Semiclassical states for fractional Schrödinger equations with critical nonlinearities

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Abstract

In this paper, we consider the following fractional Schrödinger equation ε 2 s (- [?]) s u + V (x) u = P (x) f (u) + Q (x) | u | 2 s * - 2 u in R N , where $\epsilon>0$ is a parameter, s[?](0 ,1), 2 s * = 2 N N - 2 s , $N>2$ s, (- [?]) s is the fractional Lapalacian and f is a superlinear and subcritical nonlinearity. Under a local condition imposed on the potential function, combining the penalization method and the concentration-compactness principle, we prove the existence of a positive solution for the above equations.

Semiclassical states for fractional Schrödinger equations with critical nonlinearities *

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Abstract: In this paper, we consider the following fractional Schrödinger equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = P(x)f(u) + Q(x)|u|^{2_s^*-2}u \quad \text{in } \mathbb{R}^N,$$

where $\varepsilon > 0$ is a parameter, $s \in (0,1)$, $2_s^* = \frac{2N}{N-2s}$, $N > 2s$, $(-\Delta)^s$ is the fractional Lapalacian and f is a superlinear and subcritical nonlinearity. Under a local condition imposed on the potential function, combining the penalization method and the concentration-compactness principle, we prove the existence of a positive solution for the above equations.

Key words: Concentration-compactness principle; Penalization method; Nonautonomous nonlinearities; fractional Schrödinger equation.

1 Introduction

We consider the following fractional Schrödinger equation

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u = P(x)f(u) + Q(x)|u|^{2_s^*-2}u, & x \in \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N),\ u(x) > 0 & x \in \mathbb{R}^N, \end{cases} \tag{P_ε}$$

where $\varepsilon > 0$ is a small parameter, $s \in (0,1)$, $2_s^* := \frac{2N}{N-2s}$ is the fractional critical exponent, the function f is a superlinear and subcritical nonlinearity. Here the factional Sobolev space $H^s(\mathbb{R}^N)$ is defined by

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} dxdy < \infty \right\},$$

equipped with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |u(x)|^2 dx + \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} dxdy \right)^{1/2}.$$

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$(-\Delta)^s$ stands for the usual fractional Laplacian, $(-\Delta)^s$ of a smooth function $u : \mathbb{R}^N \to \mathbb{R}$ is defined by

$$\mathcal{F}((-\Delta)^s(u))(\xi) = |\xi|^{2s}\mathcal{F}(u)(\xi),\ \xi \in \mathbb{R}^N,$$

where $\mathcal{F}$ denotes the Fourier transform, that is,

$$\mathcal{F}(w)(\xi) = \frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N} e^{-i\xi\cdot x}w(x)dx,$$

for function w in the Schwartz class. Also, $(-\Delta)^s u$ can be equivalently represented as

$$(-\Delta)^s u = -\frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\frac{(u(x+y)+u(x-y)-2u(x))}{|y|^{N+2s}}dy,\ \forall x \in \mathbb{R}^N,$$

where $C_{N,s} > 0$ is the normalizing constant, defined by

$$C_{N,s} = \left(\int_{\mathbb{R}^N}\frac{1-\cos\xi_1}{|\xi|^{N+2s}}d\xi\right)^{-1}, \xi = (\xi_1, \xi_2, \cdots, \xi_N).$$

We have from [9] that

$$\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u(x)|^2dx = \frac{C_{N,s}}{2}\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy.$$

And by taking derivative of the above equality, for any $u, v \in C_0^\infty(\mathbb{R}^N)$ we obtain

$$\int_{\mathbb{R}^{2N}}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dxdy = C_{N,s}\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}u(x)(-\Delta)^{\frac{s}{2}}v(x)dx. \tag{1.1}$$

Moreover, for any $u, v \in C_0^\infty(\mathbb{R}^N)$ we have

$$(-\Delta)^{\frac{s}{2}}(uv) = u(-\Delta)^{\frac{s}{2}}v + v(-\Delta)^{\frac{s}{2}}u - 2I_{\frac{s}{2}}(u,v), \tag{1.2}$$

where $I_{\frac{s}{2}}$ is defined in the principal value sense, as follows

$$I_{\frac{s}{2}}(u,v)(x) = P.V.\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+s}}dy.$$

Problem (P_ε) describes the so called standing waves of the nonlinear, time-dependent fractional Schrödinger equation of the form

$$i\varepsilon\frac{\partial\psi}{\partial t} = \varepsilon^{2s}(-\Delta)^s\psi + V(x)\psi - f(x,\psi). \tag{1.3}$$

Solutions of (1.3) for sufficiently small $\varepsilon > 0$ are called semiclassical states. Recently great attention has been devoted to the study of semiclassical states, see for example [1–4, 7, 12, 13, 15, 18, 19] and the references therein and most of them assume that the potential satisfies the following global condition

(V) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $0 < \inf\limits_{x\in\mathbb{R}^N} V(x) < \liminf\limits_{|x|\to+\infty} V(x) = V_\infty < +\infty$, which is first introduced by Rabinowtz in [17] in the study of a nonlinear Schrödinger equation with the nonlinear subcritical growth. There are some results for problem (P_ε) when $V(x)$ satisfies the following local condition

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ and there is constant $V_0 > 0$ such that $V_0 := \inf_{x \in \mathbb{R}^N} V(x)$;

(V_2) there is a bounded open set $\Omega \subset \mathbb{R}^N$ such that $V_0 < \min_{\partial\Omega} V$, and $M := \{x \in \Omega : V(x) = V_0\} \neq \emptyset$.

Some authors have studied the existence and concentration phenomena for potentials verifying local condition (V_1) and (V_2), see for example [2–4, 7, 13, 15] and the references therein. As far as we know, all of them concentrate on the problems with autonomous nonlinearities. Particularly, in [13] the authors consider the following equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = f(u) + u^{2_s^*-1} \quad x \in \mathbb{R}^N \tag{1.4}$$

under the conditions (V_1) and (V_2) and obtain the existence and concentration of multiple solutions, which concentrate on the minima of $V(x)$ as $\varepsilon \to 0$. Our aim is to study the existence and concentration of positive solutions for problem (P_ε) by combining a local assumption on V, and show that the penalization method introduced by del Pino and Felmer in [8] can be also applied to a general class of problems with nonautonomous nonlinearities.

Below we give some assumptions. Since we are interested in positive solutions, we assume that $f \in C^1(\mathbb{R}, \mathbb{R})$ vanishes in $(-\infty, 0)$ and satisfies the following conditions.

(f_1) $f(t) = o(t)$ as $t \to 0^+$.

(f_2) There exist constants $q, \sigma \in (2, 2_s^*)$, sufficiently large $C_0 > 0$ such that $f(t) \geq C_0 t^{q-1}$ for all $t \geq 0$, and $\lim_{t \to \infty} \frac{f(t)}{t^{\sigma-1}} = 0$.

(f_3) There exists a constant $\theta \in (2, 2_s^*)$ such that for all $t > 0$, $0 < \theta F(t) := \theta \int_0^t f(\tau)d\tau \leq t f(t)$.

(f_4) The function $\frac{f(t)}{t}$ is increasing on interval $(0, \infty)$.

The potential $V(x)$ satisfies (V_1), $P(x)$ and $Q(x)$ are assumed to satisfy the following conditions.

(P) $P \in L^\infty(\mathbb{R}^N)$ is continuous and there is a constant $\alpha > 0$ such that $P(x) \geq \alpha$ for all $x \in \mathbb{R}^N$.

(Q) $Q \in L^\infty(\mathbb{R}^N)$ is continuous and there is a constant $\beta > 0$ such that $Q(x) \geq \beta$ for all $x \in \mathbb{R}^N$.

(Ω) There is a bounded, nonempty domain $\Omega \subset \mathbb{R}^N$ such that

(Ω_1) there is $x_{min} \in \Omega$ such that $V(x_{min}) = V_0 < \min\limits_{\partial\Omega} V$, $P(x_{min}) = P_0 = \sup\limits_{\mathbb{R}^N} P$ and $Q(x_{min}) = Q_0 = \sup\limits_{\mathbb{R}^N} Q$,

or

(Ω_2) there is $x_{max} \in \Omega$ such that $P(x_{max}) = P_0 > \max\limits_{\partial\Omega} P$, $Q(x_{max}) = Q_0 > \max\limits_{\partial\Omega} Q$ and $V(x_{max}) = V_0$.

