Global well-posedness of three-dimensional incompressible Boussinesq system with temperature-dependent viscosity

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Abstract

In this paper, we focus on the global well-posedness of solutions to three-dimensional incompressible Boussinesq equations with temperature-dependent viscosity under the smallness assumption of initial velocity fields u_0 in the critical space $\dot_{B}_{3,1}^0$. The key ingredients here lie in the decomposition of the velocity fields and the regularity theory of the Stokes system, which are crucial to get rid of the smallness restricition of the initial temperature $ heta_0$. In addition, we mention that the improved decay estimates in time is also necessary.

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Key Words: Boussinesq system, Global well-posedness, decay estimates.

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1 Introduction

The Boussinesq system derived by the French mathematician Joseph Valentin Boussinesq [6] using the law of conservation of mass and momentum usually describes many physical phenomena such as large air flows, thermal convection, geophysical flows and conductive flows. The classical Boussinesq equation reads as follows:

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nabla\pi = \mu\Delta u + \theta e_N, \qquad x\in\mathbb{R}^N, t>0, \\ \partial_t\theta + u\cdot\nabla\theta = \nu\Delta\theta, \\ \mathrm{div} u = 0, \\ (u,\theta)|_{t=0} = (u_0,\theta_0), \end{cases} \tag{1.1}$$

where θ, $u = (u_1(t,x),\cdots,u_N(t,x))$ represent the temperature and velocity fields of the fluid, respectively, $\Pi = \Pi(t,x)$ is a scalar pressure function, $e_N = (0,\cdots,1)^t$ is a unit vector field in $\mathbb{R}^N$, and $\mu,\ \nu \geq 0$ represent the kinematic viscosity coefficient and the thermal diffusion coefficient respectively.

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Due to its profound physical background, there have been many progresses on the well-posedness of solutions to system (1.1). For the 2D case, Cannon-DiBenedetto [8] showed the global existence and uniqueness of smooth solutions to (1.1) for $\mu > 0$ and $\nu > 0$. Chae [9] and Hou-Li [15] independently proved the global existence of smooth solutions to (1.1) for either $\mu > 0$ and $\nu = 0$ or $\mu = 0$ and $\nu > 0$. Abidi-Hmidi [3] obtained the global well-posedness for (1.1) when the initial data satisfies $(\theta_0, u_0) \in B^0_{2,1} \times (B^{-1}_{\infty,1} \cap L^2)$ for $\mu > 0$ and $\nu = 0$. When $N \geq 3$, $e_N = (0, \cdots, 1)$, $\nu = 0$ and $\mu > 0$, Danchin and Paicu [11] proved the global well-posedness of this system with $\theta_0 \in L^{\frac{N}{3}}(\mathbb{R}^N) \cap \dot{B}^0_{N,1}(\mathbb{R}^N)$ and $u_0 \in L^{N,\infty}(\mathbb{R}^N) \cap \dot{B}^{-1+\frac{N}{p}}_{p,1}(\mathbb{R}^N)$ for $p \in [N, \infty]$ provided that

$$\|u_0\|_{L^{N,\infty}} + \mu^{-1}\|\theta_0\|_{L^{\frac{N}{3}}} \leq c\mu, \tag{1.2}$$

for some sufficiently small constant c. In addition, further results on the existence and uniqueness of Boussinesq systems with critical dissipation can be found in [4, 12, 13, 14, 23] and so on.

Compared to system (1.1), the dissipative term $\mu(\theta)$, which depends on the temperature, increases the difficulty. Recently, more progress has been made in the global well-posedness of system (1.1) with the temperature-dependent viscosity $\mu(\theta)$, especially with $\nu(\theta) = f(\theta) = 0$, i.e,

$$\begin{cases} \partial_t u - \mathrm{div}(2\mu(\theta)\mathbb{D}u) + u \cdot \nabla u + \nabla \pi = 0, \qquad x \in \mathbb{R}^3, t > 0, \\ \partial_t \theta + u \cdot \nabla \theta = 0, \\ \mathrm{div} u = 0, \\ (u, \theta)|_{t=0} = (u_0, \theta_0). \end{cases} \tag{1.3}$$

It is easy to see that system (1.3) has a scaling-invariant transformation. If (θ, u) solves equation with initial data (θ_0, u_0), then for $\forall\ \lambda > 0$,

$$(\theta, u)_\lambda(t, x) = (\theta(\lambda^2 t, \lambda x), \lambda u(\lambda^2 t, \lambda x)) \tag{1.4}$$

is a solution of (1.3) with initial data $(\theta_0(\lambda x), \lambda u_0(\lambda x))$. A functional space for data (θ_0, u_0) or for solution (θ, u) is said to be scaling-invariant of the equation if its norm is invariant under transformation (1.4). Obviously, $\dot{B}^{\frac{N}{p}}_{p,r} \times \dot{B}^{-1+\frac{N}{p}}_{p,r}$ with $(N, r) \in [1, \infty]^2$ is an example of homogeneous critical Besov spaces of Boussinesq equation (1.3).

Similar to the classical system (1.1), it is natural to ask the global well-posedness of system (1.3) under some suitable assumptions on (θ_0, u_0) in the critical spaces? Until now, there are only several results related to this topic. For two-dimensional Boussinesq equations with temperature-dependent viscosity, Abidi [1] proved the global well-posedness when the solenoidal vector fields $u_0 \in L^2 \cap \dot{B}^{-1}_{\infty,1}(\mathbb{R}^2)$ and $\theta_0 \in \dot{B}^1_{2,1}(\mathbb{R}^2)$ on the basis of the two smallness assumptions on $\|\theta_0\|_{\dot{B}^1_{2,1}}$ and $\|\mu(\theta_0) - 1\|_{L^\infty}$. Niu-Wang [19, 20] obtained similar results only under the smallness condition of $\|\mu(\theta_0) - 1\|_{L^\infty}$. For three-dimensional system (1.3) with the viscosity $\mu(\theta)$, depending on the temperature θ, Abidi-Zhang [5] obtained the global strong solutions of (1.3) under both of the smallness conditions, i.e.,

$$\|u_0\|_{\dot{B}^0_{3,1}} + \|\mu(\theta_0) - 1\|_{L^\infty} \leq \varepsilon. \tag{1.5}$$

Without the size constraint on the initial temperuature, Niu-Wang [21] proved the global well-posedness of (1.1) provided that

$$\|u_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}} \leq \varepsilon, \tag{1.6}$$

where ε is a sufficiently small constant. More results aboutglobal well-posedness of 3D Boussinesq system with temperature-dependent viscosity can be refferred to [16, 17, 22] and so on.

Throughout this paper, we shall always assume that

$$0 < \underline{\mu} \le \mu(\theta), \quad \mu(\cdot) \in \mathrm{W}^{3,\infty}(\mathbb{R}^+) \quad \text{and} \quad \mu(0) = 1. \tag{1.7}$$

The purpose of this paper is to study the solvability of (1.3) with u_0 sufficiently small in the critical Besov spaces $\dot{B}^0_{3,1}$ without any size restriction on θ_0. Our main theorem can be stated as follows:

Theorem 1.1 *Let $u_0 \in \dot{B}^0_{3,1}(\mathbb{R}^3)$ be a solenoidal vector field and $\theta_0 \in B^{\frac{3}{2}}_{2,1}(\mathbb{R}^3)$. Then there exists a positive time T^* so that* (1.3) *has a unique local-in-time solution*

$$(\theta, u) \in \mathcal{C}_b([0,T^*]; B^{\frac{3}{2}}_{2,1}(\mathbb{R}^3)) \times \mathcal{C}_b([0,T^*]; \dot{B}^0_{3,1}(\mathbb{R}^3)) \cap L^1([0,T^*]; \dot{B}^2_{3,1}(\mathbb{R}^3)).$$

Furthermore, let $q \in (3,\infty)$ and $\delta \in (\frac{1}{2}, \frac{3}{4})$. Assume that the initial data (θ_0, u_0) satisfy

$$\underline{\mu} < \mu(\theta) < \bar{\mu}, \;\; u_0 \in \dot{H}^{-2\delta}, \;\; \nabla\mu(\theta_0) \in L^q.$$

Then there exists a small positive constant ε depending on q, $\underline{\mu}$, $\bar{\mu}$, $\|\theta_0\|_{B^{\frac{3}{2}}_{2.1}}$, $\|u_0\|_{\dot{H}^{-2\delta}}$ and $\|\nabla\mu(\theta_0)\|_{L^q}$ such that when

$$\|u_0\|_{\dot{B}^0_{3,1}} \le \varepsilon. \tag{1.8}$$

the Cauchy problem (1.3) *admits a unique and global strong solution $(\theta, u, \nabla\pi)$ which satisfies for any $0 < T < \infty$ that*

$$\begin{cases} \theta \in \mathcal{C}([0,T]; B^{\frac{3}{2}}_{2,1}), \quad \nabla\mu(\theta) \in \mathcal{C}\left([0,T]; L^q\right), \\ u \in \mathcal{C}\left([0,T]; \dot{B}^0_{3,1}\right) \cap L^1\left([0,T]; \dot{B}^2_{3,1}\right), \\ \pi \in L^1\left([0,T]; L^2 \cap \dot{B}^1_{3,1}\right) \text{ and } u_t \in L^\infty\left([0,T]; \dot{B}^0_{3,1}\right), \end{cases} \tag{1.9}$$

Remark 1.1 *It should be noted here that our Theorem 1.1 holdstrue for any function $\mu(\theta)$ satisfying (1.15) and with a smallness assumption on the $\dot{B}^0_{3,1}$-norm of the initial velocity u_0, which is contrast to Abidi-Zhang [5] where they need the smallness assumptions on both $\|u_0\|_{\dot{B}^0_{3,1}}$ and $\|\mu(\theta_0) - 1\|_{L^\infty}$.*

Scheme of the proof and organization of the paper. Motivated by [5], we intend to investigate the global well-posedness of (1.1) under the assumption that u_0 is sufficiently small in the critical Besov space $\dot{B}^0_{3,1}$? The proof of global well-posedness can be devided into two steps.

To begin with, we investigated the local existence and uniqueness of solution to (1.3) when initial data u_0 being sufficiently small in the critical spaces $\dot{B}^0_{3,1}$ inspired by the idea of [2]. Then we prove that there exists some $t_0 \in (0,1)$ such that

$$\|u(t_0)\|_{\dot{B}^0_{3,1} \cap \dot{B}^2_{3,1}} \le C\|u_0\|_{\dot{B}^0_{3,1}}. \tag{1.10}$$

One can check the proof of Proposition 3.1 for more details.

Second, the key ingredient to prove the global well-posedness of system (1.3) comes from the expected uniform energy estimate of $L^1((t_0,\infty);L^\infty)$ of ∇u in general. However, because the viscosity term $\mu(\theta)$ strongly depends on the temperature θ and lack of the smallness assumption of $\mu(\theta)-1$ in L^∞, it is not applicable. Indeed, the Stokes equation transformed from the momentum equation of (1.3) can be stated as follows:

$$\Delta u-\nabla(\frac{\pi}{\mu(\theta)})=-\frac{2\mathbb{D}u\cdot\nabla\mu(\theta)}{\mu(\theta)}+\frac{\pi\nabla\mu(\theta)}{\mu(\theta)^2}-\frac{\partial_t u+u\cdot\nabla u}{\mu(\theta)}. \tag{1.11}$$

The regularity theory of Stokes equation indicates that the desired energy estimate of of $L^1((t_0,\infty);L^\infty)$ of ∇u is coupled with the *a priori* uniform estimate of $L^\infty(0,\infty;L^q)(q>3)$ of $\nabla\mu(\theta)$, which in turn heavily depends on the uniform estimate of $L^1((t_0,\infty);L^\infty)$ of ∇u. The possible solution is by means of the bootstrapping argument, which is crucial to prove the small bound of $L^1((t_0,\infty);L^\infty)$ of ∇u. According to the embedding theory, it can be transferred to the small bound of $L^\infty((t_0,\infty);L^2)$ norm of ∇u.

Now we are in a position to derive the uniform estimate of $L^\infty((t_0,\infty);L^2)$ of ∇u in terms of $\|u(t_0)\|_{\dot B^0_{3,1}}$, which is sufficiently small through (1.10). However, the dependence on θ of viscosity term $\mu(\theta)$ bring the difficulty to close the estimate of $L^\infty((t_0,\infty);L^2)$ of ∇u directly. The important observation is based on the decomposition of the velocity fields. Indeed, we split the solution u of (1.3) into two parts, i.e., $u=v+w$, where v satisfies the classical Navier-Stokes equation with constant viscosity

$$\begin{cases}\partial_t v+v\cdot\nabla v-\Delta v+\nabla\pi_v=0,\\ \operatorname{div} v=0,\\ v|_{t=t_0}=u\left(t_0\right),\end{cases} \tag{1.12}$$

and w, as well as θ is the solution of the perturbated system:

$$\begin{cases}\partial_t\theta+\operatorname{div}(\theta(v+w))=0,\\ \partial_t w+(v+w)\cdot\nabla w-\operatorname{div}(2\mu(\theta)\mathbb{D}w)+\nabla\pi_w=-w\cdot\nabla v+\operatorname{div}(2(\mu(\theta)-1)\mathbb{D}v),\\ \operatorname{div} w=0,\\ \theta|_{t=t_0}=\theta(t_0),\quad w|_{t=t_0}=0.\end{cases} \tag{1.13}$$

For system (1.12), we can easily see from [2] that for $s\in[0,2]$,

$$\|v\|_{\widetilde L^\infty([t_0,\infty);\dot B^s_{3,1})}+\|v\|_{L^1([t_0,\infty);\dot B^{s+2}_{3,1})}\le C\|u(t_0)\|_{\dot B^s_{3,1}}\le C\|u_0\|_{\dot B^0_{3,1}}. \tag{1.14}$$

According to the embedding theory $\dot B^1_{3,1}\hookrightarrow L^\infty$, we can easily prove that

$$\|\nabla v\|_{L^1([t_0,\infty);L^\infty)}\le C\|u_0\|_{\dot B^0_{3,1}}, \tag{1.15}$$

which implies that the smallness of the time-independent bounds on the $L^1(t_0,T;\dot B^1_{\infty,1})$ norm of v. More details can be referred to Proposition 5.4 later. As for system (1.13), the zero initial value of w and (1.15) are very helpful to obtain

$$\sup_{t\in[t_0,\infty]}\int_{\mathbb{R}^3}|\nabla w|^2dx\le 2C\int_{t_0}^\infty\|v\|^2_{\dot B^{s_1}_{3,1}}dt\le 2C\|u_0\|^2_{\dot B^0_{3,1}}, \tag{1.16}$$

where $s_1 \in [1,2]$. In addition, we emphasis that the improved decay estimates in time also assist the uniform estimate of (1.16), which explain that it is necessary to assume $u_0 \in H^{-2\delta}, \delta > \frac{1}{2}$ in Theorem 1.1. One can refer Proposition 5.4 for more details.

The paper is organised as follows. In Section 2, we recall some basic Littlewood-Paley theory, as well as some necessary lemmas. In Section 3, we obtain some linear estimates of the Boussinesq equations. With these estimates we will prove the local well-posedness of the solution to the system (1.3) in Section 4. The proof of Theorem 1.1 is completed in Section 5, where we obtain the decay estimates of v, w and the global well-posedness of (1.3) with smallness assumptions (1.8).

Let us complete this section with the notations we are going to use in this context.

Notations Let A, B be two operators, we denote $[A, B] = AB - BA$, the commutator between A and B. For $a \lesssim b$, we mean that there is a uniform constant C, which may be different on different lines, such that $a \leq Cb$. We shall denote by $(a|b)$ the $L^2(\mathbb{R}^3)$ inner product of a and b. For X a Banach space, we denote $\|(f,g)\|_X \triangleq \|f\|_X + \|g\|_X$. Finally, $(c_j)_{j\in\mathbb{Z}}$ (resp. $(d_j)_{j\in\mathbb{Z}}$) will be a generic element of $\ell^2(\mathbb{Z})$ (resp. $\ell^1(\mathbb{Z})$) so that $\sum_{j\in\mathbb{Z}} c_j^2 = 1$ (resp. $\sum_{j\in\mathbb{Z}} d_j = 1$).

2 Preliminaries

In this section, we introduce some notations and conventions, and recall some standard theories of Besov space which will be used throughout this paper. Since the proof of Theorem 1.1 requires a dyadic decomposition of the Fourier variables, we will first recall the Littlewood-Paley decomposition. More details can be referred to [7].

Definition 2.1 *([7]) Let $(p, r) \in [1, \infty]^2$, $s \in \mathbb{R}$ and $u \in \mathcal{S}'(\mathbb{R}^3)$, we set*

$$\|u\|_{B^s_{p,r}} \stackrel{\text{def}}{=} (2^{qs}\|\Delta_q u\|_{L^p})_{l^r}, \quad \|u\|_{\dot{B}^s_{p,r}} \stackrel{\text{def}}{=} (2^{qs}\|\dot{\Delta}_q u\|_{L^p})_{l^r}. \tag{2.1}$$

Definition 2.2 *([7]) Let $(p, \lambda, r) \in [1, \infty]^3$, $s \in \mathbb{R}$, $T \in (0, +\infty]$, and $u \in \mathcal{S}'(\mathbb{R}^3)$, we set*

$$\|u\|_{\widetilde{L}^\lambda_T(B^s_{p,r})} \stackrel{\text{def}}{=} (2^{qs}\|\Delta_q u\|_{L^\lambda_T(L^p)})_{l^r}, \quad \|u\|_{\widetilde{L}^\lambda_T(\dot{B}^s_{p,r})} \stackrel{\text{def}}{=} (2^{qs}\|\dot{\Delta}_q u\|_{L^\lambda_T(L^p)})_{l^r}. \tag{2.2}$$

Lemma 2.1 *([7]) Let $\mathcal{C}$ be the annulus $\{\xi \in \mathbb{R}^N : 3/4 \leq |\xi| \leq 8/3\}$. There exist radial functions χ and φ, valued in the interval $[0,1]$, belonging respectively to $\mathcal{D}(\mathcal{C})$, and such that*

$$\begin{aligned}
&\forall\ \xi \in \mathbb{R}^N, \quad \chi(\xi) + \sum_{j\geq 0} \varphi(2^{-j\xi}) = 1,\\
&\forall\ \xi \in \mathbb{R}^N, \quad \sum_{j\in\mathbb{Z}} \varphi(2^{-j\xi}) = 1,\\
&|j - j'| \geq 2 \Longrightarrow \mathrm{Supp}\varphi(2^{-j\xi\cdot}) \cap \mathrm{Supp}\varphi(2^{-j'\xi\cdot}) = \varnothing,\\
&j \geq 1 \Longrightarrow \mathrm{Supp}\chi \cap \mathrm{Supp}\varphi(2^{-j'\xi\cdot}) = \varnothing.
\end{aligned} \tag{2.3}$$

the set $\widetilde{\mathcal{C}} = B(0, 2/3) + \mathcal{C}$ is an annulus, and we have

$$|j - j'| \geq 5 \Longrightarrow 2^{j'}\widetilde{\mathcal{C}} \cap 2^j\mathcal{C} = \varnothing. \tag{2.4}$$

Remark 2.1 *([7])*

1. *We point out that if $s>0$ then $B^s_{p,r}\left(\mathbb{R}^3\right)=\dot{B}^s_{p,r}\left(\mathbb{R}^3\right)\cap L^p\left(\mathbb{R}^3\right)$ and*
$$\|u\|_{B^s_{p,r}}\approx\|u\|_{\dot{B}^s_{p,r}}+\|u\|_{L^p}.$$

2. *It is easy to observe that the homogeneous Besov space $\dot{B}^s_{2,2}\left(\mathbb{R}^3\right)$ coincides with the classical homogeneous Sobolev space $\dot{H}^s\left(\mathbb{R}^3\right)$.*

3. *Let $s\in\mathbb{R}, 1\leq p,r\leq\infty$, and $u\in\mathcal{S}'\left(\mathbb{R}^3\right)$. Then u belongs to $\dot{B}^s_{p,r}\left(\mathbb{R}^3\right)$ if and only if there exists $\{c_{j,r}\}_{j\in\mathbb{Z}}$ such that $\|c_{j,r}\|_{\ell^r}=1$, and*
$$\|\dot{\Delta}_j u\|_{L^p}\leq Cc_{j,r}2^{-js}\|u\|_{\dot{B}^s_{p,r}}\ \ for\ all\ j\in\mathbb{Z}.$$

Lemma 2.2 *([7]) Let $\mathcal{C}$ be an annulus and $\mathcal{B}$ a ball. There exists a constant $C>0$ such that for any nonnegative integer k, any couple (p,q) in $[1,\infty]^2$ with $q\geq p\geq 1$ and any u of L^p satisfying*

1. *If* $\operatorname{Supp}\hat{u}\subset\lambda\mathcal{B}$, *we have*
$$\sup_{|\alpha|=k}\|\partial^\alpha u\|_{L^q}\leq C^{k+1}\lambda^{k+N(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p}. \tag{2.5}$$

2. *If* $\operatorname{Supp}\hat{u}\subset\lambda\mathcal{C}$, *we have*
$$C^{-k-1}\lambda^k\|u\|_{L^p}\leq\|D^k u\|_{L^p}\leq C^{k+1}\lambda^k\|u\|_{L^p}. \tag{2.6}$$

In the rest of the paper, we shall frequently use homogeneous Bony's decomposition:
$$uv=T_u v+T'_v u=T_u v+T_v u+R(u,v), \tag{2.7}$$
where
$$\begin{aligned} T_u v&=\sum_{q\in\mathbb{Z}}\dot{S}_{q-1}u\dot{\Delta}_q v, \quad T'_v u=\sum_{q\in\mathbb{Z}}\dot{S}_{q+2}v\dot{\Delta}_q u,\\ R(u,v)&=\sum_{q\in\mathbb{Z}}\dot{\Delta}_q u\widetilde{\dot{\Delta}}_q v, \quad \widetilde{\dot{\Delta}}_q v=\sum_{|q'-q|\leq 1}\dot{\Delta}_{q'}v. \end{aligned} \tag{2.8}$$
Similarly, we can obtain inhomogeneous Bony's decomposition [7].

