, positive extra information indicates lower future claims than expected, resulting in insurers choosing to retain more policies and thereby reducing the cost of reinsurance. As a result, the utility gained by insurers from positive extra information is higher than that from negative extra information.

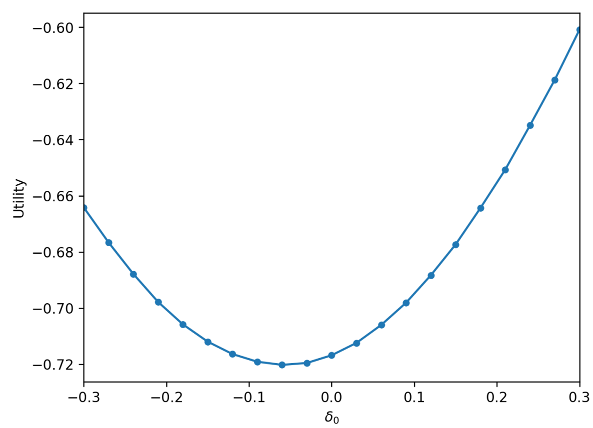


Figure 13: Effect of δ_0 on utility ($t = 0$)

Figure 14 displays the impact of negative correlation coefficient on insurers' initial utility. It can be observed that as the negative correlation strengthens, the insurers' initial utility increases. A stronger negative correlation indicates a stronger association between insurance risk and stock risk. As insurers can implement reinsurance policies, they can also mitigate the stock risk through reinsurance policies. For example, when insurers anticipate lower-than-expected future claims, it suggests a potential decline in stock prices. In such cases, insurers can balance this adverse change by selecting appropriate reinsurance strategies. Conversely, when future claims are expected to be higher than anticipated, stock prices may exhibit an upward trend. Insurers can also address this situation by coordinating between insurance risk and stock risk via reinsurance policies. Consequently, insurers' initial utility improves as the negative correlation coefficient becomes higher.

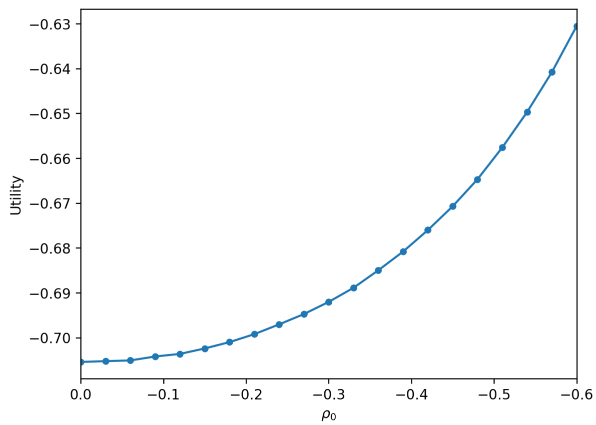


Figure 14: Effect of ρ_0 on utility ($t = 0$)

Figure 15 depicts the impact of the volatility coefficient on insurers' initial utility. The volatility coefficient, represented as v, captures the influence of volatility, denoted as L, on stock prices. It can be observed that insurers' initial utility increases with the increment of the volatility coefficient. We believe that the underlying reason for this result is similar to the factors driving the outcomes shown in Figure 11 and Figure 12. The presence of negative correlation coefficient and extra information leads to interdependencies between the insurance process, stock prices, and stock volatility. As the volatility coefficient increases, these interdependencies become stronger. With the existence of reinsurance policies, insurers can effectively balance their investment portfolios and diversify risks across different components, thereby enhancing utility.

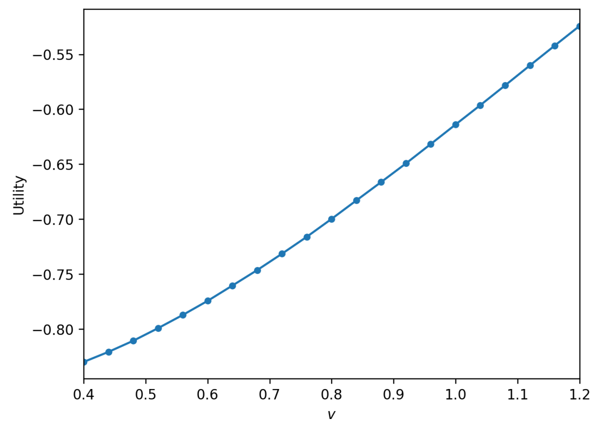


Figure 15: Effect of v on utility ($t = 0$)

6. Conclusion

In this paper, we build a model of reinsurance and investment problems involving multiple stochastic factors. Since the insurer will obtain some extra information about the future claim, we first use filtration enlargement to extract the deterministic impact of the extra information. However, due to the negative correlation between stocks and claims, extra information also contains some information about stocks and their volatility, which leads to a certain correlation between each state. Next, we obtain an HJB equation based on the CRRA utility function and dynamic programming principle, and obtain a semi-analytical solution.