The main result of this paper is stated as follows:

Theorem 1.1. *Assume that* $(V_1), (P), (Q), (\Omega)$ *and* (f_1)*–*(f_4) *hold. Then there exists* $\varepsilon_0 > 0$ *for any* $\varepsilon \in (0, \varepsilon_0)$*, problem* (P_ε) *has a positive solution* u_ε*. Furthermore, if* $\eta_\varepsilon \in \mathbb{R}^N$ *denotes its global maximum point, then*

$$u_\varepsilon(x) \leq \frac{C\varepsilon^{N+2s}}{\varepsilon^{N+2s} + |x - \eta_\varepsilon|^{N+2s}}.$$

Remark 1.2. To our best knowledge, the existence and qualitative properties of solutions for problem (P_ε) have been extensively studied when $V(x)$ satisfies the global condition (V). There are few results for problem (P_ε) when $V(x)$ satisfies a local condition as above, even in the $P = Q = 1$ case. Under a local condition imposed on V, it is necessary to create a penalization function. If $P \neq 1, Q \neq 1$, the construction of penalization function is more complicated. Especially after adding the critical nonlinearities, the problem is more difficult, so far, no one has studied this aspect. Motivated by the penalization approach used in [8], we will investigate the existence of positive solution for problem (P_ε) by supposing that V satisfies a local assumption as above. Hence, our results can be seen as an improvement and supplement to [2–4, 7, 13, 15].

Remark 1.3. Compared with the previous works, the main difficulty lies in the nonautonomous nonlinearity with the critical Sobolev growth and the potential V with a local assumption, which makes it more complicated to recover the compactness. So we focus on the essential difficulty of the problem under the assumption that $f \in C^1(\mathbb{R}^N, \mathbb{R})$. We note that if $f \in C(\mathbb{R}^N, \mathbb{R})$, Theorem 1.1 also holds true by using the method of [20] under the condition of this paper. The related specific proof can be found in [13, 21].

To establish the existence of positive solution, we will use the penalization method introduced by Del Pino and Felmer [8]. First, we need to fix some notations.

Let $K, L > 0$, $\frac{P_0}{K} + \frac{Q_0}{L} < \min\{\frac{1}{2}, \frac{\theta-2}{\theta}\}$, and $a > 0$ such that $f(a) = \frac{V_0}{K}a$ and $a^{2_s^*-1} = \frac{V_0}{L}a$, where θ and V_0 are introduced in (f_3) and (V_1) respectively. We set

$$\tilde{f}(t) = \begin{cases} f(t), & \text{if } t \leq a \\ \frac{V_0}{K}t, & \text{if } t > a \end{cases}, \qquad \tilde{g}(t) = \begin{cases} t^{2_s^*-1}, & \text{if } t \leq a \\ \frac{V_0}{L}t, & \text{if } t > a \end{cases},$$

and

$$g(x,t) = \chi_\Omega(x)(P(x)f(t) + Q(x)t^{2_s^*-1}) + (1 - \chi_\Omega(x))(P(x)\tilde{f}(t) + Q(x)\tilde{g}(t)) \qquad (1.5)$$

where χ_Ω is the charateristic function of the set Ω. Form $(f_1) - (f_4)$, it is easy to check that g satisfies the following properties,

(g_1) $\lim_{t\to 0^+}\frac{g(x,t)}{t}=0$ uniformly in $x\in\mathbb{R}^N$.

(g_2) $g(x,t)\le P_0 f(t)+Q_0 t^{2_s^*-1}$ for all $x\in\mathbb{R}^N$, $t>0$.

(g_3) There is $\theta\in(2,2_s^*)$ such that

$$0\le \theta G(x,t):=\theta\int_0^t g(x,s)ds<g(x,t)t \text{ for } \forall x\in\Omega \text{ and } \forall\, t>0$$

and

$$0\le 2G(x,t)<g(x,t)t\le\left(\frac{P_0}{K}+\frac{Q_0}{L}\right)V(x)t^2 \text{ for } \forall\, x\in\mathbb{R}^N\backslash\Omega, \text{ and } \forall\, t>0.$$

(g_4) For each $x\in\Omega$, the function $t\to\frac{g(x,t)}{t}$ is increasing in interval $(0,\infty)$ and for each $x\in\mathbb{R}^N\backslash\Omega$, the function $t\to\frac{g(x,t)}{t}$ is increasing in $(0,a)$.

Now we study the modified problem

$$\begin{cases}\varepsilon^{2s}(-\Delta)^s u+V(x)u=g(x,u), & x\in\mathbb{R}^N\\ u\in H^s(\mathbb{R}^N),\ u(x)>0, & x\in\mathbb{R}^N\end{cases}. \tag{P_ε^*}$$

Note that the positive solution of (P_ε^*) with $u(x)\le a$ for each $x\in\mathbb{R}^N\backslash\Omega$ is also the positive solution of (P_ε).

In view of the presence of potential $V(x)$, we introduce the following fractional Sobolev space

$$H_\varepsilon:=\left\{u\in H^s(\mathbb{R}^N):\int_{\mathbb{R}^N}V(x)u^2dx<\infty\right\},$$

endowed with the norm

$$\|u\|_\varepsilon^2=\int_{\mathbb{R}^N}(\varepsilon^{2s}|(-\Delta)^{\frac{s}{2}}u|^2+V(x)u^2)dx.$$

Consider the energy functional $J_\varepsilon:H_\varepsilon\to\mathbb{R}$ associated to (P_ε^*) given by

$$J_\varepsilon(u)=\frac{\varepsilon^{2s}}{2}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2dx-\int_{\mathbb{R}^N}G(x,u)dx,$$

and its Nehari manifold is defined by

$$\mathcal{N}_\varepsilon:=\{u\in H_\varepsilon\backslash\{0\}:\langle J_\varepsilon'(u),u\rangle=0\}.$$

2 Proof of the main result

We only discuss the case that V,P,Q satisfy (Ω_1), and when V,P,Q satisfy (Ω_2) the proof of the conclusion is similar to that of the case (Ω_1).

2.1 The autonomous problem (P_{V_0})

We start by considering the autonomous problem associated to (P_ε), namely,

$$(-\Delta)^s u + V_0 u = P_0 f(u) + Q_0 |u|^{2_s^*-2}u. \tag{P_{V_0}}$$

The solutions of problem (P_{V_0}) are precisely the positive critical points of the functional defined by

$$J_0(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}}u|^2 + V_0 u^2)dx - \int_{\mathbb{R}^N} P_0 F(u)dx - \frac{Q_0}{2_s^*}\int_{\mathbb{R}^N} |u|^{2_s^*}dx,$$

and its Nehari manifold is defined by

$$\mathcal{N}_0 := \{u \in H_0\backslash\{0\} : \langle J_0'(u), u\rangle = 0\}.$$

where $H_0 := \left\{u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} V_0 u^2 dx < \infty\right\}$,

The functional J_0 satisfies the mountain pass geometry, the proof is standard, and hence, it is omitted. By using a version of the mountain pass theorem without (PS) condition [22], it follows that there exists a sequence $\{u_n\} \subset H_0$ such that $J_0(u_n) \to c_0$ and $J_0'(u_n) \to 0$, and $c_0 := \inf_{\gamma\in\Gamma_0} \max_{t\in[0,1]} J_0(\gamma(t)) > 0$, where $\Gamma_0 := \{\gamma \in C([0,1], H_0) : \gamma(0) = 0 \text{ and } J_0(\gamma(1)) < 0\}$. Similarly to the arguments in [17], by (f_4), the equivalent characterization of c_0 is given by

$$c_0 = \inf_{u\in H_0\backslash\{0\}} \sup_{t\geq 0} J_0(tu) = \inf_{u\in\mathcal{N}_0} J_0(u).$$

The following lemma gives the estimate of the critical value c_0.

