# The Uniqueness of Minimizers for L2 -Subcritical Inhomogeneous Variational Problems with A Spatially Decaying Nonlinearity

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**Abstract**

By constructing various Pohozaev identities, we study the uniqueness of minimizers for L 2 -subcritical inhomogeneous variational problems with spatially decaying nonlinear terms, which contain $x$=0 as a singular point.

# The Uniqueness of Minimizers for $L^2$-Subcritical Inhomogeneous Variational Problems with A Spatially Decaying Nonlinearity

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**Abstract**

By constructing various Pohozaev identities, we study the uniqueness of minimizers for $L^2$-subcritical inhomogeneous variational problems with spatially decaying nonlinear terms, which contain $x = 0$ as a singular point.

**Keywords:** $L^2$-subcritical variational problems; Spatially decaying nonlinearity; Minimizers; Uniqueness

## 1 Introduction

In this paper, we study the minimizers of the following $L^2$-subcritical constraint inhomogeneous variational problem

$$I(M) := \inf_{\{u\in\mathcal{H}, \|u\|_2^2=1\}} E_M(u),\ M > 0, \tag{1.1}$$

where the energy functional $E_M(u)$ contains a spatially decaying nonlinearity and is defined by

$$E_M(u) := \int_{\mathbb{R}^N} \big(|\nabla u|^2 + V(x)|u|^2\big)dx - \frac{2M^{\frac{p-1}{2}}}{p+1} \int_{\mathbb{R}^N} \frac{|u|^{p+1}}{|x|^b}dx,\ N \geq 1, \tag{1.2}$$

and the space $\mathcal{H}$ is defined as

$$\mathcal{H} := \Big\{u(x) \in H^1(\mathbb{R}^N) :\ \int_{\mathbb{R}^N} V(x)|u(x)|^2 < \infty\Big\}$$

with the associated norm $\|u\|_{\mathcal{H}} = \Big\{\int_{\mathbb{R}^N} \Big(|\nabla u(x)|^2 + |u(x)|^2 + V(x)|u(x)|^2\Big)dx\Big\}^{\frac{1}{2}}$. Here positive constants $b > 0$ and $p > 0$ of (1.1) satisfy

$$0 < b < \min\{2, N\},\quad 1 < p < 1 + \frac{4-2b}{N},\quad \text{where}\quad N \geq 1. \tag{1.3}$$

We always assume that the trapping potential $V(x) \geq 0$ satisfies

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$(V1)$. $V(x) \in L^\infty_{loc}(\mathbb{R}^N) \cap C^\alpha_{loc}(\mathbb{R}^N)$ with $\alpha \in (0,1)$, $\{x \in \mathbb{R}^N : V(x) = 0\} = \{0\}$ and $\lim\limits_{|x|\to\infty} V(x) = \infty$.

The variational problem (1.1) arises in various physical contexts such as the laser beam propagation in optical fibers, Bose Einstein condensation (BEC), and nonlinear optics (cf. [1, 3, 29]), where the constant $M > 0$ usually represents the attractive interaction strength and $V(x) \geq 0$ represents the external potential. Due to the singularity of $|x|^{-b}$, the variational problem (1.1) and its associated elliptic equation have attracted a lot of attentions over the past few years, see [2, 8, 9, 13, 14, 17, 31] and the references therein.

When $b = 0$, (1.1) is a homogeneous constraint variational problem, for which there are many existing results of (1.1) (cf. [6, 15, 20, 21, 22, 23, 27, 28, 30, 33]). To be more precise, when $p > 1 + \frac{4}{N}$, one can use the energy estimates to obtain the nonexistence of minimizers for (1.1) with $b = 0$ as soon as $M > 0$ (cf. [6, 7]), which is essentially in the $L^2$-supercritical case. However, if $1 < p < 1 + \frac{4}{N}$, (1.1) with $b = 0$ is in the $L^2$-subcritical case and admits generally minimizers for all $M \in (0, \infty)$. In this case, the uniqueness, symmetry breaking and concentration behavior of minimizers were investigated recently as $M \to \infty$, see [27, 30] and the references therein. As for the case where $p = 1 + \frac{4}{N}$, (1.1) with $b = 0$ reduces to the $L^2$-critical case, which was addressed widely by Guo and his collaborators, see [20, 21, 22, 23] and the references therein.