Next we shall introduce some results about the transport equation, such as the commutator's estimates, which will be frequently used throughout the succeeding sections. The proof process can be referred to [7]. We omit them for simplicity here.

Lemma 2.3 *([7]) Let $r\in[1,\infty]$, $p\in[1,\infty]$, $-1\leq s\leq 1$ and $\operatorname{div}v=0$. Then there holds*

1. *If $s=-1$,*
$$\sup_q 2^{-q}\|[v\cdot\nabla;\dot{\Delta}_q]f\|_{L^1_t(L^p)}\lesssim\int_0^t\|\nabla v\|_{\dot{B}^0_{\infty,1}}\|f\|_{\dot{B}^{-1}_{p,\infty}}dt; \tag{2.9}$$

2. *If $-1<s<1$,*
$$(\sum_q 2^{qsr}\|[v\cdot\nabla;\dot{\Delta}_q]f\|^r_{L^1_t(L^p)})^{\frac{1}{r}}\lesssim\int_0^t\|\nabla v\|_{L^\infty}\|f\|_{\dot{B}^s_{p,r}}dt; \tag{2.10}$$

3. If $s=1$,

$$\sum_q 2^q\|[v\cdot\nabla;\dot{\Delta}_q]f\|_{L^1_t(L^p)}\lesssim\int_0^t\|\nabla v\|_{\dot{B}^0_{\infty,1}}\|f\|_{\dot{B}^1_{p,1}}dt. \tag{2.11}$$

Lemma 2.4 *([7]) Assume that* $a_0\in\dot{B}^1_{3,1}(\mathbb{R}^3)$ *and* $\nabla u\in L^1_T(\dot{B}^0_{\infty,1})$ *with* $\operatorname{div}u=0$, *and the function* $a\in\mathcal{C}([0,T];\dot{B}^1_{3,1}(\mathbb{R}^3))$ *solves*

$$\begin{cases}\partial_t a+u\cdot\nabla a=0, & (t,x)\in[0,T]\times\mathbb{R}^3,\\ a|_{t=0}=a_0.\end{cases} \tag{2.12}$$

Then there holds that for $\forall\ t\in(0,T]$

$$\|a(t)\|_{\dot{B}^1_{3,1}}\le C\|a_0\|_{\dot{B}^1_{3,1}}\exp\{C\|\nabla u\|_{L^1_t(\dot{B}^0_{\infty,1})}\}, \tag{2.13}$$

and

$$\|a-\dot{S}_k a\|_{\widetilde{L}^\infty_t(\dot{B}^1_{3,1})}\le\sum_{q\ge k}2^q\|\dot{\Delta}_q a_0\|_{L^3}+\|a_0\|_{\dot{B}^1_{3,1}}\left\{\exp\{C\|\nabla u\|_{L^1_t(\dot{B}^0_{\infty,1})}\}-1\right\}. \tag{2.14}$$

3 Linear estimates

In this section we first prove the local well-posedness of solution to (1.3) by examining some *a priori* estimates of the basic energy.

Proposition 3.1 *Let* $a\in\widetilde{L}^\infty_T(B^1_{3,1}(\mathbb{R}^3))$ *with* $1+a\ge\underline{b}$. *Let* u, v *be a solenoidal vector field such that* $(u,v)\in C([0,T];\dot{B}^0_{3,1}(\mathbb{R}^3))\cap L^1_{loc}([0,T];\dot{B}^2_{3,1}(\mathbb{R}^3))$ *solves*

$$\begin{cases}\partial_t u+v\cdot\nabla u-\operatorname{div}(2(1+a)\mathbb{D}u)+\nabla\pi=f, & (t,x)\in\mathbb{R}_+\times\mathbb{R}^3,\\ \operatorname{div}u=0,\\ u|_{t=0}=u_0.\end{cases} \tag{3.1}$$

If there exist some sufficiently small positive constant c *and some integer* $k\in\mathbb{Z}$, *there holds*

$$\|a-\dot{S}_k a\|_{\widetilde{L}^\infty_T(B^1_{3,1})}\le c, \tag{3.2}$$

then for $0\le t\le T$, *one has*

$$\begin{aligned}\|u\|_{\widetilde{L}^\infty_t(\dot{B}^0_{3,1})}+\|u\|_{L^1_t(\dot{B}^2_{3,1})}+\|\nabla\pi\|_{L^1_t(\dot{B}^0_{3,1})}&\lesssim\|u_0\|_{\dot{B}^0_{3,1}}+\|f\|_{L^1_t(\dot{B}^0_{3,1})}\\&+\int_0^t\|u\|_{\dot{B}^0_{3,1}}\|v\|_{\dot{B}^2_{3,1}}d\tau+2^{4k}\|\dot{S}_k a\|^2_{L^\infty_t(L^3)}\|u\|_{L^1_t(\dot{B}^0_{3,1})}.\end{aligned} \tag{3.3}$$

Proof We first get from $1+a\ge\underline{b}$ that

$$1+\dot{S}_k a=1+a+(\dot{S}_k a-a)\ge\frac{\underline{b}}{2}.$$

Correspondingly, we rewrite the u equation of $(3.1)_1$ as

$$\partial_t u - v\cdot\nabla u - \operatorname{div}(2(1+\dot S_k a)\mathbb{D}u) + \nabla\pi = \operatorname{div}(2(a-\dot S_k a)\mathbb{D}u) + f. \tag{3.4}$$

We now decompose the proof of Proposition 3.1 into the following steps:

Step 1. The estimate of $\|u\|_{\widetilde{L}^\infty_t(\dot B^0_{3,1})} + \|u\|_{L^1_t(\dot B^2_{3,1})}$.

Let $\mathbb{P} = I + \nabla(-\Delta)^{-1}\operatorname{div}$ be the Leray projection operator, Applying $\dot\Delta_j\mathbb{P}$ to (3.4) gives

$$\partial_t\dot\Delta_j u - \dot\Delta_j\mathbb{P}(v\cdot\nabla u) - \dot\Delta_j\mathbb{P}\{\operatorname{div}(2(1+\dot S_k a)\mathbb{D}u)\} = \dot\Delta_j\mathbb{P}(\operatorname{div}(2(a-\dot S_k a)\mathbb{D}u) + f). \tag{3.5}$$

By applying a standard commutator process, we find

$$\begin{aligned}\dot\Delta_j\mathbb{P}\{\operatorname{div}(2(1+\dot S_k a)\mathbb{D}u)\} &= \dot\Delta_j\mathbb{P}\{2(1+\dot S_k a)\Delta u + 2\nabla\dot S_k a\mathbb{D}u\}\\ &= (1+\dot S_k a)\dot\Delta_j\mathbb{P}\Delta u - [\dot S_k a;\dot\Delta_j\mathbb{P}]\Delta u + \dot\Delta_j\mathbb{P}\{2\nabla\dot S_k a\mathbb{D}u\},\end{aligned} \tag{3.6}$$

where

$$(1+\dot S_k a)\dot\Delta_j\mathbb{P}\Delta u = \operatorname{div}((1+\dot S_k a)\dot\Delta_j\nabla u) - \nabla\dot S_k a\cdot\dot\Delta_j\nabla u. \tag{3.7}$$

As a consequence, we obtain

$$\begin{aligned}\partial_t\dot\Delta_j u - v\cdot\nabla\dot\Delta_j u - \operatorname{div}((1+\dot S_k a)\dot\Delta_j\nabla u) &= \dot\Delta_j\mathbb{P}\{\operatorname{div}(2(a-\dot S_k a)\mathbb{D}u\} + \dot\Delta_j\mathbb{P}f\\ &- \nabla\dot S_k a\cdot\dot\Delta_j\nabla u - [\dot S_k a;\dot\Delta_j\mathbb{P}]\Delta u + \dot\Delta_j\mathbb{P}\{2\nabla\dot S_k a\mathbb{D}u\} + [v\cdot\nabla;\dot\Delta_j\mathbb{P}]u.\end{aligned} \tag{3.8}$$

Multiplying the above equation by $|\dot\Delta_j u|\dot\Delta_j u$ and then integrating the resulting equations on $x\in\mathbb{R}^3$ leads to

$$\begin{aligned}&\frac{1}{3}\frac{d}{dt}\|\dot\Delta_j u\|^3_{L^2} + \int_{\mathbb{R}^3} v\cdot\nabla\frac{|\dot\Delta_j u|^2}{3}dx - \int_{\mathbb{R}^3}\operatorname{div}((1+\dot S_k a)\dot\Delta_j\nabla u)|\dot\Delta_j u|\dot\Delta_j u dx\\ &\lesssim \|\dot\Delta_j u\|^2_{L^3}\{\|\dot\Delta_j(\operatorname{div}(2(a-\dot S_k a)\mathbb{D}u)\|_{L^3} + \|\dot\Delta_j f\|_{L^3} + \|\nabla\dot S_k a\cdot\dot\Delta_j\nabla u\|_{L^3}\\ &\quad + \|[\dot S_k a;\dot\Delta_j\mathbb{P}]\Delta u\|_{L^3} + \|\dot\Delta_j\{2\nabla\dot S_k a\mathbb{D}u\}\|_{L^3} + \|[v\cdot\nabla;\dot\Delta_j\mathbb{P}]u\|_{L^3}\}.\end{aligned} \tag{3.9}$$

Yet applying Bernstein-type inequation ([7]) ensures that for some positive constant $\bar c$

$$-\int_{\mathbb{R}^3}\operatorname{div}((1+\dot S_k a)\dot\Delta_j\nabla u)\cdot|\dot\Delta_j u|\dot\Delta_j u dx \geq \bar c 2^{2j}\|\dot\Delta_j u\|^3_{L^3}. \tag{3.10}$$

Notice that $\operatorname{div} v = 0$ guarantees

$$\begin{aligned}\frac{d}{dt}\|\dot\Delta_j u\|_{L^3} + 2^{2j}\|\dot\Delta_j u\|_{L^3} &\lesssim \|\dot\Delta_j(\operatorname{div}(2(a-\dot S_k a)\mathbb{D}u)\|_{L^3} + \|\nabla\dot S_k a\cdot\dot\Delta_j\nabla u\|_{L^3}\\ &+ \|\dot\Delta_j f\|_{L^3} + \|[\dot S_k a;\dot\Delta_j\mathbb{P}]\Delta u\|_{L^3} + \|\dot\Delta_j\{2\nabla\dot S_k a\mathbb{D}u\}\|_{L^3} + \|[v\cdot\nabla;\dot\Delta_j\mathbb{P}]u\|_{L^3}.\end{aligned} \tag{3.11}$$

Intergrating the above inequality over $[0,t]$ and summing up the resulting inequality over $j\in\mathbb{Z}$, we achieve

$$\begin{aligned}\|u\|_{\widetilde{L}^\infty_t(\dot B^0_{3,1})} + \|u\|_{L^1_t(\dot B^2_{3,1})} &\lesssim \|u_0\|_{\dot B^0_{3,1}} + \|\operatorname{div}(2(a-\dot S_k a)\mathbb{D}u\|_{L^1_t(\dot B^0_{3,1})}\\ &+ \|f\|_{L^1_t(\dot B^0_{3,1})} + \|\nabla\dot S_k a\|_{L^\infty}\|u\|_{L^1_t(\dot B^1_{3,1})} + \sum_{j\in\mathbb{Z}}\|[\dot S_k a;\dot\Delta_j\mathbb{P}]\Delta u\|_{L^1_t(L^3)}\\ &+ \|\nabla\dot S_k a\mathbb{D}u\|_{L^1_t(\dot B^0_{3,1})} + \sum_{j\in\mathbb{Z}}\|[v\cdot\nabla;\dot\Delta_j\mathbb{P}]u\|_{L^1_t(L^3)}.\end{aligned} \tag{3.12}$$

In what follows, we shall deal with the right-hand side of (3.12). Firstly, applying Lemma 2.1 and Lemma 2.2 yields

$$\begin{aligned}&\|\operatorname{div}(2(a-\dot S_k a)\mathbb{D}u\|_{L^1_t(\dot B^0_{3,1})}\lesssim\|a-\dot S_k a\|_{\widetilde L^\infty_t(\dot B^1_{3,1})}\|u\|_{L^1_t(\dot B^2_{3,1})},\\&\|\nabla\dot S_k a\mathbb{D}u\|_{L^1_t(\dot B^0_{3,1})}\lesssim\|\nabla\dot S_k a\|_{\widetilde L^\infty_t(\dot B^1_{3,\infty}\cap L^\infty)}\|u\|_{L^1_t(\dot B^1_{3,1})},\end{aligned}\tag{3.13}$$

For $[\dot S_k a;\dot\Delta_j\mathbb{P}]\Delta u$, we get, by means of homogeneous Bony decomposition [7], that

$$[\dot S_k a;\dot\Delta_j\mathbb{P}]\Delta u=[T_{\dot S_k a};\dot\Delta_j\mathbb{P}]\Delta u+T'_{\dot\Delta_j\Delta u}\dot S_k a-\dot\Delta_j\mathbb{P}(T_{\Delta u}\dot S_k a)-\dot\Delta_j\mathbb{P}\mathcal{R}(\Delta u,\dot S_k a).$$

Hence, due to Lemma 2.2, implies that

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\|[T_{\dot S_k a};\dot\Delta_j\mathbb{P}]\Delta u\|_{L^3}&\lesssim\sum_{j\in\mathbb{Z}}2^{-j}\sum_{|j-l|\le4}\|\nabla\dot S_{l-1}\dot S_k a\|_{L^\infty}\|\dot\Delta_l\Delta u\|_{L^3}\\&\lesssim\sum_{j\in\mathbb{Z}}\sum_{|j-l|\le4}2^{l-j}\|\nabla\dot S_k a\|_{L^\infty}2^{-l}\|\dot\Delta_l\Delta u\|_{L^3}\\&\lesssim\|\nabla\dot S_k a\|_{L^\infty}\|u\|_{\dot B^1_{3,1}}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\|T'_{\dot\Delta_j\Delta u}\dot S_k a\|_{L^3}&\lesssim\sum_{j\in\mathbb{Z}}\sum_{l\ge j-2}\|\dot S_{l+2}\dot\Delta_j\Delta u\|_{L^3}\|\dot\Delta_l\dot S_k a\|_{L^\infty}\\&\lesssim\sum_{j\in\mathbb{Z}}2^{-j}\|\dot\Delta_j\Delta u\|_{L^3}\sum_{l\ge j-2}2^{j-l}2^l\|\dot\Delta_l\dot S_k a\|_{L^\infty}\\&\lesssim\|\nabla\dot S_k a\|_{L^\infty}\|u\|_{\dot B^1_{3,1}}.\end{aligned}$$

The same estimate holds for $\dot\Delta_j(T_{\Delta u}\dot S_k a)$. Notice that

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\|\dot\Delta_j\mathcal{R}(\Delta u,\dot S_k a)\|_{L^3}&\lesssim\sum_{j\in\mathbb{Z}}2^j\|\dot\Delta_j\mathcal{R}(\Delta u,\dot S_k a)\|_{L^{\frac32}}\\&\lesssim\sum_{j\in\mathbb{Z}}\sum_{l\ge j-3}2^{j-l}2^{-l}\|\dot\Delta_l\Delta u\|_{L^3}2^{2l}\|\dot\Delta_l\dot S_k a\|_{L^3}\\&\lesssim\|\nabla\dot S_k a\|_{\dot B^1_{3,\infty}}\|u\|_{\dot B^1_{3,1}}.\end{aligned}$$

Therefore, using the interpolation inequality $\|u\|_{\dot B^1_{3,1}}\lesssim\|u\|^{\frac12}_{\dot B^0_{3,1}}\|u\|^{\frac12}_{\dot B^2_{3,1}}$ and Young's inequality, we deduce for any η that

$$\sum_{j\in\mathbb{Z}}\|[\dot S_k a;\dot\Delta_j\mathbb{P}]\Delta u\|_{L^1_t(L^3)}\lesssim\|\nabla\dot S_k a\|^2_{\widetilde L^\infty_t(\dot B^1_{3,\infty}\cap L^\infty)}\|u\|_{L^1_t(\dot B^0_{3,1})}+\eta\|u\|_{L^1_t(\dot B^2_{3,1})}.\tag{3.14}$$

The estimate of $[v\cdot\nabla;\dot\Delta_j\mathbb{P}]u$, we get, by virtue of Lemma 2.3, that

$$\sum_{j\in\mathbb{Z}}\|[v\cdot\nabla;\dot\Delta_j\mathbb{P}]u\|_{L^1_t(L^3)}\lesssim\int_0^t\|\nabla v\|_{\dot B^0_{\infty,1}}\|u\|_{\dot B^0_{3,1}}\,d\tau.\tag{3.15}$$

Substituting (3.13), (3.14) and (3.15) into (3.12) and taking η small enough, we achieve

$$\begin{aligned}
&\|u\|_{\widetilde{L}_t^\infty(\dot{B}^0_{3,1})}+\|u\|_{L^1_t(\dot{B}^2_{3,1})}\lesssim\|u_0\|_{\dot{B}^0_{3,1}}+\|a-\dot{S}_ka\|_{\widetilde{L}_t^\infty(\dot{B}^1_{3,1})}\|u\|_{L^1_t(\dot{B}^2_{3,1})}\\
&\quad+\|f\|_{L^1_t(\dot{B}^0_{3,1})}+\|\nabla\dot{S}_ka\|^2_{\widetilde{L}_t^\infty(\dot{B}^1_{3,\infty}\cap L^\infty)}\|u\|_{L^1_t(\dot{B}^0_{3,1})}+\int_0^t\|v\|_{\dot{B}^2_{3,1}}\|u\|_{\dot{B}^0_{3,1}}\,d\tau.
\end{aligned}\tag{3.16}$$

Step 2. The estimate of $\|\nabla\pi\|_{L^1_t(\dot{B}^0_{3,1})}$.

