Blowing-up solutions for the Moore-Gibson-Thompson equation with visco-elastic memory and an external force

Mokhtar Kirane[1], DRAIFIA ALA[2], and Jlali Lotfi[3]

[1]Khalifa University
[2]Universite Amar Telidji Laghouat Departement de Mathematiques
[3]Imam Muhammad Ibn Saud Islamic University

October 3, 2023

Abstract

The Moore-Gibson-Thompson equation with a viscoelastic memory and a forcing is considered. The existence and uniqueness of a local solution is obtained via the Faedo-Galerkin's method. Furthermore, blowing-up solutions with or without a positive initial energy exist due to the nonlinear forcing.

Blowing-up solutions for the Moore-Gibson-Thompson equation with a visco-elastic memory and a non-linear external force

Mokhtar Kirane[1,5,*], Draifia Alaeddine[3,4], Jlali Lotfi[5]

[1] *Department of Mathematics, Faculty of Art and Sciences, Khalifa University of Science and Technology, P.O. Box: 127788, Abu Dhabi, United Arab Emirates*

[3] *Institute of Sciences, Department of mathematics and Informatics, University Center Aflou, Laghouat, Algeria*

[4] *Laboratory of mathematics, Informatics and Systems (LAMIS), Larbi Tebessi University, 12002 Tebessa, Algeria*

[5] *Department of Mathematics and Statistics, College of Sciences, Imam Moammad IbnS aud Islamic University (IMSIU), Riyadh, Saudi Arabia*

Abstract

The Moore-Gibson-Thompson equation with a viscoelastic memory and a forcing is considered. The existence and uniqueness of a local solution is obtained via the Faedo-Galerkin's method. Furthermore, blowing-up solutions with or without a positive initial energy exist due to the nonlinear forcing.

Keywords: Moore-Gibson-Thompson equation; Viscoelastic memory; Faedo-Galerkin's method; Blow-up.
2000 MSC: 35L20, 35L70, 58G16, 93D15.

1. Introduction

We present the existence of local solutions, and the occurence of blowing-up solutions of the Moore-Gibson-Thompson (MGT in short) equation with a

*Corresponding author
Email address: mokhtar.kirane@ku.ac.ae (Mokhtar Kirane[1,5,*])

viscoelastic memory and a nonlinear forcing

$$v_{ttt} + \alpha v_{tt} - c^2 \Delta v - \beta \Delta v_t + \int_0^t \mathcal{K}(t-\sigma)\Delta v(\sigma) d\sigma = |v|^{p-2} v, \tag{1}$$

for $P := (x,t) \in \Omega_T := \Omega \times (0,T)$, with initial position and velocity

$$v(x,0) = \varphi(x),\ v_t(x,0) = \omega(x),\ v_{tt}(x,0) = \Upsilon(x), \quad x \in \Omega, \tag{2}$$

and supplemented with the boundary condition

$$v(P) = 0, \quad P \in \Gamma \times (0,T),\, T < \infty, \tag{3}$$

where the bounded domain $\Omega \subset \mathbb{R}^N$ is smooth and with boundary Γ; the kernel $\mathcal{K} : [0,+\infty) \longrightarrow [0,+\infty)$, $\varphi(x), \omega(x)$ and $\Upsilon(x)$ are given functions; $\alpha > 0$ and $\beta > 0$ are real numbers. In acoustics, the variable v denotes a scalar acoustic velocity potential, c^2 the speed of sound, β describes damping effect associated with an acoustic environment, see Lebon and Cloot [14] for futher details; the memory effects of an acoustic environment is described by $\int_0^t \mathcal{K}(t-s)\Delta v(\sigma)\, d\sigma$.

Before we present our results, let us dwell on some existing literature. In [17], Caixeta et al. studied global solutions of the equation

$$\tau v_{ttt} + \alpha v_{tt} + c^2 A v + \beta A v_t = f\left(v, v_t, v_{tt}\right), \quad P \in \Omega_T,$$

where A is a self-adjoint, positive operator densely defined in a Hilbert space H and f is nonlinear. In [13],

$$\tau v_{ttt} + \alpha v_{tt} + c^2 A v + \beta A v_t - \int_0^t \mathcal{K}(t-\sigma) A w\left(\sigma\right) d\sigma = 0, \quad P \in \Omega_T, \tag{4}$$

has been studied; it is shown that the memory causes energy decay whenever $\mathcal{K}$ satisfies some conditions.

The blow-up for the MGT equation with viscoelastic memory and nonlinear forcing is considered as a new problem. The works [2]-[17], examine blow-up results of $(1)-(3)$; No blow-up result when $\Xi(0) \geq 0$ or $\Xi(0) < 0$ ($\Xi(t)$ is the

energy of the system) is mentioned. We will then study blowing-up solutions of problem$(1)-(3)$ when $\Xi(0)\geq 0$ and when $\Xi(0)<0$.
The outline of the paper is as follows. In Section 2, we recall some spaces, and we define a weak solution of problem $(1)-(3)$ and establish some useful inequalities which will be used in the sequel. Section 3 is devoted to the study of local existence and uniqueness of the weak solution of $(1)-(3)$ via Galerkin's method. Finally, we present the main results on blowing-up solutions and estimates of the blow-up time.

2. Assumptions and main results

The norm of $L^p(\Omega)$ is denoted by $\|v\|_p := \left(\int_\Omega |v|^p\,dx\right)^{\frac{1}{p}}$.
The scalar product in $L^2(\Omega)$ is defined by

$$(v(x),u(x)) = \int_\Omega v(x)u(x)\,dx.$$

Set

$$\mathcal{V}(\Omega_T) := \Big\{v : v \in H_0^1(\Omega_T), v_t \in H^1(\Omega_T)\Big\},$$

and

$$\mathcal{W}(\Omega_T) := \Big\{u : u \in V(\Omega_T), u(x,T) = 0\Big\}.$$

Definition 2.1. *A function $v \in \mathcal{V}(\Omega_T)$ is called a generalized solution of problem $(1)-(3)$ if it satisfies, for each $u \in \mathcal{W}(\Omega_T)$,*

$$\begin{aligned}
- &\quad (v_{tt}(t), u_t(t))_{2,\Omega_T} - \alpha\,(v_t(t), u_t(t))_{2,\Omega_T} \\
+ &\quad c^2\,(\nabla v(t), \nabla u(t))_{2,\Omega_T} + \beta\,(\nabla v_t(t), \nabla u(t))_{2,\Omega_T} \\
- &\quad \left(\int_0^t \mathcal{K}(t-s)\nabla v(\sigma)ds, \nabla u(t)\right)_{2,\Omega_T} \\
= &\quad \alpha(\omega(x), u(0))_2 + (\Upsilon(x), u(0))_2 + \left(|v|^{p-2}\,v(t), u(t)\right)_{2,\Omega_T}.
\end{aligned} \tag{5}$$

Lemma 2.2. *[1] If $1 \leq q \leq \dfrac{2N}{N-2}, N \geq 3$, then the embedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$ is continuous.*

Lemma 2.3. *[16] Let* $4 < p \leq \dfrac{4(N-1)}{N-2}$, $N \geq 3$. *Then there exists a constant* $C_p = C_p(p, N, \Omega) > 0$ *such that*

$$\begin{aligned}\left\| |\upsilon_1|^{p-2}\upsilon_1 - |\upsilon_2|^{p-2}\upsilon_2 \right\|_2^2 &\leq C_p\Big(1 + \left(\|\upsilon_1\|_{H^1(\Omega)} + \|\upsilon_2\|_{H^1(\Omega)}\right)^{\frac{1}{N}} \\ &+ \left(\|\upsilon_1\|_{H^1(\Omega)} + \|\upsilon_2\|_{H^1(\Omega)}\right)^{p-2}\Big)^2 \|\upsilon_1 - \upsilon_2\|_{H^1(\Omega)}^2,\end{aligned}$$

for $\upsilon_1, \ \upsilon_2 \in H^1(\Omega)$.

Lemma 2.4. *[15] Let* $\varrho > 0$ *and let* $0 \leq B(t) \in C^2(0, \infty)$ *be such that*

$$B''(t) - 4(\varrho + 1) B'(t) + 4(\varrho + 1) B(t) \geq 0.$$

If $B'(0) > r_* B(0) + k_0$, *then*

$$B'(t) > k_0,$$

for $t > 0$, *where* k_0 *is a constant,* $r_* := 2(\varrho + 1) - 2\sqrt{(\varrho + 1)\varrho}$ *is the smallest root of the equation*

$$r^2 - 4(\varrho + 1) r + 4(\varrho + 1) = 0.$$

Lemma 2.5. *[15] If* $M(t)$ *is a non-increasing function on* $[t_*, \infty)$, $t_* \geq 0$, *and satisfies*

$$M'(t)^2 \geq \nu + R M(t)^{2+\frac{1}{\varrho}}, \quad \text{for } t \geq t_*, \ \nu > 0,$$

where R *is a real number, then*

$$\lim_{t \to T_{max}-} M(t) = 0,$$

for some $T_{max} < \infty$. *An upper bound of* T_{max} *is given in the following cases:*

(i) *When* $R < 0$ *and* $M(t_*) < \min\left\{1, \left(\dfrac{\nu}{-R}\right)^{1/2}\right\}$, *then*

$$T_{max} \leq t_* + \frac{1}{\sqrt{-R}} \ln \frac{\left(\dfrac{\nu}{R}\right)^{1/2}}{\left(\dfrac{\nu}{-R}\right)^{1/2} - M(t_*)}.$$

(ii) *When* $R = 0$, *then*

$$T_{max} \leq t_* + \frac{M(t_*)}{\sqrt{\nu}}.$$

(iii) *When* $R > 0$, *then*

$$T_{max} \leq \frac{M\,(t_*)}{\sqrt{\nu}},$$

or

$$T_{max} \leq t_* + 2^{\dfrac{3\varrho+1}{2\varrho}} \frac{\varrho\gamma}{\sqrt{\nu}} \left(1 - (1 + \gamma M(t_*))^{\dfrac{1}{2\varrho}} \right), \quad \gamma := \left(\frac{R}{\nu}\right)^{\varrho/(2+\varrho)}.$$

Our results need some assumptions:
$(\mathbf{H}_1)$ We assume that $\mathcal{K} \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ satisfies

$$\mathcal{K}'(t) \leq 0,\ \mathcal{K}''(t) \geq 0,\ \bar{\mathcal{K}} := \int_0^\infty \mathcal{K}(\sigma)d\sigma < \gamma^2,\ t \geq 0.$$

$(\mathbf{H}_2)$ The exponent p satisfies

$$4 < p,\ \text{if } N = 1, 2; \quad 4 < p \leq \frac{4\,(N-1)}{N-2},\ \text{if } N \geq 3.$$

3. Existence and uniqueness

The existence and uniqueness of the weak solution of problem $(1) - (3)$ is obtained using the Faedo–Galerkin method.

3.1. ***Existence***

Theorem 3.1. *Let* $\varphi(x) \in H_0^1(\Omega)$, $\omega(x) \in H^1(\Omega)$ *and* $\Upsilon(x) \in L^2(\Omega)$. *Under* $(\mathbf{H}_1) - (\mathbf{H}_2)$, *there is at least one generalized solution of* $(1) - (3)$ *in* $\mathcal{V}(\Omega_T)$.