When $b \neq 0$, (1.1) contains the singular nonlinear term $\frac{|u|^{p+1}}{|x|^b}$. Note that similar inhomogeneous problems were analyzed recently in [10, 11, 30] and the references therein. However. the above mentioned works focused mainly on the case where $m(x)$ satisfies $m(x) \in L^\infty(\mathbb{R}^N)$ without any singular point. On the other hand, Ardila and Dinh obtained recently in [2] the existence of minimizers and the stability of the standing waves, for which they studied the associated constraint variational problem (1.1), in the $L^2$-subcritical case where the harmonic potential satisfies $V(x) = \gamma^2|x|^2(\gamma > 0)$, $b > 0$ and $p > 0$ satisfy (1.3).

Inspired by the above works, in this paper we shall mainly studied the uniqueness of positive minimizers for (1.1) in the subcritical case. We assume that $u_M$ is a minimizer of $I(M)$ for any $M > 0$. It then follows from the variational theory that $u_M$ satisfies the following Euler-Lagrange equation

$$-\Delta u_M + V(x)u_M - M^{\frac{p-1}{2}} \frac{u_M^p}{|x|^b} = \mu_M u_M \ \text{ in } \ \mathbb{R}^N, \tag{1.4}$$

where $\mu_M \in \mathbb{R}$ is a suitable Lagrange multiplier and satisfies

$$\mu_M = I(M) - \frac{p-1}{p+1} M^{\frac{p-1}{2}} \int_{\mathbb{R}^N} \frac{|u_M|^{p+1}}{|x|^b} dx. \tag{1.5}$$

Note that $E_M(u) = E_M(|u|)$ holds for any $u \in \mathcal{H}$[26, Theorem 6.17], which implies that $|u_M|$ is also a minimizer of $I(M)$. By the strong maximum principle, we can further derive from (1.4) that $|u_M| > 0$ holds in $\mathbb{R}^N$. Therefore, without loss of generality, we only consider the positive minimizers $u_M > 0$ of $I(M)$.

We next introduce the following associted limit equation

$$-\Delta w + w - \frac{w^p}{|x|^b} = 0 \ \text{ in } \ \mathbb{R}^N, \quad w \in H^1(\mathbb{R}^N). \tag{1.6}$$

Note that (1.6) admits a unique positive solution, which is radially symmetric (cf. [4, 17, 18, 25, 32]). We always denote this unique positive solution by $w(x)$ in this paper.

Under the assumptions $(V1)$ and (1.3), let $u_k$ be a positive minimizer of $I(M_k)$, where $M_k \to \infty$ as $k \to \infty$. Then we have the following conclusions (cf. [16]):

1. $u_k$ satisfies

$$w_k(x) := \epsilon_k^{\frac{N}{2}} u_k(\epsilon_k x) \to \frac{w(x)}{\sqrt{a^*}} \text{ uniformly in } L^\infty(\mathbb{R}^N) \text{ as } k \to \infty, \tag{1.7}$$

where $\epsilon_k := \left(\frac{M_k}{a^*}\right)^{-\frac{p-1}{4-N(p-1)-2b}} > 0$, $a^* := \|w\|_2^2 > 0$, and $w > 0$ is the unique positive solution of (1.6).

2. $u_k$ decays exponentially in the sense that for sufficiently large $k > 0$,

$$w_k(x) \le Ce^{-\sqrt{\theta}|x|} \text{ and } |\nabla w_k(x)| \le Ce^{-\theta|x|} \text{ as } |x| \to \infty, \tag{1.8}$$

where $0 < \theta < 1$ and $C > 0$ are independent of $k > 0$.