In order to estimate the pressure function π, we get by taking div to $(3.1)_1$ that

$$\Delta\pi=-\operatorname{div}(v\cdot\nabla u)+\operatorname{div}\operatorname{div}(2\dot{S}_ka\mathbb{D}u)+\operatorname{div}\operatorname{div}(2(a-\dot{S}_ka)\mathbb{D}u)+\operatorname{div}f.\tag{3.17}$$

Taking L^2 inner product of the above equation with $|\dot{\Delta}_j\pi|\dot{\Delta}_j\pi$ and using a similar argument as (3.12), we find

$$\begin{aligned}
\|\nabla\pi\|_{L^1_t(\dot{B}^0_{3,1})}&\lesssim\|\operatorname{div}(v\cdot\nabla u)\|_{L^1_t(\dot{B}^{-1}_{3,1})}+\|(\dot{S}_ka-a)\nabla\pi\|_{L^1_t(\dot{B}^0_{3,1})}+\|f\|_{L^1_t(\dot{B}^0_{3,1})}\\
&\quad+\|\nabla\dot{S}_ka\mathbb{D}u\|_{L^1_t(\dot{B}^0_{3,1})}+\|\operatorname{div}(\dot{S}_ka\Delta u)\|_{L^1_t(\dot{B}^{-1}_{3,1})}+\|(a-\dot{S}_ka)\mathbb{D}u\|_{L^1_t(\dot{B}^1_{3,1})}.
\end{aligned}\tag{3.18}$$

We now estimate term by term in (3.18). Due to $\operatorname{div}v=\operatorname{div}u=0$. Then applying Lemma 2.2 yields

$$\begin{aligned}
\|\operatorname{div}(v\cdot\nabla u)\|_{L^1_t(\dot{B}^0_{3,1})}&\lesssim\|\nabla u\cdot\nabla v\|_{L^1_t(\dot{B}^{-1}_{3,1})}\lesssim\|T_{\nabla u}\nabla v\|_{L^1_t(\dot{B}^{-1}_{3,1})}\\
&\quad+\|T_{\nabla v}\nabla u\|_{L^1_t(\dot{B}^{-1}_{3,1})}+\|\mathcal{R}(u,v)\|_{L^1_t(\dot{B}^1_{3,1})}\lesssim\int_0^t\|u\|_{\dot{B}^0_{3,1}}\|v\|_{\dot{B}^2_{3,1}}\,d\tau.
\end{aligned}\tag{3.19}$$

Similarly, we can deduce that

$$\begin{aligned}
&\|\nabla\dot{S}_ka\mathbb{D}u\|_{L^1_t(\dot{B}^0_{3,1})}\lesssim\|\nabla\dot{S}_ka\|_{\widetilde{L}_t^\infty(\dot{B}^1_{3,\infty}\cap L^\infty)}\|u\|_{L^1_t(\dot{B}^1_{3,1})},\\
&\|(a-\dot{S}_ka)\mathbb{D}u\|_{L^1_t(\dot{B}^0_{3,1})}\lesssim\|a-\dot{S}_ka\|_{\widetilde{L}_t^\infty(\dot{B}^1_{3,1})}\|u\|_{L^1_t(\dot{B}^2_{3,1})}.
\end{aligned}\tag{3.20}$$

While thanks to Bony's decomposition and $\operatorname{div}u=0$, one has

$$\operatorname{div}(\dot{S}_ka\Delta u)=T_{\nabla\dot{S}_ka}\Delta u+T_{\Delta u}\nabla\dot{S}_ka+\operatorname{div}\mathcal{R}(\dot{S}_ka,\Delta u).$$

Hence we obtain

$$\begin{aligned}
\|\operatorname{div}(\dot{S}_ka\Delta u)\|_{L^1_t(\dot{B}^{-1}_{3,1})}&\lesssim\|T_{\nabla\dot{S}_ka}\Delta u\|_{L^1_t(\dot{B}^{-1}_{3,1})}+\|T_{\Delta u}\nabla\dot{S}_ka\|_{L^1_t(\dot{B}^{-1}_{3,1})}\\
&\quad+\|\mathcal{R}(\dot{S}_ka,\Delta u)\|_{L^1_t(\dot{B}^0_{3,1})}\lesssim\|\nabla\dot{S}_ka\|_{\widetilde{L}_t^\infty(\dot{B}^1_{3,\infty}\cap L^\infty)}\|u\|_{L^1_t(\dot{B}^1_{3,1})}.
\end{aligned}\tag{3.21}$$

Substituting (3.19), (3.20) and (3.21) into (3.18), we obtain

$$\begin{aligned}
\|\nabla\pi\|_{L^1_t(\dot{B}^0_{3,1})}&\lesssim\int_0^t\|u\|_{\dot{B}^0_{3,1}}\|v\|_{\dot{B}^2_{3,1}}\,d\tau+\|\nabla\dot{S}_ka\|_{\widetilde{L}_t^\infty(\dot{B}^1_{3,\infty}\cap L^\infty)}\|u\|_{L^1_t(\dot{B}^1_{3,1})}\\
&\quad+\|a-\dot{S}_ka\|_{\widetilde{L}_t^\infty(\dot{B}^1_{3,1})}\|u\|_{L^1_t(\dot{B}^2_{3,1})}+\|f\|_{L^1_t(\dot{B}^0_{3,1})}.
\end{aligned}\tag{3.22}$$

According to Lemma 2.1 and Lemma 2.2, we may get

$$\|\nabla\dot{S}_ka\|_{\widetilde{L}_t^\infty(\dot{B}^1_{3,\infty}\cap L^\infty)}\lesssim2^{2k}\|\dot{S}_ka\|_{L_t^\infty(L^3)},\tag{3.23}$$

which combining (3.2) and (3.16), we achieve

$$\begin{aligned}\|u\|_{\widetilde{L}^\infty_t(\dot{B}^0_{3,1})}+\|u\|_{L^1_t(\dot{B}^2_{3,1})}+\|\nabla\pi\|_{L^1_t(\dot{B}^0_{3,1})}\lesssim&\|u_0\|_{\dot{B}^0_{3,1}}+\|f\|_{L^1_t(\dot{B}^0_{3,1})}\\&+\int_0^t\|u\|_{\dot{B}^0_{3,1}}\|v\|_{\dot{B}^2_{3,1}}d\tau+2^{4k}\|\dot{S}_ka\|^2_{L^\infty_t(L^3)}\|u\|_{L^1_t(\dot{B}^0_{3,1})}.\end{aligned}\tag{3.24}$$

This completes the proof of Proposition 3.1.

Remark 3.1 *In the particular case when $v=u$ in Proposition 3.1, it holds true, i.e.,*

$$\begin{aligned}&\|u\|_{\widetilde{L}^\infty_t(\dot{B}^0_{3,1})}+\|u\|_{L^1_t(\dot{B}^2_{3,1})}+\|\nabla\pi\|_{L^1_t(\dot{B}^0_{3,1})}\\\le&C\exp\{t2^{4k}\|\dot{S}_ka\|^2_{L^\infty_t(L^3)}\}\left\{\|u_0\|_{\dot{B}^0_{3,1}}+\int_0^t\|u\|_{\dot{B}^0_{3,1}}\|u\|_{\dot{B}^2_{3,1}}d\tau+\|f\|_{L^1_t(\dot{B}^0_{3,1})}\right\}.\end{aligned}\tag{3.25}$$

Proposition 3.2 *Let $a\in\widetilde{L}^\infty_T(B^1_{3,1}(\mathbb{R}^3))$ with $1+a\ge\underline{b}>0$. We assume that $u_0\in\dot{H}^{-2\delta}(\mathbb{R}^3)$ with $\delta<\frac{3}{4}$. Let u, v be two solenoidal vector fields satisfying $\nabla v\in L^1_T(\dot{B}^1_{3,1}(\mathbb{R}^3))$, $u\in C([0,T];\dot{H}^{-2\delta}(\mathbb{R}^3))\cap L^1_{loc}([0,T];\dot{H}^{2(1-\delta)}(\mathbb{R}^3))$, and*

$$\begin{cases}\partial_tu+v\cdot\nabla u-\operatorname{div}(2(1+a)\mathbb{D}u)+\nabla\pi=f,\quad(t,x)\in\mathbb{R}_+\times\mathbb{R}^3,\\\operatorname{div}u=0,\\u|_{t=0}=u_0.\end{cases}\tag{3.26}$$

If there exists some sufficiently small positive constant c and some integer $k\in\mathbb{Z}$, satisfies

$$\|a-\dot{S}_ka\|_{\widetilde{L}^\infty_T(B^1_{3,1})}\le c,\tag{3.27}$$

then for $0<t\le T$, one has

$$\begin{aligned}\|u\|_{\widetilde{L}^\infty_t(\dot{H}^{-2\delta})}+\|u\|_{\widetilde{L}^1_t(\dot{H}^{2(1-\delta)})}\lesssim&\{\|u_0\|_{\dot{H}^{-2\delta}}+\|f\|_{L^1(\dot{H}^{-2\delta})}\}\\&\times\exp\{\|\nabla v\|_{L^1_t(\dot{B}^1_{3,1})}+t2^{4k}\|\dot{S}_ka\|^2_{\widetilde{L}^\infty_t(L^3)}\}.\end{aligned}\tag{3.28}$$

Proof We first rewrite the u equation of (3.26) as

$$\partial_tu-v\cdot\nabla u-\operatorname{div}(2(1+\dot{S}_ka)\mathbb{D}u)+\nabla\pi=\operatorname{div}(2(a-\dot{S}_ka)\mathbb{D}u)+f.\tag{3.29}$$

Let $\mathbb{P}=I+\nabla(-\Delta)^{-1}\operatorname{div}$ be the Leray projection operator, Applying $\dot{\Delta}_j\mathbb{P}$ to (3.29) gives

$$\partial_t\dot{\Delta}_ju-\dot{\Delta}_j\mathbb{P}(v\cdot\nabla u)-\dot{\Delta}_j\mathbb{P}\{\operatorname{div}(2(1+\dot{S}_ka)\mathbb{D}u)\}=\dot{\Delta}_j\operatorname{div}(2(a-\dot{S}_ka)\mathbb{D}u)+f).\tag{3.30}$$

Multiplying the above equation by $\dot{\Delta}_ju$ and then integrating the resulting equations on $x\in\mathbb{R}^3$ leads to

$$\begin{aligned}\frac{d}{dt}\|\dot{\Delta}_ju\|_{L^2}+2^{2j}\|\dot{\Delta}_ju\|_{L^2}\lesssim&\|[v\cdot\nabla;\dot{\Delta}_j\mathbb{P}]u\|_{L^2}+2^j\|[\dot{S}_ka;\dot{\Delta}_j\mathbb{P}]\mathbb{D}u\|_{L^2}\\&+\|\dot{\Delta}_j\operatorname{div}(2(a-\dot{S}_ka)\mathbb{D}u)\|_{L^2}+\|\dot{\Delta}_jf\|_{L^2}.\end{aligned}\tag{3.31}$$

In what follows, we shall deal with the right-hand side of (3.31). Firstly applying homogeneous Bony's decomposition yields

$$[v\cdot\nabla;\dot{\Delta}_j\mathbb{P}]u=[T_v\cdot\nabla;\dot{\Delta}_j\mathbb{P}]u+T'_{\nabla\dot{\Delta}_ju}v-\dot{\Delta}_j\mathbb{P}(T_{\nabla u}v)-\dot{\Delta}_j\mathbb{P}\operatorname{div}\mathcal{R}(u,v).$$

It follows again from the above estimate, which implies that

$$\begin{aligned}\|[T_v\cdot\nabla;\dot\Delta_j\mathbb{P}]u\|_{L^2}&\lesssim\sum_{|j-k|\le4}2^{-j}\|\nabla\dot S_{k-1}v\|_{L^\infty}\|\dot\Delta_k\nabla u\|_{L^2}\\&\lesssim\sum_{|j-k|\le4}2^{k-j}\|\nabla v\|_{L^\infty}\|\dot\Delta_k u\|_{L^2}\\&\lesssim 2^{2j\delta}\sum_{|j-k|\le4}2^{(k-j)(1+2\delta)}\|\nabla v\|_{L^\infty}2^{-2k\delta}\|\dot\Delta_k u\|_{L^2}\\&\lesssim c_j2^{2j\delta}\|\nabla v\|_{L^\infty}\|u\|_{\dot H^{-2\delta}}.\end{aligned}$$

The same estimate holds for $T'_{\nabla\dot\Delta_j u}v$. Note that

$$\begin{aligned}\|\dot\Delta_j\mathbb{P}(T_{\nabla u}v)\|_{L^2}&\lesssim\sum_{|k-j|\le4}\|\dot S_{k-1}\nabla u\|_{L^2}\|\dot\Delta_k v\|_{L^\infty}\\&\lesssim 2^{2j\delta}\sum_{|k-j|\le4}2^{2(k-j)\delta}2^{-2k\delta}\|\dot S_{k-1}u\|_{L^2}\|\dot\Delta_k\nabla v\|_{L^\infty}\\&\lesssim c_j2^{2j\delta}\|u\|_{\dot H^{-2\delta}}\|\nabla v\|_{L^\infty}.\end{aligned}$$

For $\dot\Delta_j\mathbb{P}\,\mathrm{div}\,\mathcal{R}(u,v)$, we have

$$\begin{aligned}\|\dot\Delta_j\mathbb{P}\,\mathrm{div}\,\mathcal{R}(u,v)\|_{L^2}&\lesssim 2^{2j}\|\dot\Delta_j\mathbb{P}\mathcal{R}(u,v)\|_{L^{\frac65}}\\&\lesssim 2^{2j}\sum_{k\ge j-3}\|\dot\Delta_k u\|_{L^2}\|\dot\Delta_k v\|_{L^3}\\&\lesssim 2^{2j\delta}\sum_{k\ge j-3}2^{(2-2\delta)(j-k)}2^{-2k\delta}\|\dot\Delta_k u\|_{L^2}2^k\|\dot\Delta_k\nabla v\|_{L^3}\\&\lesssim c_j2^{2j\delta}\|u\|_{\dot H^{-2\delta}}\|\nabla v\|_{\dot B^1_{3,1}}.\end{aligned}$$

From which, we obtain

$$\|[v\cdot\nabla;\dot\Delta_j\mathbb{P}]u\|_{L^1_t(L^2)}\lesssim c_j2^{2j\delta}\int_0^t\|\nabla v\|_{\dot B^1_{3,1}}\|u\|_{\dot H^{-2\delta}}d\tau. \tag{3.32}$$

The estimate of $[\dot S_k a;\dot\Delta_j\mathbb{P}]\mathbb{D}u$, we get, by virtue of Lemma 2.2, that

$$\|[\dot S_k a;\dot\Delta_j\mathbb{P}]\mathbb{D}u\|_{L^1_t(L^2)}\lesssim c_j2^{j(2\delta-1)}\|\nabla\dot S_k a\|_{\widetilde L^\infty_t(\dot B^1_{3,\infty}\cap L^\infty)}\|u\|_{L^1_t(\dot H^{1-2\delta})}. \tag{3.33}$$

Along the same line, for $\delta<\frac34$, one has

$$\|\dot\Delta_j\,\mathrm{div}(2(a-\dot S_k a)\mathbb{D}u)\|_{L^1_t(L^2)}\lesssim c_j2^{2j\delta}\|a-\dot S_k a\|_{\widetilde L^\infty_T(\dot B^1_{3,1})}\|u\|_{L^1_t(\dot H^{2(1-\delta)})}. \tag{3.34}$$

Substituting (3.32)-(3.34) into (3.31), and using the interpolation inequality $\|u\|_{\dot H^{1-2\delta}}\lesssim\|u\|^{\frac12}_{\dot H^{-2\delta}}\|u\|^{\frac12}_{\dot H^{2(1-\delta)}}$, we write

$$\begin{aligned}\|u\|_{\widetilde L^\infty_t(\dot H^{-2\delta})}+\|u\|_{\widetilde L^1_t(\dot H^{2(1-\delta)})}&\le\|u_0\|_{\dot H^{-2\delta}}+C\int_0^t\|\nabla v\|_{\dot B^1_{3,1}}\|u\|_{\dot H^{-2\delta}}d\tau\\&\quad+C\|f\|_{L^1(\dot H^{-2\delta})}+C2^{4k}\|\dot S_k a\|^2_{\widetilde L^\infty_t(L^3)}\int_0^t\|u\|_{\dot H^{-2\delta}}d\tau\\&\quad+C\|a-\dot S_k a\|_{\widetilde L^\infty_T(\dot B^1_{3,1})}\|u\|_{L^1_t(\dot H^{2(1-\delta)})}+\frac12\|u\|_{\widetilde L^1_t(\dot H^{2(1-\delta)})},\end{aligned} \tag{3.35}$$

which along with (3.27) ensures that

$$\begin{aligned}\|u\|_{\widetilde{L}^\infty_t(\dot{H}^{-2\delta})}+\|u\|_{\widetilde{L}^1_t(\dot{H}^{2(1-\delta)})} \leq \|u_0\|_{\dot{H}^{-2\delta}} + C\int_0^t \|\nabla v\|_{\dot{B}^1_{3,1}}\|u\|_{\dot{H}^{-2\delta}}d\tau \\ + C\|f\|_{L^1(\dot{H}^{-2\delta})} + C2^{4k}\|\dot{S}_k a\|^2_{\widetilde{L}^\infty_t(L^3)}\int_0^t \|u\|_{\dot{H}^{-2\delta}}d\tau.\end{aligned} \tag{3.36}$$

Applying Gronwall's lemma to (3.36) leads to (3.28).

Corollary 3.1 *Under the assumptions of Proposition 3.2, we can find some $t_1 \in (\tau, t)$ and there holds*

$$\begin{aligned}&\|u\|_{\widetilde{L}^\infty([\tau,t];L^2)} + \|u\|_{\widetilde{L}^1([\tau,t];\dot{H}^2)} \\ \lesssim &\{\|u_0\|_{\dot{H}^{-2\delta}}/t_1^\delta + \|f\|_{L^1([\tau,t];L^2)}\}\exp\left\{\|v\|_{L^1_t(\dot{B}^2_{3,1})} + t2^{4k}\|\dot{S}_k a\|^2_{\widetilde{L}^\infty_t(L^3)}\right\}.\end{aligned} \tag{3.37}$$

Proof Thanks to Proposition 3.2, we can find some $0 < \tau < t_1 < t$ such that $u(\tau) \in L^2$. Moreover, we deduce by a similar proof of Proposition 3.2,

$$\begin{aligned}&\|u\|_{\widetilde{L}^\infty([\tau,t];L^2)} + \|u\|_{\widetilde{L}^1([\tau,t];\dot{H}^2)} \\ \lesssim &\{\|u(\tau)\|_{L^2} + \|f\|_{L^1([\tau,t];L^2)}\}\exp\left\{\|v\|_{L^1_t(\dot{B}^2_{3,1})} + t2^{4k}\|\dot{S}_k a\|^2_{\widetilde{L}^\infty_t(L^3)}\right\} \\ \lesssim &\{\|u\|_{L^1([0,t_1];L^2)}/t_1 + \|f\|_{L^1([\tau,t];L^2)}\}\exp\left\{\|v\|_{L^1_t(\dot{B}^2_{3,1})} + t2^{4k}\|\dot{S}_k a\|^2_{\widetilde{L}^\infty_t(L^3)}\right\} \\ \lesssim &\{\|u_0\|_{\dot{H}^{-2\delta}}/t_1^\delta + \|f\|_{L^1([\tau,t];L^2)}\}\exp\left\{\|v\|_{L^1_t(\dot{B}^2_{3,1})} + t2^{4k}\|\dot{S}_k a\|^2_{\widetilde{L}^\infty_t(L^3)}\right\},\end{aligned}$$

from which, we can obtain the (3.37).

4 Local well-posedness of (1.3)

The goal of this section is to prove the following local well-posedness result of (1.3).

Theorem 4.1 *Let $u_0 \in \dot{B}^0_{3,1}(\mathbb{R}^3)$ be a solenoidal vector field and $\theta_0 \in B^1_{3,1}(\mathbb{R}^3)$. There exists a small positive constant ε_0, such that if*

$$\|u_0\|_{\dot{B}^0_{3,1}} \leq \varepsilon_0, \tag{4.1}$$

then (1.3) *has a local solution $(\theta, u, \nabla\pi)$ satisfying for $T > 1$ that*

$$\theta \in \mathcal{C}_b([0,T]; B^1_{3,1}(\mathbb{R}^3)),\ \ u \in \mathcal{C}_b([0,T]; \dot{B}^0_{3,1}(\mathbb{R}^3)) \cap L^1([0,T]; \dot{B}^2_{3,1}(\mathbb{R}^3)). \tag{4.2}$$

Furthermore, if $\theta_0 \in B^{\frac{3}{2}}_{2,1}$, then the solution is unique.

4.1 Local existence

We begin with the proof of local existence of solutions in Theorem 4.1 by solving an approximation problem, and then perform the uniform estimates for the approximate solutions. Finally, the existence part of the Theorem 4.1 is reached by a compactness argument.

Step 1. Construction of smooth approximate solutions.

For $n \in \mathbb{N}$, let
$$\theta_0^n = \dot{S}_n\theta_0 - \dot{S}_{-n}\theta_0 \quad \text{and} \quad u_0^n = \dot{S}_n u_0 - \dot{S}_{-n} u_0.$$
Then [1] ensures that the system (1.3) with the initial data (θ_0^n, u_0^n) generates a unique local-in-time smooth solution $(\theta^n, u^n, \nabla\pi^n)$.

Step 2. Uniform estimates to the approximate solutions.

We shall prove that there exists a positive time T_n^* such that $(\theta^n, u^n, \nabla\pi^n)$ is uniformly bounded in the space

$$E_T = \widetilde{L}^\infty_T(B^1_{3,1}) \times (\widetilde{L}^\infty_T(\dot{B}^0_{3,1}) \cap L^1_T(\dot{B}^2_{3,1})) \times L^1_T(\dot{B}^0_{3,1}).$$

In order to do so, let $u_L^n(t) = e^{t\Delta}u_0^n$. Then it is easy to observe that

$$\|u_L^n\|_{\widetilde{L}^\infty_t(\mathbb{R}^+;\dot{B}^0_{3,1})} \lesssim \|u_0\|_{\dot{B}^0_{3,1}} \quad \text{and} \quad \|u_L^n\|_{L^1_t(\dot{B}^2_{3,1})} \lesssim \sum_{j\in\mathbb{Z}}(1 - e^{-ct2^{2j}})\|u_0\|_{L^3}. \tag{4.3}$$

Let $u^n = u_L^n + \bar{u}^n$. Then $(\theta^n, \bar{u}^n, \nabla\pi^n)$ solves

$$\begin{cases} \partial_t\theta^n + (u_L^n + \bar{u}^n)\cdot\nabla\theta^n = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \partial_t\bar{u}^n + u_L^n\cdot\nabla\bar{u}^n - \operatorname{div}(2(1+a^n)\mathbb{D}\bar{u}^n) + \nabla\pi^n = H_n, \\ \operatorname{div}\bar{u}^n = 0, \\ (a^n, \bar{u}^n)|_{t=0} = (a_0^n, 0), \end{cases} \tag{4.4}$$

where
$$H_n = -\bar{u}^n\cdot\nabla\bar{u}^n - \bar{u}^n\cdot\nabla u_L^n - u_L^n\cdot\nabla u_L^n + \operatorname{div}(2a^n\mathbb{D}u_L^n).$$
For notational simiplicity, we denote by

$$a^n \stackrel{\text{def}}{=} \mu(\theta^n) - 1 \quad \text{and} \quad \partial_t a^n + (u_L^n + \bar{u}^n)\cdot\nabla a^n = 0. \tag{4.5}$$

Now let us turn to the uniform estimates of $(\theta^n, \bar{u}^n, \pi^n)$. Firstly, as $\theta_0 \in B^1_{3,1}(\mathbb{R}^3)$, we define $m \in \mathbb{Z}$ by
$$k \stackrel{\text{def}}{=} \inf\{q \in \mathbb{Z} | \sum_{j\geq q} 2^j\|\Delta_j(\mu(\theta_0) - 1)\|_{L^3} \leq c_0\underline{b}\}, \tag{4.6}$$
for some sufficiently small positive constant c_0.