Lemma 2.1. *Suppose that $(f_1) - (f_4)$ hold, then*

$$0 < c_0 < \frac{s}{NQ_0^{\frac{N-2s}{2s}}} S_*^{\frac{N}{2s}},$$

where S_* is the best Sobolev constant

$$S_* = \inf_{u\in H^s(\mathbb{R}^N)\backslash\{0\}} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 dx}{(\int_{\mathbb{R}^N} |u|^{2_s^*}dx)^{\frac{2}{2_s^*}}}.$$

Proof. We define

$$\tilde{u}_\varepsilon(x) = \psi(x)U_\varepsilon(x),\ x \in \mathbb{R}^N,$$

where $U_\varepsilon(x) = \varepsilon^{-\frac{N-2s}{2}} u^*(\frac{x}{\varepsilon})$, $u^*(x) = \frac{\bar{u}(x/S_*^{\frac{1}{2s}})}{|\bar{u}|_{2_s^*}}$, where $\bar{u}(x) = \kappa(\mu^2 + |x - x_0|^2)^{-\frac{N-2s}{2}}$, with $\kappa \in \mathbb{R}\setminus\{0\}, \mu > 0$, and $x_0 \in \mathbb{R}^N$, and $\psi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \psi \leq 1$ in $\mathbb{R}^N$, $\psi(x) \equiv 1$ in $B_r(0)$, and $\psi \equiv 0$ in $\mathbb{R}^N\backslash B_{2r}(0)$. Define $v_\varepsilon(x) = \frac{\tilde{u}_\varepsilon(x)}{|\tilde{u}_\varepsilon(x)|_{2_s^*}}$, from [13], we have the following estimates:

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_\varepsilon(x)|^2 dx \leq S_* + O(\varepsilon^{N-2s}), \tag{2.1}$$

and

$$\int_{\mathbb{R}^N} |v_\varepsilon(x)|^2 dx = \begin{cases} O(\varepsilon^{2s}), & N > 4s \\ O(\varepsilon^{2s} \ln \frac{1}{\varepsilon}), & N = 4s \ , \\ O(\varepsilon^{N-2s}), & N < 4s \end{cases} \tag{2.2}$$

$$\int_{\mathbb{R}^N} |v_\varepsilon(x)|^q dx \geq C\varepsilon^{\frac{2N-(N-2s)q}{2}}. \tag{2.3}$$

By the definition of v_ε and (f_2), we have

$$\begin{aligned} J_0(tv_\varepsilon) &= \frac{t^2}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_\varepsilon|^2 dx + \frac{t^2}{2}\int_{\mathbb{R}^N} V_0|v_\varepsilon|^2 dx - \int_{\mathbb{R}^N} P_0 F(tv_\varepsilon)dx - \frac{Q_0}{2_s^*}t^{2_s^*} \\ &\leq \frac{t^2}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_\varepsilon|^2 dx + \frac{t^2}{2}\int_{\mathbb{R}^N} V_0|v_\varepsilon|^2 dx - C_0P_0t^q\int_{\mathbb{R}^N} |v_\varepsilon|^q dx - \frac{Q_0}{2_s^*}t^{2_s^*}. \end{aligned}$$

We consider the function

$$g(t) = \frac{t^2}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_\varepsilon|^2 dx + \frac{t^2}{2}\int_{\mathbb{R}^N} V_0|v_\varepsilon|^2 dx - C_0P_0t^q\int_{\mathbb{R}^N} |v_\varepsilon|^q dx - \frac{Q_0}{2_s^*}t^{2_s^*}. \tag{2.4}$$

It is clear that $g(t) > 0$ for $t > 0$ small enough, and $g(t) \to -\infty$ as $t \to +\infty$. Hence there exists $t_\varepsilon > 0$ such that $\max_{t\geq 0} g(t) = g(t_\varepsilon)$, and

$$0 = g'(t_\varepsilon) = t_\varepsilon\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_\varepsilon|^2 dx + \int_{\mathbb{R}^N} V_0|v_\varepsilon|^2 dx - qC_0P_0t_\varepsilon^{q-2}\int_{\mathbb{R}^N} |v_\varepsilon|^q dx - Q_0t_\varepsilon^{2_s^*-2}\right).$$

Therefore,

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_\varepsilon|^2 dx + \int_{\mathbb{R}^N} V_0|v_\varepsilon|^2 dx = qC_0P_0t_\varepsilon^{q-2}\int_{\mathbb{R}^N} |v_\varepsilon|^q dx + Q_0t_\varepsilon^{2_s^*-2},$$

which implies

$$0 < t_\varepsilon \leq \frac{1}{Q_0^{\frac{1}{2_s^*-2}}}\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_\varepsilon|^2 dx + \int_{\mathbb{R}^N} V_0|v_\varepsilon|^2 dx\right)^{\frac{1}{2_s^*-2}}.$$

It is easy to verifies that J_0 satisfies the mountain-pass geometry conditions, and we get

$$0 < \delta \leq J_0(t_\varepsilon v_\varepsilon) \leq \frac{t_\varepsilon^2}{2}\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_\varepsilon|^2 dx + \int_{\mathbb{R}^N} V_0|v_\varepsilon|^2 dx\right).$$

Hence we have a lower bound and a upper bound for t_ε, independent of ε, let

$$\bar{g}(t) = \frac{t^2}{2}\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_\varepsilon|^2 dx + \int_{\mathbb{R}^N} V_0|v_\varepsilon|^2 dx\right) - \frac{Q_0}{2_s^*}t^{2_s^*},$$

then $t_\varepsilon = \frac{1}{Q_0^{\frac{1}{2_s^*-2}}}(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_\varepsilon|^2 + V_0|v_\varepsilon|^2 dx)^{\frac{1}{2_s^*-2}}$ is the maximum point of $\bar{g}(t)$. Hence, by (2.1)-(2.3), and the elementary inequality $(a+b)^p \leq a^p + p(a+b)^{p-1}b$ for $a, b > 0$ and $p \geq 1$, we obtain

$$g(t_\varepsilon) = \bar{g}(t_\varepsilon) - C_0P_0t_\varepsilon^q\int_{\mathbb{R}^N} |v_\varepsilon|^q dx$$

$$
\begin{aligned}
&\le \bar{g}\left(\frac{1}{Q_0^{\frac{1}{2_s^*-2}}}(\int_{\mathbb{R}^N}(|(-\Delta)^{\frac{s}{2}}v_\varepsilon|^2+V_0|v_\varepsilon|^2)dx)^{\frac{1}{2_s^*-2}}\right)-C_0P_0t_\varepsilon^q\int_{\mathbb{R}^N}|v_\varepsilon|^q dx\\
&\le \frac{s}{NQ_0^{\frac{N-2s}{2s}}}\left(\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}v_\varepsilon|^2dx+\int_{\mathbb{R}^N}V_0|v_\varepsilon|^2dx\right)^{\frac{N}{2s}}-C_0C|v_\varepsilon|_q^q\\
&\le \frac{s}{NQ_0^{\frac{N-2s}{2s}}}\left(S_*+O(\varepsilon^{N-2s})+\int_{\mathbb{R}^N}V_0|v_\varepsilon|^2dx\right)^{\frac{N}{2s}}-C_0C|v_\varepsilon|_q^q\\
&\le \frac{s}{NQ_0^{\frac{N-2s}{2s}}}S_*^{\frac{N}{2s}}+O(\varepsilon^{N-2s})+C|v_\varepsilon|_2^2-C_0C|v_\varepsilon|_q^q.
\end{aligned}
$$

Next we distinguish the following cases.

(i) If $N>4s$, then $\frac{N}{N-2s}<2$, we have $q>\frac{N}{N-2s}$, by (2.2) and (2.3) we get

$$
\sup_{t>0} g(t)\le \frac{s}{NQ_0^{\frac{N-2s}{2s}}}S_*^{\frac{N}{2s}}+O(\varepsilon^{N-2s})+O(\varepsilon^{2s})-O(\varepsilon^{\frac{2N-(N-2s)q}{2}}).
$$

since $\frac{2N-(N-2s)q}{2}<2s<N-2s$, we get the conclusion for ε sufficiently small.

(ii) If $N=4s$, then $2<q<2_s^*=4$, by (2.2) and (2.3) we obtain

$$
\begin{aligned}
\sup_{t>0} g(t)&\le \frac{s}{NQ_0^{\frac{N-2s}{2s}}}S_*^{\frac{N}{2s}}+O(\varepsilon^{N-2s})+O(\varepsilon^{2s}\ln\frac{1}{\varepsilon})-O(\varepsilon^{4s-sq})\\
&\le \frac{s}{NQ_0^{\frac{N-2s}{2s}}}S_*^{\frac{N}{2s}}+O(\varepsilon^{2s}(1+\ln\frac{1}{\varepsilon}))-O(\varepsilon^{4s-sq})\\
&< \frac{s}{NQ_0^{\frac{N-2s}{2s}}}S_*^{\frac{N}{2s}}.
\end{aligned}
$$

Since

$$
\lim_{\varepsilon\to0^+}\frac{\varepsilon^{4s-sq}}{\varepsilon^{2s}(1+\ln\frac{1}{\varepsilon})}=+\infty,
$$

we get the conclusion for ε sufficiently small.

(iii) If $2s<N<4s$ and $\frac{N}{N-2s}<q<2_s^*$, by (2.2) and (2.3) we have

$$
\sup_{t>0} g(t)\le \frac{s}{NQ_0^{\frac{N-2s}{2s}}}S_*^{\frac{N}{2s}}+O(\varepsilon^{N-2s})-O(\varepsilon^{\frac{2N-(N-2s)q}{2}}).
$$

In view of $\frac{2N-(N-2s)q}{2}<N-2s$, we get the conclusion for ε sufficiently small.