Proof. Let $\{\psi_l(x)\}_{l\geq 1}$ be an orthonormal basis of $H_0^1(\Omega)$. We look for an approximating solution

$$v^{(\kappa)}(x,t) = \sum_{=1}^{l=\kappa} G_l(t)\psi_l(x), \tag{6}$$

with

$$G_l(t) = (\psi_l(x), v^{(\kappa)}(P))_2, \quad l = 1, ..., \kappa.$$

They can be found, for $k = 1, ..., \kappa$, via the relations

$$\begin{aligned} &\left(v_{ttt}^{(\kappa)}, \psi_k(x)\right)_2 + \alpha \left(v_{tt}^{(\kappa)}, \psi_k(x)\right)_2 + c^2 \left(\nabla v^{(\kappa)}, \nabla \psi_k(x)\right)_2 + b \left(\nabla v_t^{(\kappa)}, \nabla \psi_k \left(x\right)\right)_2 \\ &- \left(\int_0^t \mathcal{K}(t-s) \nabla v^{(\kappa)}(s) ds, \nabla \psi_k(x)\right)_2 = \left(\left|v^{(\kappa)}\right|^{p-2} v^{\kappa}(t), \psi_k\left(x\right)\right)_2. \end{aligned} \tag{7}$$

System (7) is supplemented with initial conditions

$$G_l(0) = (\psi_l, \varphi(x))_2\,, \; G_l'(0) = (\psi_l, \omega\left(x\right))_2\,, \; G_l''(t) = (\psi_l, \Upsilon(x))_2\,.$$

Carathéodory theorem [2] ensures the existence of solutions $G_l(t),\ l = 1, ..., \kappa$, for $t \in [0, t_\kappa)$. In order to prolong the solution on $[0, T]$ some bounds for $v^{(\kappa)}$ independent of κ are needed.
Set

$$\Sigma^{(\kappa)}(t) := \left\|v^{(\kappa)}(t)\right\|_{H_0^1(\Omega)}^2 + \left\|v_t^{(\kappa)}(t)\right\|_{H^1(\Omega)}^2 + \left\|v_{tt}^{(\kappa)}(t)\right\|_2^2. \tag{8}$$

For, we have

$$\begin{aligned} & \left(v_{ttt}^{(\kappa)}(t), v_t^{(\kappa)}(t)\right)_{2,\Omega_\tau} + \alpha \left(v_{tt}^{(\kappa)}(t), v_t^{(\kappa)}(t)\right)_{2,\Omega_\tau} \\ & + c^2 \left(\nabla v^{(\kappa)}(t), \nabla v_t^{(\kappa)}(t)\right)_{2,\Omega_\tau} + b \left(\nabla v_t^{(\kappa)}(t), \nabla v_t^{(\kappa)}(t)\right)_{2,\Omega_\tau} \\ & - \left(\int_0^t \mathcal{K}(t-s) \nabla v^{(\kappa)}\left(s\right) ds, \nabla v_t^{(\kappa)}(t)\right)_{2,\Omega_\tau} \\ = \; & \left(\left|v^{(\kappa)}\right|^{p-2} v^{(\kappa)}(t), v_t^{(\kappa)}(t)\right)_{2,\Omega_\tau}. \end{aligned} \tag{9}$$

Now, we evaluate each term of (9). integrating by parts, we get

$$\begin{aligned}
& \frac{1}{2}\left(c^2 - \int_0^\tau \mathcal{K}(s)\,ds\right)\left\|\nabla v^{(\kappa)}(\tau)\right\|_2^2 + \frac{\alpha}{2}\left\|v_\tau^{(\kappa)}(\tau)\right\|_2^2 \\
+ \; & \left(v_{\tau\tau}^{(\kappa)}(\tau), v_\tau^{(\kappa)}(\tau)\right)_2 + \frac{1}{2}\left(\mathcal{K}\circ\nabla v^{(\kappa)}\right)(\tau) \\
= \; & \frac{c^2}{2}\left\|\nabla v^{(\kappa)}(0)\right\|_2^2 + \frac{\alpha}{2}\left\|v_t^{(\kappa)}(0)\right\|_2^2 + \left(v_{tt}^{(\kappa)}(0)\,, v_t^{(\kappa)}(0)\right)_2 \\
- \; & \beta\int_0^\tau \left\|\nabla v_t^{(\kappa)}(t)\right\|_2^2 dt + \int_0^\tau \left\|v_{tt}^{(\kappa)}(t)\right\|_2^2 dt \\
+ \; & \frac{1}{2}\int_0^\tau \left(\mathcal{K}'\circ\nabla v^{(\kappa)}\right)(t)\,dt - \frac{1}{2}\int_0^\tau \mathcal{K}(t)\left\|\nabla v^{(\kappa)}(t)\right\|_2^2 dt \\
+ \; & \left(\left|v^{(\kappa)}\right|^{p-2} v^{(\kappa)}(t), v_t^{(\kappa)}(t)\right)_{L^2(\Omega_\tau)},
\end{aligned} \tag{10}$$

where

$$\left(\mathcal{K}\circ\nabla v^{(\kappa)}\right)(t) := \int_\Omega\int_0^t \mathcal{K}(t-s)\left|\nabla v^{(\kappa)}P - \nabla v^{(\kappa)}(x,s)\right|^2 dsdx.$$

Using ε-Young's inequality with $\varepsilon = \vartheta_1$, we obtain

$$-\frac{\vartheta_1}{2}\left\|v_{\tau\tau}^{(\kappa)}(\tau)\right\|_2^2 - \frac{1}{2\vartheta_1}\left\|v_\tau^{(\kappa)}(\tau)\right\|_2^2 \le \left(v_{\tau\tau}^{(\kappa)}(\tau), v_\tau^{(\kappa)}(\tau)\right)_2. \tag{11}$$

It holds

$$\left(v_{tt}^{(\kappa)}(0), v_t^{(\kappa)}(0)\right)_2 \le \frac{1}{2}\left\|v_{tt}^{(\kappa)}(0)\right\|_2^2 + \frac{1}{2}\left\|v_t^{(\kappa)}(0)\right\|_2^2. \tag{12}$$

Using Cauchy-Schwarz inequality, Young's inequality, the embedding $H^1(\Omega) \hookrightarrow L^{2(p-1)}(\Omega)$, and (8), we obtain

$$\begin{aligned}
& \left(\left|v^{(\kappa)}\right|^{p-2} v^{(\kappa)}(t), v_t^{(\kappa)}(t)\right)_{L^2(\Omega_\tau)} \\
\le \; & C_T\left\|v^{(\kappa)}(0)\right\|_2^{2(p-1)} + C_T\int_0^\tau \left(\Sigma^{(\kappa)}(t)\right)^{p-1} dt + \int_0^\tau \Sigma^{(\kappa)}(t)\,dt.
\end{aligned} \tag{13}$$

Using inequalities (11)-(13) into (10), we obtain

$$
\begin{aligned}
& \frac{\left(c^2-\bar{\mathcal{K}}\right)}{2}\left\|\nabla v^{(\kappa)}(\tau)\right\|_2^2+\left(\frac{\alpha}{2}-\frac{1}{2\vartheta_1}\right)\left\|v_\tau^{(\kappa)}(\tau)\right\|_2^2 \\
- & \frac{\vartheta_1}{2}\left\|v_{\tau\tau}^{(\kappa)}(\tau)\right\|_2^2+\frac{1}{2}\left(\mathcal{K}\circ\nabla v^{(\kappa)}\right)(\tau) \\
\leq & C_T\left\|v^{(\kappa)}(0)\right\|_2^{2(p-1)}+\frac{c^2}{2}\left\|\nabla v^{(\kappa)}(0)\right\|_2^2 \\
+ & \frac{(\alpha+1)}{2}\left\|v_t^{(\kappa)}(0)\right\|_2^2+\frac{1}{2}\left\|v_{tt}^{(\kappa)}(0)\right\|_2^2 \\
+ & C_T\int_0^\tau\left(\Sigma^{(\kappa)}(t)\right)^{p-1}dt+2\int_0^\tau\Sigma^{(\kappa)}(t)\,dt. \qquad (14)
\end{aligned}
$$

For, multiplying each equation of (7) by $G_k''(t)$, add them up from 1 to m and then integrate on $(0,\tau)$, $\tau\leq T$, we obtain

$$
\begin{aligned}
& \left(v_{ttt}^{(\kappa)}(t),v_{tt}^{(\kappa)}(t)\right)_{2,\Omega_\tau}+\alpha\left(v_{tt}^{(\kappa)}(t),v_{tt}^{(\kappa)}(t)\right)_{2,\Omega_\tau} \\
+ & c^2\left(\nabla v^{(\kappa)}(t),\nabla v_{tt}^{(\kappa)}(t)\right)_{2,\Omega_\tau}+b\left(\nabla v_t^{(\kappa)}(t),\nabla v_{tt}^{(\kappa)}(t)\right)_{2,\Omega_\tau} \\
- & \left(\int_0^t\mathcal{K}(t-s)\nabla v^{(\kappa)}(s)\,ds,\nabla v_{tt}^{(\kappa)}(t)\right)_{2,\Omega_\tau} \\
= & \left(\left|v^{(\kappa)}\right|^{p-2}v^{(\kappa)}(t),v_{tt}^{(\kappa)}(t)\right)_{2,\Omega_\tau}. \qquad (15)
\end{aligned}
$$

An integration by parts leads to

$$\begin{aligned}
& c^2\left(\nabla v^{(\kappa)}(\tau), \nabla v_\tau^m(\tau)\right)_2+\frac{\beta}{2}\left\|\nabla v_\tau^{(\kappa)}(\tau)\right\|_2^2 \\
+ & \frac{\mathcal{K}\tau)}{2}\left\|\nabla v^{(\kappa)}(\tau)\right\|_2^2+\frac{1}{2}\left(-\mathcal{K}^{\prime} \circ \nabla v^{(\kappa)}\right)(\tau)+\frac{1}{2}\left\|v_{\tau\tau}^{(\kappa)}(\tau)\right\|_2^2 \\
- & \int_0^\tau \mathcal{K}(\tau-s)\left(\nabla v^{(\kappa)}(s), \nabla v_\tau^{(\kappa)}(\tau)\right)_2 ds \\
= & \frac{\mathcal{K}(0)}{2}\left\|\nabla v^{(\kappa)}(0)\right\|_2^2+c^2\left(\nabla v^{(\kappa)}(0), \nabla v_t^{(\kappa)}(0)\right)_2+\frac{\beta}{2}\left\|\nabla v_t^{(\kappa)}(0)\right\|_2^2 \\
+ & \frac{1}{2}\left\|v_{tt}^{(\kappa)}(0)\right\|_2^2+\frac{1}{2} \int_0^\tau \mathcal{K}^{\prime}(t)\left\|\nabla v^{(\kappa)}(t)\right\|_2^2 dt \\
+ & c^2 \int_0^\tau\left\|\nabla v_t^{(\kappa)}(t)\right\|_2^2 dt-\alpha \int_0^\tau\left\|v_{tt}^{(\kappa)}(t)\right\|_2^2 dt \\
- & \frac{1}{2} \int_0^\tau\left(\mathcal{K}^{\prime\prime} \circ \nabla v^{(\kappa)}\right)(t)\, dt+\left(\left|v^{(\kappa)}\right|^{p-2} v^{(\kappa)}(t), v_{tt}^{(\kappa)}(t)\right)_{2, \Omega_\tau}.
\end{aligned} \tag{16}$$