Notice that $\operatorname{div}(u_L^n + \bar{u}^n) = 0$ and (4.5), applying Lemma 2.4 to θ^n equation of $(4.4)_1$ leads to
$$\|\theta^n\|_{\widetilde{L}^\infty_t(B^1_{3,1})} \lesssim \|\theta_0\|_{B^1_{3,1}} \exp\left\{\|\nabla u_L^n\|_{L^1_t(\dot{B}^1_{3,1})} + \|\nabla\bar{u}^n\|_{L^1_t(B^1_{3,1})}\right\},$$
$$\|a^n\|_{L^\infty_t(L^3\cap L^\infty)} \lesssim \|\theta_0^n\|_{L^3\cap L^\infty} \lesssim \|\theta_0\|_{B^1_{3,1}} \quad \text{and} \quad \|a^n\|_{B^1_{3,1}} \leq \|\theta^n\|_{B^1_{3,1}}. \tag{4.7}$$
While thanks to Lemma 2.2, one has

$$\|\bar{u}^n\cdot\nabla\bar{u}^n\|_{\dot{B}^0_{3,1}} \lesssim \|\bar{u}^n\|_{\dot{B}^0_{3,1}}\|\bar{u}^n\|_{\dot{B}^2_{3,1}}, \qquad \|\bar{u}^n\cdot\nabla u_L^n\|_{\dot{B}^0_{3,1}} \lesssim \|\bar{u}^n\|_{\dot{B}^0_{3,1}}\|u_L^n\|_{\dot{B}^2_{3,1}},$$

$$\|u_L^n\cdot\nabla u_L^n\|_{\dot{B}^0_{3,1}} \lesssim \|u_L^n\|_{\dot{B}^0_{3,1}}\|u_L^n\|_{\dot{B}^2_{3,1}} \quad \text{and} \quad \|\operatorname{div}2(a^n\mathbb{D}u_L^n)\|_{\dot{B}^0_{3,1}} \lesssim \|a^n\|_{\dot{B}^1_{3,1}}\|u_L^n\|_{\dot{B}^2_{3,1}}.$$

Therefore, applying Proposition 3.1 to (4.4), we arrive at

$$
\begin{aligned}
&\|\bar{u}^n\|_{\widetilde{L}^\infty_t(\dot{B}^0_{3,1})}+\|\bar{u}^n\|_{L^1_t(\dot{B}^2_{3,1})}+\|\nabla\pi^n\|_{L^1_t(\dot{B}^0_{3,1})}\lesssim\int_0^t\mathcal{D}(t)\|\bar{u}^n\|_{\dot{B}^0_{3,1}}d\tau\\
&+\int_0^t\|\bar{u}^n\|_{\dot{B}^0_{3,1}}\|\bar{u}^n\|_{\dot{B}^2_{3,1}}d\tau+\int_0^t\|u^n_L\|_{\dot{B}^0_{3,1}}\|u^n_L\|_{\dot{B}^2_{3,1}}d\tau+\int_0^t\|a^n\|_{\dot{B}^1_{3,1}}\|u^n_L\|_{\dot{B}^2_{3,1}}d\tau.
\end{aligned}
\tag{4.8}
$$

with

$$
\mathcal{D}(t)\overset{\mathrm{def}}{=}2^{4k}\|\theta_0\|^2_{L^3}+\|u^n_L\|_{\dot{B}^2_{3,1}}.
$$

Let $Z^n(t)\overset{\mathrm{def}}{=}\|\bar{u}^n\|_{\widetilde{L}^\infty_t(\dot{B}^0_{3,1})}+\|\bar{u}^n\|_{L^1_t(\dot{B}^0_{3,1})}+\|\nabla\pi^n\|_{L^1_t(\dot{B}^0_{3,1})}$. Then applying Gronwall's inequality to (4.8) leads to

$$
Z^n(t)\leq C\exp\{\int_0^t\mathcal{D}(t)d\tau\}\Big\{Z^n(t)^2+\Big(\|u^n_L\|_{L^\infty_t(\dot{B}^0_{3,1})}+\|a^n\|_{L^\infty_t(\dot{B}^1_{3,1})}\Big)\|u^n_L\|_{L^1_t(\dot{B}^2_{3,1})}\Big\},\tag{4.9}
$$

under the assumption that

$$
\|a^n-\dot{S}_ka^n\|_{\widetilde{L}^\infty_t(\dot{B}^1_{3,1})}\leq 2c_0\underline{b}.\tag{4.10}
$$

However, thanks to (4.3), one has

$$
\int_0^t\mathcal{D}(t)d\tau=t2^{4k}\|\theta_0\|^2_{L^3}+\|u_0\|_{\dot{B}^0_{3,1}}\overset{\mathrm{def}}{=}W(t),
$$

so that (4.9) ensures

$$
Z^n(t)\lesssim\exp\{W(t)\}\left\{Z^n(t)^2+\|\theta_0\|_{\dot{B}^1_{3,1}}\exp\Big\{C(\|u_0\|_{\dot{B}^0_{3,1}}+Z^n(t))\Big\}\sum_{j\in\mathbb{Z}}(1-e^{-ct2^{2j}})\|\dot{\Delta}_ju_0\|_{L^3}\right\},
$$

under the assumption (4.10), or equivalently, we have

$$
Z^n(t)\leq C_1\exp\{C_1W(t)\}Z^n(t)^2+\Theta(t)\exp\{C_1Z^n(t)\}\tag{4.11}
$$

with $\Theta(t)$ being determined by

$$
\Theta(t)\overset{\mathrm{def}}{=}C_1\exp\Big\{C_1(W(t)+\|u_0\|_{\dot{B}^0_{3,1}}+\|\theta_0\|_{\dot{B}^1_{3,1}})\Big\}\sum_{j\in\mathbb{Z}}(1-e^{-ct2^{2j}})\|\dot{\Delta}_ju_0\|_{L^3},
$$

for some large enough constant C_1.

On the other hand, applying Lemma 2.4 to the transport equation of $(4.4)_1$ together with (4.6), (4.7) and

$$
e^x-1\leq xe^x\quad\text{for}\quad x\geq 0,
$$

gives rise to

$$
\begin{aligned}
&\|a^n-\dot{S}_ka^n\|_{\widetilde{L}^\infty_t(\dot{B}^1_{3,1})}\\
\leq&\sum_{j\geq k}2^j\|\dot{\Delta}_j(\mu(\theta_0)-1)\|_{L^3}+\|\theta_0\|_{\dot{B}^1_{3,1}}\{\exp\{C(Z^n(t)+\|u^n_L\|_{L^1_t(\dot{B}^0_{3,1})})\}-1\}\\
\leq&c_0\underline{b}+C\|\theta_0\|_{\dot{B}^1_{3,1}}(Z^n(t)+\|u^n_L\|_{L^1_t(\dot{B}^0_{3,1})})\exp\{C(Z^n(t)+\|u^n_L\|_{L^1_t(\dot{B}^0_{3,1})})\},
\end{aligned}
$$

which along with (4.3) and the definition of a implies

$$\begin{aligned}\|a^n-\dot S_k a^n\|_{\widetilde L^\infty_t(\dot B^1_{3,1})}\leq c_0\underline{b}+&C_2\|\theta_0\|_{\dot B^1_{3,1}}\exp\{C_2(Z^n(t)+\|u_0\|_{\dot B^0_{3,1}})\}\\ &\times(Z^n(t)+\sum_{j\in\mathbb{Z}}(1-e^{-ct2^{2j}})\|\dot\Delta_j u_0\|_{L^3}).\end{aligned}\tag{4.12}$$

Now for some sufficiently small $\varepsilon_0\geq 0$, we take

$$0<T_1\leq\min\left\{1,\varepsilon_0(4C_1\exp\{C_1(W(1)+\|u_0\|_{\dot B^0_{3,1}}+\|\theta_0\|_{\dot B^1_{3,1}})\})^{-1}\right\},$$

and

$$\sum_{j\in\mathbb{Z}}(1-e^{-cT_12^{2j}})\|u_0\|_{L^3}\leq\frac{\varepsilon_0}{4C_1\exp\{C_1(W(1)+\|u_0\|_{\dot B^0_{3,1}}+\|\theta_0\|_{\dot B^1_{3,1}})\}}.$$

Then it follows from (4.11) that for $t\leq T_1$

$$Z^n(t)\leq C_1\exp\{C_1W(1)\}Z^n(t)^2+\varepsilon_0\exp\{C_1Z^n(t)\},$$

provided that

$$C_2\|\theta_0\|_{\dot B^1_{3,1}}\exp\{C_2(Z^n(t)+\|u_0\|_{\dot B^0_{3,1}})\}(Z^n(t)+\sum_{j\in\mathbb{Z}}(1-e^{-cT_12^{2j}})\|u_0\|_{L^3})\leq c_0\underline{b}.\tag{4.13}$$

Let $T_2(\varepsilon_0)\overset{\text{def}}{=}\sup\{t\in[0,T_1(\varepsilon_0)]|Z^n(t)\leq 4\varepsilon_0\}$. Without loss of generality, we may assume that ε_0 is so small that

$$16C_1\varepsilon_0\exp\{C_1W(T_2)\}\leq 1\quad\text{and}\quad 4\varepsilon_0\exp\{C_2(4\varepsilon_0+\|u_0\|_{\dot B^0_{3,1}}+\|\theta_0\|_{\dot B^1_{3,1}})\}\leq c_0\underline{b}.$$

Then for $t\leq T_2(\varepsilon_0)$, we infer from (4.1) that (4.13) holds, and moreover (4.11) ensures that

$$Z^n(t)\leq 4C_1\varepsilon_0\exp\{C_1W(t)\}Z^n(t)+e\varepsilon_0,$$

which along with (4.12) shows that there holds (4.10) for $t=T_2(\varepsilon_0)$ and

$$Z^n(t)\leq\frac{4}{3}e\varepsilon_0\quad\text{for}\quad t\leq T_2(\varepsilon_0).$$

This together with (4.3) and (4.7) ensures

$$\{\theta^n,u^n,\nabla\pi^n\}_{n\in\mathbb{N}}\text{ is uniformly bounded in }E_{T_2(\varepsilon_0)}.\tag{4.14}$$

Step 3. The local existence part of Theorem 4.1 with large data.

Thanks to (4.12), we can repeat the compactness arguement in Step 3 to the proof of Theorem 5.1 in [10] to conclude that there exists a subsequence of $\{\theta^n,u^n,\nabla\pi^n\}_{n\in\mathbb{N}}$ which converges to a solution $(\theta,u,\nabla\pi)$, which belongs to $\mathcal{C}_b([0,T_2];B^1_{3,1}(\mathbb{R}^3))\times\mathcal{C}_b([0,T_2];\dot B^0_{3,1}(\mathbb{R}^3))\cap L^1([0,T_2];\dot B^2_{3,1}(\mathbb{R}^3))\times L^1([0,T_2];\dot B^0_{3,1}(\mathbb{R}^3))$ of (1.3) on $[0,T_2(\varepsilon_0)]$. Moreover, for some integer k, there holds

$$\|a-\dot S_k a\|_{\widetilde L^\infty_{T_2(\varepsilon_0)}(B^1_{3,1})}\leq 2c_0\underline{b}.\tag{4.15}$$

Step 4. Large time well-posedness of (1.3) for $\|u_0\|_{\dot B^0_{3,1}}$ small.

In the case when $\|u_0\|_{\dot{B}^0_{3,1}}$ is sufficiently small, we denote $(\theta^n, u^n, \nabla\pi^n)$ to be the unique solution of (1.3) with the initial data (a_0^n, u_0^n), and

$$X^n(t) \stackrel{\text{def}}{=} \|u^n\|_{\widetilde{L}^\infty_t(\dot{B}^0_{3,1})} + \|u^n\|_{L^1_t(\dot{B}^2_{3,1})} + \|\nabla\pi^n\|_{L^1_t(\dot{B}^0_{3,1})}.$$

Then if there holds (4.10), applying Remark 3.1 and using (4.7) gives rise to

$$X^n(t) \leq C_3 \exp\left\{C_3 t 2^{4k}\|\theta_0\|_{L^3}^2\right\}\{\|u_0\|_{\dot{B}^0_{3,1}} + X^n(t)^2\}. \tag{4.16}$$

While thanks to (4.6) and (4.7), applying Lemma 2.4 to the a^n equation leads to

$$\begin{aligned} \|a^n - \dot{S}_k a^n\|_{\widetilde{L}^\infty_t(\dot{B}^1_{3,1})} &\leq c_0\underline{b} + \|\theta_0\|_{\dot{B}^1_{3,1}}(\exp\{C_3X^n(t)\} - 1) \\ &\leq c_0\underline{b} + C_3X^n(t)\|\theta_0\|_{\dot{B}^1_{3,1}}\exp\{C_3X^n(t)\}, \end{aligned} \tag{4.17}$$

provided that

$$C_3X^n(t)\|\theta_0\|_{\dot{B}^1_{3,1}}\exp\{C_3X^n(t)\} \leq c_0\underline{b}. \tag{4.18}$$

Without loss of generality, we may assume that $T^* \leq 1$. We define

$$T^* \stackrel{\text{def}}{=} \sup\left\{T > 0 : X^n(T) \leq 4C_3\|u_0\|_{\dot{B}^0_{3,1}}\exp\{C_3t2^{4k}\|\theta_0\|^2_{L^3}\}\right\}. \tag{4.19}$$

Then we infer that for $T \leq T^*$

$$C_3\exp\{C_3t2^{4k}\|\theta_0\|^2_{L^3}\}X^n(t) \leq \frac{1}{4}, \tag{4.20}$$

which together with (4.16) leads to

$$X^n(T) \leq \frac{4}{3}C_3\|u_0\|_{\dot{B}^0_{3,1}}\exp\{C_3t2^{4k}\|\theta_0\|^2_{L^3}\}. \tag{4.21}$$

This contradicts (4.19), and thus $T^* > 1$.

On the other hand, applying Lemma 2.4 applied to the transport equation of $(4.4)_1$ yields

$$\|\theta^n\|_{\widetilde{L}^\infty_T(B^1_{3,1})} \leq C_4\|\theta_0\|_{B^1_{3,1}}\exp\{C_4X^n(t)\} \quad \text{for } T \leq T^*.$$

Therefore, $(\theta^n, u^n, \nabla\pi^n)$ is uniformly bounded in E_{T^*}. With this bound of $(\theta^n, u^n, \nabla\pi^n)$, we can repeat the argument in Step 3 to conclude that the lifespan to the solution $(\theta, u, \nabla\pi)$ obtained there is greater than 1. Futhermore there holds (4.2). This completes the existence part of Theorem 4.1.

4.2 Local uniqueness

To prove the uniqueness of solutions in Theorem 4.1, we need the following Propositions:

Proposition 4.1 *Let $a \in L^\infty_T(B^{\frac{1}{2}}_{2,1})$, $f \in L^1([0,T]; B^{-\frac{1}{2}}_{2,1})$ and $b \in L^\infty_T(B^{\frac{1}{2}}_{2,1})$ with $1 + a \geq \underline{b} > 0$ and $1 + b \geq \underline{b} > 0$. Let u, v be two solenoidal vector field which satisfy $u \in \mathcal{C}([0,T]; B^{-\frac{1}{2}}_{2,1}) \cap L^1([0,T]; B^{\frac{3}{2}}_{2,1})$, $v \in L^1([0,T]; B^1_{\infty,1})$, and*

$$\begin{cases} \partial_t u + v\cdot\nabla u - \operatorname{div}((1+a)\mathbb{D}u) + \nabla\pi = f, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0. \end{cases} \tag{4.22}$$

Then there holds

$$
\begin{aligned}
\|u\|_{\widetilde{L}^\infty_t(B^{-\frac12}_{2,1})}+\|u\|_{L^1_t(B^{\frac32}_{2,1})}\lesssim \|u_0\|_{B^{-\frac12}_{2,1}}+\int_0^t\|u\|_{B^{-\frac12}_{2,1}}\|v\|_{B^1_{\infty,1}}d\tau\\
+\|u\|_{L^1_t(L^2)}+\int_0^t\|a\|^2_{B^1_{\infty,1}}\|u\|_{B^{-\frac12}_{2,1}}d\tau+\|f\|_{L^1_t(B^{-\frac12}_{2,1})}.
\end{aligned}
\tag{4.23}
$$

Proof Applying Δ_j to $(4.22)_1$ and using a standard commutator process. we easily obtain

$$\partial_t\Delta_j u+v\cdot\Delta_j u-\mathrm{div}((1+a)\Delta_j\mathbb{D}u)+\Delta_j\nabla\pi=[v\cdot\nabla;\Delta_j]u-\mathrm{div}[a;\Delta_j]\mathbb{D}u+\Delta_j f.$$

Let $u^H\overset{\text{def}}{=}u-\Delta_{-1}u$. Then multiplying the above equation by $\Delta_j u$ and then integrating the resulting equations on $x\in\mathbb{R}^3$, which leads to

$$\frac{d}{dt}\|\Delta_j u\|_{L^2}+2^{2j}\|\Delta_j u^H\|_{L^2}\le\|[v\cdot\nabla;\Delta_j]u\|_{L^2}+2^j\|[a;\Delta_j]\mathbb{D}u\|_{L^2}+\|\Delta_j f\|_{L^2}. \tag{4.24}$$

Intergrating the above inequality over $[0,T]$ and multiplying it by $2^{-\frac{j}{2}}$, and then summing up the resulting inequality over $j\in\mathbb{Z}$, we achieve

$$
\begin{aligned}
\|u\|_{\widetilde{L}^\infty_t(B^{-\frac12}_{2,1})}+\|u^H\|_{L^1_t(B^{\frac32}_{2,1})}\lesssim\|u_0\|_{B^{-\frac12}_{2,1}}+\sum_{j\in\mathbb{Z}}2^{-\frac{j}{2}}\|[v\cdot\nabla;\Delta_j]u\|_{L^1_t(L^2)}\\
+\sum_{j\in\mathbb{Z}}2^{\frac{j}{2}}\|[a;\Delta_j]\mathbb{D}u\|_{L^1_t(L^2)}+\|f\|_{L^1_t(B^{-\frac12}_{2,1})}.
\end{aligned}
\tag{4.25}
$$

In what follows, we shall deal with the right-hand side of (4.25). Firstly applying Lemma 2.3 yields

$$\sum_{j\in\mathbb{Z}}2^{-\frac{j}{2}}\|[v\cdot\nabla;\Delta_j]u\|_{L^1_t(L^2)}\lesssim\int_0^t\|u\|_{B^{-\frac12}_{2,1}}\|v\|_{B^1_{\infty,1}}d\tau. \tag{4.26}$$

For $[a;\Delta_j]\mathbb{D}u$, the homogenous Bony's decomposition implies

$$[a;\Delta_j]\mathbb{D}u=[T_a;\Delta_j]\mathbb{D}u+T'_{\Delta_j\mathbb{D}u}a-\Delta_j(T_{\mathbb{D}u}a)-\Delta_j\mathcal{R}(\mathbb{D}u,a).$$

It follows again from the above estimates, which implies that

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}}2^{\frac{j}{2}}\|[T_a;\Delta_j]\mathbb{D}u\|_{L^2}&\lesssim\sum_{j\in\mathbb{Z}}2^{-\frac{j}{2}}\sum_{|j-k|\le4}\|\nabla S_{k-1}a\|_{L^\infty}\|\Delta_k\nabla u\|_{L^2}\\
&\lesssim\sum_{j\in\mathbb{Z}}\sum_{|j-k|\le4}2^{-\frac12(j-k)}\|\nabla a\|_{L^\infty}2^{-\frac{k}{2}}\|\Delta_k\nabla u\|_{L^2}\\
&\lesssim\|\nabla a\|_{L^\infty}\|u\|_{B^{\frac12}_{2,1}}.
\end{aligned}
$$

The same estimate holds true for $T'_{\Delta_j\mathbb{D}u}a$. Note that

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}}2^{\frac{j}{2}}\|T'_{\Delta_j\mathbb{D}u}a\|_{L^2}&\lesssim\sum_{j\in\mathbb{Z}}2^{\frac{j}{2}}\sum_{k\ge j-2}\|S_{k+2}\Delta_j\nabla u\|_{L^2}\|\Delta_k a\|_{L^\infty}\\
&\lesssim\sum_{j\in\mathbb{Z}}2^{-\frac{j}{2}}\|\Delta_j\nabla u\|_{L^2}\sum_{k\ge j-2}2^{j-k}2^k\|\Delta_k b\|_{L^\infty}\\
&\lesssim\|a\|_{B^1_{\infty,1}}\|u\|_{B^{\frac12}_{2,1}}.
\end{aligned}
$$

Similarly, we have

$$\begin{aligned}\sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}}\|\Delta_j(T'_{\mathbb{D}u}a)\|_{L^2} &\lesssim \sum_{j\in\mathbb{Z}}\sum_{k\geq j-3} 2^{\frac{j}{2}}\|S_{k+2}\nabla u\|_{L^2}\|\Delta_k a\|_{L^\infty}\\ &\lesssim \sum_{j\in\mathbb{Z}}\sum_{k\geq j-3} 2^{\frac{1}{2}(j-k)}2^{-\frac{k}{2}}\|S_{k+2}\nabla u\|_{L^2}2^k\|\Delta_k a\|_{L^\infty}\\ &\lesssim \|a\|_{B^1_{\infty,1}}\|u\|_{B^{\frac{1}{2}}_{2,1}}.\end{aligned}$$

Therefore, we can deduce that

$$\sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}}\|[a;\Delta_j\mathbb{P}]\mathbb{D}u\|_{L^1_t(L^2)} \lesssim \int_0^t \|a\|_{B^1_{\infty,1}}\|u\|_{B^{\frac{1}{2}}_{2,1}}\,d\tau. \tag{4.27}$$

Plugging the above estimates into (4.25) and using the interpolation inequality $\|u\|_{B^{\frac{1}{2}}_{2,1}} \lesssim \|u\|^{\frac{1}{2}}_{B^{-\frac{1}{2}}_{2,1}}\|u\|^{\frac{1}{2}}_{B^{\frac{3}{2}}_{2,1}}$ and Young's inequality, we achieve

$$\begin{aligned}\|u\|_{\widetilde{L}^\infty_t(B^{-\frac{1}{2}}_{2,1})} + \|u^H\|_{L^1_t(B^{\frac{3}{2}}_{2,1})} \lesssim&\|u_0\|_{B^{-\frac{1}{2}}_{2,1}} + \int_0^t \|u\|_{B^{-\frac{1}{2}}_{2,1}}\|v\|_{B^1_{\infty,1}}\,d\tau\\ &+\int_0^t \|a\|^2_{B^1_{\infty,1}}\|u\|_{B^{-\frac{1}{2}}_{2,1}}\,d\tau + \|f\|_{L^1_t(B^{-\frac{1}{2}}_{2,1})},\end{aligned} \tag{4.28}$$

which along with $\|u\|_{L^1_t(B^{\frac{3}{2}}_{2,1})} \lesssim \|u^H\|_{L^1_t(B^{\frac{3}{2}}_{2,1})} + \|u\|_{L^1_t(L^2)}$ leads to (4.23).