(iv) If $2s<N<4s$ and $2<q\le\frac{N}{N-2s}$, from (2.2) and (2.3) we obtain

$$
\sup_{t>0} g(t)\le \frac{s}{NQ_0^{\frac{N-2s}{2s}}}S_*^{\frac{N}{2s}}+O(\varepsilon^{N-2s})-C_0O(\varepsilon^{\frac{2N-(N-2s)q}{2}}),
$$

and for $C_0=\varepsilon^{-\theta}$ with $\theta>\frac{4s-(N-2s)q}{2}$, we also get the conclusion. Hence, $c_0<\frac{s}{NQ_0^{\frac{N-2s}{2s}}}S_*^{\frac{N}{2s}}$. $\square$

Lemma 2.2. *Assume that $\{u_n\} \subset H_0$ is a $(PS)_c$ sequence for J_0 with $c < \frac{s}{NQ_0^{\frac{N-2s}{2s}}} S_*^{\frac{N}{2s}}$ and such that $u_n \rightharpoonup 0$. Then, one of the following alternatives occurs:*

(a) $u_n \to 0$ in H_0, or

(b) There exists a sequence $\{z_n\} \subset \mathbb{R}^N$ and constants $R, \eta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(z_n)} |u_n|^2 dx \geq \eta > 0.$$

Proof. Suppose (b) is not satisfied. Then for any $R > 0$, we have

$$\lim_{n\to\infty} \sup_{z\in\mathbb{R}^N} \int_{B_R(z)} |u_n|^2 dx = 0.$$

Since $\{u_n\}$ is bounded, from [11], it follows that $u_n \to 0$ in $L^p(\mathbb{R}^N)$, $\forall p \in (2, 2_s^*)$. Hence,

$$\int_{\mathbb{R}^N} f(u_n)u_n dx = \int_{\mathbb{R}^N} F(u_n)dx = o_n(1).$$

Moreover, from $J_0(u_n) \to c > 0$ and $\langle J_0'(u_n), u_n\rangle \to 0$, we have that

$$\frac{1}{2}\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u_n|^2 + V_0|u_n|^2)dx - \frac{Q_0}{2_s^*}\int_{\mathbb{R}^N} |u_n|^{2_s^*} dx \to c, \tag{2.5}$$

and

$$\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u_n|^2 + V_0|u_n|^2)dx = Q_0 \int_{\mathbb{R}^N} |u_n|^{2_s^*} dx + o_n(1). \tag{2.6}$$

Since $\{u_n\}$ is bounded, up to a subsequence, we get

$$\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u_n|^2 + V_0|u_n|^2)dx \to l \geq 0.$$

If $l > 0$, by (2.5) and (2.6), we obtain $c = \frac{s}{N} l$. On the other hand,

$$l \geq \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx \geq S_*(|u_n|_{2_s^*}^{2_s^*})^{2/2_s^*},$$

taking the limit as $n \to \infty$, it follows that $l \geq \frac{1}{Q_0^{\frac{N-2s}{2s}}} S_*^{\frac{N}{2s}}$. Hence,

$$c = \frac{s}{N} l \geq \frac{s}{NQ_0^{\frac{N-2s}{2s}}} S_*^{\frac{N}{2s}},$$

which is a contradiction. Therefore, $l = 0$. It implies that $u_n \to 0$ in H_0. □

Lemma 2.3. *Suppose that $(f_1) - (f_4)$ hold, then problem (P_{V_0}) has a positive ground state solution.*

Proof. Assume $\{u_n\} \subset H_0$ is a $(PS)_{c_0}$ sequence, from the condition of nonlinearity f, we can easily check that $\{u_n\}$ is bounded in H_0. Thus, up to a subsequence, $u_n \rightharpoonup u$ weakly in H_0. Moreover, u is a critical point of J_0. If $u \neq 0$, it remains to show that $J_0(u) = c_0$. By Fatou's Lemma, we have that

$$
\begin{aligned}
c_0 &\leq J_0(u) - \frac{1}{\theta}\langle J_0'(u), u\rangle \\
&= \left(\frac{1}{2} - \frac{1}{\theta}\right) \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + V_0 u^2) dx \\
&\quad + \frac{1}{\theta} \int_{\mathbb{R}^N} P_0(f(u)u - \theta F(u)) dx + \left(\frac{1}{\theta} - \frac{1}{2_s^*}\right) \int_{\mathbb{R}^N} Q_0 |u|^{2_s^*} dx \\
&\leq \liminf_{n\to\infty} \left\{ \left(\frac{1}{2} - \frac{1}{\theta}\right) IR(|(-\Delta)^{\frac{s}{2}} u_n|^2 + V_0 u_n^2) dx \right. \\
&\quad \left. + \frac{1}{\theta} \int_{\mathbb{R}^N} P_0(f(u_n)u_n - \theta F(u_n)) dx + \left(\frac{1}{\theta} - \frac{1}{2_s^*}\right) \int_{\mathbb{R}^N} Q_0 |u_n|^{2_s^*} dx \right\} \\
&= \liminf_{n\to\infty} \left(J_0(u_n) - \frac{1}{\theta}\langle J_0'(u_n) u_n\rangle \right) \\
&= c_0.
\end{aligned}
$$

Hence, we proved that $J_0(u) = c_0$.

Now, we consider the case $u = 0$. In fact, $u_n \nrightarrow 0$ in H_0. If $u_n \to 0$ in H_0, we have $J_0(u_n) \to 0$, this is contradicts to $c_0 > 0$. By Lemma 2.2, there exists a sequence $\{z_n\} \subset \mathbb{R}^N$ and constants $R, \eta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(z_n)} |u_n|^2 dx \geq \eta > 0.$$

Set $w_n(x) = u_n(x+z_n)$, again $\{w_n\}$ is a $(PS)_{c_0}$ sequence of J_0. Thus $\{w_n\}$ is bounded in H_0, and there exists $w \in H_0$ such that $w_n \rightharpoonup w$ in H_0 with $w \neq 0$, then the conclusion follows as in the first case.

Using $-u^-$ as a test function, we have

$$0 = \langle (J_0'(u), -u^-\rangle = \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u^-|^2 + V_0 |u^-|^2) dx.$$

This implies that $u^- = 0$. Noting that $u \not\equiv 0$, we claim that $u > 0$ in $\mathbb{R}^N$. In fact, if $u(x_0) = 0$ for some $x_0 \in \mathbb{R}^N$, then $(-\Delta)^s u(x_0) = 0$ and by the representation formula

$$(-\Delta)^s u(x) = -\frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} dy,$$

we obtain that, at x_0,

$$\int_{\mathbb{R}^N} \frac{u(x_0+y) + u(x_0-y)}{|y|^{N+2s}} dy = 0,$$

yielding $u \equiv 0$, which leads a contradiction.The proof is completed. □

2.2 The modified problem (P_ε^*)

In this section, we shall prove the existence of positive solution for the modified problem (P_ε^*).

Similar to section 2.1, by the condition of nonlinearity g, we can easily prove that the functional J_ε verifies the mountain pass geometry. Thus there exists a sequence $\{u_n\} \subset H_\varepsilon$ such that $J_\varepsilon(u_n) \to c_\varepsilon$ and $J'_\varepsilon(u_n) \to 0$, where $c_\varepsilon := \inf_{\gamma\in\Gamma_\varepsilon} \max_{t\in[0,1]} J_\varepsilon(\gamma(t)) > 0$, where $\Gamma_\varepsilon := \{\gamma \in C([0,1], H_\varepsilon) : \gamma(0) = 0 \text{ and } J_\varepsilon(\gamma(1)) < 0\}$. As in the previous section 2.1, we have the equivalent characterization of c_ε

$$c_\varepsilon = \inf_{u\in H_\varepsilon\setminus\{0\}} \sup_{t\geq 0} J_\varepsilon(tu) = \inf_{u\in\mathcal{N}_\varepsilon} J_\varepsilon(u).$$

The main feature of the modified functional is that it satisfies the local compactness condition, which can be stated in the following results.