Young's inequality with $\varepsilon=\vartheta_2$ allows to get

$$-\frac{c^2\vartheta_2}{2}\left\|\nabla v^{(\kappa)}(\tau)\right\|_2^2-\frac{c^2}{2\vartheta_2}\left\|\nabla v_\tau^{(\kappa)}(\tau)\right\|_2^2 \leq c^2\left(\nabla v^{(\kappa)}(\tau), \nabla v_\tau^{(\kappa)}(\tau)\right)_2. \tag{17}$$

Using Young's inequality with $\varepsilon=\vartheta_2$, and $-\int_0^\tau \mathcal{K}(\sigma)d\sigma \geq -\bar{\mathcal{K}}$, we obtain

$$\begin{aligned}
& -\frac{\vartheta_2\bar{\mathcal{K}}}{2}\left\|\nabla v^{(\kappa)}(\tau)\right\|_2^2-\frac{\vartheta_2}{2}\left(\mathcal{K} \circ \nabla v^{(\kappa)}\right)(\tau)-\frac{\bar{\mathcal{K}}}{\vartheta_2}\left\|\nabla v_\tau^{(\kappa)}(\tau)\right\|_2^2 \\
\leq & -\int_0^\tau \mathcal{K}(\tau-s)\left(\nabla v^{(\kappa)}(s), \nabla v_\tau^{(\kappa)}(\tau)_2\, ds.\right.
\end{aligned} \tag{18}$$

It holds

$$c^2\left(\nabla v^{(\kappa)}(0), \nabla v_t^m(0)\right)_2 \leq \frac{c^2}{2}\left\|\nabla v^{(\kappa)}(0)\right\|_2^2+\frac{c^2}{2}\left\|\nabla v_t^{(\kappa)}(0)\right\|_2^2. \tag{19}$$

Using Cauchy-Schwarz inequality, Young's inequality, $H^1(\Omega) \hookrightarrow L^{2(p-1)}(\Omega)$, and (8), we obtain

$$\begin{aligned}
& \left(\left|v^{(\kappa)}\right|^{p-2} v^{(\kappa)}(t), v_{tt}^{(\kappa)}(t)\right)_{2, D_\tau} \\
\leq & C_T\left\|v^{(\kappa)}(0)\right\|_2^{2(p-1)}+C_T \int_0^\tau\left(\Sigma^{(\kappa)}(t)\right)^{p-1} dt+\int_0^\tau \Sigma^{(\kappa)}(t)\, dt.
\end{aligned} \tag{20}$$

Combining inequalities (17) − (20) with (16), we obtain

$$
\begin{aligned}
& -\frac{\vartheta_2(c^2+\bar{\mathcal{K}})}{2}\left\|\nabla v^{(\kappa)}(\tau)\right\|_2 + \left(\frac{\beta}{2}-\frac{c^2}{2\vartheta_2}-\frac{\bar{\mathcal{K}}}{\vartheta_2}\right)\left\|\nabla v_\tau^{(\kappa)}(\tau)\right\|_2^2 \\
+ \;& \frac{1}{2}\left\|v_{\tau\tau}^{(\kappa)}(\tau)\right\|_2^2 - \frac{\vartheta_2}{2}\left(\mathcal{K}\circ\nabla v^{(\kappa)}\right)(\tau) \\
\leq \;& C_T\left\|v^{(\kappa)}(0)\right\|_2^{2(p-1)} + \frac{\mathcal{K}(0)+c^2}{2}\left\|\nabla v^{(\kappa)}(0)\right\|_2^2 \\
+ \;& \frac{(c^2+b)}{2}\left\|\nabla v_t^{(\kappa)}(0)\right\|_2^2 + \frac{1}{2}\left\|v_{tt}^{(\kappa)}(0)\right\|_2^2 \\
+ \;& C_T\int_0^\tau \left(\Sigma^{(\kappa)}(t)\right)^{p-1} dt + \left(c^2+1\right)\int_0^\tau \Sigma^{(\kappa)}(t)\,dt. \qquad (21)
\end{aligned}
$$

Multiplying (14) by λ_1, (21) by λ_2 and making use of the following inequalities

$$
\left\|v^{(\kappa)}(\tau)\right\|_2^2 \leq \int_0^\tau \Sigma^{(\kappa)}(t)\,dt + \left\|v^{(\kappa)}(0)\right\|_2^2,
$$

$$
\alpha_1\left\|\nabla v^{(\kappa)}(\tau)\right\|_2^2 \leq \alpha_1\int_0^\tau \Sigma^{(\kappa)}(t)\,dt + \alpha_1\left\|\nabla v^{(\kappa)}(0)\right\|_2^2,
$$

and

$$
\alpha_2\left\|v_\tau^{(\kappa)}(\tau)\right\|_2^2 \leq \alpha_2\int_0^\tau \Sigma^{(\kappa)}(t)\,dt + \alpha_2\left\|v_t^{(\kappa)}(0)\right\|_2^2,
$$

where

$$
\alpha_1 = \frac{\lambda_2\vartheta_2\left(c^2+\bar{\mathcal{K}}\right)}{2} > 0, \quad \vartheta_2 = \frac{2\left(c^2+2\bar{\mathcal{K}}\right)}{\beta} > 0,
$$

$$
\alpha_2 = \frac{\lambda_1}{2\vartheta_2} > 0, \quad \vartheta_1 = \frac{1}{8\vartheta_2} > 0,
$$

we obtain

$$
\begin{aligned}
& \left\|v^{(\kappa)}(\tau)\right\|_2^2+\frac{\lambda_1\left(c^2-\bar{\mathcal{K}}\right)}{2}\left\|\nabla v^{(\kappa)}(\tau)\right\|_2^2+\frac{\lambda_1 \alpha}{2}\left\|v_\tau^{(\kappa)}(\tau)\right\|_2^2 \\
+\quad & \frac{\lambda_2 b}{4}\left\|\nabla v_\tau^{(\kappa)}(\tau)\right\|_2^2+\frac{\lambda_2}{4}\left\|v_{\tau\tau}^{(\kappa)}(\tau)\right\|_2^2 \\
\leq\quad & \left\|v^{(\kappa)}(0)\right\|_2^2+C_1\left\|\nabla v^{(\kappa)}(0)\right\|_2^2+C_2\left\|v_t^{(\kappa)}(0)\right\|_2^2 \\
+\quad & \frac{\lambda_2(c^2+b)}{2}\left\|\nabla v_t^{(\kappa)}(0)\right\|_2^2+\frac{(\lambda_1+\lambda_2)}{2}\left\|v_{tt}^{(\kappa)}(0)\right\|_2^2 \\
+\quad & (\lambda_1+\lambda_2) C_T\left\|v^{(\kappa)}(0)\right\|_2^{2(p-1)} \\
+\quad & C_3 \int_0^\tau \Sigma^{(\kappa)}(t)\, dt+(\lambda_1+\lambda_2) C_T \int_0^\tau\left(\Sigma^{(\kappa)}(t)\right)^{p-1} dt,
\end{aligned}
$$

where

$$
\begin{aligned}
C_1 &:= \frac{\lambda_1 c^2+\lambda_2(\mathcal{K}(0)+c^2)+2\alpha_1}{2}>0, \\
C_2 &:= \frac{\lambda_1(\alpha+1)+2\alpha_2}{2}>0, \\
C_3 &:= 2\lambda_1+\lambda_2(c^2+1)+1+\alpha_1+\alpha_2>0.
\end{aligned}
$$

We have

$$
\begin{aligned}
\Sigma^{(\kappa)}(\tau) \quad \leq\quad & \omega\left(\left\|v^{(\kappa)}(0)\right\|_{H_0^1(\Omega)}^2+\left\|v_t^{(\kappa)}(0)\right\|_{H^1(\Omega)}^2+\left\|v_{tt}^{(\kappa)}(0)\right\|_2^2+\left\|v^{(\kappa)}(0)\right\|_2^{2(p-1)}\right) \\
+\quad & \omega \int_0^\tau \Sigma^{(\kappa)}(t)\, dt+\omega \int_0^\tau\left(\Sigma^{(\kappa)}(t)\right)^{p-1} dt, \qquad (22)
\end{aligned}
$$

where

$$
\omega:=\frac{\max\left\{1, C_1, C_2, \lambda_2(c^2+b)/2,(\lambda_1+\lambda_2)/2,(\lambda_1+\lambda_2) C_T, C_3\right\}}{\min\left\{1, \lambda_1\left(c^2-\bar{\mathcal{K}}\right)/2, \lambda_1 \alpha/2, \lambda_2 b/4, \lambda_2/4\right\}}>0 .
$$

By solving (22) and then integrating on $(0, \tau)$, we obtain

$$
\int_0^\tau \Sigma^{(\kappa)}(t)\, dt \leq \omega T\left(\|v(0)\|_{H_0^1(\Omega)}^2+\|v_t(0)\|_{H^1(\Omega)}^2+\left\|v_{tt}^{(\kappa)}(0)\right\|_2^2+\|v(0)\|_2^{2(p-1)}\right).
$$

We deduce from (8) that

$$
\left\|v^{(\kappa)}(t)\right\|_{H_0^1(\Omega_T)}^2+\left\|v_t^{(\kappa)}(t)\right\|_{H^1(\Omega_T)}^2+\left\|v_{tt}^{(\kappa)}(t)\right\|_{2,\Omega_T}^2 \leq C.
$$

Whereupon, the sequence $\left\{v^{(\kappa)}\right\}_{\kappa\geq 1} \subset \mathcal{V}(\Omega_T)$ is bounded; so, a subsequence (with the same notation) can be extracted that converges weakly in $\mathcal{V}(\Omega_T)$ to $v(P)$. Let us show that $v(P)$ is a generalized solution of $(1)-(3)$. Since $\lim\limits_{\kappa\longrightarrow\infty}\left\|v^{(\kappa)}(P)-v(P)\right\|_{2,\Omega_T} = 0$ and $\lim\limits_{\kappa\longrightarrow\infty}\left\|v^{(\kappa)}(x,0)-\varphi(x)\right\|_{2,\Omega} = 0$, then $v(x,0)=\varphi(x)$. Now to prove (5), we multiply each of the relations (7) by a function $\Psi_l(t) \in W_2^1(0,T)$, $\Psi_l(T)=0$, then add up the obtained equalities ranging from $l=1$ to $l=\kappa$, and integrate on $(0,T)$. If we set $\phi^{(\kappa)} := \sum\limits_{l=1}^{l=\kappa}\Psi_l(t)\psi_l(x)$, then we have

$$
\begin{aligned}
& -\left(v_{tt}^{(\kappa)}(t),\phi_t^{(\kappa)}(t)\right)_{2,\Omega_T} - \alpha\left(v_t^{(\kappa)}(t),\phi_t^{(\kappa)}(t)\right)_{2,\Omega_T} \\
+ \;& c^2\left(\nabla v^{(\kappa)},\nabla\phi^{(\kappa)}(t)\right)_{2,\Omega_T} + \beta\left(\nabla v_t^m,\nabla\phi^{(\kappa)}(t)\right)_{2,\Omega_T} \\
- \;& \left(\int_0^t \mathcal{K}(t-s)\nabla v^{(\kappa)}(s)ds,\nabla\phi^{(\kappa)}(t)\right)_{2,\Omega_T} \\
= \;& \alpha\left(v_t^{(\kappa)}(0),\phi^{(\kappa)}(0)\right)_2 + \left(v_{tt}^{(\kappa)}(0),\phi^{(\kappa)}(0)\right)_2 \\
+ \;& \left(\left|v^{(\kappa)}\right|^{p-2} v^{(\kappa)}(t),\phi^{(\kappa)}(t)\right)_{2,\Omega_\tau}.
\end{aligned}
$$