In order to get the uniqueness of solutions in Theorem 1.1, we need to recall the following Proposition 4.2 in [2], where we omitted the details.

Proposition 4.2 *Let $F \in B^{-\frac{1}{2}}_{2,1}(\mathbb{R}^3)$ and $\nabla\pi \in B^{-\frac{1}{2}}_{2,1}(\mathbb{R}^3)$ solving*

$$\Delta\pi = \operatorname{div} F. \tag{4.29}$$

Then it holds true that

$$\|\nabla\pi\|_{B^{-\frac{1}{2}}_{2,1}} \lesssim \|F\|_{B^{-\frac{3}{2}}_{2,1}} + \|\operatorname{div} F\|_{B^{-\frac{3}{2}}_{2,1}}. \tag{4.30}$$

Proof of the uniqueness in Theorem 4.1 Let $(\theta^i, u^i, \nabla\pi^i)$ with $i = 1, 2$ be two solutions of (1.3). We denote

$$(\delta\theta, \delta u, \nabla\delta\pi) \stackrel{def}{=} (\theta^2 - \theta^1, u^2 - u^1, \nabla\pi^2 - \nabla\pi^1).$$

Then the system for $(\delta\theta, \delta u, \nabla\delta\pi)$ reads

$$\begin{cases}\partial_t\delta\theta + u^2\cdot\nabla\delta\theta = -\delta u\cdot\nabla\theta^1,\\ \partial_t\delta u + u^2\cdot\nabla\delta u - \operatorname{div}(2(1+a^2)\mathbb{D}\delta u) + \nabla\delta\pi = \delta F,\\ \operatorname{div}\delta u = 0,\\ (\delta\theta,\delta u)|_{t=0} = (0,0),\end{cases} \tag{4.31}$$

where δF is determined by

$$\delta F \stackrel{\text{def}}{=} -\delta u \cdot \nabla u^1 + 2\,\mathrm{div}(\delta a \mathbb{D} u^1).$$

For δu, we first write the momentum equation of $(4.31)_2$ as

$$\partial_t \delta u + u^2 \cdot \nabla \delta u - \mathrm{div}(2(1+S_k a^2)\mathbb{D}\delta u) + \nabla \delta\pi = H \tag{4.32}$$

with

$$H \stackrel{\text{def}}{=} \mathrm{div}\{2(a^2 - S_k a^2)\mathbb{D}\delta u\} - \delta u \cdot \nabla u^1 + 2\,\mathrm{div}(\delta a \mathbb{D} u^1).$$

Applying Proposition 4.1, we yields that for $\forall\ 0<t<T$

$$\|\delta u\|_{\widetilde{L}^\infty_t(B^{-\frac12}_{2,1})} + \|\delta u\|_{L^1_t(B^{\frac32}_{2,1})} \lesssim \exp\{2^{2k}T\}\|H\|_{L^1_t(B^{-\frac12}_{2,1})}. \tag{4.33}$$

When taking div to the δu equation of $(4.31)_2$, it leads to

$$\Delta \pi = \mathrm{div}\, G \tag{4.34}$$

with

$$\begin{aligned} G = &\,\mathrm{div}\{2(a^2 - S_k a^2)\mathbb{D}\delta u\} - u^2\cdot\nabla\delta u - \delta u \cdot \nabla u^1 + 2\,\mathrm{div}(\delta a \mathbb{D} u^1) \\ &+ \mathrm{div}(2(1+S_k a^2)\mathbb{D}\delta u) \stackrel{\text{def}}{=} \sum_{i=1}^5 G_i. \end{aligned}$$

Then with the helpf of (4.30), we get by applying Proposition 4.2 that

$$\|\nabla \delta\pi\|_{L^1_t(B^{-\frac12}_{2,1})} \lesssim \|G\|_{L^1_t(B^{-\frac32}_{2,1})} + \|\,\mathrm{div}\, G\|_{L^1_t(B^{-\frac32}_{2,1})}. \tag{4.35}$$

According to Lemma 2.2 and Bony's decomposition, one can see

$$\|G_1\|_{L^1_t(B^{-\frac12}_{2,1})} \lesssim \|a^2 - S_k a^2\|_{L^\infty_t(B^{\frac32}_{2,1})} \|\delta u\|_{L^1_t(B^{\frac32}_{2,1})}. \tag{4.36}$$

Noticing that $\mathrm{div}\,\delta u = \mathrm{div}\, u^2 = 0$, one gets by applying Remark 2.1 that for any $\eta>0$

$$\|G_2\|_{L^1_t(B^{-\frac12}_{2,1})} + \|G_3\|_{L^1_t(B^{-\frac12}_{2,1})} \le \eta \|\delta u\|_{L^1_t(B^{\frac32}_{2,1})} + C_\eta \int_0^t \|\delta u\|_{B^{-\frac12}_{2,1}} (\|u^1\|^2_{\dot B^1_{3,1}} + \|u^2\|^2_{\dot B^1_{3,1}}) d\tau, \tag{4.37}$$

where we used the fact that $\|\delta u\|_{\dot B^{\frac12}_{2,1}} \lesssim \|\delta u\|_{B^{\frac12}_{2,1}} \lesssim \|\delta u\|^{\frac12}_{B^{-\frac12}_{2,1}} \|\delta u\|^{\frac12}_{B^{\frac32}_{2,1}}$. Similarly, we can deduce

$$\begin{aligned} \|G_4\|_{L^1_t(B^{-\frac12}_{2,1})} \lesssim & \|\delta a \mathbb{D} u^1\|_{L^1_t(\dot B^{\frac12}_{2,1})} \lesssim \|T_{\delta a}\mathbb{D} u^1\|_{L^1_t(\dot B^{\frac12}_{2,1})} + \|T_{\mathbb{D} u^1}\delta a\|_{L^1_t(\dot B^{\frac12}_{2,1})} \\ & + \|\mathcal{R}(\delta a, \mathbb{D} u^1)\|_{L^1_t(\dot B^{\frac12}_{2,1})} \lesssim \|\delta a\|_{L^\infty_t(L^3)} \|\mathbb{D} u^1\|_{L^1_t(\dot B^{\frac12}_{6,1})} + \\ & \|\nabla u^1\|_{L^1_t(L^\infty)} \|\delta a\|_{L^\infty_t(\dot B^{\frac12}_{2,1})} \lesssim \int_0^t \|\delta a\|_{\dot B^{\frac12}_{2,1}} \|u^1\|_{\dot B^2_{3,1}} d\tau, \end{aligned} \tag{4.38}$$

where based on the fact that $\dot B^{-\frac12}_{2,1} \hookrightarrow B^{-\frac12}_{2,1}$, and utilizing the Bony's decomposition, we obtain that for any $\eta>0$

$$\begin{aligned} \|G_5\|_{L^1_t(B^{-\frac32}_{2,1})} + \|\,\mathrm{div}\, G_5\|_{L^1_t(B^{-\frac32}_{2,1})} \lesssim & \|(1+S_k a^2)\mathbb{D}\delta u\|_{L^1_t(B^{-\frac12}_{2,1})} + \|\Delta \delta u \cdot \nabla S_k a^2\|_{L^1_t(B^{-\frac32}_{2,1})} \\ + \|\nabla S_k a^2 \mathbb{D}\delta u\|_{L^1_t(B^{-\frac12}_{2,1})} \lesssim & \eta \|\delta u\|_{L^1_t(B^{\frac32}_{2,1})} + C_\eta 2^{2k} \int_0^t (1 + \|S_k a^2\|^2_{B^1_{3,1}}) \|\delta u\|_{B^{-\frac12}_{2,1}} d\tau. \end{aligned} \tag{4.39}$$

Taking k sufficiently large and t small enough, one may achieve for any $k \geq k_0$,

$$\|a^2 - S_k a^2\|_{\widetilde{L}^\infty_t(B^{\frac{3}{2}}_{2,1})} \leq c. \tag{4.40}$$

Therefore, plugging (4.36)-(4.39) into (4.33) and (4.35), rescpectively, and taking c sufficiently small, it follows that

$$\begin{aligned}
&\|\delta u\|_{\widetilde{L}^\infty_t(\dot{B}^{-\frac{1}{2}}_{2,1})} + \|\delta u\|_{L^1_t(B^{-\frac{3}{2}}_{2,1})} + \|\nabla\delta\pi\|_{L^1_t(B^{-\frac{1}{2}}_{2,1})} \\
\lesssim&\eta\|\delta u\|_{L^1_t(B^{\frac{3}{2}}_{2,1})} + C_\eta\int_0^t \|\delta u\|_{B^{-\frac{1}{2}}_{2,1}}\left(\|u^1\|^2_{\dot{B}^1_{3,1}} + \|u^2\|^2_{\dot{B}^1_{3,1}}\right)d\tau \\
&+\int_0^t \|\delta a\|_{\dot{B}^{\frac{1}{2}}_{2,1}}\|u^1\|_{\dot{B}^2_{3,1}}d\tau + C_\eta 2^{2k}\int_0^t\left(1+\|S_k a^2\|^2_{B^1_{3,1}}\right)\|\delta u\|_{B^{-\frac{1}{2}}_{2,1}}d\tau.
\end{aligned} \tag{4.41}$$

On the other hand, we get from $(4.31)_1$ that

$$\partial_t\delta a + u^2\cdot\nabla\delta a = \delta u\cdot\nabla a^1,$$

by means of the classical results of transport equation (see [7] for example), we obtain that

$$\|(\delta a,\delta\theta)\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{1}{2}}_{2,1})} \lesssim \exp\{C\|u^2\|_{L^1_t(\dot{B}^1_{\infty,1})}\}\|\delta u\|_{L^1_t(\dot{B}^{\frac{3}{2}}_{2,1})}\|\theta^1\|_{L^\infty_t(\dot{B}^{\frac{3}{2}}_{2,1})}. \tag{4.42}$$

Substituting the above inequality into (4.41) and taking η small enough, we obtain

$$\begin{aligned}
&\|\delta u\|_{\widetilde{L}^\infty_t(\dot{B}^{-\frac{1}{2}}_{2,1})} + \|\delta u\|_{L^1_t(B^{-\frac{3}{2}}_{2,1})} + \|\nabla\delta\pi\|_{L^1_t(B^{-\frac{1}{2}}_{2,1})} \lesssim \int_0^t \|\delta u\|_{B^{-\frac{1}{2}}_{2,1}}\left(\|u^1\|^2_{\dot{B}^1_{3,1}} + \|u^2\|^2_{\dot{B}^1_{3,1}}\right)d\tau \\
&+2^{2k}\int_0^t\left(1+\|\theta^2\|^2_{B^1_{3,1}}\right)\|\delta u\|_{B^{-\frac{1}{2}}_{2,1}}d\tau + \int_0^t \|\delta u\|_{L^1_t(B^{\frac{3}{2}}_{2,1})}\|u^1\|_{\dot{B}^2_{3,1}}d\tau.
\end{aligned}$$

Applying Gronwall's inequality to the above inequality, we obtain that $\delta u = \nabla\delta\pi = 0$, which together with (4.42) implies that $\delta\theta = 0$ for all $t \in [0,T)$ with T small. The inductive argument implies that $\delta u = \nabla\delta\pi = \delta\theta = 0$ for all $t > 0$. This completes the proof of uniqueness of Theorem 4.1.

Corollary 4.1 *Under the assumptions of Theorem 1.1, we can find some $t_0 \in (t_1, 1)$ such that $u(t_0) \in \dot{H}^{-2\delta}\cap\dot{H}^2(\mathbb{R}^3)\cap\dot{B}^s_{3,1}$, for $s \in [0,2]$, moreover, there holds* (1.10) *and*

$$\|u(t_0)\|_{\dot{H}^{-2\delta}\cap\dot{H}^2} \lesssim \|u_0\|_{\dot{H}^{-2\delta}}(1+1/t_1^\delta)\exp\left\{\|u_0\|_{\dot{B}^0_{3,1}} + 2^{4k}\|\theta_0\|^2_{L^3}\right\}. \tag{4.43}$$

Proof Let (θ, u, π) be the unique solution of (1.3) constructed in Section 4.1. While thanks to (3.28), one has

$$\|u\|_{\widetilde{L}^\infty_1(\dot{H}^{-2\delta})} + \|u\|_{\widetilde{L}^1_1(\dot{H}^{2(1-\delta)})} \lesssim \|u_0\|_{\dot{H}^{-2\delta}}\exp\{\|u_0\|_{\dot{B}^0_{3,1}} + 2^{4k}\|\theta_0\|^2_{L^3}\}.$$

Then for any $0 < t_1 < 1 < T^*$, with T^* being determined by Theorem 4.1, we deduce that

$$\|u\|_{\widetilde{L}^\infty([t_1,1];L^2)} + \|u\|_{\widetilde{L}^1([t_1,1];\dot{H}^2)} \lesssim \|u_0\|_{\dot{H}^{-2\delta}}/t_1^\delta\exp\left\{\|u_0\|_{\dot{B}^0_{3,1}} + 2^{4k}\|\theta_0\|^2_{L^3}\right\}. \tag{4.44}$$

This together with (4.21) concludes the proof of Corollary 4.1.

5 Global well-posedness of (1.3)

The goal of this section is to prove the global well-posedness part of Theorem 1.1 provided that $\|u_0\|_{\dot{B}^0_{3,1}}$ is sufficiently small. As a convention in the remaining of this section, we shall always denote t_0 to be the positive time determined by Corollary 4.1. We shall prove (1.3) has a unique global solution (θ, u, π) on $[t_0, \infty)$ with $u = v + w$ and v solving (1.12), w solving (1.13).

Strategy of the proof of Theorem 1.1: Based on the result of Theorem 4.1, we conclude that: for given $\theta_0 \in B^{\frac{3}{2}}_{2,1}(\mathbb{R}^3)$ and $u_0 \in \dot{B}^0_{3,1}(\mathbb{R}^3)$ with $\|u_0\|_{\dot{B}^0_{3,1}}$ sufficiently small, (1.3) has a unique local solution (θ, u) satisfying $\theta \in \mathcal{C}([0, T^*); B^{\frac{3}{2}}_{2,1}(\mathbb{R}^3))$ and $u \in \mathcal{C}([0, T^*); \dot{B}^0_{3,1}(\mathbb{R}^3)) \cap L^1_{loc}([0, T^*); \dot{B}^2_{3,1}(\mathbb{R}^3))$ for some $T^* \geq 1$, and we can find some $t_0 \in (0, 1)$ such that

$$\|u(t_0)\|_{\dot{B}^0_{3,1} \cap \dot{B}^2_{3,1}} \leq C \|u_0\|_{\dot{B}^0_{3,1}}. \tag{5.1}$$

Notice from (5.1) that $\|u(t_0)\|_{\dot{B}^0_{3,1} \cap \dot{B}^2_{3,1}}$ is very small provided that $\|u_0\|_{\dot{B}^0_{3,1}}$ is sufficiently small. Moreover, applying Corollary 4.1, it infers that

$$\|u(t_0)\|_{\dot{H}^{-2\delta} \cap \dot{H}^2} \lesssim \|u_0\|_{\dot{H}^{-2\delta}} (1 + 1/t_1^{\delta}) \exp\left\{ \|u_0\|_{\dot{B}^0_{3,1}} + 2^{4k} \|\theta_0\|^2_{L^3} \right\}. \tag{5.2}$$

Then we shall prove the global well-posedness of u on $[t_0, \infty)$ with $u = v + w$, where v solves the following system

$$\begin{cases} \partial_t v + v \cdot \nabla v - \Delta v + \nabla \pi_v = 0, \\ \operatorname{div} v = 0, \\ v|_{t=t_0} = u\,(t_0)\,, \end{cases} \tag{5.3}$$

and the perturbation $w = u - v$ satsifying

$$\begin{cases} \partial_t \theta + \operatorname{div}(\theta(v + w)) = 0, \\ \partial_t w + (v + w) \cdot \nabla w - \operatorname{div}(2\mu(\theta)\mathbb{D}w) + \nabla \pi_w = -w \cdot \nabla v + \operatorname{div}(2(\mu(\theta) - 1)\mathbb{D}v), \\ \operatorname{div} w = 0, \\ \theta|_{t=t_0} = \theta\,(t_0)\,, \quad w|_{t=t_0} = 0, \end{cases} \tag{5.4}$$

which can be reached through energy estimate in the L^2 framework. The detailed information of v is presented in Proposition 5.1, and that of w is in Subsection 5.3. Our aim of what follows is to prove that $T^* = \infty$.

In order to get the global solution of system (5.3), we need to recall the following proposition which is introduced in Proposition 5.1 of [2] and we skip the proof for simplicity.

Proposition 5.1 *([2].) Let (v, π_v) be a unqiue global solution of* (5.3) *which satisfies* (1.8). *Then for $s \in [0, 2]$, it holds true that*

$$\|v\|_{\widetilde{L}^\infty([t_0,\infty); \dot{B}^s_{3,1})} + \|v\|_{L^1([t_0,\infty); \dot{B}^{s+2}_{3,1})} \leq C \|u_0\|_{\dot{B}^0_{3,1}}. \tag{5.5}$$

and

$$\|\partial_t v\|_{L^\infty([t_0,\infty); \dot{B}^0_{3,1})} + \|\partial_t v\|_{L^1([t_0,\infty); \dot{B}^2_{3,1})} \leq C \|u_0\|_{\dot{B}^0_{3,1}}. \tag{5.6}$$

Since v has been obtained above, we will pay attention to the global well-posedness of w in (5.4) to complete the proof of Theorem 1.1. For simplicity, in what follows, we just present the *a priori* estimates for smooth enough solutions of system (5.4) on $[0, T^*)$. Before estimating w, we first need the following large time-decay estimate of v.