3. The Lagrange multiplier $\mu_k$ satisfies

$$\lim_{k\to\infty} \epsilon_k^2 \mu_k = -1. \tag{1.9}$$

To state our results, we need additional assumptions of $V(x)$, for which we define

**Definition 1.1.** *The function $h(x) > 0$ in $\mathbb{R}^N$ is homegeneous of degree $d \in \mathbb{R}^+$ (with respect to the orign), if there exists some $d > 0$ such that*

$$h(tx) = t^d h(x) \text{ in } \mathbb{R}^N \text{ for any } t > 0. \tag{1.10}$$

This definition implies that the homogeneous function $h(x) \in C(\mathbb{R}^N)$ of degree $d > 0$ satisfies $0 \le h(x) \le C|x|^d$ in $\mathbb{R}^N$, where $C = \max_{x\in\partial B_1(0)} h(x)$. Moreover, 0 is the unique minimum point of $h(x)$, if $\lim_{|x|\to\infty} h(x) = \infty$. Following the above definition and the assumption $(V1)$, we next assume that

$(V2)$. $V(x) \in C^1(\mathbb{R}^2)$ satisfies $\{x \in \mathbb{R}^N : V(x) = 0\}$=$\{0\}$,

$$|V(x)) \le Ce^{\alpha|x|}, \quad |\nabla V(x)| \le Ce^{\alpha|x|} \text{ for some } \alpha > 0 \text{ as } |x| \to \infty, \tag{1.11}$$

and for $m = 1, 2, \cdots, N$,

$$V(x) = h(x) + o(|x|^d), \frac{\partial V(x)}{\partial x_m} = \frac{\partial h(x)}{\partial x_m} + o(|x|^{d-1}) \text{ as } |x| \to 0, \tag{1.12}$$

where $0 \le h(x) \in C^1(\mathbb{R}^N)$ is a homogeneous funtion of degree $d > 0$ and satisfies $\lim_{|x|\to\infty} h(x) = +\infty$.

Motivated by [5, 12, 20], we study the following uniqueness of positive minimizers for $I(M)$ as $M \to \infty$.

**Theorem 1.1.** *Assume that $N \ge 3$, $0 < b < \min\{2, \frac{N}{2}\}$ and $1 < p < 1 + \frac{4-2b}{N}$, and suppose $V(x)$ satisfies $(V1)$ and $(V2)$. Then there exists a unique positive minimizer of $I(M)$ as $M \to \infty$.*

We note that the restriction $N \ge 3$ in Theorem 1.1 can be removed if the nondegeneracy (2.2) below still holds for any dimension $N \ge 1$. Similar to [5, 12, 20], Theorem 1.1 is proved by establishing local Pohozaev identities. However, the calculation involved in the proof is more complicated due to the existence of inhomogeneous nonlinear terms. In addition, we remark that the standard elliptic regularity theory should be used with caution for singular terms near the origin. The rest of this paper is devoted to the proof of Theorem 1.1.

## 2 Proof of Local Uniqueness

In this section, we shall complete the proof of Theorem 1.1. Inspired by [5, 12], we first define the linear operator

$$\mathcal{L} = -\Delta + 1 - pw^{p-1}|x|^{-b} \ \text{ in } \mathbb{R}^N, \tag{2.1}$$

where $w = w(|x|)$ is the unique positive solution of (1.6). Note from [17] that

$$ker(\mathcal{L}) = \{0\} \text{ if } N \geq 3. \tag{2.2}$$

**Proof of Theorem 1.1**. We prove it by contradiction. Suppose that there exist two different positive minimizers $u_{1,k}$ and $u_{2,k}$ of $I(M_k)$ as $k \to \infty$. It then follows from (1.4) that $u_{i,k}$ solves the following Euler-Lagrange equation

$$-\Delta u_{i,k} + V(x)u_{i,k} - M_k^{\frac{p-1}{2}} \frac{|u_{i,k}|^p}{|x|^b} = \mu_{i,k} u_{i,k} \quad \text{in } \mathbb{R}^N, \quad i = 1, 2, \tag{2.3}$$

where $\mu_{i,k} \in \mathbb{R}$ is a suitable Lagrange multiplier. Define

$$\bar{u}_{i,k}(x) := \sqrt{a^*}\epsilon_k^{\frac{N}{2}} u_{i,k}(x) \ \text{ and } \ \widehat{u}_{i,k}(x) := \bar{u}_{i,k}(\epsilon_k x), \text{ where } i = 1, 2. \tag{2.4}$$