Since

$$
\begin{aligned}
& -\left(\int_0^t \mathcal{K}(t-s)\left(\nabla v^{(\kappa)}(s)-\nabla v(s)\right)ds,\nabla\phi^{\kappa}(t)\right)_{2,\Omega_T} \\
\leq \;& \frac{\sup\left(\mathcal{K}(t)\right)T}{\sqrt{2}}\left\|\nabla\phi^{(\kappa)}(t)\right\|_{2,\Omega_T}\left\|\nabla v^{(\kappa)}(t)-\nabla v(t)\right\|_{2,\Omega_T},
\end{aligned}
$$

and

$$
\lim_{\kappa\longrightarrow\infty}\left\|v^{(\kappa)}(t)-v(t)\right\|_{H^1(\Omega_T)} = 0,
$$

then,

$$
\lim_{\kappa\longrightarrow\infty}\left[\left(\int_0^t \mathcal{K}(t-s)\nabla v^{\kappa}(s)\,ds,\nabla\phi^{\kappa}(t)\right)_{2,\Omega_T} - \left(\int_0^t \mathcal{K}(t-s)\nabla v(s)\,ds,\nabla\phi(t)\right)_{2,\Omega_T}\right] = 0.
$$

As the function $z \longmapsto |z|^{p-2} z$ is continuous,

$$
\left|v^{(\kappa)}\right|^{p-2} v^{(\kappa)}(t) \longrightarrow |v|^{p-2} v(t), \quad \text{and a.e. in } \Omega_T. \tag{23}
$$

Also,

$$\left\| \left| v^{(\kappa)} \right|^{p-2} v^{(\kappa)}(t) \right\|_{2,\Omega_T}^2 \leq C, \tag{24}$$

for some constant C. Using [[16], Lemma 1.3], it follows from (23) and (24) that

$$\left(\left| v^{(\kappa)} \right|^{p-2} v^{(\kappa)}(t), \phi^{(\kappa)}(t) \right)_{2,\Omega_T} \longrightarrow \left(|v|^{p-2} v(t), \phi(t) \right)_{2,\Omega_T}, \text{ as } \kappa \longrightarrow \infty.$$

Hence, the limit v satisfies (5) for every $\phi^{(\kappa)}(P) := \sum_{l=1}^{l=\kappa} \Psi_l(t)\psi_l(x)$. We denote by $\mathbb{Q}_\kappa$ all functions of the form $\phi^{(\kappa)}(P) := \sum_{l=1}^{l=\kappa} \Psi_l(t)\psi_l(x)$, with $\Psi_l(t) \in W_2^1(0,T), \Psi_l(T) = 0$. As $\bigcup_{\kappa=1}^{\infty} \mathbb{Q}_\kappa \subset \mathcal{W}(\Omega_T)$ is dense, then (5) holds for all $u \in \mathcal{W}(\Omega_T)$. Hence, $v(P)$ is a generalized solution, in $\mathcal{V}(\Omega_T)$, of problem $(1) - (3)$. □

3.2. *Uniqueness*

Theorem 3.2. *Under* $(\mathbf{H}_1) - (\mathbf{H}_2)$*, problem* $(1) - (3)$ *admits, in* $\mathcal{V}(\Omega_T)$*, a unique generalized solution.*

Proof. Assume that $v_1, v_2 \in \mathcal{V}(\Omega_T)$ are two different solutions of $(1) - (3)$. Then $v := v_1 - v_2$ solves

$$v_{ttt} + \alpha v_{tt} - c^2 \varrho v - b\Delta v_t + \int_0^t \mathcal{K}(t-s)\Delta v(\sigma)d\sigma = |v_1|^{p-2} v_1 - |v_2|^{p-2} v_2, \tag{25}$$

$$v(x,0) = v_t(x,0) = v_{tt}(x,0) = 0,$$

$$v(P) = 0, \quad P \in \partial\Omega \times (0,T).$$

So,

$$\begin{aligned} & -\left(v_{tt}(t), u_t(t)\right)_{L^2(\Omega_T)} - \alpha\left(v_t(t), u_t(t)\right)_{2,\Omega_T} \\ + \;& c^2\left(\nabla v(t), \nabla u(t)\right)_{2,\Omega_T} + b\left(\nabla v_t(t), \nabla u(t)\right)_{2,\Omega_T} \\ - \;& \left(\int_0^t \mathcal{K}(t-s)\nabla v(s)\, ds, \nabla u(t) \right)_{2,\Omega_T} \\ = \;& \left(|v_1|^{p-2} v_1(t) - |v_2|^{p-2} v_2(t), u \right)_{2,\Omega_T}. \end{aligned} \tag{26}$$

Define the function $u(P)$ by

$$u(P) := \begin{cases} \displaystyle\int_t^\tau v(x,s)\,ds, & 0 \le t \le \tau, \\ \\ 0, & \tau \le t \le T. \end{cases} \tag{27}$$

It holds $u \in \mathcal{W}(\Omega_T)$ and $u_t(P) = -v(P)$, $0 \le t \le \tau$. Integration by parts allows to write

$$\begin{aligned} & \frac{\alpha}{2}\|v(\tau)\|_2^2 + (v_\tau(\tau), v(\tau))_2 + \frac{c^2}{2}\|\nabla u(0)\|_2^2 \\ = & \left(\int_0^t \mathcal{K}(t-s)\nabla v(s)ds, \nabla u(t)\right)_{2,\Omega_T} \\ + & \int_0^\tau \|v_t(t)\|_2^2\,dt - b\int_0^\tau \|\nabla u_t(t)\|_2^2\,dt \\ + & \left(|v_1|^{p-2}v_1 - |v_2|^{p-2}v_2, u\right)_{2,\Omega_T}. \end{aligned} \tag{28}$$

We set

$$\Sigma(t) := \|v(t)\|_{H_0^1(\Omega}^2 + \|v_t(t)\|_{H^1(\Omega)}^2 + \|v_{tt}(t)\|_2^2 + \|\nabla\vartheta(t)\|_2^2, \tag{29}$$

where

$$\vartheta(P) := \int_0^t v\,(x,s)\,ds. \tag{30}$$

Using (27) and (30), we get

$$u(P) = \vartheta\,(x,\tau) - \vartheta(P), \quad \nabla u\,(x,0) = \nabla\vartheta\,(x,\tau),$$

and

$$\int_0^\tau \|\nabla u(t)\|_2^2\,dt \le 2\tau\|\nabla\vartheta(\tau)\|_2^2 + 2\int_0^\tau \|\nabla\vartheta\,(t)\|_2^2\,dt. \tag{31}$$

Using (27) and (29), we get

$$\int_0^\tau \|u(t)\|_2^2\,dt \le \frac{T^2}{2}\int_0^\tau \Sigma(t)\,dt. \tag{32}$$

It holds

$$-\frac{1}{2}\|v_\tau(\tau)\|_2^2 - \frac{1}{2}\|v(\tau)\|_2^2 \le (v_\tau(\tau), v(\tau))_2. \tag{33}$$

Cauchy-Schwarz inequality, (27) and (31), lead to

$$\begin{aligned}
&\left(\int_0^t \mathcal{K}(t-s)\nabla v(s)ds, \nabla u(t)\right)_{2,\Omega_T} \\
\leq\ & \frac{\sup(\mathcal{K}^2(t))T^2+4}{4}\int_0^\tau \Sigma(t)\,dt + \tau\left\|\nabla\vartheta(\tau)\right\|_2^2 . \qquad (34)
\end{aligned}$$

Using Cauchy-Schwarz inequality, Lemma 2.3, (27), and (29), we obtain

$$\begin{aligned}
&\left(\left|v_1\right|^{p-2} v_1 - \left|v_2\right|^{p-2} v_2, u\ \right)_{2,\Omega_T} \\
\leq\ & C_T' \int_0^\tau \left(\Sigma(t)\right)^{p-1}\,dt + \frac{(2+T^2)}{2}\int_0^\tau \Sigma(t)\,dt. \qquad (35)
\end{aligned}$$

Combining (33)-(35), and (28), results in

$$\begin{aligned}
&\frac{(\alpha-1)}{2}\left\|v(\tau)\right\|_2^2 - \frac{1}{2}\left\|v_\tau(\tau)\right\|_2^2 + \left(\frac{c^2}{2}-\tau\right)\left\|\nabla\vartheta(\tau)\right\|_2^2 \\
\leq\ & C_T' \int_0^\tau (\Sigma(t))^{p-1}\,dt + \frac{\sup(\mathcal{K}^2(t))T^2+12+2T^2}{4}\int_0^\tau \Sigma(s)\,ds. \qquad (36)
\end{aligned}$$

Now, multiplying (25) by v_t, then integrating over Ω_τ , we get

$$\begin{aligned}
&\frac{1}{2}\left(c^2 - \int_0^\tau \mathcal{K}(s)ds\right)\left\|\nabla v(\tau)\right\|_2^2 + \frac{\alpha}{2}\left\|v_\tau(\tau)\right\|_2^2 \\
+\ & \left(v_{\tau\tau}(\tau), v_\tau(\tau)\ \right)_2 + \frac{1}{2}\left(\mathcal{K}\circ\nabla v\right)(\tau) \\
=\ & -\beta\int_0^\tau \left\|\nabla v_t(t)\right\|_2^2\,dt + \int_0^\tau \left\|v_{tt}(t)\right\|_2^2\,dt \\
+\ & \frac{1}{2}\int_0^\tau \left(\mathcal{K}'\circ\nabla v\right)(s)\,ds - \frac{1}{2}\int_0^\tau \mathcal{K}(s)\left\|\nabla v(s)\right\|_2^2\,ds \\
+\ & \left(\left|v_1\right|^{p-2} v_1(t) - \left|v_2\right|^{p-2} v_2(t), v_t(t)\right)_{2,\Omega_T} . \qquad (37)
\end{aligned}$$

It holds

$$-\frac{\vartheta_3}{2}\left\|v_{\tau\tau}(\tau)\right\|_2^2 - \frac{1}{2\vartheta_3}\left\|v_\tau(\tau)\right\|_2^2 \leq \left(v_{\tau\tau}(\tau), v_\tau\,(\tau)\ \right)_2 . \qquad (38)$$

Using Cauchy-Schwarz inequality, Lemma 2.3 and (29), we obtain

$$\left(\left|v_1\right|^{p-2} v_1 - \left|v_2\right|^{p-2} v_2, v_t\right)_{2,\Omega_T} \leq C_T' \int_0^\tau \left(\Sigma(s)\right)^{p-1}\,ds + \frac{3}{2}\int_0^\tau \Sigma(s)\,ds. \qquad (39)$$