Lemma 5.1 *([18]) Let $v_0 \in \dot{B}^s_{3,1} \cap \dot{H}^{-2\delta}(\mathbb{R}^3)$ be a solenoidal vector field for some $\delta \in (\frac{1}{2}, \frac{3}{4})$ and $s \in [0, 2]$. Assume that the function v solves*

$$\begin{cases} \partial_t v + v \cdot \nabla v - \Delta v + \nabla \pi = 0, \\ \operatorname{div} v = 0, \\ v|_{t=t_0} = u(t_0), \end{cases} \tag{5.7}$$

then there holds for all $t_0 \le t \le T$,

$$\|v(t)\|_{\dot{B}^s_{3,1}} \le C(1+t)^{-\frac{2s+4\delta+1}{4}}. \tag{5.8}$$

Proposition 5.2 *([18]) Let (v, π_v) be the unqiue global solution of (1.12) which satisfies (1.8). Then for $s \in [0, 2]$ and $\beta(s) \overset{\Delta}{=} \frac{2s+4\delta+1}{4}$, there holds*

$$\|t^{\beta(s)} v\|_{\widetilde{L}^\infty([t_0,t];\dot{B}^s_{3,1})} + \|t^{\beta(s)-} v\|_{\widetilde{L}^1([t_0,t];\dot{B}^{s+2}_{3,1})} \le C, \tag{5.9}$$

and

$$\|t^{\beta(2)-} v_t\|_{\widetilde{L}^\infty([t_0,t];\dot{B}^0_{3,1})} + \|t^{\beta(2)-} v_t\|_{\widetilde{L}^1([t_0,t];\dot{B}^2_{3,1})} \le C. \tag{5.10}$$

Proposition 5.3 *([21, 18]) Under the assumptions of Theorem 1.1, we have for any $t \in [t_0, T]$,*

$$\|(u, v, w)\|^2_{L^2} \le C\mathcal{H}_0 \langle t\rangle^{-2\delta} \quad \text{with} \quad \langle t\rangle \overset{\text{def}}{=} 1 + t, \tag{5.11}$$

where

$$\mathcal{H}_0 \overset{\text{def}}{=} 1 + \|u(t_0)\|^2_{\dot{H}^{-2\delta}} + \|u(t_0)\|^2_{H^2}(1 + \|\theta_0\|^2_{L^2} + \|u(t_0)\|^2_{L^2}). \tag{5.12}$$

Proof: We first get from the classical Navier-Stokes equations (1.12) that

$$\|v\|^2_{L^2} \le C\mathcal{H}_0 \langle t\rangle^{-2\delta}, \tag{5.13}$$

and the proof of u is rather standard. Notice that $w = u - v$, one has

$$\|w\|^2_{L^2} \le C\mathcal{H}_0 \langle t\rangle^{-2\delta}.$$

5.1 Higher Regularity of w

As a convention, in the remainder of this subsection we will always denote $s_1 \in [1, 2]$. The following regularity results on the Stokes equations will be useful for our derivation of higher order *a priori* estimates, and the proof process can be referred to [21].

Lemma 5.2 *([21]) For positive constants $\underline{\mu}, \bar{\mu}$, and $q \in (3, \infty)$, in addition, assume that $\mu(\theta)$ satisfies*

$$\nabla \mu(\theta) \in L^q, \quad 0 < \underline{\mu} \leq \mu(\theta) \leq \bar{\mu} < \infty. \tag{5.14}$$

Then, if $F \in L^r$ with $r \in [\frac{2q}{q+2}, q]$, there exists some positive C depending only on $\underline{\mu}, \bar{\mu}, q$ and r such that the unique weak solution $(w, \pi) \in H^1 \times L^2$ to the Cauchy problem

$$\begin{cases} -\operatorname{div}(2\mu(\theta)\mathbb{D}w) + \nabla \pi = F, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \operatorname{div} w = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ w(x) \longrightarrow 0, & |x| \longrightarrow 0 \end{cases} \tag{5.15}$$

satisfies

$$\|\nabla^2 w\|_{L^2} \lesssim \|F\|_{L^2} + \|\nabla \mu(\theta)\|_{L^q}^{\frac{q}{q-3}} \|\nabla w\|_{L^2}, \tag{5.16}$$

and

$$\|\nabla^2 w\|_{L^r} \lesssim \|F\|_{L^r} + \|\nabla \mu(\theta)\|_{L^q}^{\frac{q(5r-6)}{2r(q-3)}} \{\|\nabla w\|_{L^2} + \|(-\Delta)^{-1} \operatorname{div} F\|_{L^2}\}. \tag{5.17}$$

Proposition 5.4 *There exists some positive constant ε_0 depending only on $q, \bar{\mu}, \underline{\mu}$ and M, such that if (θ, u, π) is a smooth solution of (1.3) on $\mathbb{R}^3 \times [0, T]$ satisfying*

$$\sup_{t \in [0,T]} (\|\nabla \mu(\theta)\|_{L^q} + \|\nabla \theta\|_{L^3}) \leq 4M \quad \text{and} \quad \sup_{t \in [t_0,T]} \|\nabla u\|_{L^2}^2 \leq 4C \|u_0\|_{\dot{B}^0_{3,1}}^2, \tag{5.18}$$

where $M \overset{\text{def}}{=} \|\nabla \mu(\theta_0)\|_{L^q} + \|\theta_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}}$, the following estimates holds:

$$\sup_{t \in [0,T]} (\|\nabla \mu(\theta)\|_{L^q} + \|\nabla \theta\|_{L^3}) \leq 2M \quad \text{and} \quad \sup_{t \in [t_0,T]} \|\nabla u\|_{L^2}^2 \leq 2C \|u_0\|_{\dot{B}^0_{3,1}}^2, \tag{5.19}$$

provided that $\|u_0\|_{\dot{B}^0_{3,1}} \leq \varepsilon_0$.

Before proving Proposition 5.4, we establish some necessary *a priori* estimates from Lemma 5.3 to Lemma 5.7.

Corollary 5.1 *Under the assumptions of Proposition 5.4, one has for $q \in (3, \infty)$,*

$$\|\nabla^2 w\|_{L^2} \lesssim \|\partial_t w\|_{L^2} + \|\nabla w\|_{L^2} + \|\nabla w\|_{L^2}^3 + \|v\|_{\dot{B}^s_{3,1}}. \tag{5.20}$$

Proof The momentum equations can be rewritten as follows,

$$-\operatorname{div}(\mu(\theta)\mathbb{D}w) + \nabla \pi = -\partial_t w - (v + w) \cdot \nabla w - w \cdot \nabla v + \operatorname{div}\left(2(\mu(\theta) - 1)\mathbb{D}v\right). \tag{5.21}$$

By virtue of Lemma 5.2, one has

$$\begin{aligned} \|\nabla^2 w\|_{L^2} \lesssim & \|\nabla w\|_{L^2} + \|\partial_t w\|_{L^2} + \|(v + w) \cdot \nabla w\|_{L^2} \\ & + \|w \cdot \nabla v\|_{L^2} + \|\operatorname{div}\left(2(\mu(\theta) - 1)\mathbb{D}v\right)\|_{L^2} \end{aligned} \tag{5.22}$$

Then, by virtue of Proposition 5.1 and the Gagliardo-Nirenberg inequality, we obtain that

$$\begin{aligned}\|\nabla^2 w\|_{L^2} &\lesssim \|\nabla w\|_{L^2} + \|\partial_t w\|_{L^2} + \|w\|_{L^6}\|\nabla w\|_{L^3} + \|v\|_{L^\infty}\|\nabla w\|_{L^2}\\ &\quad + \|w\|_{L^6}\|\nabla v\|_{L^3} + \|(\mu(\theta)-1)\Delta v\|_{L^2} + \|\mathbb{D}v\cdot\nabla\mu(\theta)\|_{L^2}\\ &\lesssim \|\nabla w\|_{L^2} + \|\partial_t w\|_{L^2} + \|\nabla w\|_{L^2}^{\frac{3}{2}}\|\nabla^2 w\|_{L^2}^{\frac{1}{2}} + \|u_0\|_{\dot{B}^0_{3,1}}\|\nabla w\|_{L^2}\\ &\quad + \|\mu(\theta)-1\|_{L^6}\|\Delta v\|_{L^3} + \|\nabla v\|_{L^6}\|\mu'(\theta)\nabla\theta\|_{L^3}.\end{aligned}$$

By Young's inequality, we can deduce for $s_1 \in [1,2]$,

$$\|\nabla^2 w\|_{L^2} \lesssim \|\partial_t w\|_{L^2} + \|\nabla w\|_{L^2} + \|\nabla w\|_{L^2}^3 + \|v\|_{\dot{B}^{s_1}_{3,1}}.$$

5.2 The *a priori* estimates related to the System (1.13)

Next, we are going to derive some *a priori* time-weighted estimates. Noticing that the regularity of initial velocity in $\dot{B}^0_{3,1}$ is not enough, we intend to obtain some higher order estimates, which are independent of time.

Lemma 5.3 *Under the assumptions of Proposition 5.4, one has*

$$\sup_{t\in[t_0,T]}\int_{\mathbb{R}^3}|w|^2dx + \int_{t_0}^T\int_{\mathbb{R}^3}|\nabla w|^2dxdt \le C\|u_0\|^2_{\dot{B}^0_{3,1}}. \tag{5.23}$$

where C is independent of t.

Proof: Firstly, we get by using standard energy estimate to the w equation of (1.13) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 &\le \left|\int_{\mathbb{R}^3} w\cdot\nabla v|wdx\right| + \left|\int_{\mathbb{R}^3} 2(\mu(\theta)-1)\mathbb{D}v : \mathbb{D}wdx\right|\\ &\le \|w\|_{L^2}^2\|\nabla v\|_{L^\infty} + \|\mu(\theta)-1\|_{L^6}\|\nabla w\|_{L^2}\|\nabla v\|_{L^3}\\ &\le \frac{1}{2}\|\nabla w\|_{L^2}^2 + C\|w\|_{L^2}^2\|\nabla v\|_{L^\infty} + C\|\theta_0\|_{L^6}^2\|\nabla v\|_{L^3}^2.\end{aligned} \tag{5.24}$$

Hence, one has

$$\frac{d}{dt}\|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 \lesssim \|v\|_{\dot{B}^2_{3,1}}\|w\|_{L^2}^2 + \|v\|^2_{\dot{B}^1_{3,1}}. \tag{5.25}$$

Integrating in time over $[t_0,T]$ yields

$$\|w\|^2_{L^\infty([t_0,T];L^2)} + \|\nabla w\|^2_{L^2([t_0,T];L^2)} \lesssim \|u_0\|^2_{\dot{B}^0_{3,1}}.$$

This completes the proof of Lemma 5.3.

Corollary 5.2 *Under the assumptions of Theorem 1.1, we have*

$$\|t^\delta w\|^2_{L^\infty([t_0,T];L^2)} + \|t^{\delta_-}\nabla w\|^2_{L^2([t_0,T];L^2)} \le C\mathcal{H}_0, \tag{5.26}$$

where $\mathcal{H}_0$ is given by (5.12).

Proof Multiplying (5.25) by $t^{2\delta-}$ and then integrating the resulting inequality over $[t_0, T]$, we get, by applying (5.11), that

$$\begin{aligned}
&\|t^{\delta-}w\|^2_{L^\infty([t_0,T];L^2)} + \|t^{\delta-}\nabla w\|^2_{L^2([t_0,T];L^2)} \\
&\lesssim \left\{\int_{t_0}^T t^{(-1)-}\|t^\delta w(t)\|^2_{L^2}dt + \int_{t_0}^T t^{2\delta-}\|v\|^2_{\dot{B}^1_{3,1}}dt\right\}\exp\{\int_{t_0}^T \|v\|_{\dot{B}^2_{3,1}}dt\}.
\end{aligned} \tag{5.27}$$

Hence, one has

$$\|t^\delta w\|^2_{L^\infty([t_0,T];L^2)} + \|t^{\delta-}\nabla w\|^2_{L^2([t_0,T];L^2)} \le C\mathcal{H}_0.$$

We completes the proof of the Corollary 5.2.

Lemma 5.4 *Suppose* (θ, w, π) *is the unique local strong solution to* (5.4) *satisfying* (1.8). *Then it follows that*

$$\frac{1}{\underline{\nu}}\int_{t_0}^T\int_{\mathbb{R}^3}|w_t|^2dxdt + \sup_{t\in[t_0,T]}\int_{\mathbb{R}^3}|\nabla w|^2dx \le 2C\|u_0\|^2_{\dot{B}^0_{3,1}}. \tag{5.28}$$

Proof Multiplying the momentum equations by w_t and integrating over $\mathbb{R}^3$, it yields

$$\begin{aligned}
&\int_{\mathbb{R}^3}|w_t|^2dx + \frac{d}{dt}\int_{\mathbb{R}^3}\mu(\theta)\mathbb{D}w : \mathbb{D}wdx \\
&\le \left|\int_{\mathbb{R}^3}(w\cdot\nabla v)|w_tdx\right| + \left|\int_{\mathbb{R}^3}\operatorname{div}(2(\mu(\theta)-1)\mathbb{D}v)|\partial_t wdx\right| \\
&\quad + \left|\int_{\mathbb{R}^3}\partial_t\mu(\theta)\mathbb{D}w : \mathbb{D}wdx\right| + \left|\int_{\mathbb{R}^3}(v+w)\cdot\nabla w|w_tdx\right|.
\end{aligned} \tag{5.29}$$

Applying the Gagliardo-Nirenberg inequality, we know that

$$\begin{aligned}
\left|\int_{\mathbb{R}^3}(w\cdot\nabla v)|w_tdx\right| &\le \|w_t\|_{L^2}\|w\|_{L^2}\|\nabla v\|_{L^\infty} \\
&\le \frac{1}{8}\|w_t\|^2_{L^2} + C\|w\|^2_{L^2}\|\nabla v\|^2_{L^\infty},
\end{aligned}$$

and

$$\begin{aligned}
\left|\int_{\mathbb{R}^3}\operatorname{div}(2(\mu(\theta)-1)\mathbb{D}v)|\partial_t wdx\right| &\le \|(\mu(\theta)-1)\Delta v + \mathbb{D}v\cdot\nabla\mu(\theta)\|_{L^2}\|w_t\|_{L^2} \\
&\le C\left\{\|\mu(\theta)-1\|_{L^6}\|\Delta v\|_{L^3} + \|\nabla v\|_{L^6}\|\nabla\theta\|_{L^3}\right\}\|w_t\|_{L^2} \\
&\le \frac{1}{8}\|w_t\|^2_{L^2} + C(\|\theta_0\|_{L^6}, M)\|v\|^2_{\dot{B}^{s_1}_{3,1}}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\left|\int_{\mathbb{R}^3}\partial_t\mu(\theta)\mathbb{D}w : \mathbb{D}wdx\right| &\le \left|\int_{\mathbb{R}^3}(v+w)\cdot\nabla\mu(\theta)\mathbb{D}w : \mathbb{D}wdx\right| \\
&\le C\|\nabla\theta\|_{L^3}\|\nabla w\|_{L^2}\|\nabla w\|_{L^6}\left(\|w\|_{L^\infty} + \|v\|_{L^\infty}\right) \\
&\le C(M)\|\nabla w\|^{\frac{3}{2}}_{L^2}\|\nabla^2 w\|^{\frac{3}{2}}_{L^2} + C(M)\|\nabla w\|_{L^2}\|\nabla^2 w\|_{L^2}\|v\|_{L^\infty}.
\end{aligned}$$

Here we have used the fact that

$$\partial_t\mu(\theta) + (v+w)\cdot\nabla\mu(\theta) = 0,$$

which is a consequence of mass equation by means of the fact $\operatorname{div} u = 0$. Notice that

$$\left|\int_{\mathbb{R}^3} (v+w)\cdot\nabla w | w_t dx\right| \le \|w_t\|_{L^2}\left\{\|\nabla w\|_{L^3}\|w\|_{L^6} + \|\nabla w\|_{L^2}\|v\|_{L^\infty}\right\}$$
$$\le \|w_t\|_{L^2}\|\nabla w\|_{L^2}^{\frac{3}{2}}\|\nabla^2 w\|_{L^2}^{\frac{1}{2}} + \|w_t\|_{L^2}\|\nabla w\|_{L^2}\|v\|_{L^\infty}.$$

Hence, by Young's inequality and Corollary 5.1, it infers that

$$\begin{aligned}
&\int_{\mathbb{R}^3} |w_t|^2 dx + \frac{d}{dt}\int_{\mathbb{R}^3} \mu(\theta)\mathbb{D}w : \mathbb{D}w dx \\
\le&\frac{1}{2}\|w_t\|_{L^2}^2 + C\|\nabla w\|_{L^2}^{\frac{3}{2}}\left\{\|\partial_t w\|_{L^2} + \|\nabla w\|_{L^2} + \|\nabla w\|_{L^2}^3 + \|v\|_{\dot{B}^{s_1}_{3,1}}\right\}^{\frac{3}{2}} \\
&+ C\|w\|_{L^2}^2\|\nabla v\|_{L^\infty}^2 + C\|\nabla w\|_{L^2}\|v\|_{L^\infty}\left\{\|\partial_t w\|_{L^2} + \|\nabla w\|_{L^2} + \|\nabla w\|_{L^2}^3 + \|v\|_{\dot{B}^{s_1}_{3,1}}\right\} \\
&+ C\|v\|_{\dot{B}^{s_1}_{3,1}}^2 + C\|w_t\|_{L^2}\|\nabla w\|_{L^2}^{\frac{3}{2}}\left\{\|\partial_t w\|_{L^2} + \|\nabla w\|_{L^2} + \|\nabla w\|_{L^2}^3 + \|v\|_{\dot{B}^{s_1}_{3,1}}\right\}^{\frac{1}{2}},
\end{aligned} \tag{5.30}$$

from which together with (5.5) it yields that

$$\begin{aligned}
&\int_{\mathbb{R}^3} |w_t|^2 dx + \frac{d}{dt}\int_{\mathbb{R}^3} \mu(\theta)\mathbb{D}w : \mathbb{D}w dx \\
\le&\frac{7}{8}\|w_t\|_{L^2}^2 + C\|\nabla w\|_{L^2}^6 + C\|\nabla w\|_{L^2}^4 + C\|\nabla w\|_{L^2}^3 + C\|v\|_{\dot{B}^{s_1}_{3,1}}^2 + C\|w\|_{L^2}^2\|\nabla v\|_{L^\infty}^2 \\
\le&\frac{7}{8}\|w_t\|_{L^2}^2 + C\|\nabla w\|_{L^2}^6 + C\|\nabla w\|_{L^2}^2 + C\|v\|_{\dot{B}^{s_1}_{3,1}}^2 + C\|w\|_{L^2}^2\|\nabla v\|_{L^\infty}^2.
\end{aligned} \tag{5.31}$$

Integrating with respect to time on $[t_0, T]$ it gives that

$$\begin{aligned}
&\int_{t_0}^T\int_{\mathbb{R}^3} |w_t|^2 dxdt + \sup_{t\in[t_0,T]}\int_{\mathbb{R}^3} |\nabla w|^2 dx \\
\le& C\int_{t_0}^T \|\nabla w\|_{L^2}^6 dt + C\int_{t_0}^T \|\nabla w\|_{L^2}^2 dt + C\int_{t_0}^T \|v\|_{\dot{B}^{s_1}_{3,1}}^2 dt + C\sup_{t\in[t_0,T]}\|w\|_{L^2}^2\int_{t_0}^T \|\nabla v\|_{L^\infty}^2 dt.
\end{aligned}$$

Applying Gronwall's inequality,

$$\int_{t_0}^T\int_{\mathbb{R}^3} |w_t|^2 dxdt + \sup_{t\in[t_0,T]}\int_{\mathbb{R}^3} |\nabla w|^2 dx \le C\|u_0\|_{\dot{B}^0_{3,1}}^2 \cdot \exp\{C\int_{t_0}^T \|\nabla w\|_{L^2}^4 dt\}.$$

Such that if

$$\|u_0\|_{\dot{B}^0_{3,1}} \le \varepsilon_1 \quad \text{and} \quad \sup_{t\in[t_0,T]}\|\nabla w\|_{L^2}^2 \le 4C\|u_0\|_{\dot{B}^0_{3,1}}^2 \le 1, \tag{5.32}$$

then

$$\begin{aligned}
\int_{t_0}^T \|\nabla w\|_{L^2}^4 dt &\le \sup_{t\in[t_0,T]}\|\nabla w\|_{L^2}^2 \cdot \int_{t_0}^T \|\nabla w\|_{L^2}^2 dt \\
&\le 4C\|u_0\|_{\dot{B}^0_{3,1}}^4 \le \|u_0\|_{\dot{B}^0_{3,1}}^2.
\end{aligned} \tag{5.33}$$

Hence, we arrive at

$$\int_{t_0}^T\int_{\mathbb{R}^3} |w_t|^2 dxdt + \sup_{t\in[t_0,T]}\int_{\mathbb{R}^3} |\nabla w|^2 dx \le C\|u_0\|_{\dot{B}^0_{3,1}}^2 \cdot \exp\{C\|u_0\|_{\dot{B}^0_{3,1}}^2\}. \tag{5.34}$$

Choosing some small positive constant $\varepsilon_1 = \min\{\sqrt{\frac{1}{4C}}, \sqrt{\frac{\ln 2}{C}}\}$, we easily deduce (5.34) according to (5.32).