To reduce the difficulty of solving, we use deep learning to solve the HJB equation. First, we convert the PDE related to the semi-analytical solution into a 2BSDE system, then forward discretization is performed, and the functional relationship between the state and the 2BSDE system is replaced by a deep neural network. Then, we use the neural network and the 2BSDE system to obtain an approximate optimal solution.

Finally, we conduct a sensitivity analysis. We have found that due to the negative correlation between insurance risk and stock risk, insurers can diversify the part of risk between insurance and stock by implementing more aggressive reinsurance and investment strategies. However, whether insurers can truly benefit from this depends on their ability to discern the authenticity of extra information. Furthermore, the impact of extra information on insurers is asymmetric, as both positive and negative extra information can be advantageous to insurers, as insurers will gain more benefits from positive extra information.

7. References

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8. Appendix

8.1. Proof of Theorem 3.4

Proof. (i) To simplify the symbol, we use $J(\cdot)$ as the simplified expression. The control uses the similar expression $\overline{u}(\cdot)$ and $\overline{u}^*(\cdot)$. We apply Itô lemma to any $J \in C^{1,2,2,2,2}(\Omega)$. For any $(t, y, r_n, l, m) \in \Omega$, $\overline{u} \in A^{\mathbb{G}}$, $s \in [t, T]$, and $\tau \in [t, +\infty]$, we can derive that

$$\begin{aligned} & J\left(s \wedge \tau, Y^{\overline{u}^*}(s \wedge \tau), r_n(s \wedge \tau), M(s \wedge \tau), L(s \wedge \tau)\right) \\ & = J(t, y, r_n, m, l) + \int_t^{s\wedge\tau} \left(\frac{\partial J(z)}{\partial t} + \mathscr{L}^{\overline{u}} J(z)\right) dz + \int_t^{s\wedge\tau} \frac{\partial J(z)}{\partial y} \overline{u}^\top \sigma dW(z) \\ & - \int_t^{s\wedge\tau} \frac{\partial J(z)}{\partial r_n} \sigma_{r_n} dW_{r_n}(z) + \int_t^{s\wedge\tau} \frac{\partial J(z)}{\partial m} \frac{1}{T_0 - z} dW_0(z) \\ & + \int_t^{s\wedge\tau} \frac{\partial J(z)}{\partial L} \sigma_L \sqrt{L} \rho_S \rho_0 dW_0(z) + \int_t^{s\wedge\tau} \frac{\partial J(z)}{\partial L} \sigma_L \sqrt{L} \rho_S \sqrt{1 - \rho_0^2} dW_S(z) \\ & + \int_t^{s\wedge\tau} \frac{\partial J(z)}{\partial L} \sigma_L \sqrt{L} \rho_S \sqrt{1 - \rho_S^2} dW_L(z), \end{aligned} \tag{8.58}$$

where $s \wedge \tau \triangleq \min\{s, \tau\}$.

We define a non-negative function $\varphi(z) : [0, T] \to [0, +\infty)$

$$\begin{aligned} \varphi(z) = & \left(\frac{\partial J(z)}{\partial y}\right)^2 \left(\overline{u}^\top \sigma \sigma^\top \overline{u}\right) + \left(\frac{\partial J(z)}{\partial r_n}\right)^2 \sigma_{r_n}^2 \\ & + \left(\frac{\partial J(z)}{\partial m}\right)^2 \frac{1}{T_0 - z}^2 + \left(\frac{\partial J(z)}{\partial l}\right)^2 \sigma_L^2 l. \end{aligned} \tag{8.59}$$

For $n = 1, 2, ...$ choose a stopping time τ_n satisfying

$$\tau_n = \inf\left\{s \in [t, T] : \int_s^t \varphi(z) \geq n\right\}. \tag{8.60}$$

Note that $\tau_n \to T$ when $n \to \infty$, the stopping time processes

$$\begin{aligned}&\int_t^{s\wedge\tau} \frac{\partial J(z)}{\partial y}\overline{u}^\top \sigma dW(z), \int_t^{s\wedge\tau} \frac{\partial J(z)}{\partial r_n}\sigma_{r_n} dW_{r_n}(z),\\ &\int_t^{s\wedge\tau} \frac{\partial J(z)}{\partial m}\frac{1}{T_0 - z}dW_0(z), \int_t^{s\wedge\tau} \frac{\partial J(z)}{\partial L}\sigma_L \sqrt{L}\rho_S\rho_0 dW_0(z),\\ &\int_t^{s\wedge\tau} \frac{\partial J(z)}{\partial L}\sigma_L \sqrt{L}\rho_S \sqrt{1-\rho_0^2}dW_S(z), \int_t^{s\wedge\tau} \frac{\partial J(z)}{\partial L}\sigma_L \sqrt{L}\rho_S \sqrt{1-\rho_S^2}dW_L(z)\end{aligned} \tag{8.61}$$

are true martingales. Taking expectations for (8.58), we can obtain

$$\begin{aligned}&\mathbb{E}\left[J(s\wedge\tau, Y^{\overline{u}}(s\wedge\tau), r_n(s\wedge\tau), M(s\wedge\tau), L(s\wedge\tau))\right]\\ &= J(t,y,r_n,m,l) + \mathbb{E}\left[\int_t^{s\wedge\tau}\left(\frac{\partial J(z)}{\partial t} + \mathscr{L}^{\overline{u}}J(z)\right)dz\right].\end{aligned} \tag{8.62}$$