Lemma 2.4. *Let $c > 0$ and $\{u_n\}$ be a $(PS)_c$ sequence for J_ε, then $\{u_n\}$ is bounded.*

Proof. Assume that $\{u_n\}$ is a $(PS)_c$ sequence for J_ε, then $J_\varepsilon(u_n) \to c$, and $J'_\varepsilon(u_n) \to 0$. Therefore, we have

$$\begin{aligned}
c + o_n(1) &= J_\varepsilon(u_n) - \frac{1}{\theta}\langle J'_\varepsilon(u_n), u_n\rangle \\
&= \left(\frac{1}{2} - \frac{1}{\theta}\right) \int_{\mathbb{R}^N} (\varepsilon^{2s}|(-\Delta)^{\frac{s}{2}} u_n|^2 + V(x)u_n^2)dx \\
&\quad + \frac{1}{\theta}\int_{\mathbb{R}^N} [g(x,u_n)u_n - \theta G(x,u_n)]dx \\
&\geq \frac{\theta-2}{2\theta}\|u_n\|_\varepsilon^2 + \frac{1}{\theta}\int_{\mathbb{R}^N\setminus\Omega} [g(x,u_n)u_n - \theta G(x,u_n)]dx \\
&\geq \frac{\theta-2}{2\theta}\|u_n\|_\varepsilon^2 - \frac{\theta-2}{2\theta}\int_{\mathbb{R}^N\setminus\Omega} \left(\frac{P_0}{K} + \frac{Q_0}{L}\right) V(x)u_n^2 dx \\
&\geq \frac{\theta-2}{2\theta}\|u_n\|_\varepsilon^2 - \frac{\theta-2}{2\theta}\left(\frac{P_0}{K} + \frac{Q_0}{L}\right)\|u_n\|_\varepsilon^2 \\
&= \left(1 - \left(\frac{P_0}{K} + \frac{Q_0}{L}\right)\right)\frac{\theta-2}{2\theta}\|u_n\|_\varepsilon^2.
\end{aligned}$$

From $\theta > 2$ and $\frac{P_0}{K} + \frac{Q_0}{L} < \min\{\frac{1}{2}, \frac{\theta-2}{\theta}\}$, we have that $\{u_n\}$ is bounded. $\square$

Lemma 2.5. *([14]) Let $\{u_n\}$ be a bounded $(PS)_c$ sequence for J_ε, then for each $\zeta > 0$, there is a number $R = R(\zeta) > 0$ such that*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B_R(0)} \left(\varepsilon^{2s}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dy + V(x)|u_n|^2\right) dx < \zeta.$$

To establish the local compactness results for J_ε, we need an extension of a concentration compactness result proposed by Lions.

Lemma 2.6. *([16]) Let $\Omega \subseteq \mathbb{R}^N$ be an open subset and let $\{u_n\}$ be a sequence in $H^s(\Omega)$ weakly converging to u as $n \to \infty$ and such that*

$$|(-\Delta)^{\frac{s}{2}} u_n(x)|^2 dx \overset{*}{\rightharpoonup} \mu, \ |u_n(x)|^{2_s^*} dx \overset{*}{\rightharpoonup} \nu \ in\ \mathcal{M}(\mathbb{R}^N).$$

Then, either $u_n \to u$ in $L^{2_s^*}_{loc}(\mathbb{R}^N)$ or there exists a (at most countable) set of distinct points $\{x_j\}_{j\in J} \subset \overline{\Omega}$ and positive numbers $\{\nu_j\}_{j\in J}$ such that

$$\nu = |u(x)|^{2_s^*} dx + \sum_{j\in J} \nu_j \delta_{x_j}. \tag{2.7}$$

Moreover, if Ω is bounded, then there exists a positive measure $\tilde{\mu} \in \mathcal{M}(\mathbb{R}^N)$ with spt $\tilde{\mu} \subset \overline{\Omega}$ and positive numbers $\{\mu_j\}_{j\in J}$ such that

$$\mu = |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx + \tilde{\mu} + \sum_{j\in J} \mu_j \delta_{x_j}, \ S_* \nu_j^{2/2_s^*} \le \mu_j \tag{2.8}$$

Lemma 2.7. *The functional J_ε satisfies the Palais-Smale condition at any level $c < \varepsilon^N \frac{s}{NQ_0^{\frac{N-2s}{2s}}} S_*^{\frac{N}{2s}}$.*

Proof. Let $\{u_n\} \subset H_\varepsilon$ be such that $J_\varepsilon(u_n) \to c < \varepsilon^N \frac{s}{NQ_0^{\frac{N-2s}{2s}}} S_*^{\frac{N}{2s}}$, and $J'_\varepsilon(u_n) \to 0$. Then from Lemma 2.4, we have that $\{u_n\}$ is bounded in H_ε. Since $\langle J'_\varepsilon(u_n), u_n\rangle \to 0$, we get

$$\|u_n\|_\varepsilon^2 = \int_{\mathbb{R}^N} g(x,u_n)u_n dx + o_n(1). \tag{2.9}$$

Up to a subsquence, we may assume that

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } H_\varepsilon, \\ u_n \to u & \text{strongly in } L^s_{Loc}(\mathbb{R}^N) \text{ for any } s \in [1.2_s^*), \\ u_n(x) \to u(x) & \text{for } a.e.\ x \in \mathbb{R}^N. \end{cases} \tag{2.10}$$

Hence it is standard to check that for any $\varphi \in C_0^\infty(\mathbb{R}^N) \subset H_\varepsilon$,

$$\begin{aligned} \int_{\mathbb{R}^N} \varepsilon^{2s} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} \varphi dx &\to \int_{\mathbb{R}^N} \varepsilon^{2s} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi dx, \\ \int_{\mathbb{R}^N} V(x) u_n \varphi dx &\to \int_{\mathbb{R}^N} V(x) u \varphi dx, \\ \int_{\mathbb{R}^N} g(x,u_n)\varphi dx &\to \int_{\mathbb{R}^N} g(x,u)\varphi dx. \end{aligned} \tag{2.11}$$

By (2.11), the density of $C_0^\infty(\mathbb{R}^N)$ in H_ε, and $J'_\varepsilon(u_n) \to 0$, we obtain that the weak limit u is a critical point of J_ε, then

$$\|u\|_\varepsilon^2 = \int_{\mathbb{R}^N} g(x,u)u dx. \tag{2.12}$$

Next, we will prove two claims.

Claim 1 $\lim_{n\to\infty}\int_{\mathbb{R}^N} g(x,u_n)u_n dx = \int_{\mathbb{R}^N} g(x,u)u dx$.

this claim, (2.9) and (2.12) imply that $\|u_n\|_\varepsilon^2 \to \|u\|_\varepsilon^2$, which yields that $u_n \to u$ in H_ε.

To prove this claim, from Lemma 2.5, we know that: for each $\zeta > 0$, there exists $R = R(\zeta) > 0$ such that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N\setminus B_R(0)}\left(\varepsilon^{2s}\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dy + V(x)|u_n|^2\right)dx < \zeta. \tag{2.13}$$

By (2.13), $(g_2), (f_1), (f_2)$ and the Sobolev embedding, for n large enough, we get that

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B_R(0)} g(x,u_n)u_n dx &\le C_1\int_{\mathbb{R}^N\setminus B_R(0)}(u_n^2 + u_n^{2_s^*})dx\\ &\le C_2(\zeta + \zeta^{2_s^*/2}).\end{aligned} \tag{2.14}$$

On the other hand, choosing R large enough, we may assume that

$$\int_{\mathbb{R}^N\setminus B_R(0)} g(x,u)u dx \le \zeta.$$

Therefore, from the last inequality and (2.14), we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N\setminus B_R(0)} g(x,u_n)u_n dx = \int_{\mathbb{R}^N\setminus B_R(0)} g(x,u)u dx. \tag{2.15}$$

By the definition of g, we obtain that

$$g(x,u_n)u_n \le P_0 f(u_n)u_n + Q_0 a^{2_s^*} + \left(\frac{P_0}{K} + \frac{Q_0}{L}\right)V_0 u_n^2 \ \forall x \in \mathbb{R}^N\setminus\Omega.$$

Since the set $B_R(0)\cap(\mathbb{R}^N\setminus\Omega)$ is bounded, we can use the above estimates, $(g_1), (g_2)$,(2.11) and Lebesgue's theorem to conclude that

$$\lim_{n\to\infty}\int_{B_R(0)\cap(\mathbb{R}^N\setminus\Omega)} g(x,u_n)u_n dx = \int_{B_R(0)\cap(\mathbb{R}^N\setminus\Omega)} g(x,u)u dx. \tag{2.16}$$

Claim 2 $u_n \to u$ in $L^{2_s^*}(\Omega)$. If Claim 2 holds, by $(g_2), (f_1), (f_2)$, (2.11) and Lebesgue's theorem, we can obtain that

$$\lim_{n\to\infty}\int_{B_R(0)\cap\Omega} g(x,u_n)u_n dx = \int_{B_R(0)\cap\Omega} g(x,u)u dx.$$

Hence, Claim 1 follows from the above expression, (2.15) and (2.16).