Inequalities (38) and (39) in (37) allow to write

$$\begin{aligned} & \frac{\left(c^2-\bar{\mathcal{K}}\right)}{2}\left\|\nabla v\left(\tau\right)\right\|_2^2+\left(\frac{\alpha}{2}-\frac{1}{2\vartheta_3}\right)\left\|v_\tau(\tau)\right\|_2^2 \\ - & \frac{\vartheta_3}{2}\left\|v_{\tau\tau}(\tau)\right\|_2^2+\frac{1}{2}\left(\mathcal{K}\circ\nabla v\right)(\tau) \\ \leq & C_T'\int_0^\tau\left(\Sigma(s)\right)^{p-1}\,ds+\frac{5}{2}\int_0^\tau\Sigma(s)\,ds. \end{aligned} \tag{40}$$

Clearly, we have

$$\begin{aligned} & c^2\left(\nabla v(\tau),\nabla v_\tau(\tau)\right)_2+\frac{\beta}{2}\left\|\nabla v_\tau(\tau)\right\|_2^2 \\ + & \frac{\mathcal{K}(\tau)}{2}\left\|\nabla v(\tau)\right\|_2^2+\frac{1}{2}\left(-\mathcal{K}'\circ\nabla v\right)(\tau)+\frac{1}{2}\left\|v_{\tau\tau}(\tau)\right\|_2^2 \\ - & \int_0^\tau\mathcal{K}\left(\tau-\sigma\right)\left(\nabla v(\sigma),\nabla v_\tau(\tau)\right)_2\,d\sigma \\ = & \frac{1}{2}\int_0^\tau\mathcal{K}'(s)\left\|\nabla v(s)\right\|_2^2\,ds+c^2\int_0^\tau\left\|\nabla v_t(t)\right\|_2^2\,dt \\ - & \alpha\int_0^\tau\left\|v_{tt}(t)\right\|_2^2\,dt-\frac{1}{2}\int_0^\tau\left(\mathcal{K}''\circ\nabla v\right)(s)\,ds \\ + & \left(\left|v_1\right|^{p-2}v_1(t)-\left|v_2\right|^{p-2}v_2(t),v_{tt}(t)\ \right)_{2,\Omega_T}. \end{aligned} \tag{41}$$

It holds,

$$-\frac{\vartheta_4 c^2}{2}\left\|\nabla v(\tau)\right\|_2^2-\frac{c^2}{2\vartheta_4}\left\|\nabla v_\tau(\tau)\right\|_2^2\leq c^2\left(\nabla v(\tau),\nabla v_\tau(\tau)\right)_2. \tag{42}$$

Appealing to Young's inequality with $\varepsilon=\vartheta_5$ and using $-\int_0^\tau h\left(s\right)\,ds\geq-\bar{\mathcal{K}}$, we get

$$\begin{aligned} & -\frac{\vartheta_5\bar{\mathcal{K}}}{2}\left\|\nabla v(\tau)\right\|_2^2-\frac{\vartheta_5}{2}\left(\mathcal{K}\circ\nabla v\right)(\tau)-\frac{\bar{\mathcal{K}}}{\vartheta_5}\left\|\nabla v_\tau(\tau)\right\|_2^2 \\ \leq & -\int_0^\tau h\left(\tau-s\right)\left(\nabla v(s),\nabla v_\tau(\tau)\right)_2\,ds. \end{aligned} \tag{43}$$

Using Cauchy-Schwarz inequality, Lemma 2.3, and (29), we obtain

$$\left(\left|v_1\right|^{p-2}v_1-\left|v_2\right|^{p-2}v_2,v_{tt}\right)_{2,\Omega_T}\leq C_T'\int_0^\tau\left(\Sigma(s)\right)^{p-1}\,ds+\frac{3}{2}\int_0^\tau\Sigma(s)\,ds. \tag{44}$$

Combining inequalities (42) $-$ (44) in (41), we get

$$
\begin{aligned}
& -\left(\frac{\vartheta_4 c^2}{2}+\frac{\vartheta_5 \bar{\mathcal{K}}}{2}\right)\|\nabla v(\tau)\|_2^2+\left(\frac{\beta}{2}-\frac{c^2}{2\vartheta_4}-\frac{\bar{\mathcal{K}}}{\vartheta_5}\right)\|\nabla v_\tau(\tau)\|_2^2 \\
+ & \frac{1}{2}\|v_{\tau\tau}(\tau)\|_2^2-\frac{\vartheta_5}{2}(\mathcal{K}\circ\nabla v)(\tau) \\
\leq & C_T' \int_0^\tau (\Sigma(s))^{p-1}\, ds+\frac{(2c^2+3)}{2}\int_0^\tau \Sigma(s)\, ds. \qquad (45)
\end{aligned}
$$

Multiplying (36) by λ_3, (40) by λ_4, (45) by λ_5, and using the following inequalities

$$\alpha_4\|v(\tau)\|_2^2 \leq \alpha_4 \int_0^\tau \Sigma(s)\, ds,$$

$$\alpha_5\|\nabla v(\tau)\|_2^2 \leq \alpha_5 \int_0^\tau \Sigma(s)\, ds,$$

and

$$\alpha_6\|v_\tau(\tau)\|_2^2 \leq \alpha_6 \int_0^\tau \Sigma(s)\, ds,$$

where

$$\alpha_4 := \frac{\lambda_3}{2}>0, \alpha_5 := \frac{\lambda_5(\vartheta_4 c^2+\vartheta_5\bar{\mathcal{K}})}{2}>0, \quad \vartheta_4=\frac{3c^2}{\beta}>0,$$

$$\alpha_6 := \frac{\lambda_3}{2}+\frac{\lambda_4}{2\vartheta_3}>0, \quad \vartheta_3=\vartheta_5=\frac{\lambda_5}{2\lambda_4}=\frac{6\bar{\mathcal{K}}}{\beta}>0,$$

we get

$$
\begin{aligned}
\frac{\lambda_3\alpha}{2}\|v(\tau)\|_2^2 \quad + & \quad \frac{\lambda_4\left(c^2-\bar{\mathcal{K}}\right)}{2}\|\nabla v(\tau)\|_2^2+\frac{\lambda_4\alpha}{2}\|v_\tau(\tau)\|_2^2+\frac{\lambda_5 b}{6}\|\nabla v_\tau(\tau)\|_2^2 \\
+ & \quad \frac{\lambda_5}{4}\|v_{\tau\tau}(\tau)\|_2^2+\frac{\lambda_4}{4}(h\circ\nabla v)(\tau)+\frac{\lambda_3\left(c^2-2\tau\right)}{2}\|\nabla\vartheta(\tau)\|_2^2 \\
\leq & \quad (\lambda_3+\lambda_4+\lambda_5)\, C_T' \int_0^\tau (\Sigma(s))^{p-1}\, ds+C_5\int_0^\tau \Sigma(s)\, ds, \quad (46)
\end{aligned}
$$

where

$$C_5 := C_4+\alpha_4+\alpha_5+\alpha_6>0.$$

Since τ is arbitrary, we assume that $c^2 - 2\tau > 0$; thus inequality (46) takes the form

$$\Sigma(\tau) \leq \omega' \int_0^\tau (\Sigma(s))^{p-1}\, ds + \omega' \int_0^\tau \Sigma(s)\, ds, \tag{47}$$

where

$$\omega' := \frac{\max\left\{(\lambda_3 + \lambda_4 + \lambda_5)\, C_T', C_5\right\}}{\min\left\{\frac{\lambda_3 \alpha}{2}, \frac{\lambda_4\left(c^2 - \bar{\mathcal{K}}\right)}{2}, \frac{\lambda_4 \alpha}{2}, \frac{\lambda_5 b}{6}, \frac{\lambda_5}{4}, \frac{\lambda_3\left(c^2 - \tau\right)}{2}\right\}} > 0.$$

Then, solving (47) (see [18]), we obtain

$$\|v(\tau)\|_{H_0^1(\Omega)}^2 + \|v_\tau(\tau)\|_{H^1(\Omega)}^2 + \|v_{\tau\tau}(\tau)\|_2^2 + \|\nabla\vartheta(\tau)\|_2^2 \leq 0, \quad \tau \in \left[0, \frac{c^2}{2}\right].$$

Similarly, for $\tau \in \left[\frac{(N-1)\, c^2}{2}, \frac{Nc^2}{2}\right]$ to cover $[0,T]$, one shows that $v = 0$ on $[0,T]$. $\square$

4. Blowing-up Solutions

Definition 4.1. *A blowing-up solution u of problem* $(1)-(3)$ *is a solution that exists till a finite time T_{max} such that*

$$\lim_{t \to T_{max}^-} \|\nabla v(t)\|_2 = \infty. \tag{48}$$

The energy of problem $(1)-(3)$ is

$$\begin{aligned}
\Xi(t) \;:\; &= \frac{k}{2}\|v_{tt}\|_2^2 + \frac{\alpha}{2}\|v_t\|_2^2 + (v_{tt}, v_t)_2 + kc^2(\nabla v, \nabla v_t)_2 \\
&+ \frac{k\beta}{2}\|\nabla v_t\|_2^2 + \frac{k}{2}(-\mathcal{K}' \circ \nabla v)(t) + \frac{k}{2}\mathcal{K}(t)\|\nabla v\|_2^2 \\
&+ \frac{1}{2}\left(c^2 - \int_0^t \mathcal{K}(\sigma)\, d\sigma\right)\|\nabla v\|_2^2 - k\int_0^t \mathcal{K}(t-s)\,(\nabla v(\sigma), \nabla v_t(t))_2\, d\sigma \\
&+ \frac{1}{2}(\mathcal{K} \circ \nabla v)(t) - \frac{1}{p}\|v\|_p^p,
\end{aligned} \tag{49}$$

where the constant k satisfies

$$\frac{1}{\alpha} < 1 < k < \frac{\beta}{2c^2}. \tag{50}$$

Lemma 4.2. *Let u be a solution of problem* $(1)-(3)$*, then on* $[0,t)$

$$\begin{aligned} \Xi'(t) &= (1-k\alpha)\left\|v_{tt}\right\|_2^2+(kc^2-\beta)\left\|\nabla v_t\right\|_2^2 \\ &- \frac{k}{2}\left(\mathcal{K}''\circ\nabla v\right)(t)+\frac{1}{2}\left(\mathcal{K}'\circ\nabla v\right)(t) \\ &+ \frac{k}{2}\mathcal{K}'(t)\left\|\nabla v\right\|_2^2-\frac{1}{2}\mathcal{K}(t)\left\|\nabla v\right\|_2^2 \\ &\leq 0. \end{aligned} \tag{51}$$

Proof. Multiplying equation of (1) by v_t+kv_{tt} and integrating by parts, we obtain (51). □

Remark 4.3. *Integrating* (51) *over* $(0,t)$*, it comes*

$$\begin{aligned} \Xi(t) &= \Xi(0)+(1-k\alpha)\int_0^t\left\|v_{\sigma\sigma}\right\|_2^2\,d\sigma+(kc^2-\beta)\int_0^t\left\|\nabla v_\sigma\right\|_2^2\,d\sigma \\ &- \frac{k}{2}\int_0^t\left(\mathcal{K}''\circ\nabla v\right)(\sigma)\,d\sigma+\frac{1}{2}\int_0^t\left(\mathcal{K}'\circ\nabla v\right)(\sigma)\,d\sigma \\ &+ \frac{k}{2}\int_0^t\mathcal{K}'(\sigma)\left\|\nabla v\right\|_2^2\,d\sigma-\frac{1}{2}\int_0^t\mathcal{K}(\sigma)\left\|\nabla v\right\|_2^2\,d\sigma. \end{aligned} \tag{52}$$