As a byproduct of the above estimates, we have the following result.

Lemma 5.5 *Suppose* (θ, w, π) *is the unique local strong solution to* (5.4) *satisfying* (1.8). *Then under the assumptions of Proposition 5.4, we have*

$$\int_{t_0}^{T} t\|w_t\|_{L^2}^2 dt + \sup_{t\in[t_0,T]} t\|\nabla w\|_{L^2}^2 \leq C\mathcal{H}_0, \tag{5.35}$$

where $\mathcal{H}_0$ *is given by* (5.12).

Proof: Multiplying (5.31) by t, as shown in the last proof, one has

$$\begin{aligned}\int_{t_0}^{T} t\|w_t\|_{L^2}^2 dt + \sup_{t\in[t_0,T]} t\|\nabla w\|_{L^2}^2 \leq C\int_{t_0}^{T} \|\nabla w\|_{L^2}^2 dt + C\int_{t_0}^{T} t\|\nabla w\|_{L^2}^6 dt \\ + C\int_{t_0}^{T} t\|\nabla w\|_{L^2}^2 dt + C\int_{t_0}^{T} t\|v\|_{\dot{B}_{3,1}^{s_1}}^2 dt + C\sup_{t\in[t_0,T]} \|w\|_{L^2}^2 \int_{t_0}^{T} t\|\nabla v\|_{L^\infty}^2 dt.\end{aligned} \tag{5.36}$$

According to Lemma 5.4 and Proposition 5.2,

$$\int_{t_0}^{T} \|\nabla w\|_{L^2}^2 dt \leq \|u_0\|_{\dot{B}_{3,1}^{0}}^2 \quad \text{and} \quad \int_{t_0}^{T} t\|v\|_{\dot{B}_{3,1}^{s_1}}^2 dt \leq C, \tag{5.37}$$

for $s_1 \in [1,2]$. With the help of (5.26), for $\delta > \frac{1}{2}$, we obtain that

$$\begin{aligned}\int_{t_0}^{T} t\|w_t\|_{L^2}^2 dt + \sup_{t\in[t_0,T]} t\|\nabla w\|_{L^2}^2 \leq & C(\|u_0\|_{\dot{B}_{3,1}^{0}}^2 + \int_{t_0}^{T} t^{1-2\delta_-}\|t^{\delta_-}\nabla w\|_{L^2}^2 dt \\ & + 1)\exp\left\{C\int_{t_0}^{T} \|\nabla w\|_{L^2}^4 dt\right\} \leq C\mathcal{H}_0.\end{aligned} \tag{5.38}$$

This completes the proof of Lemma 5.5.

Lemma 5.6 *Suppose* (θ, w, π) *is the unique local strong solution to* (5.4) *satisfying* (1.8). *Then under the assumptions of Proposition 5.4, we have*

$$\sup_{t\in[t_0,T]} t\|w_t\|_{L^2}^2 + \int_{t_0}^{T} t\|\nabla w_t\|_{L^2}^2 dt \leq C\mathcal{H}_0, \tag{5.39}$$

and

$$\sup_{t\in[t_0,T]} t^2\|w_t\|_{L^2}^2 + \int_{t_0}^{T} t^2\|\nabla w_t\|_{L^2}^2 dt \leq C\mathcal{H}_0^2, \tag{5.40}$$

where $\mathcal{H}_0$ *is given by* (5.12).

Proof Taking t-derivative of the momentum equations, it follows that

$$\begin{aligned}&\partial_{tt}w+(v+w)\cdot\nabla w_t-\operatorname{div}(2\mu(\theta)\mathbb{D}w_t)+\nabla\pi_t=-(v_t+w_t)\cdot\nabla w\\&\quad+\operatorname{div}(2\partial_t\mu(\theta)\mathbb{D}w)-\partial_t(w\cdot\nabla v)+\operatorname{div}(2\partial_t\mu(\theta)\mathbb{D}v)+\operatorname{div}\left(2(\mu(\theta)-1)\mathbb{D}v_t\right).\end{aligned}\tag{5.41}$$

Multiplying (5.41) by w_t and integrating over $\mathbb{R}^3$, we get after integration by parts that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|w_t|^2dx+2\int_{\mathbb{R}^3}\mu(\theta)\mathbb{D}w_t:\mathbb{D}w_tdx\\&=-\int_{\mathbb{R}^3}(v_t+w_t)\cdot\nabla w|w_tdx-2\int_{\mathbb{R}^3}\partial_t\mu(\theta)\mathbb{D}w\cdot\nabla w_tdx-\int_{\mathbb{R}^3}\partial_t(w\cdot\nabla v)|w_tdx\\&\quad-2\int_{\mathbb{R}^3}\partial_t\mu(\theta)\mathbb{D}v\cdot\nabla w_tdx+\int_{\mathbb{R}^3}\operatorname{div}\left(2(\mu(\theta)-1)\mathbb{D}v_t\right)|w_tdx\overset{\text{def}}{=}\sum_{i=1}^{5}J_i.\end{aligned}\tag{5.42}$$

Now, we will use the Gagliardo-Nirenberg inequality and Corollary 5.1 to estimate each term on the right-hand side of (5.42). First, notice that

$$\begin{aligned}J_1&=-\int_{\mathbb{R}^3}(v_t+w_t)\cdot\nabla w|w_tdx\\&\le C\|v_t\|_{L^3}\|\nabla w\|_{L^2}\|w_t\|_{L^6}+C\|w_t\|_{L^3}\|w_t\|_{L^6}\|\nabla w\|_{L^2}\\&\le C\|v_t\|_{L^3}\|\nabla w\|_{L^2}\|\nabla w_t\|_{L^2}+C\|w_t\|_{L^2}^{\frac{1}{2}}\|\nabla w_t\|_{L^2}^{\frac{3}{2}}\|\nabla w\|_{L^2}\\&\le\eta\|\nabla w_t\|_{L^2}^2+C\|\nabla w\|_{L^2}^2\|v\|_{\dot{B}^{s_1}_{3,1}}^2+C\|w_t\|_{L^2}^2\|\nabla w\|_{L^2}^4.\end{aligned}\tag{5.43}$$

for any $\eta>0$. Second, utilizing the equation for $\mu(\theta)$, we know

$$\begin{aligned}J_2&=-2\int_{\mathbb{R}^3}(v+w)\cdot\nabla\mu(\theta)\mathbb{D}w\cdot\nabla w_tdx\\&\le C(\|v\|_{L^\infty}+\|w\|_{L^\infty})\|\mu'(\theta)\nabla\theta\|_{L^3}\|\nabla w\|_{L^6}\|\nabla w_t\|_{L^2}\\&\le C(M)\left(\|v\|_{L^\infty}+\|\nabla w\|_{L^2}^{\frac{1}{2}}\|\nabla^2 w\|_{L^2}^{\frac{1}{2}}\right)\|\nabla^2 w\|_{L^2}\|\nabla w_t\|_{L^2}\\&\le\eta\|\nabla w_t\|_{L^2}^2+C\|\nabla^2 w\|_{L^2}^2\|v\|_{L^\infty}^2+C\|\nabla w\|_{L^2}\|\nabla^2 w\|_{L^2}^3.\end{aligned}\tag{5.44}$$

Third, according to Corollary 5.1 and assumption (5.18), it holds that

$$J_3=-\int_{\mathbb{R}^3}w_t\cdot\nabla v|w_tdx-\int_{\mathbb{R}^3}w\cdot\nabla v_t|w_tdx,$$

and

$$-\int_{\mathbb{R}^3}w_t\cdot\nabla v|w_tdx\le C\|w_t\|_{L^2}\|w_t\|_{L^6}\|\nabla v\|_{L^3}\le\eta\|\nabla w_t\|_{L^2}^2+C\|w_t\|_{L^2}^2\|v\|_{\dot{B}^{s_1}_{3,1}}^2,$$

and

$$-\int_{\mathbb{R}^3}w\cdot\nabla v_t|w_tdx\le C\|w_t\|_{L^6}\|\nabla v_t\|_{L^3}\|w\|_{L^2}\le\eta\|\nabla w_t\|_{L^2}^2+\|w\|_{L^2}^2\|v_t\|_{\dot{B}^1_{3,1}}^2.$$

Hence, we deduce that

$$|J_3|\le\eta\|\nabla w_t\|_{L^2}^2+C\|w_t\|_{L^2}^2\|v\|_{\dot{B}^{s_1}_{3,1}}^2+C\|w\|_{L^2}^2\|v_t\|_{\dot{B}^1_{3,1}}^2.\tag{5.45}$$

Finally, taking into account the equation for $\mu(\theta)$ again, we arrive at

$$\begin{aligned}
J_4+J_5=&2\int_{\mathbb{R}^3}(v+w)\cdot\nabla\mu(\theta)\mathbb{D}v\cdot\nabla w_t dx-2\int_{\mathbb{R}^3}(\mu(\theta)-1)\mathbb{D}v_t\cdot\nabla w_t dx\\
&\le C\left(\|v\|_{L^\infty}\|\nabla v\|_{L^6}+\|\nabla v\|_{L^\infty}\|w\|_{L^6}\right)\|\mu'(\theta)\nabla\theta\|_{L^3}\|\nabla w_t\|_{L^2}\\
&\quad+C\|\mu(\theta)-1\|_{L^6}\|\nabla v_t\|_{L^3}\|\nabla w_t\|_{L^2}\\
&\le \eta\|\nabla w_t\|_{L^2}^2+C(\eta,M,\|\theta_0\|_{L^6})\left(\|\nabla w\|_{L^2}^2\|v\|_{\dot B^{s_1}_{3,1}}^2+\|v_t\|_{\dot B^1_{3,1}}^2+\|v\|_{\dot B^{s_1}_{3,1}}^4\right).
\end{aligned} \tag{5.46}$$

Substituting (5.43)-(5.46) into (5.42) and applying Corollary 5.1, for any $\eta>0$ and $t\in[t_0,T]$, we have

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^3}|w_t|^2dx+\|\nabla w_t\|_{L^2}^2\le&\ \eta\|\nabla w_t\|_{L^2}^2+C\|w_t\|_{L^2}^4+C\|v\|_{\dot B^{s_1}_{3,1}}^4\\
&+C\|v_t\|_{\dot B^1_{3,1}}^2+C\|\nabla w\|_{L^2}^4+C\|\nabla w\|_{L^2}^6+C\|\nabla w\|_{L^2}^{10}.
\end{aligned} \tag{5.47}$$

Taking η small enough and multiplying (5.47) by t one easily deduces that

$$\begin{aligned}
\frac{d}{dt}\|\sqrt{t}w_t\|_{L^2}^2+\|\sqrt{t}\nabla w_t\|_{L^2}^2\lesssim&\ \|w_t\|_{L^2}^2+\|\sqrt{t}w_t\|_{L^2}^2\|w_t\|_{L^2}^2+t\|v\|_{\dot B^{s_1}_{3,1}}^4\\
&+t\|v_t\|_{\dot B^1_{3,1}}^2+\|\sqrt{t}\nabla w\|_{L^2}^2\{\|\nabla w\|_{L^2}^2+\|\nabla w\|_{L^2}^4+\|\nabla w\|_{L^2}^8\}.
\end{aligned}$$

Integrating with respect to time on $[t_0,T]$ and using Gronwall's inequality, it follows that

$$\begin{aligned}
\sup_{t\in[t_0,T]}t\|w_t\|_{L^2}^2+\int_{t_0}^T t\|\nabla w_t\|_{L^2}^2dt\lesssim&\ \{\|w_t(t_0)\|_{L^2}^2+\int_{t_0}^T t(\|v\|_{\dot B^{s_1}_{3,1}}^4+\|v_t\|_{\dot B^1_{3,1}}^2)dt\\
+\int_{t_0}^T\|\sqrt{t}\nabla w\|_{L^2}^2&\left(\|\nabla w\|_{L^2}^2+\|\nabla w\|_{L^2}^4+\|\nabla w\|_{L^2}^8\right)dt\}\exp\left\{C\int_{t_0}^T\|w_t\|_{L^2}^2dt\right\}.
\end{aligned} \tag{5.48}$$

Whereas taking L^2-norm of the w_t at $t=t_0$ and using (5.1)-(5.4), it gives rise to

$$\begin{aligned}
\|w_t(t_0)\|_{L^2}\le&\|(v+w)\cdot\nabla w(t_0)\|_{L^2}+\|\operatorname{div}(2\mu(\theta)\mathbb{D}w)(t_0)\|_{L^2}+\|w\cdot\nabla v(t_0)\|_{L^2}\\
&+\|\operatorname{div}(2(\mu(\theta)-1)\mathbb{D}v)(t_0)\|_{L^2}\\
\le&\|\nabla\mu(\theta)(t_0)\|_{L^q}\|\mathbb{D}u(t_0)\|_{L^{\frac{q-2}{2q}}}+\|\mu(\theta_0)-1\|_{L^\infty}\|\Delta u(t_0)\|_{L^2}\\
\le&M\|\nabla u(t_0)\|_{L^2}^{\frac{q-3}{q}}\|\nabla^2u(t_0)\|_{L^2}^{\frac{3}{q}}+\|\theta_0\|_{L^\infty}\|\Delta u(t_0)\|_{L^2}\\
\le&C(M,\|\theta_0\|_{L^\infty})\|u(t_0)\|_{H^2}.
\end{aligned} \tag{5.49}$$

Thanks to (5.9) and (5.10), we conclude that

$$\int_{t_0}^T t\|v\|_{\dot B^{s_1}_{3,1}}^4dt+\int_{t_0}^T t\|v_t\|_{\dot B^1_{3,1}}^2dt\le C. \tag{5.50}$$

According to Lemma 5.4 and 5.5,

$$\begin{aligned}
&\int_{t_0}^T\|\sqrt{t}\nabla w\|_{L^2}^2\left(\|\nabla w\|_{L^2}^2+\|\nabla w\|_{L^2}^4+\|\nabla w\|_{L^2}^8\right)dt\\
\le&\sup_{t\in[t_0,T]}\|\sqrt{t}\nabla w\|_{L^2}^2\cdot\left(1+\sup_{t\in[t_0,T]}\|\nabla w\|_{L^2}^2+\sup_{t\in[t_0,T]}\|\nabla w\|_{L^2}^6\right)\cdot\int_{t_0}^T\|\nabla w\|_{L^2}^2dt\\
\le&C\mathcal{H}_0.
\end{aligned} \tag{5.51}$$

Plugging (5.51) and (5.50) into (5.48), we have

$$\sup_{t\in[t_0,T]} t\|w_t\|_{L^2}^2 + \int_{t_0}^T t\|\nabla w_t\|_{L^2}^2 dt \le C\mathcal{H}_0. \tag{5.52}$$

On the other hand, multiplying (5.47) by t^2, one has

$$\begin{aligned}\frac{d}{dt}\|tw_t\|_{L^2}^2+\|t\nabla w_t\|_{L^2}^2 &\lesssim \|\sqrt{t}w_t\|_{L^2}^2 + \|tw_t\|_{L^2}^2\|w_t\|_{L^2}^2 + t^2\|v\|_{\dot B^{s_1}_{3,1}}^4 \\ &\quad + t^2\|v_t\|_{\dot B^1_{3,1}}^2 + \|\sqrt{t}\nabla w\|_{L^2}^4\{1+\|\nabla w\|_{L^2}^2 + \|\nabla w\|_{L^2}^6\}.\end{aligned}$$

Thanks to (5.9) and (5.10), we conclude that

$$\int_{t_0}^T t^2\|v\|_{\dot B^{s_1}_{3,1}}^4 dt + \int_{t_0}^T t^2\|v_t\|_{\dot B^1_{3,1}}^2 dt \le C. \tag{5.53}$$

Owing to Corollary 5.2, we get for $\delta > \frac{1}{2}$,

$$\int_{t_0}^T \|\sqrt{t}\nabla w\|_{L^2}^4 dt \le \sup_{t\in[t_0,T]} \|\sqrt{t}\nabla w\|_{L^2}^2 \int_{t_0}^T t^{1-2\delta_-}\|t^{\delta_-}\nabla w\|_{L^2}^2 dt \le C\mathcal{H}_0^2.$$

From which and Gronwall's inequality, we can deduce

$$\sup_{t\in[t_0,T]} t^2\|w_t\|_{L^2}^2 + \int_{t_0}^T t^2\|\nabla w_t\|_{L^2}^2 dt \le C\mathcal{H}_0^2. \tag{5.54}$$

This completes the proof of Lemma 5.6.

5.3 The $L^1([t_0,T];\dot B^1_{\infty,1}(\mathbb{R}^3))$ estimate for w

The goal of this subsection is to present the *a priori* $L^1([t_0,T];\dot B^1_{\infty,1}(\mathbb{R}^3))$ estimate for w, which is the most important ingredient used in the proof of Theorem 1.1.

Lemma 5.7 *Let $p \in (2, \frac{6\delta_-}{1+\delta_-})$ for $\delta \in (\frac{1}{2}, \frac{3}{4})$ and $\alpha_1 = \frac{(r-3)p}{3r-3p+pr}$. Assume that (θ, w, π) is the unique local strong solution to* (5.4) *satisfying* (1.8). *Then under the assumptions of Proposition 5.4, we have*

$$\|\nabla w\|_{L^1([t_0,T];\dot B^0_{\infty,1})} \le C\mathcal{H}_0^2\|u_0\|_{\dot B^0_{3,1}}^{\frac{3\alpha_1(p-2)}{2p}}., \tag{5.55}$$

where $\mathcal{H}_0$ is given by (5.12).

Proof: By virtue of Lemma 5.2, one has for $r \in (3, \min\{q,6\})$

$$\|\nabla^2 w\|_{L^r} \lesssim \|\nabla w\|_{L^2} + \|F\|_{L^r} + \|(-\Delta)^{-1}\operatorname{div} F\|_{L^2} \tag{5.56}$$

with

$$F = -\partial_t w - (v+w)\cdot\nabla w - w\cdot\nabla v + \operatorname{div}(2(\mu(\theta)-1)\mathbb{D}v).$$

Thus, we can obtain that for any $t \in [t_0, T]$ and $\eta > 0$

$$
\begin{aligned}
\|F\|_{L^r} \lesssim & \|w_t\|_{L^r} + \|(v+w)\cdot\nabla w\|_{L^r} + \|w\cdot\nabla v\|_{L^r} + \|\operatorname{div}(2(\mu(\theta)-1)\mathbb{D}v)\|_{L^r} \\
\lesssim & \|w_t\|_{L^2}^{\frac{6-r}{2r}} \|\nabla w_t\|_{L^2}^{\frac{3r-6}{2r}} + \|v+w\|_{L^6}\|\nabla w\|_{L^{\frac{6r}{6-r}}} + \|w\|_{L^6}\|\nabla v\|_{\frac{6r}{6-r}} \\
& + \|(\mu(\theta)-1)\Delta v\|_{L^r} + \|\mathbb{D}v\cdot\nabla\mu(\theta)\|_{L^r} \\
\lesssim & \|w_t\|_{L^2}^{\frac{6-r}{2r}} \|\nabla w_t\|_{L^2}^{\frac{3r-6}{2r}} + \|u_0\|_{\dot B^0_{3,1}}\|\nabla w\|_{L^2}^{\frac{r}{5r-6}}\|\nabla^2 w\|_{L^r}^{\frac{4r-6}{5r-6}} + \|\nabla w\|_{L^2}^{\frac{6(r-1)}{5r-6}}\|\nabla^2 w\|_{L^r}^{\frac{4r-6}{5r-6}} \\
& + \|\nabla w\|_{L^2}\|u_0\|_{\dot B^0_{3,1}} + \|\theta_0\|_{L^r}\|\Delta v\|_{\dot B^1_{3,1}} + \|\nabla\mu(\theta)\|_{L^q}\|\nabla v\|_{L^{\frac{rq}{q-r}}} \\
\lesssim & \eta\|\nabla^2 w\|_{L^r} + \|w_t\|_{L^2}^{\frac{6-r}{2r}} \|\nabla w_t\|_{L^2}^{\frac{3r-6}{2r}} + \|\nabla w\|_{L^2} + \|\nabla w\|_{L^2}^{\frac{6(r-1)}{r}} + \|v\|_{\dot B^{s_1}_{3,1}} + \|v\|_{\dot B^3_{3,1}},
\end{aligned}
$$

and utilizing the fact that $\operatorname{div} w = \operatorname{div} v = 0$, one has

$$
\begin{aligned}
\|(-\Delta)^{-1}\operatorname{div} F\|_{L^2} \lesssim & \|(v+w)\cdot\nabla w\|_{L^{\frac{6}{5}}} + \|\theta w\cdot\nabla v\|_{L^{\frac{6}{5}}} + \|(\mu(\theta)-1)\mathbb{D}v\|_{L^2} \\
\lesssim & \|v\|_{L^3}\|\nabla w\|_{L^2} + \|w\|_{L^3}\|\nabla w\|_{L^2} + \|w\|_{L^2}\|\nabla v\|_{L^3} + \|\mu(\theta)-1\|_{L^2}\|\nabla v\|_{L^\infty} \\
\lesssim & \|\nabla w\|_{L^2} + \|w\|_{L^2}^{\frac12}\|\nabla w\|_{L^2}^{\frac32} + C(\|\theta_0\|_{L^2})(1+\|w\|_{L^2})\|v\|_{\dot B^{s_1}_{3,1}},
\end{aligned}
$$

where we used $L^{\frac{6}{5}} \hookrightarrow W^{-1,2}$. Substituting the above inequality into (5.56) and taking $\eta > 0$ sufficiently small, we arrive at

$$
\begin{aligned}
\|\nabla^2 w\|_{L^r} \lesssim & \|\nabla w\|_{L^2} + \|\nabla w\|_{L^2}^{\frac{6(r-1)}{r}} + (1+\|w\|_{L^2})\|v\|_{\dot B^{s_1}_{3,1}} \\
& + \|v\|_{\dot B^3_{3,1}} + \|w_t\|_{L^2}^{\frac{6-r}{2r}} \|\nabla w_t\|_{L^2}^{\frac{3r-6}{2r}} + \|w\|_{L^2}^{\frac12}\|\nabla w\|_{L^2}^{\frac32},
\end{aligned}
\tag{5.57}
$$

and it is easy to observe that

$$
\begin{aligned}
\int_{t_0}^T \|\nabla^2 w\|_{L^r} dt \lesssim & \int_{t_0}^T \|\nabla w\|_{L^2} dt + \int_{t_0}^T \|\nabla w\|_{L^2}^{\frac{6(r-1)}{r}} dt + \int_{t_0}^T (1+\|w\|_{L^2})\|v\|_{\dot B^{s_1}_{3,1}} dt \\
& + \int_{t_0}^T \|v\|_{\dot B^3_{3,1}} dt + \int_{t_0}^T \|w_t\|_{L^2}^{\frac{6-r}{2r}} \|\nabla w_t\|_{L^2}^{\frac{3r-6}{2r}} dt + \int_{t_0}^T \|w\|_{L^2}^{\frac12}\|\nabla w\|_{L^2}^{\frac32} dt.
\end{aligned}
\tag{5.58}
$$