It remains to prove Claim 2. By Phrokorovs theorem (see Bogachev [6], Theorem 8.6.2) we may suppose that there are positive measures μ, ν such that

$$|(-\Delta)^{\frac{s}{2}}u_n(x)|^2 dx \overset{*}{\rightharpoonup} \mu, \ |u_n(x)|^{2_s^*}dx \overset{*}{\rightharpoonup} \nu. \tag{2.17}$$

Hence, by Lemma 2.6 we have an at most countable index set of distinct points $\{x_j\}_{j\in J}$, $\{\mu_j\}_{j\in J}$,$\{\nu_j\}_{j\in J} \subset (0,\infty)$, and positive measures $\tilde{\mu}$ with support contained in Ω such that

$$\mu = |(-\Delta)^{\frac{s}{2}}u(x)|^2dx + \tilde{\mu} + \sum_{j\in J}\mu_j\delta_{x_j},\ \nu = |u(x)|^{2_s^*}dx + \sum_{j\in J}\nu_j\delta_{x_j} \text{ and } S_*\nu_j^{2/2_s^*} \leq \mu_j, \tag{2.18}$$

for all $j \in J$, where δ_{x_j} is the Dirac mass at $x_j \in \mathbb{R}^N$.

It suffices to show that $\{x_j\}_{j\in J} \cap \Omega = \emptyset$. If not, suppose that $x_j \in \Omega$ for some $j \in J$. For $\rho > 0$, define the function $\psi_\rho(x) := \psi_\rho(\frac{x-x_j}{\rho})$ where $\psi \in C_0^\infty(\overline{\mathbb{R}^N}, [0,1])$ is such that $\psi \equiv 1$ on $B_{\frac{1}{2}}(0), \psi \equiv 0$ on $\mathbb{R}^N \backslash B_1(0)$. We assume that ρ is chosen in such way that the support of ψ_ρ is contained in Ω.

Since $\{u_n\psi_\rho\}$ is bounded, $\langle J'_\varepsilon(u_n), u_n\psi_\rho\rangle = o_n(1)$, we have

$$\int_{\mathbb{R}^N} \varepsilon^{2s}(-\Delta)^{\frac{s}{2}}u_n(-\Delta)^{\frac{s}{2}}u_n\psi_\rho dx \leq \int_{\mathbb{R}^N} P_0 f(u_n)u_n\psi_\rho dx + \int_{\mathbb{R}^N} Q_0|u_n|^{2_s^*}\psi_\rho dx + o_n(1). \tag{2.19}$$

Using the fact that ψ_ρ has compact support and f has subcritical growth, we have

$$\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N} P_0 f(u_n)u_n\psi_\rho dx = \lim_{\rho\to 0}\int_{\mathbb{R}^N} P_0 f(u)u\psi_\rho dx = 0.$$

And, by (1.1) and (1.2), we can write

$$\begin{aligned}
&\int_{\mathbb{R}^N} \varepsilon^{2s}(-\Delta)^{\frac{s}{2}}u_n(-\Delta)^{\frac{s}{2}}u_n\psi_\rho dx \\
={}& \int_{\mathbb{R}^N} \varepsilon^{2s}u_n(x)(-\Delta)^{\frac{s}{2}}u_n(x)(-\Delta)^{\frac{s}{2}}\psi_\rho(x)dx + \int_{\mathbb{R}^N} \varepsilon^{2s}|(-\Delta)^{\frac{s}{2}}u_n(x)|^2\psi_\rho(x)dx \\
&- 2\int_{\mathbb{R}^N} \varepsilon^{2s}(-\Delta)^{\frac{s}{2}}u_n(x)\int_{\mathbb{R}^N} \frac{(u_n(x)-u_n(y))(u_n(x)\psi_\rho(x)-u_n(y)\psi_\rho(y))}{|x-y|^{N+s}}dxdy.
\end{aligned} \tag{2.20}$$

Now, we show that

$$\lim_{\rho\to 0}\lim_{n\to\infty}\left|\int_{\mathbb{R}^N} u_n(x)(-\Delta)^{\frac{s}{2}}u_n(x)(-\Delta)^{\frac{s}{2}}\psi_\rho(x)dx\right| = 0 \tag{2.21}$$

and

$$\lim_{\rho\to 0}\lim_{n\to\infty}\left|\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}}u_n(x)\int_{\mathbb{R}^N} \frac{(u_n(x)-u_n(y))(u_n(x)\psi_\rho(x)-u_n(y)\psi_\rho(y))}{|x-y|^{N+s}}dxdy\right| = 0. \tag{2.22}$$

If (2.21) and (2.22) hold, we can use (2.17), (2.18), and take limits as $n \to \infty$ and $\rho \to 0$ in (2.20) to obtain that $Q_0\nu_j \geq \varepsilon^{2s}\mu_j$. The proof of (2.21) and (2.22) is standard which can found in [5, Lemma 2.8 and Lemma 2.9], and we omit it. Then from the last statement in (2.18), we get $\nu_j \geq \varepsilon^N \frac{1}{Q_0^{\frac{N}{2s}}} S_*^{\frac{N}{2s}}$, and hence we can use (g_3), (f_2), (f_3), (P), (Q) and (Ω_1) to obtain

$$\begin{aligned}
c &= J_\varepsilon(u_n) - \frac{1}{2}\langle J'_\varepsilon(u_n), u_n\rangle + o_n(1) &\quad (2.23)\\
&= \int_{\mathbb{R}^N\setminus\Omega} \left(\frac{1}{2} g(x,u_n)u_n - G(x,u_n)\right) dx \\
&\quad + \int_\Omega P(x)\left(\frac{1}{2} f(u_n)u_n - F(u_n)\right) dx + \frac{s}{N}\int_\Omega Q(x) u_n^{2_s^*} dx + o_n(1) \\
&\geq \alpha \int_\Omega \left(\frac{1}{2} - \frac{1}{\theta}\right) f(u_n)u_n dx + \frac{s}{N}\int_\Omega (Q(x)-Q_0)u_n^{2_s^*} dx + \frac{s}{N}\int_\Omega Q_0 u_n^{2_s^*} dx + o_n(1) \\
&\geq \alpha \left(\frac{1}{2} - \frac{1}{\theta}\right) C_0 \int_\Omega u_n^q dx - \frac{s}{N}\int_\Omega (Q_0 - Q(x))u_n^{2_s^*} dx + \frac{s}{N}\int_\Omega Q_0 u_n^{2_s^*} dx + o_n(1) \\
&\geq \frac{s}{N}\int_\Omega Q_0 u_n^{2_s^*} dx + o_n(1). &\quad (2.24)
\end{aligned}$$

Since $Q \in L^\infty(\mathbb{R}^N), Q_0 = \sup_{\mathbb{R}^N} Q$ and $\{u_n\}$ is bounded, then the last inequality holds for sufficiently large C_0. In [13], although it is not stated in the condition (f_2) that the C_0 used is also sufficiently large. Taking the limit and from (2.18) one has

$$c \geq \frac{s}{N} Q_0 \sum_{\{j\in J: x_j\in\Omega\}} \psi_\rho(x_j)\nu_j = \frac{s}{N} Q_0 \sum_{\{j\in J: x_j\in\Omega\}} \nu_j \geq \varepsilon^N \frac{s}{N Q_0^{\frac{N-2s}{2s}}} S_*^{\frac{N}{2s}}. \quad (2.25)$$

which yields a contradiction. Hence, Claim 2 holds true and the lemma is proved. □

Lemma 2.8. *The functional J_ε possesses a positive critical point $u_\varepsilon \in H_\varepsilon$ such that $J_\varepsilon(u_\varepsilon) = c_\varepsilon$ for ε small.*

Proof. Let $x_0 \in \Omega$ be such that $V(x_0) = V_0$. By Lemma 2.3, we know that problem (P_{V_0}) has a positive ground state solution. Let $w \in H_0$ be a least energy solution of problem (P_{V_0}), then,

$$c_0 := J_0(w) = \inf_{u\in H_0\setminus\{0\}} \sup_{t\geq 0} J_0(tu) = \inf_{u\in\mathcal{N}_0} J_0(u).$$

Set $\tilde{w}(x) := w\left(\frac{x-x_0}{\varepsilon}\right)$. Then $c_\varepsilon \leq \sup_{t>0} J_\varepsilon(t\tilde{w}) = J_\varepsilon(t_0\tilde{w})$ for some $t_0 > 0$. We have

$$\begin{aligned}
J_\varepsilon(t_0\tilde{w}) &= \varepsilon^N \left[\frac{t_0^2}{2}\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} w|^2 + V(x_0+\varepsilon x)w^2)dx - \int_{\mathbb{R}^N} G(x_0+\varepsilon x, t_0 w)dx\right] \\
&= \varepsilon^N \left[J_0(t_0 w) + \frac{t_0^2}{2}\int_{\mathbb{R}^N} (V(x_0+\varepsilon x) - V_0)w^2 dx + \int_{\mathbb{R}^N} P_0 F(t_0 w)dx \right.\\
&\qquad \left. + \frac{Q_0}{2_s^*}\int_{\mathbb{R}^N} |t_0 w|^{2_s^*} dx - \int_{\mathbb{R}^N} G(x_0+\varepsilon x, t_0 w)dx\right].
\end{aligned}$$