For a solution v of $(1)-(3)$, we let

$$\Psi(t) := \Psi_1(t)+\Psi_2(t), \tag{53}$$

where

$$\begin{aligned} \Psi_1(t) \;:&= (v_t,v)_2+\int_0^T\left|\frac{d}{ds}(v_s,v)_2\right|ds \\ &+\left|(v_t(0),v(0))_2\right|+(T-t)\int_0^T\left|\frac{d}{ds}\left|(v_{ss},v)_2\right|\right|ds \\ &+\frac{\alpha}{2}\left\|v\right\|_2^2+\alpha(T-t)\int_0^T\left|\frac{d}{ds}\left|(v_t,v)_2\right|\right|ds \\ &+\frac{\beta}{2}\int_0^t\left\|\nabla v\right\|_2^2\,ds-\frac{3}{2}\int_0^t\left\|v_s\right\|_2^2\,ds+\frac{3}{2}\int_0^T\left\|v_s\right\|_2^2\,ds, \end{aligned}$$

and

$$\begin{aligned} \Psi_2(t) \;:&= k_1\int_0^t\int_0^s\left\|\nabla v\right\|_2^2 d\mu\,ds+k_1(T-t)\int_0^T\left\|\nabla v\right\|_2^2\,ds \\ &+k_2\int_0^t\int_0^s\left\|\nabla v_\mu\right\|_2^2 d\mu\,ds+k_2(T-t)\int_0^T\left\|\nabla v_s\right\|_2^2\,ds, \end{aligned}$$

$k_1 > 0$, $k_2 > 0$ are constants.

Lemma 4.4. *Suppose that the hypotheses* $(\mathbf{H}_1) - (\mathbf{H}_2)$ *hold and*

$$\bar{\mathcal{K}} < \frac{p\,(p-2)}{(p-1)^2+1}c^2 < c^2, \tag{54}$$

then we have

$$\frac{p\beta}{2}\|\nabla v\|_2^2 + \frac{p\beta}{2}\int_0^t \|\nabla v_s\|_2^2\,ds - p\Xi\,(0) \leq \Psi''(t). \tag{55}$$

Proof. Form (53) and using the solution v of problem $(1)-(3)$, we get

$$\begin{aligned} \Psi''(t) &= (k_1 - c^2)\|\nabla v\|_2^2 + \alpha\|v_t\|_2^2 + k_2\|\nabla v_t\|_2^2 \\ &+ \|v\|_p^p + \int_0^t \mathcal{K}(t-s)\,(\nabla v(s), \nabla v(t))_2\,ds. \end{aligned} \tag{56}$$

Multiplying Eq. (52) by p and summing up in(56), we obtain

$$\begin{aligned} \Psi''(t) &= -p\Xi(0) + \frac{pk}{2}\|v_{tt}\|_2^2 + p(k\alpha-1)\int_0^t \|v_{ss}\|_2^2\,ds \\ &+p(\beta - kc^2)\int_0^t \|\nabla v_s\|_2^2\,ds + \left(\alpha + \frac{p\alpha}{2}\right)\|v_t\|_2^2 \\ &+\left(k_1 + \frac{(p-2)c^2}{2} - \frac{p}{2}\left(\int_0^t \mathcal{K}(s)\,ds\right)\right)\|\nabla v\|_2^2 \\ &+\left(k_2 + \frac{pk\beta}{2}\right)\|\nabla v_t\|_2^2 + p\,(v_{tt}, v_t)_2 \\ &+pkc^2\,(\nabla v, \nabla v_t)_2 + \frac{p}{2}\,(\mathcal{K}\circ\nabla v)\,(t) \\ &+\int_0^t \mathcal{K}(t-s)\,(\nabla v(s), \nabla v(t))_2\,ds \\ &-pk\int_0^t \mathcal{K}(t-s)\,(\nabla v(s), \nabla v_t(t))_2\,ds \\ &+\frac{pk}{2}\mathcal{K}(t)\|\nabla v\|_2^2 + \frac{pk}{2}\,(-\mathcal{K}'\circ\nabla v)\,(t) \\ &+\frac{p}{2}\int_0^t \mathcal{K}(s)\|\nabla v\|_2^2\,ds - \frac{pk}{2}\int_0^t h'(s)\|\nabla v\|_2^2\,ds \\ &-\frac{p}{2}\int_0^t (\mathcal{K}'\circ\nabla v)\,(s)\,ds + \frac{pk}{2}\int_0^t (\mathcal{K}''\circ\nabla v)\,(s)\,ds. \end{aligned} \tag{57}$$

Young's inequality with $\varepsilon = k$, allows to write get

$$p\,(v_{tt}, v_t)_2 \geq -\frac{pk}{2}\,\|v_{tt}\|_2^2 - \frac{p}{2k}\,\|v_t\|_2^2\,. \tag{58}$$

It holds

$$pkc^2\,(\nabla v, \nabla v_t)_2 \geq -\frac{pkc^2}{2}\,\|\nabla v\|_2^2 - \frac{pkc^2}{2}\,\|\nabla v_t\|_2^2\,. \tag{59}$$

Young's inequality, with $\varepsilon = \dfrac{p}{2}$, allows to get

$$\begin{aligned} &\int_0^t \mathcal{K}(t-\sigma)\,(\nabla v(\sigma), \nabla v(t))_2\,d\sigma \\ \geq\;& \left(1-\frac{1}{p}\right)\left(\int_0^t \mathcal{K}\,(s)\;ds\right)\|\nabla v\|_2^2 - \frac{p}{4}\,(\mathcal{K}\circ\nabla v)\,(t). \end{aligned} \tag{60}$$

Using $-\int_0^t \mathcal{K}(s)\,ds \geq -c^2$, we obtain

$$\begin{aligned} &-pk\int_0^t \mathcal{K}(t-s)\,(\nabla u(s), \nabla u_t(t))_2\;ds \\ \geq\;& -\frac{p}{4}\,(\mathcal{K}\circ\nabla v)\,(t) - pk\left(k+\frac{1}{2}\right)c^2\,\|\nabla v_t\|_2^2 - \frac{pkc^2}{2}\,\|\nabla v\|_2^2\,. \end{aligned} \tag{61}$$

Combining inequalities (58) – (61) into (57) and using (50), (54), we get

$$\begin{aligned} &\Psi''(t) - \left(k_1 - pkc^2\right)\|\nabla v\|_2^2 \\ \geq\;& -p\Xi\,(0) + p\left(\beta - kc^2\right)\int_0^t \|\nabla v_s\|_2^2\;ds + \left(k_2 - pk^2c^2\right)\|\nabla v_t\|_2^2\,. \end{aligned}$$

Let $k_1 := pkc^2 + \dfrac{p\beta}{2}$ and $k_2 := pk^2c^2$, then, for $p > 4$, (55) is obtained. $\square$

Lemma 4.5. *Suppose* $(\mathbf{H}_1) - (\mathbf{H}_2)$ *and* (54) *hold and that either one of the following condition is satisfied*

(i) $\Xi\,(0) < 0$,

(ii) $\Xi\,(0) = 0$, *and*

$$\Psi'\,(0) > \frac{\beta}{2}\,\|\nabla v\,(0)\|_2^2 + \left|(v_{tt}\,(0)\,, v\,(0))_2\right| + \alpha\left|(v_t\,(0)\,, v\,(0))_2\right|. \tag{62}$$

(iii) $\Xi(0) > 0$ *and*

$$\begin{aligned}\Psi'(0) &> r_*\left[\Psi(0)+k_0\right] \\ &+ \frac{\beta}{2}\left\|\nabla v(0)\right\|_2^2 + \left|\left(v_{tt}(0), v(0)\right)_{L^2(\Omega)}\right| + \alpha\left|\left(v_t(0), v(0)\right)_2\right|, \quad (63)\end{aligned}$$

where

$$r_* := \frac{p-\sqrt{p(p-4)}}{2},$$

and

$$k_0 := \frac{\beta}{2}\left\|\nabla v(t)\right\|_2^2 + \left|\left(v_{tt}(t), v(t)\right)_2\right| + \alpha\left|\left(v_t(t), v(t)\right)_2\right| + \Xi(t). \tag{64}$$

Then

$$\Psi'(t) \geq \frac{\beta}{2}\left\|\nabla v(t)\right\|_2^2 + \left|(v_{tt}(t), v(t))_2\right| + \alpha\left|(v_t(t), v(0))_2\right|, \tag{65}$$

for $t > t_*$, and

$$t^* := \max\left\{0, \frac{\Lambda}{p\Xi(0)}\right\}, \tag{66}$$

where we have set

$$\Lambda := \Psi'(0) - \frac{\beta}{2}\left\|\nabla v(0)\right\|_2^2 - \left|\left(v_{tt}(0), v(0)\right)_2\right| - \alpha\left|\left(v_t(0), v(0)\right)_2\right|,$$

and $t_* = t^*$ *in case* (**i**) *and* $t_* = 0$ *in case* (**ii**) *and* (**iii**).

Proof. (**i**) If $\Xi(0) < 0$, then $-p\Xi(t) > 0$.
Using (55), we get

$$\Psi''(t) \geq -p\Xi(t); \tag{67}$$

whereupon

$$\Psi'(t) - \Psi'(0) \geq -p\Xi(0)t. \tag{68}$$

Then (68) is written

$$\begin{aligned}\Psi'(t) \geq \frac{\beta}{2}\left\|\nabla v(0)\right\|_2^2 + \left|(v_{tt}(0), v(0))_2\right| + \alpha\left|(v_t(0), v(0))_2\right| \\ + \left(-p\Xi(0)t + \Psi'(0) - \frac{\beta}{2}\left\|\nabla v(0\right\|_2^2 - \left|\left(v_{tt}(0), v(0)\right)_2\right| - \alpha\left|(v_t(0), v(0))_2\right|\right).\end{aligned} \tag{69}$$

Let

$$t \geq \frac{\Psi'(0) - \frac{\beta}{2} \|\nabla v(0)\|_2^2 - |(v_{tt}(0), v(0))_2| - \alpha |(v_t(0), v(0))_2|}{p\Xi(0)}. \tag{70}$$

Substituting (70) into (69), using the definition of t^* given in (66), we obtain directly (65).

(ii) If $\Xi(0) = 0$, then using (55), we obtain

$$\Psi''(t) \geq 0, \ t \geq 0; \tag{71}$$

whereupon

$$\Psi'(t) - \Psi'(0) \geq 0. \tag{72}$$

Then (72) is written in form

$$\begin{aligned} \Psi'(t) \geq & \frac{\beta}{2} \|\nabla v(0)\|_2^2 + |(v_{tt}(0), v(0))_2| + \alpha |(v_t(0), v(0))_2| \\ + & \left\{ \Psi'(0) - \frac{\beta}{2} \|\nabla v(0)\|_2^2 - |(v_{tt}(0), v(0))_2| - \alpha |(v_t(0), v(0))_2| \right\}. \end{aligned} \tag{73}$$

Furthermore, if (62) holds, hence we obtain (65) for $t \geq 0$.