Combining with (3.38) and (3.39), we have

$$\int_t^{s\wedge\tau}\left(\frac{\partial J(z)}{\partial t} + \mathscr{L}^{\overline{u}}J(z)\right)dz \le 0, \overline{u} \in A^{\mathbb{G}}. \tag{8.63}$$

Therefore, for every $\overline{u} \in A^{\mathbb{G}}$

$$\begin{aligned}&\mathbb{E}\left[J(s\wedge\tau, Y^{\overline{u}}(s\wedge\tau), r_n(s\wedge\tau), M(s\wedge\tau), L(s\wedge\tau))\right]\\ &\le J(t,y,r_n,m,l).\end{aligned} \tag{8.64}$$

According to (3.35), we can get

$$\begin{aligned}&\sup\left|J(s\wedge\tau, Y^{\overline{u}}(s\wedge\tau), r_n(s\wedge\tau), M(s\wedge\tau), L(s\wedge\tau))\right|\\ &\le K_3\Big(1 + \sup_{s\in[t,T]}\left|Y^{\overline{u}}(s)\right|^2 + \sup_{s\in[t,T]}|r_n(s)|^2 + \sup_{s\in[t,T]}|M(s)|^2 + \sup_{s\in[t,T]}|L(s)|^2\Big).\end{aligned} \tag{8.65}$$

Combining (3.32) with (8.65), and sending n to infinity in (8.64), we have that

$$\mathbb{E}\left[J(s, Y^{\overline{u}}(s), r_n(s), M(s), L(s))\right] \le J(t,y,r_n,m,l). \tag{8.66}$$

Since $J(\cdot)$ is continuous with respect to t and Y, sending s tends to T, we can again apply the control convergence theorem to obtain

$$\mathbb{E}\left[U\left(Y^{\overline{u}}(T)\right)\right] \le J(t,y,r_n,m,l), \tag{8.67}$$

for all $\overline{u} \in A^{\mathbb{G}}$. Since $\overline{u}$ is arbitrary, we can infer that $V \leq J$ for all (t, y, r_n, m, l), and the equality holds at $\overline{u}^*$.

(ii) Applying Itô lemma to $J(t, y, r_n, l, m)$ between $t \in [0, T]$ and $s \in [t, T]$, we can get

$$\begin{aligned} &\mathbb{E}\left[J(s, Y^{\overline{u}}(s), r_n(s), M(s), L(s))\right] \\ &= J(t, y, r_n, m, l) + \mathbb{E}\left[\int_t^s \left(\frac{\partial J(z)}{\partial t} + \mathscr{L}^{\overline{u}} J(z)\right) dz\right]. \end{aligned} \tag{8.68}$$

According to the definition $\overline{u}^* \in A^{\mathbb{G}}$, we have

$$-\frac{\partial J(t, y, r_n, m, l)}{\partial t} - \mathscr{L}^{\overline{u}^*} J(t, y, r_n, m, l) = 0. \tag{8.69}$$

Then

$$J(s, Y^{\overline{u}^*}(s), r_n(s), M(s), L(s)) = J(t, y, r_n, m, l). \tag{8.70}$$

Applying the control convergence theorem, we have

$$J(t, y, r_n, m, l) = \mathbb{E}\left[U\left(Y^{\overline{u}^*}(T)\right)\right]. \tag{8.71}$$

We find that $J \leq V$. Combining the result in (i), we can derive that $\overline{u}^*$ is the optimal control and $J = V$.

□

8.2. Proof of Proposition 3.5

Proof. Substituting the (3.42) into (3.37) we can derive that

$$\begin{aligned} \overline{u}^*(t) = &\frac{y}{\gamma}\Sigma^{-1}\sigma\Lambda + \frac{y}{\gamma}\frac{h_r}{h}\Sigma^{-1}\sigma\sigma_r \\ &+ \frac{y}{\gamma}\frac{h_m}{h}\Sigma^{-1}\sigma\sigma_t + \frac{y}{\gamma}\frac{h_l}{h}\sigma_L\sqrt{L}\Sigma^{-1}\sigma\sigma_{L_1}, \end{aligned} \tag{8.72}$$

where $\Sigma = \sigma^\top\sigma$. Substituting (2.21), (2.22) and (3.34) into (8.72), we can derive (3.43). In addition, combining (3.40), (3.41), (3.43) and (8.72), we have the equation (3.44). □