By virtue of Corollary 5.2, we get for $\delta_- > \frac12$,

$$
\int_{t_0}^T \|\nabla w\|_{L^2} dt \le \|t^{\delta_-}\nabla w\|_{L^2([t_0,T];L^2)} \Big(\int_{t_0}^T t^{-2\delta_-} dt\Big)^{\frac12} \le C\mathcal{H}_0,
$$

and

$$
\int_{t_0}^T \|\nabla w\|_{L^2}^{\frac{6(r-1)}{r}} dt \le \sup_{t\in[t_0,T]} \|\nabla w\|_{L^2}^{\frac{4r-6}{r}} \cdot \int_{t_0}^T \|\nabla w\|_{L^2}^2 dt \le C\|u_0\|_{B^0_{3,1}}^2.
$$

Similarly, applying Lemma 5.3 and Proposition 5.2, we arrive at

$$
\sup_{t\in[t_0,T]} (1+\|w\|_{L^2}) \int_{t_0}^T \|v\|_{\dot B^{s_1}_{3,1}} dt + \int_{t_0}^T \|v\|_{\dot B^3_{3,1}} dt \le C,
$$

and

$$
\begin{aligned}
& \int_{t_0}^T \|\partial_t w\|_{L^2}^{\frac{6-r}{2r}} \|\nabla w_t\|_{L^2}^{\frac{3(r-2)}{2r}} dt \le \int_{t_0}^T \|t\partial_t w\|_{L^2}^{\frac{6-r}{2r}} \|t\nabla w_t\|_{L^2}^{\frac{3(r-2)}{2r}} \cdot t^{-1} dt \\
\le & \Big(\sup_{t\in[t_0,T]} \|t\partial_t w\|_{L^2}\Big)^{\frac{6-r}{2r}} \cdot \Big(\int_{t_0}^T \|t\nabla w_t\|_{L^2}^2 dt\Big)^{\frac{3(r-2)}{4r}} \cdot \Big(\int_{t_0}^T t^{-\frac{4r}{r+6}} dt\Big)^{\frac{6+r}{4r}} \\
\le & C\mathcal{H}_0^2.
\end{aligned}
$$

The same estimate holds true for $\int_{t_0}^T \|w\|_{L^2}^{\frac12}\|\nabla w\|_{L^2}^{\frac32}dt$, i.e.,

$$\begin{aligned}\int_{t_0}^T \|w\|_{L^2}^{\frac12}\|\nabla w\|_{L^2}^{\frac32}dt &\le C\sup_{t\in[t_0,T]}\|t^{\delta}w\|_{L^2}^{\frac12}(\int_{t_0}^T \|t^{\delta_-}\nabla w\|_{L^2}^2dt)^{\frac34}(\int_{t_0}^T t^{-8\delta_-}dt)^{\frac14}\\ &\le C\mathcal{H}_0^2.\end{aligned}$$

Therefore, according to (5.58), we can deduce

$$\int_{t_0}^T \|\nabla^2 w\|_{L^r}dt \le C\mathcal{H}_0^2. \tag{5.59}$$

On the other hand, applying the Gagliardo-Nirenberg inequality, we have

$$\|\nabla w\|_{\dot B^0_{\infty,1}} \lesssim \|\nabla w\|_{L^p}^{\alpha_1}\|\nabla^2 w\|_{L^r}^{1-\alpha_1} \quad \text{for} \quad 0=\frac{\alpha_1}{p}+(1-\alpha_1)(\frac1r-\frac13). \tag{5.60}$$

Let p be a positive constant which will be determined later on. By virtue of Corollary 5.1, we infer that

$$\begin{aligned}\int_{t_0}^T \|\nabla w\|_{L^6}^2dt &\lesssim \int_{t_0}^T \|\nabla^2 w\|_{L^2}^2dt \lesssim \int_{t_0}^T \|\nabla w\|_{L^2}^2dt+\int_{t_0}^T \|\partial_t w\|_{L^2}^2dt\\ &+\int_{t_0}^T \|\nabla^2 w\|_{L^2}^6dt+\int_{t_0}^T \|v\|^2_{\dot B^{s_1}_{3,1}}dt \lesssim \int_{t_0}^T \|\nabla w\|_{L^2}^2dt+\int_{t_0}^T \|\partial_t w\|_{L^2}^2dt\\ &+\sup_{t\in[t_0,T]}\|\nabla w\|_{LL^2}^4\int_{t_0}^T \|\nabla w\|_{L^2}^2dt+\int_{t_0}^T \|v\|^2_{\dot B^{s_1}_{3,1}}dt \lesssim \|u_0\|^2_{\dot B^0_{3,1}},\end{aligned} \tag{5.61}$$

where we used the embedding inequality $\dot H^1 \hookrightarrow L^6$ in the first inequality. Notice that

$$\|t^{(1-\alpha_2)\delta_-}\nabla w\|_{L^2([t_0,T];L^p)} \lesssim \|\nabla w\|^{\alpha_2}_{L^2([t_0,T];L^6)}\|t^{\delta_-}\nabla w\|^{1-\alpha_2}_{L^2([t_0,T];L^2)}, \tag{5.62}$$

where α_2 satisfies

$$\frac1p=\frac{\alpha_2}{6}+\frac{1-\alpha_2}{2}, \quad \text{or} \quad \alpha_2=\frac{3(p-2)}{2p}.$$

Taking $p\in(2,\frac{6\delta_-}{1+\delta_-})$ for $\delta\in(\frac12,\frac34)$, we get, by using Holder's inequality, that

$$\begin{aligned}\|\nabla w\|_{L^1(L^p)} &\le C\|t^{\frac{6-p}{2p}\delta_-}\nabla w\|_{L^2([t_0,T];L^p)}(\int_{t_0}^T t^{\frac{p-6}{p}\delta_-}dt)^{\frac12}\\ &\le C\mathcal{H}_0^2\|u_0\|^{\frac{3(p-2)}{2p}}_{\dot B^0_{3,1}},\end{aligned}$$

which along with (5.59) yields

$$\begin{aligned}\|\nabla w\|_{L^1([t_0,T];\dot B^0_{\infty,1})} &\le \|\nabla w\|^{\alpha_1}_{L^1([t_0,T];L^p)}\|\nabla^2 w\|^{1-\alpha_1}_{L^1([t_0,T];L^r)}\\ &\le C\mathcal{H}_0^2\|u_0\|^{\frac{3\alpha_1(p-2)}{2p}}_{\dot B^0_{3,1}}.\end{aligned}$$

This completes the proof of Lemma 5.7.

With Lemma 5.3-5.7 at hand, we are in a position to prove Proposition 5.4.

Proof Since $\mu(\theta)$ satisfies

$$\partial_t(\mu(\theta)) + u\cdot\nabla\mu(\theta) = 0, \tag{5.63}$$

the standard caclulations show that

$$\frac{d}{dt}\|\nabla\mu(\theta)\|_{L^q} \le q\|\nabla u\|_{L^\infty}\|\nabla\mu(\theta)\|_{L^q}, \tag{5.64}$$

which together with Gronwall's inequality and (5.55) yields

$$\begin{aligned}\sup_{t\in[0,T]}\|\nabla\mu(\theta)\|_{L^q} &\le \|\nabla\mu(\theta_0)\|_{L^q}\exp\{q\int_{t_0}^{T}\|\nabla u\|_{L^\infty}dt + q\int_0^{t_0}\|\nabla u\|_{L^\infty}dt\} \\ &\le \|\nabla\mu(\theta_0)\|_{L^q}\exp\left\{C\|u_0\|_{\dot B^0_{3,1}} + C\mathcal{H}_0^2\|u_0\|_{\dot B^0_{3,1}}^{\frac{3\alpha_1(p-2)}{2p}}\right\}.\end{aligned} \tag{5.65}$$

Hence, one has

$$\sup_{t\in[t_0,T]}\|\nabla\mu(\theta)\|_{L^q} \le 2\|\nabla\mu(\theta_0)\|_{L^q},$$

provided that

$$\|u_0\|_{\dot B^0_{3,1}} \le \varepsilon_2 \quad \text{and} \quad \varepsilon_2 + \varepsilon_2^{\frac{3\alpha_1(p-2)}{2p}}\mathcal{H}_0^2 \overset{\text{def}}{=} C^{-1}\ln 2. \tag{5.66}$$

Similarly, we can obtain

$$\|\nabla\theta\|_{L^3} \le 2\|\theta_0\|_{\dot B^{\frac32}_{2,1}}.$$

Choosing $\varepsilon = \min\{1,\varepsilon_1,\varepsilon_2\}$, we directly obtain (5.19) from (5.64)-(5.66). Then The proof of Proposition 5.4 is completed.

5.4 Proof of Theorem 1.1

In this part, we will finish the proof of Theorem 1.1. To begin with, we rewrite the momentum equation in $(1.3)_2$ as

$$\partial_t u + u\cdot\nabla u - \mu(\theta)\Delta u + \nabla\pi = 2\mathbb{D}u\nabla\mu(\theta). \tag{5.67}$$

Applying the operator $\dot\Delta_j\mathbb{P}$ to (5.67), we derive

$$\begin{aligned}\partial_t\dot\Delta_j u + u\cdot\nabla\dot\Delta_j u - \operatorname{div}\{\mu(\theta)\dot\Delta_j\nabla u\} &= [u\cdot\nabla;\dot\Delta_j\mathbb{P}]u \\ - [\mu(\theta);\dot\Delta_j\mathbb{P}]\Delta u &- \nabla\mu(\theta)\cdot\dot\Delta_j\nabla u + \dot\Delta_j\mathbb{P}(2\mathbb{D}u\nabla\mu(\theta)).\end{aligned} \tag{5.68}$$

Utilizing that $\operatorname{div} u = 0$ and $\mu(\theta)\ge\underline{\mu}$ and multiplying (5.68) by $|\dot\Delta_j u|\dot\Delta_j u$ and then integrating the resulting equality over $\mathbb{R}^3$, we obtain

$$\begin{aligned}\frac{d}{dt}\|\dot\Delta_j u\|_{L^3} + 2^{2j}\|\dot\Delta_j u\|_{L^3} &\lesssim \|[u\cdot\nabla;\dot\Delta_j\mathbb{P}]u\|_{L^3} + \|[\mu(\theta);\dot\Delta_j\mathbb{P}]\Delta u\|_{L^3} \\ &\quad + \|\nabla\mu(\theta)\cdot\dot\Delta_j\nabla u\|_{L^3} + \|\dot\Delta_j\mathbb{P}(2\mathbb{D}u\nabla\mu(\theta))\|_{L^3}.\end{aligned} \tag{5.69}$$

Intergrating the above inequality over $[t_0,T]$ and multiplying (5.69) by 2^j, then summing up the resulting inequality over $j\in\mathbb{Z}$, we achieve

$$\begin{aligned}&\|u\|_{\widetilde L^\infty([t_0,T];\dot B^0_{3,1})} + \|u\|_{L^1([t_0,T];\dot B^2_{3,1})} \\ &\lesssim \|u(t_0)\|_{\dot B^0_{3,1}} + \sum_{j\in\mathbb{Z}}\|[u\cdot\nabla;\dot\Delta_j\mathbb{P}]u\|_{L^1([t_0,T];L^3)} + \sum_{j\in\mathbb{Z}}\|[\mu(\theta);\dot\Delta_j\mathbb{P}]\Delta u\|_{L^1([t_0,T];L^3)} \\ &\quad + \sum_{j\in\mathbb{Z}}\|\nabla\mu(\theta)\cdot\dot\Delta_j\nabla u\|_{L^1([t_0,T];L^3)} + \|\mathbb{D}u\nabla\mu(\theta)\|_{L^1([t_0,T];\dot B^0_{3,1})}.\end{aligned} \tag{5.70}$$

In what follows, we shall deal with the right-hand side of (5.70). By virtue of Lemma 2.3,

$$\sum_{j\in\mathbb{Z}}\|[u\cdot\nabla;\dot{\Delta}_j\mathbb{P}]u\|_{L^1([t_0,T];L^3)}\lesssim\int_{t_0}^{T}\|\nabla u\|_{L^\infty}\|u\|_{\dot{B}^0_{3,1}}dt. \tag{5.71}$$

For $[\mu(\theta);\dot{\Delta}_j\mathbb{P}]\Delta u$, we denote $f\overset{\text{def}}{=}\mu(\theta)-1$, then the homogenous Bony's decomposition implies

$$[\mu(\theta);\dot{\Delta}_j\mathbb{P}]\Delta u=[f;\dot{\Delta}_j\mathbb{P}]\Delta u=[T_f;\dot{\Delta}_j\mathbb{P}]\Delta u+T'_{\dot{\Delta}_j\Delta u}f-\dot{\Delta}_j\mathbb{P}(T_{\Delta u}f)-\dot{\Delta}_j\mathbb{P}\mathcal{R}(\Delta u,f).$$

It follows again from the above estimate

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\|[T_f;\dot{\Delta}_j\mathbb{P}]\Delta u\|_{L^3}&\lesssim\sum_{j\in\mathbb{Z}}\sum_{|j-k|\le4}2^{k-j}\|\nabla\dot{S}_{k-1}f\|_{L^q}2^{-k}\|\dot{\Delta}_k\Delta u\|_{L^{q_1^*}}\\&\lesssim\sum_{j\in\mathbb{Z}}\sum_{|j-k|\le4}\|\nabla f\|_{L^q}2^{\frac{3k}{q}}\|\dot{\Delta}_k\nabla u\|_{L^3}\\&\lesssim\|\nabla f\|_{L^q}\|u\|_{\dot{B}^{1+\frac{3}{q}}_{3,1}},\end{aligned}$$

where $\frac{1}{q}+\frac{1}{q^*}=\frac{1}{3}$. Similarly, one has

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\|T'_{\dot{\Delta}_j\Delta u}f\|_{L^3}&\lesssim\sum_{j\in\mathbb{Z}}\sum_{k\ge j-2}\|\dot{S}_{k+2}\dot{\Delta}_j\Delta u\|_{L^{q_1^*}}\|\dot{\Delta}_kf\|_{L^q}\\&\lesssim\sum_{j\in\mathbb{Z}}2^{-j}\|\dot{\Delta}_j\Delta u\|_{L^{q_1^*}}\sum_{k\ge j-2}2^{j-k}\|\dot{\Delta}_k\nabla f\|_{L^q}\\&\lesssim\|u\|_{\dot{B}^{1+\frac{3}{q}}_{3,1}}\|\nabla f\|_{L^q}.\end{aligned}$$

Notice that

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\|\dot{\Delta}_j\mathbb{P}(T_{\Delta u}f)\|_{L^2}&\lesssim\sum_{j\in\mathbb{Z}}\sum_{|k-j|\le4}2^{-k}\|\dot{S}_{k-1}\Delta u\|_{L^{q_1^*}}2^k\|\dot{\Delta}_kf\|_{L^q}\\&\lesssim\|u\|_{\dot{B}^{1+\frac{3}{q}}_{3,1}}\|\nabla f\|_{L^q}.\end{aligned}$$

The same estimate holds true for $\dot{\Delta}_j\mathbb{P}\mathcal{R}(\Delta u,f)$. Therefore, we obtain

$$\sum_{j\in\mathbb{Z}}\|[\mu(\theta);\dot{\Delta}_j\mathbb{P}]\Delta u\|_{L^1([t_0,T];L^3)}\lesssim\int_{t_0}^{T}\|u\|_{\dot{B}^{1+\frac{3}{q}}_{3,1}}\|\nabla\mu(\theta)\|_{L^q}dt. \tag{5.72}$$

Along the same line, one has

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}\|\nabla\mu(\theta)\cdot\dot{\Delta}_j\nabla u\|_{L^1([t_0,T];L^3)}+\|\mathbb{D}u\cdot\nabla\mu(\theta)\|_{L^1([t_0,T];\dot{B}^0_{3,1})}\\&\lesssim\int_{t_0}^{T}\|u\|_{\dot{B}^{1+\frac{3}{q}}_{3,1}}\|\nabla\mu(\theta)\|_{L^q}dt+\int_{t_0}^{T}\|\nabla u\|_{L^\infty}\|\nabla\mu(\theta)\|_{\dot{B}^0_{3,1}}dt\\&\lesssim\int_{t_0}^{T}\|u\|^{\frac{q-3}{2q}}_{\dot{B}^0_{3,1}}\|u\|^{\frac{q+3}{2q}}_{\dot{B}^2_{3,1}}\|\nabla\mu(\theta)\|_{L^q}dt+\int_{t_0}^{T}\|\nabla u\|_{L^\infty}\|\theta\|_{\dot{B}^1_{3,1}}dt,\end{aligned} \tag{5.73}$$

where we use the interpolation inequality $\|u\|_{\dot{B}^{1+\frac{3}{q}}_{3,1}}\lesssim\|u\|^{\frac{q-3}{2q}}_{\dot{B}^0_{3,1}}\|u\|^{\frac{q+3}{2q}}_{\dot{B}^2_{3,1}}$. By virtue of Lemma 2.4, one has

$$\|\theta\|_{\widetilde{L}^\infty_T(\dot{B}^1_{3,1})}\le\|\theta_0\|_{\dot{B}^1_{3,1}}\exp\left\{C\int_0^T\|\nabla u\|_{\dot{B}^0_{\infty,1}}dt\right\}. \tag{5.74}$$

Yet due to Lemma 5.7 and (5.5), one has

$$\int_0^T \|\nabla u\|_{\dot{B}^0_{\infty,1}} dt \leq C\|u_0\|_{\dot{B}^0_{3,1}} + C\mathcal{H}_0^2 \|u_0\|_{\dot{B}^0_{3,1}}^{\frac{3\alpha_1(p-2)}{2p}}. \tag{5.75}$$

Plugging the above estimates into (5.70) and applying Young's inequality, we arrive at

$$\begin{aligned} \|u\|_{\widetilde{L}^\infty([t_0,T];\dot{B}^0_{3,1})} + \|u\|_{L^1([t_0,T];\dot{B}^2_{3,1})} &\lesssim (\|u_0\|_{\dot{B}^0_{3,1}} + \|\theta_0\|_{\dot{B}^1_{3,1}}) \\ &\times \exp\left\{\{1 + T\|\nabla\mu(\theta_0)\|_{L^q}^{\frac{2q}{q-3}}\} \exp\left\{\mathcal{H}_0^2\right\}\right\}. \end{aligned} \tag{5.76}$$

The similary estimates holds true for $\nabla\pi$ and u_t. Hence, direct calculations leads to

$$\begin{aligned} \|u_t\|_{\widetilde{L}^\infty([t_0,T];\dot{B}^0_{3,1})} + \|\nabla\pi\|_{L^1([t_0,T];\dot{B}^0_{3,1})} &\lesssim (\|u_0\|_{\dot{B}^0_{3,1}} + \|\theta_0\|_{\dot{B}^1_{3,1}}) \\ &\times \exp\left\{\{1 + T\|\nabla\mu(\theta_0)\|_{L^q}^{\frac{2q}{q-3}}\} \exp\left\{\mathcal{H}_0^2\right\}\right\}, \end{aligned} \tag{5.77}$$

which completes the proof of Theorem 1.1.

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