(iii) When $\Xi(0) > 0$, making use of (53), we get

$$\begin{aligned} \Psi'(t) & - \frac{\beta}{2} \|\nabla v(0)\|_2^2 - |(v_{tt}(0), v(0))_2| - \alpha |(v_t(0), v(0))_2| \\ & \leq (v_{tt}, v)_2 - |(v_{tt}(0), v(0))_2| - \int_0^T \left| \frac{d}{ds} |(v_{ss}, v)_2| \right| ds \\ & + \frac{\beta}{2} \|\nabla v\|_2^2 - \frac{\beta}{2} \|\nabla v(0)\|_2^2 + \alpha (v_t, v)_2 \\ & - \alpha |(v_t(0), v(0))_2| - \alpha \int_0^T \left| \frac{d}{ds} |(v_t, v)_2| \right| ds. \end{aligned} \tag{74}$$

Using

$$(v_{tt}, v)_2 - |(v_{tt}(0), v(0))_2| - \int_0^T \left| \frac{d}{ds} |(v_{ss}, v)_2| \right| ds \leq 0 \tag{75}$$

and

$$\alpha (v_t, v)_2 - \alpha |(v_t(0), v(0))_2| - \alpha \int_0^T \left| \frac{d}{ds} |(v_t, v)_2| \right| ds \leq 0 \tag{76}$$

into (74), we obtain

$$\begin{aligned}\Psi'(t) &- \frac{\beta}{2}\|\nabla v(0)\|_2^2 - |(v_{tt}(0), v(0))_2| - \alpha\,|(v_t(0), v(0))_2| \\ &\leq \Psi(t) + \frac{\beta}{2}\|\nabla v\|_2^2 .\end{aligned}$$

Then

$$\begin{aligned}p\left(\Psi'(t) - \Psi(t)\right) &- \frac{p\beta}{2}\|\nabla v\left(0\right)\|_2^2 - p\left|\left(v_{tt}\left(0\right), v\left(0\right)\right)_2\right| - p\alpha\left|\left(v_t\left(0\right), v\left(0\right)\right)_2\right| \\ &\leq \frac{p\beta}{2}\|\nabla v\|_2^2 .\end{aligned} \tag{77}$$

Using (55), we get

$$\frac{p\beta}{2}\|\nabla v\|_2^2 \leq \Psi''(t) + p\Xi(0). \tag{78}$$

Using (78) into (77), we get

$$\begin{aligned}p\left(\Psi'(t) - \Psi(t)\right) &- \frac{p\beta}{2}\|\nabla v(0)\|_2^2 - p\,|(v_{tt}(0), v(0))_2| - p\alpha\,|(v_t(0), v(0))_2| \\ &\leq \Psi''(t) + p\Xi\left(0\right).\end{aligned}$$

Then

$$\Psi''(t) - p\Psi'(t) + p\Psi(t) + pk_0 \geq 0,$$

where k_0 is defined in (64).
Let

$$B(t) := \Psi(t) + k_0,$$

then $B'(t) = \Psi'(t), B''(t) = \Psi''(t)$, and $B(t)$ satisfies

$$B''(t) - pB'(t) + pB\left(t\right) \geq 0. \tag{79}$$

Therefore, using Lemma 2.4 in (79) and (63), (65) is obtained. □

Theorem 4.6. *Suppose that the hypotheses* $(\mathbf{H}_1) - (\mathbf{H}_2)$ *and* (54) *hold and that either one of the following conditions is satisfied*
(i) $\Xi\left(0\right) < 0$,
(ii) $\Xi\left(0\right) = 0$ *and* (62) *holds,*

(iii) $0 < \Xi(0) < \dfrac{\vartheta}{\left(\dfrac{(p-2)^2}{2} - 2\right)\left(\dfrac{1}{2} - \dfrac{1}{p-2}\right)}$,

where

$$\vartheta := \left(\frac{p-4}{4}\right)^2 \Big(\Psi'(t_*) - \frac{\beta}{2}\left\|\nabla v(0)\right\|_2^2 - |(v_{tt}(0), v(0))_2| - \alpha\left|(v_t(0), v(0))_2\right|\Big)^2 \Lambda(t_*)^{1/\xi},$$

and (63) *holds. Then the solution v blows-up at finite time T^*.*

In case (**i**),

$$T^* \leq t_* - \frac{\Lambda(t_*)}{\Lambda'(t_*)}.$$

Furthermore, when $\Lambda\left(t_*\right) < \min\left\{1, \left(\dfrac{\nu}{-\beta}\right)^{1/2}\right\}$, *then*

$$T^* \leq t_* + \frac{1}{\sqrt{-\beta}} \ln \frac{\left(\dfrac{\nu}{-\beta}\right)^{1/2}}{\left(\dfrac{\nu}{-\beta}\right)^{1/2} - \Lambda\left(t_*\right)}.$$

In case (**ii**)

$$T^* \leq t_* - \frac{\Lambda\left(t_*\right)}{\Lambda'(t_*)},$$

or

$$T^* \leq t_* + \frac{\Lambda\left(t_*\right)}{\Lambda'(t_*)}.$$

In case (**iii**)

$$T^* \leq \frac{\Lambda\left(t_0\right)}{\sqrt{\mu}},$$

or

$$T_{max} \leq t_* + \frac{\xi c}{\sqrt{\mu}} 3^{(3\xi+1)/2\xi}\Big(1 - (1 + c\Lambda(t_*)\Big)^{1/2\xi},$$

where $c := (\beta/\mu)^{\xi/(2+\xi)}$, $\xi := (p-4)/4$, *and* $\Lambda(t)$, μ *and* β *are given here after in* (80), (94) *and* (95), *respectively.*

Note that in ***case*** (**i**), $t_* = t^*$ *is given in* (66) *and* $t_* = 0$ *in* ***case*** (**ii**) *and* (**iii**).

Proof. Let

$$\begin{aligned}\Lambda(t) \quad : \quad &= \Big(\Psi(t) + (T-t)\frac{\beta}{2}\left\|\nabla v(0)\right\|_2^2 + (T-t)\left|(v_{tt}(0), v(0))_2\right| \\ &+ (T-t)\left|(v_t(0), v(0))_2\right|\Big)^{-\xi}. \end{aligned} \tag{80}$$

It holds

$$\Lambda'(t) = -\xi\Lambda(t)^{1+\frac{1}{\xi}}\left(\Psi'(t) - \frac{\beta}{2}\left\|\nabla v(0)\right\|_2^2 - \left|(v_{tt}(0), v(0))_2\right| - \left|(v_t(0), v(0))_2\right|\right).$$

Then

$$\Lambda''(t) = -\xi\Lambda(t)^{1+\frac{2}{\xi}}Q(t), \tag{81}$$

for

$$\begin{aligned}Q(t) \quad : \quad &= \Psi''(t)\Big(\Psi(t) + (T-t)\frac{\beta}{2}\left\|\nabla v(0)\right\|_2^2 \\ &+ (T-t)\left|(v_{tt}(0), v(0))_2\right| + (T-t)\left|(v_t(0), v(0))_2\right|\Big) \\ &-(1+\xi)\Big(\Psi'(t) - \frac{\beta}{2}\left\|\nabla v(0)\right\|_2^2 - \left|(v_{tt}(0), v(0))_2\right| \\ &- \left|v_t(0), v(0))_2\right|\Big)^2. \end{aligned} \tag{82}$$

From (53), we have

$$\begin{aligned}\Psi(t) \quad &+ \quad (T-t)\frac{\beta}{2}\left\|\nabla v(0)\right\|_2^2 + (T-t)\left|(v_{tt}(0), v(0))_2\right| \\ &+ \quad (T-t)\left|(v_t(0), v(0))_2\right| \\ &\geq \quad \frac{\beta}{2}\int_0^t \left\|\nabla v\right\|_2^2 ds, \end{aligned} \tag{83}$$

and from (55), we have

$$\Psi''(t) \geq -p\Xi(0) + \frac{p\beta}{2}\int_0^t \left\|\nabla v_s\right\|_2^2 ds. \tag{84}$$

From (53), we get

$$\begin{aligned}
&\Psi'(t) - \frac{\beta}{2} \|\nabla v(0)\|_2^2 - |(v_{tt}(0), v(0))_2| - \alpha |(v_t(0), v(0))_2| \\
\leq\ & (v_{tt}, v)_2 - |(v_{tt}(0), v(0))_2| - \int_0^T \left| \frac{d}{ds} |(v_{ss}, v)_2| \right| ds \\
+\ & \frac{\beta}{2} \|\nabla v\|_2^2 - \frac{\beta}{2} \|\nabla v(0)\|_2^2 + \alpha (v_t, v)_2 \\
-\ & \alpha |(v_t(0), v(0))_2| - \alpha \int_0^T \left| \frac{d}{ds} |(v_t, v)_2| \right| ds.
\end{aligned}$$

Using (75) and (76), and

$$\frac{\beta}{2} \|\nabla v(t)\|_2^2 - \frac{\beta}{2} \|\nabla v(0)\|_2^2 = \beta \int_0^t (\nabla v(s), \nabla v_s(s))_2 \, ds,$$

we get

$$\begin{aligned}
\Psi'(t) \ &-\ \frac{\beta}{2} \|\nabla v(0)\|_2^2 - |(v_{tt}(0), v(0))_2| - \alpha |(v_t(0), v(0))_2| \\
&\leq\ \beta \int_0^t (\nabla v(s), \nabla v_s(s))_2 \, ds.
\end{aligned}$$

Then

$$\begin{aligned}
&\left(\Psi'(t) - \frac{\beta}{2} \|\nabla v(0)\|_2^2 - |(v_{tt}(0), v(0))_2| - \alpha |(v_t(0), v(0))_2| \right)^2 \qquad (85) \\
\leq\ & 4 \left(\frac{\beta}{2} \int_0^t (\nabla v(s), \nabla v_s(s))_2 \, ds \right)^2.
\end{aligned}$$

Multiplying (85) by $-(1+\xi)$, we obtain

$$\begin{aligned}
-\ & (1+\xi) \left(\Psi'(t) - \frac{\beta}{2} \|\nabla v(0)\|_2^2 - |(v_{tt}(0), v(0))_2| - \alpha |(v_t(0), v(0))_2| \right)^2 \\
\geq\ & -4(1+\xi) \left(\frac{\beta}{2} \int_0^t (\nabla v(s), \nabla v_s(s))_2 \, ds \right)^2. \qquad (86)
\end{aligned}$$

Substituting (83), (84) and (86) into (82), we get

$$
\begin{aligned}
Q(t) \quad \geq \quad & -p\Xi(0)\Big(\Psi(t) + (T-t)\frac{\beta}{2}\left\|\nabla v(0)\right\|_2^2 \\
& + (T-t)\left|(v_{tt}(0), v(0))_2\right| + (T-t)\left|(v_t(0), v(0))_2\right|\Big) \\
& + p\left(\frac{\beta}{2}\int_0^t \left\|\nabla v_s\right\|_2^2 \, ds\right)\left(\frac{\beta}{2}\int_0^t \left\|\nabla v\right\|_2^2 \, ds\right) \\
& - 4\left(1+\xi\right)\left(\frac{\beta}{2}\int_0^t \left(\nabla v\left(s\right), \nabla v_s\left(s\right)\right)_2 \, ds\right)^2 .
\end{aligned}
$$