From the Lebesgue's theorem we have $\int_{\mathbb{R}^N}(V(x_0+\varepsilon x)-V_0)w^2dx \to 0$ as $\varepsilon \to 0^+$. From (f_1), (f_2) and (g_2) we have $G(x,t_0w) \le C_1|w|^2 + C_2|w|^{2_s^*}$. Again, by the Lebesgue's theorem the following convergence holds

$$\int_{\mathbb{R}^N} G(x_0+\varepsilon x, t_0w)dx \to \int_{\mathbb{R}^N} P_0F(t_0w) + \frac{Q_0}{2_s^*}\int_{\mathbb{R}^N} |t_0w|^{2_s^*}dx,$$

as $\varepsilon \to 0^+$. Hence

$$c_\varepsilon \le J_\varepsilon(t_0\tilde{w}) = \varepsilon^N(J_0(t_0w)+o(1)) \le \varepsilon^N(c_0+o(1)). \tag{2.26}$$

By Lemma 2.1, we get $c_\varepsilon < \varepsilon^N \frac{s}{NQ_0^{\frac{N-2s}{2s}}} S_*^{\frac{N}{2s}}$ for ε small enough. Since J_ε satisfies the mountain-pass geometry conditions. Thus there exists a sequence $\{u_n\} \subset H_0$ such that $J_\varepsilon(u_n) \to c_\varepsilon$ and $J'_\varepsilon(u_n) \to 0$. From Lemma 2.7, we obtain J_ε satisfies the Palais-Smale condition at level c_ε. Then, the functional J_ε possesses a nontrivial critical point $u_\varepsilon \in H_\varepsilon$ such that $J_\varepsilon(u_\varepsilon) = c_\varepsilon$, similar to the proof of Lemma 2.3, we get $u_\varepsilon \in H_\varepsilon$ is a positive critical point such that $J_\varepsilon(u_\varepsilon) = c_\varepsilon$. □

2.3 Proof of theorem 1.1

Next we shall prove our main result. The idea is to show that the solution obtained in Lemma 2.8 satisfy the estimate $u_\varepsilon \le a$, $\forall x \in \Omega$ for ε small enough. This fact implies that the solution is indeed solution of the original problem (P_ε).

Lemma 2.9. *There is $C > 0$ such that*

$$\int_{\mathbb{R}^N} (\varepsilon^{2s}|(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2 + V(x)|u_\varepsilon|^2)dx \le C\varepsilon^N.$$

Proof. Indeed, we have $\langle J'_\varepsilon(u_\varepsilon), u_\varepsilon\rangle = 0$, that is,

$$\int_{\mathbb{R}^N} (\varepsilon^{2s}|(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2 + V(x)|u_\varepsilon|^2)dx = \int_{\mathbb{R}^N} g(x,u_\varepsilon)u_\varepsilon dx.$$

By (2.26) and (g_3), we have

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^N} (\varepsilon^{2s}|(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2 + V(x)|u_\varepsilon|^2)dx \\
&= J_\varepsilon(u_\varepsilon) + \int_{\mathbb{R}^N} G(x,u_\varepsilon)dx \\
&\le \varepsilon^N(c_0+o(1)) + \frac{1}{\theta}\int_\Omega g(x,u_\varepsilon)u_\varepsilon dx + \frac{1}{2}\left(\frac{P_0}{K}+\frac{Q_0}{L}\right)\int_{\mathbb{R}^N} V(x)|u_\varepsilon|^2dx \\
&\le C_1\varepsilon^N + \left(\frac{1}{\theta}+\frac{1}{2}\left(\frac{P_0}{K}+\frac{Q_0}{L}\right)\right)\int_{\mathbb{R}^N} (\varepsilon^{2s}|(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2 + V(x)|u_\varepsilon|^2)dx.
\end{aligned}$$

Therefore, we have

$$\left(\frac{1}{2}-\frac{1}{\theta}-\frac{1}{2}\left(\frac{P_0}{K}+\frac{Q_0}{L}\right)\right)\int_{\mathbb{R}^N}(\varepsilon^{2s}|(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2+V(x)|u_\varepsilon|^2)dx\le C\varepsilon^N.$$

Moreover $\frac{1}{2}-\frac{1}{\theta}-\frac{1}{2}\left(\frac{P_0}{K}+\frac{Q_0}{L}\right)=\frac{1}{2}\left(\frac{\theta-2}{\theta}-\left(\frac{P_0}{K}+\frac{Q_0}{L}\right)\right)>0$ and the proof is completed. □

Lemma 2.10. *If $\varepsilon_n\to 0^+$ and $\{x_n\}\subset\bar{\Omega}$ are such that $u_{\varepsilon_n}(x_n)\ge\gamma>0$, then $\lim\limits_{n\to\infty}V(x_n)=V_0$.*

Proof. Assume by contradiction, passing to a subsequence, that $x_n\to\bar{x}\in\bar{\Omega}$ and $V(\bar{x})>V_0$. Let $v_n(x):=u_{\varepsilon_n}(x_n+\varepsilon_n x)$. Obviously, $v_n\in H_\varepsilon$ satisfy the following equation

$$(-\Delta)^s v_n+V(x_n+\varepsilon_n x)v_n=g(x_n+\varepsilon_n x,v_n)\ \text{ in }\mathbb{R}^N. \tag{2.27}$$

The associated energy functional is given by

$$J_n(u)=\frac{1}{2}\int_{\mathbb{R}^N}(|(-\Delta)^{\frac{s}{2}}u|^2+V(\varepsilon_n x+x_n)u^2)dx-\int_{\mathbb{R}^N}G(\varepsilon_n x+x_n,u)dx.$$

From Lemma 2.9, we have that $\{v_n\}$ is bounded in H_ε and therefore $v_n\rightharpoonup v$ in H_ε for some $v\in H_\varepsilon$. From (f_1)–(f_3), it is easy to get that $J_0(tv_n)>0$ for $t>0$ small enough and $J_0(tv_n)\to-\infty$ as $t\to\infty$. Thus, there exists $t_n>0$ such that $J_0(t_nv_n)=\max\limits_{t\ge0}J_0(tv_n)$. Set $\tilde{v}_n:=t_nv_n$, therefore, $c_0\le J_0(\tilde{v}_n)$. Since $\{v_n\}$ satisfy equation (2.27), we have $J_n(v_n)=\max_{t\ge0}J_n(tv_n)$, thus

$$\begin{aligned}
c_0\le J_0(\tilde{v}_n)&=\frac{1}{2}\int_{\mathbb{R}^N}(|(-\Delta)^{\frac{s}{2}}\tilde{v}_n|^2+V_0\tilde{v}_n^2)dx-\int_{\mathbb{R}^N}P_0F(\tilde{v}_n)dx-\frac{Q_0}{2_s^*}\int_{\mathbb{R}^N}|\tilde{v}_n|^{2_s^*}dx\\
&\le\frac{1}{2}\int_{\mathbb{R}^N}(|(-\Delta)^{\frac{s}{2}}\tilde{v}_n|^2+V(\varepsilon_nx+x_n)\tilde{v}_n^2)dx-\int_{\mathbb{R}^N}P(\varepsilon_nx+x_n)F(\tilde{v}_n)dx\\
&\quad-\frac{1}{2_s^*}\int_{\mathbb{R}^N}Q(\varepsilon_nx+x_n)|\tilde{v}_n|^{2_s^*}dx\\
&\le\frac{t_n^2}{2}\int_{\mathbb{R}^N}(|(-\Delta)^{\frac{s}{2}}v_n|^2+V(\varepsilon_nx+x_n)v_n^2)dx-\int_{\mathbb{R}^N}G(\varepsilon_nx+x_n,t_nv_n)dx\\
&=J_n(t_nv_n)\le J_n(v_n)=\varepsilon_n^{-N}J_{\varepsilon_n}(u_{\varepsilon_n})\le c_0+o(1),
\end{aligned}$$

which implies $\lim_{n\to\infty}J_0(\tilde{v}_n)=c_0$, where the last inequality is from (2.26). Moreover, $\{\tilde{v}_n\}$ is bounded and $\tilde{v}_n\rightharpoonup\tilde{v}$. We claim that $\{\tilde{v}_n\}$ satisfies the following limits

$$J_0(\tilde{v}_n)\to c_0\text{ and }J_0'(\tilde{v}_n)\to 0.$$

In fact, using Ekeland's variational Principle in [10], there exists a sequence $\{\nu_n\}\subset\mathcal{N}_0$ satisfying $\nu_n=\tilde{v}_n+o_n(1)$, $J_0(\nu_n)\to c_0$ and $J_0'(\nu_n)-\lambda_n\Phi'(\nu_n)=o_n(1)$, where λ_n is a real number and $\Phi(\nu_n)=\langle J_0'(\nu_n),\nu_n\rangle$. Thus, by the definition of $\Phi(\nu_n)$ and $\{\nu_n\}\subset\mathcal{N}_0$, we have that

$$\langle\Phi'(\nu_n),\nu_n\rangle=\int_{\mathbb{R}^N}P_0(f(\nu_n)\nu_n-f'(\nu_n)|\nu_n|^2)dx-(2_s^*-2)\int_{\mathbb{R}^N}Q_0|\nu_n|^{2_s^*}dx$$

$$\leq \int_{\mathbb{R}^N} P_0(f(\nu_n)\nu_n - f'(\nu_n)|\nu_n|^2)dx. \tag{2.28}$$