Let us set

$$
\begin{aligned}
\mathcal{A} \quad : & = \frac{\beta}{2}\int_0^t \left\|\nabla v\right\|_2^2 \, ds, \\
\mathcal{B} \quad : & = \frac{\beta}{2}\int_0^t \left(\nabla v\left(s\right), \nabla v_s\left(s\right)\right)_2 \, ds, \\
\mathcal{C} \quad : & = \frac{\beta}{2}\int_0^t \left\|\nabla v_s\right\|_2^2 \, ds.
\end{aligned}
$$

Then, we have

$$
Q(t) \geq -p\Xi(0)\Lambda(t)^{-\frac{1}{\xi}} + p(\mathcal{A}\mathcal{C} - \mathcal{B}^2). \tag{87}
$$

Observe that, for all $\rho, \eta \in \mathbb{R}$ and $t > 0$,

$$
\begin{aligned}
\mathcal{A}\rho^2 + 2\mathcal{B}\rho\eta + \mathcal{C}\eta^2 \quad = \quad & \rho^2\frac{\beta}{2}\int_0^t \left\|\nabla v\right\|_2^2 \, ds + \beta\rho\eta\int_0^t \left(\nabla v\left(s\right), \nabla v_s(s)\right)_2 \, ds \\
+ \quad & \frac{\beta}{2}\eta^2\int_0^t \left\|\nabla v_s\right\|_2^2 \, ds.
\end{aligned}
$$

Whereupon

$$
\mathcal{A}\rho^2 + 2\mathcal{B}\rho\eta + \mathcal{C}\eta^2 = \frac{\beta}{2}\int_0^t \left\|\rho\nabla v + \eta\nabla v_s\right\|_2^2 \, ds \geq 0.
$$

Consequently,

$$
\mathcal{B}^2 - \mathcal{A}\mathcal{C} \leq 0.
$$

Hence, thanks to (87),

$$
Q(t) \geq -p\Xi(0)\Lambda(t)^{-\frac{1}{\xi}}, \quad t \geq t_*. \tag{88}
$$

Therefore, using (88) and (81), we get

$$\Lambda''(t) \leq \left(\frac{(p-2)^2}{4} - 1\right) \Xi(0)\Lambda(t)^{1+\frac{1}{\xi}},\ t \geq t_*. \tag{89}$$

Note that by Lemma 4.5, $\Lambda'(t) < 0$ for $t \geq t_*$. We have

$$\Lambda''(t)\Lambda'(t) \geq \left(\frac{(p-2)^2}{4} - 1\right) \Xi\left(0\right) \Lambda'\left(t\right) M(t)^{1+\frac{1}{\xi}}, \quad t \geq t_*. \tag{90}$$

Integrating (90) on (t_*, t), it comes

$$\int_{t_*}^{t} \Lambda''\left(s\right) \Lambda'(s) ds \geq \left(\frac{(p-2)^2}{4} - 1\right) \Xi(0) \int_{t_*}^{t} \Lambda'\left(s\right) \Lambda\left(s\right)^{1+\frac{1}{\xi}} ds. \tag{91}$$

By using (81) in (80), we obtain

$$\begin{aligned}\int_{t_*}^{t} \Lambda''\Lambda'(s)ds &= \frac{1}{2}\Lambda'^2(t) - \frac{1}{2}\left(\frac{p-4}{4}\right)^2 \Bigg[\Psi'(t_*) - \frac{\beta}{2} \left\|\nabla v(0)\right\|^2_{L^2(\Omega)} \\ &\quad - \left|\left(v_{tt}(0), v(0)\right)_{L^2(\Omega)}\right| - \alpha \left|\left(v_t(0), v(0)\right)_{L^2(\Omega)}\right| \Bigg]^2 \Lambda(t_*)^{2+\frac{2}{\xi}}. \end{aligned} \tag{92}$$

We also have

$$\begin{aligned}\int_{t_*}^{t} \Lambda'\left(s\right) \Lambda\left(s\right)^{1+\frac{1}{\xi}} ds &= \int_{t_*}^{t} \frac{\xi}{2\xi+1} \frac{d}{ds}\left[\Lambda(s)^{\frac{2\xi+1}{\xi}}\right] ds \\ &= \frac{\xi}{2\xi+1} M(t)^{\frac{2\xi+1}{\xi}} - \frac{\xi}{2\xi+1} \Lambda\left(t_*\right)^{\frac{2\xi+1}{\xi}} \\ &= \left(\frac{1}{2} - \frac{1}{p-2}\right) M(t)^{\frac{2\xi+1}{\xi}} - \left(\frac{1}{2} - \frac{1}{p-2}\right) \Lambda\left(t_*\right)^{\frac{2\xi+1}{\xi}}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\left(\frac{(p-2)^2}{4} - 1\right) \Xi(0) \int_{t_*}^{t} \Lambda'(s)\Lambda(s)^{1+\frac{1}{\xi}} ds \\ =\ &\left(\frac{(p-2)^2}{4} - 1\right) \left(\frac{1}{2} - \frac{1}{p-2}\right) \Xi(0)\Lambda(t)^{2+\frac{1}{\xi}} \\ -\ &\left(\frac{(p-2)^2}{4} - 1\right) \left(\frac{1}{2} - \frac{1}{p-2}\right) \Xi(0)\Lambda\left(t_*\right)^{2+\frac{1}{\xi}}.\end{aligned} \tag{93}$$

By substituting (92) and (93) in (91), it comes

$$\begin{aligned} & \frac{1}{2}\Lambda'(t)^2 - \frac{1}{2}\left(\frac{p-4}{4}\right)^2 \left[\Psi'(t_*) - \frac{\beta}{2}\left\|\nabla v(t)\right\|^2_{L^2(\Omega)}\right. \\ - & \left.\left|(v_{tt}(0), v(0))_{L^2(\Omega)}\right| - \alpha\left|(v_t(t), v(t))_{L^2(\Omega)}\right|\right]^2 \Lambda(t_*)^{2+\frac{2}{\xi}} \\ \geq & \left(\frac{(p-2)^2}{4} - 1\right)\left(\frac{1}{2} - \frac{1}{p-2}\right)\Xi(t)\Lambda(t)^{2+\frac{1}{\xi}} \\ - & \left(\frac{(p-2)^2}{4} - 1\right)\left(\frac{1}{2} - \frac{1}{p-2}\right)\Xi(0)\Lambda(t_*)^{2+\frac{1}{\xi}}. \end{aligned}$$

Thus

$$\Lambda'(t)^2 \geq \mu + \beta\Lambda(t)^{2+\frac{1}{\gamma}},$$

where

$$\begin{aligned} \mu \quad : \quad = & \left(\left(\frac{p-4}{4}\right)^2 \left(\Psi'(t_*) - \frac{\beta}{2}\left\|\nabla v(0)\right\|_2^2\right.\right. \\ & - \left|(v_{tt}(0), v(0))_2\right| - \alpha\left|(v_t(0), v(0))_2\right|\Big)^2 \\ & \left. - \left(\frac{(p-2)^2}{2} - 2\right)\left(\frac{1}{2} - \frac{1}{p-2}\right)E(0)\Lambda(t_*)^{-\frac{1}{\gamma}}\right)\Lambda(t_*)^{2+\frac{2}{\gamma}} \\ > \quad & 0, \end{aligned} \tag{94}$$

and

$$\beta := \left(\frac{(p-2)^2}{2} - 2\right)\left(\frac{1}{2} - \frac{1}{p-2}\right)\Xi(0). \tag{95}$$

□

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgement:

This work was supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (grant number IMSIU-RP23077).

References

[1] R. Adams, J. Fournier, Sobolev Spaces, Academic Press, New York, 2003.

[2] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, New York, 2010.

[3] A. H. Caixeta, I. Lasiecka and V. N.D. Cavalcanti, Global attractors for a third order in time nonlinear dynamics, J. Differential Equ. (1) 261 (2016) 113-147.

[4] A. Draifia, Global existence and general decay of Moore–Gibson–Thompson equation with not necessarily decreasing kernel. Bol. Soc. Paran. Mat. doi:10.5269/bspm. 51505.

[5] A. Draifia, Existence and uniqueness of the weak solution for the Moore-Gibson-Thompson equation with viscoelastic memory and non-classical condition, J. Nonl. Evol. Equ. Appl. (1) (2020) 1–31.

[6] P. M. Jordan, Second-sound phenomena in inviscid, thermally relaxing gases, Discrete Contin. Dyn. Syst. Ser. B 19 (2014) 2189-2205.

[7] P. M. Jordan, Nonlinear acoustic phenomena in viscous thermally relaxing fluids: Shock bifurcation and the emergence of diffusive solitons, in The 9th Int. Conf. on Theoretical and Computational Acoustics (ICTCA, 2009), Dresden, Germany, 9 September 2009. See also J. Acoust. Soc. Amer. 124, 2491(2009).

[8] B. Kaltenbacher, Mathematics of nonlinear acoustics, Evolution Equations and Control Theory, 4 (4) (2015) 447-491.

[9] B. Kaltenbacher, I. Lasiecka and R. Marchand, Wellposedness and exponential decay rates for the Moore-Gibson-Thompson equation arising in high intensity ultrasound, Control Cybernet. 40 (4) (2011) 971-988.

[10] B. Kaltenbacher, I. Lasiecka and M. Pospieszalska, Well-posedness and exponential decay of the energy in the nonlinear Jordan-Moore-Gibson-Thompson equation arising in high intensity ultrasound. Math. Models Methods Appl. Sci., 22 (11) (2012) 1250035, (34 pages).

[11] I. Lasiecka, Global solvability of Moore–Gibson–Thompson equation with memory arising in nonlinear acoustics, J. Evol. Equ. 2016 Springer International Publishing, DOI 10.1007/s00028-016-0353-3.

[12] I. Lasiecka, X. Wang, Moore–Gibson–Thompson equation with memory, part II: general decay of energy. J. Differ. Equ. 259 (12) (2015) 7610-7635.

[13] I. Lasiecka, X. Wang, Moore–Gibson–Thompson equation with memory, part I: exponential decay of energy, Z. Angew. Math. Phys. 67 (17) (2016).

[14] G. Lebon, A. Cloot, Propagation of ultrasonic sound waves in dissipative dilute gases and extended irreversible thermodynamics. Wave Motion. 11 (1989) 23-32.

[15] M.R Li, L.Y Tsai, Existence and nonexistence of global solutions of some systems of semilinear wave equations. Nonlinear Anal Theory, Methods and Applications. 54 (2003) 1397-1415.

[16] J. L. Lions, Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires. Dunod/Gauthier-Villars, Paris, 1969.

[17] R. Marchand, T. McDevitt and R. Triggiani, An abstract semigroup approach to the third-order Moore-Gibson-Thompson partial differential equation arising in high-intensity ultrasound: structural decomposition, spectral analysis, exponential stability, Math. Methods Appl. Sci. 35 (15) (2012) 1896-1929.

[18] F. K. Moore, W. E. Gibson, Propagation of weak disturbances in a gas subject to relaxation effects, J. Aero/Space Sci. 27 (1960) 117-127.