Since $\{\nu_n\}$ is bounded and $\nu_n \not\to 0$, Lemma 2.2 guarantees the existence of a sequence $\{y_n\} \subset \mathbb{R}^N$ such that $\tilde{\nu}_n = \nu_n(\cdot + y_n)$ is a bounded sequence in H_0 and $\tilde{\nu}_n \rightharpoonup \tilde{\nu}$ for some $\tilde{\nu} \neq 0$. Hence, there exists a subset $\Lambda \subset \mathbb{R}^N$ having positive measure, such that $\tilde{\nu} > 0$ a.e. in Λ. Assume by contradiction that $\limsup_{n\to\infty}\langle \Phi'(\nu_n), \nu_n\rangle = 0$. Then, taking into account (2.28), (f_4) and Fatou's Lemma, we get $0 > \int_\Lambda (f(\tilde{\nu})\tilde{\nu} - f'(\tilde{\nu})|\tilde{\nu}|^2) \geq 0$ which gives a contradiction. Hence $\limsup_{n\to\infty}\langle \Phi'(\nu_n), \nu_n\rangle < 0$, implying that $\lambda_n = o_n(1)$, thus $J_0(\nu_n) \to c_0$, $J_0'(\nu_n) \to 0$. Without loss of generalization, we may assume that $J_0(\tilde{v}_n) \to c_0$, $J_0'(\tilde{v}_n) \to 0$. Thus, $J_0'(\tilde{v}) = 0$, by Fatou's Lemma we get

$$\begin{aligned}
c_0 \leq J_0(\tilde{v}) &= J_0(\tilde{v}) - \frac{1}{\theta}\langle J_0'(\tilde{v}), \tilde{v}\rangle \\
&= \left(\frac{1}{2} - \frac{1}{\theta}\right)\left(\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}}\tilde{v}|^2 + V_0\tilde{v}^2)dx\right) \\
&\quad + P_0 \int_{\mathbb{R}^N} \left(\frac{1}{\theta} f(\tilde{v})\tilde{v} - F(\tilde{v})\right) dx + \left(\frac{1}{\theta} - \frac{1}{2_s^*}\right) \int_{\mathbb{R}^N} Q_0 |\tilde{v}|^{2_s^*} dx \\
&\leq \liminf_{n\to\infty} \left(\frac{1}{2} - \frac{1}{\theta}\right)\left(\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}}\tilde{v}_n|^2 + V_0\tilde{v}_n^2)dx\right) \\
&\quad + P_0 \int_{\mathbb{R}^N} \left(\frac{1}{\theta} f(\tilde{v}_n)\tilde{v} - F(\tilde{v}_n)\right) dx + \left(\frac{1}{\theta} - \frac{1}{2_s^*}\right) \int_{\mathbb{R}^N} Q_0 |\tilde{v}_n|^{2_s^*} dx \\
&= \liminf_{n\to\infty} \left(J_0(\tilde{v}_n) - \frac{1}{\theta}\langle J_0'(\tilde{v}_n), \tilde{v}_n\rangle\right) \leq c_0.
\end{aligned}$$

Then, $\lim\limits_{n\to\infty} \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}}\tilde{v}_n|^2 + V_0\tilde{v}_n^2)dx = \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}}\tilde{v}|^2 + V_0\tilde{v}^2)dx$. Hence, $\tilde{v}_n \to \tilde{v}$ in $H^s(\mathbb{R}^N)$.

Taking into account that $V(\bar{x}) > V_0, P(\bar{x}) \leq P_0, Q(\bar{x}) \leq Q_0, \tilde{v}_n \to \tilde{v}$ and Fatou's Lemma, we obtain

$$\begin{aligned}
c_0 = J_0(\tilde{v}) &< \liminf_{n\to\infty} \left\{ \frac{1}{2} \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}}\tilde{v}_n|^2 + V(\varepsilon_n x + x_n)\tilde{v}_n^2)dx \right. \\
&\quad \left. - \int_{\mathbb{R}^N} P(\varepsilon_n x + x_n)F(\tilde{v}_n)dx - \frac{1}{2_s^*}\int_{\mathbb{R}^N} Q(\varepsilon_n x + x_n)|\tilde{v}_n|^{2_s^*}dx \right\} \\
&\leq \liminf_{n\to\infty} \left\{ \frac{t_n^2}{2} \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}}v_n|^2 + V(\varepsilon_n x + x_n)v_n^2)dx \right. \\
&\quad \left. - \int_{\mathbb{R}^N} G(\varepsilon_n x + x_n, t_n v_n)dx \right\} \\
&= \liminf_{n\to\infty} J_n(t_n v_n) \leq \liminf_{n\to\infty} J_n(v_n) = \liminf_{n\to\infty} \varepsilon_n^{-N} J_{\varepsilon_n}(u_{\varepsilon_n}) \leq c_0 + o(1),
\end{aligned}$$

which yields a contradiction. Thus the proof is completed. □

Lemma 2.11. *There holds* $\lim\limits_{\varepsilon\to 0^+} m_\varepsilon = 0$, *where* $m_\varepsilon := \max\limits_{\partial\Omega} u_\varepsilon$.

Proof. Assume by contradiction that $m_\varepsilon \nrightarrow 0$. Let $x_\varepsilon \in \partial\Omega \subset \bar{\Omega}$ be such that $u_\varepsilon(x_\varepsilon) = m_\varepsilon$. Therefore, up to a subsequence we have $u_{\varepsilon_n}(x_{\varepsilon_n}) \geq \gamma > 0$ and $x_{\varepsilon_n} \to x_0 \in \partial\Omega$, by Lemma 2.10 we have

$$\min_{\partial\Omega} V \leq \lim_{n\to\infty} V(x_{\varepsilon_n}) = V(x_0) = V_0 < \min_{\partial\Omega} V,$$

which gets a contradiction. $\square$

Proof of theorem 1.1. Let u_ε be a positive critical point for J_ε. From Lemma 2.11, there exists ε_0 such that for any $\varepsilon \in (0, \varepsilon_0)$, $m_\varepsilon < a$. Therefore $u_\varepsilon(x) < a$ for $x \in \partial\Omega$. Thus, in view of the maximum principle, we obtain

$$u_\varepsilon(x) \leq a \text{ for } x \in \Omega.$$

Taking $(u_\varepsilon - a)_+ = \max\{u_\varepsilon - a, 0\}$ as a test function for J_ε, we have

$$\begin{aligned} 0 = J'_\varepsilon(u_\varepsilon)((u_\varepsilon - a)_+) = &\int_{\mathbb{R}^N\setminus\Omega} \varepsilon^{2s}|(-\Delta)^{\frac{s}{2}}(u_\varepsilon - a)_+|^2 dx \\ &+ \int_{\mathbb{R}^N\setminus\Omega} c(x)(u_\varepsilon - a)_+^2 + c(x)a(u_\varepsilon - a)_+ dx \end{aligned} \tag{2.29}$$

where $c(x) = V(x) - \frac{g(x,u_\varepsilon)}{u_\varepsilon}$. Moreover, for $x \in \mathbb{R}^N\setminus\Omega$, we get $\frac{g(x,u_\varepsilon)}{u_\varepsilon} \leq \left(\frac{P_0}{K} + \frac{Q_0}{L}\right) V_0$. Hence, $c(x) > 0$ for $x \in \mathbb{R}^N\setminus\Omega$. So every term in the last identity of (2.29) is 0. Therefore, $(u_\varepsilon - a)_+ = 0$ and $u_\varepsilon(x) \leq a$ for $x \in \mathbb{R}^N\setminus\Omega$. Thus, $g(x, u_\varepsilon) = P(x)f(u_\varepsilon) + Q(x)|u_\varepsilon|^{2_s^*-2}u_\varepsilon$ and u_ε is a solution of (P_ε). Similar arguments as [13], we can obtained that if $\eta_\varepsilon \in \mathbb{R}^N$ denotes the global maximum point of u_ε, then

$$u_\varepsilon(x) \leq \frac{C\varepsilon^{N+2s}}{\varepsilon^{N+2s} + |x - \eta_\varepsilon|^{N+2s}}.$$

$\square$

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