It then follows from (1.7) and (1.8) that

$$\widehat{u}_{i,k}(x) = \bar{u}_{i,k}(\epsilon_k x) \to w(x) \text{ uniformly in } L^\infty(\mathbb{R}^N) \text{ as } k \to \infty, \tag{2.5}$$

and

$$|\widehat{u}_{i,k}(x)| = |\bar{u}_{i,k}(\epsilon_k x)| \leq Ce^{-\sqrt{\theta}|x|} \ \text{ and } \ |\nabla \widehat{u}_{i,k}(x)| = |\nabla \bar{u}_{i,k}(\epsilon_k x)| \leq Ce^{-\theta|x|} \text{ as } |x| \to \infty, \tag{2.6}$$

where $0 < \theta < 1$ and $C > 0$ are independent of $k$. For $i = 1, 2$, $\bar{u}_{i,k}$ solves the equation

$$-\epsilon_k^2 \Delta \bar{u}_{i,k}(x) + \epsilon_k^2 V(x)\bar{u}_{i,k}(x) - \epsilon_k^b \bar{u}_{i,k}^p |x|^{-b} = \mu_{i,k}\epsilon_k^2 \bar{u}_{i,k}(x) \quad \text{in} \quad \mathbb{R}^N. \tag{2.7}$$

Since $u_{1,k} \not\equiv u_{2,k}$, we consider

$$\bar{\xi}_k(x) = \frac{u_{2,k} - u_{1,k}}{\|u_{2,k} - u_{1,k}\|_{L^\infty(\mathbb{R}^N)}} = \frac{\bar{u}_{2,k} - \bar{u}_{1,k}}{\|\bar{u}_{2,k} - \bar{u}_{1,k}\|_{L^\infty(\mathbb{R}^N)}}. \tag{2.8}$$

Stimulated by [5], we first claim that for any $x_0 \in \mathbb{R}^N$, there exsits a small $\delta > 0$ such that

$$\int_{\partial B_\delta(x_0)} \Big[\epsilon_k^2 |\nabla \bar{\xi}_k|^2 + \frac{1}{2}|\bar{\xi}_k|^2 + \epsilon_k^2 V(x)|\bar{\xi}_k|^2\Big] \mathrm{d}S = O(\epsilon_k^N) \ \text{ as } \ k \to \infty. \tag{2.9}$$

Similar to [5, 20], denote

$$\begin{aligned} \bar{D}_k^{s-1}(x) &:= \frac{\bar{u}_{2,k}^s(x) - \bar{u}_{1,k}^s(x)}{s(\bar{u}_{2,k}^s(x) - \bar{u}_{1,k})} \\ &= \frac{\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}[t\bar{u}_{2,k} + (1-t)\bar{u}_{1,k}]^s \mathrm{d}t}{s(\bar{u}_{2,k} - \bar{u}_{1,k})} \\ &= \int_0^1 [t\bar{u}_{2,k} + (1-t)\bar{u}_{1,k}]^{s-1} \mathrm{d}t \end{aligned} \tag{2.10}$$

We then obtain from (2.7) and (2.8) that

$$-\epsilon_k^2 \Delta \bar{\xi}_k + \bar{C}_k \bar{\xi}_k = \bar{g}_k(x) \quad \text{in} \quad \mathbb{R}^N, \tag{2.11}$$

where the cofficient $\bar{C}_k$ satisfies

$$\bar{C}_k = -\epsilon_k^2 \mu_{1,k} - p\epsilon_k^b |x|^{-b} \bar{D}_k^{p-1} + \epsilon_k^2 V(x), \tag{2.12}$$

and the inhomogeneous term $\bar{g}_k$ satisfies

$$\begin{aligned} \bar{g}_k(x) :&= \frac{\epsilon_k^2 \bar{u}_{2,k}(\mu_{2,k} - \mu_{1,k})}{\|\bar{u}_{2,k} - \bar{u}_{1,k}\|_{L^\infty(\mathbb{R}^N)}} \\ &= -\frac{(p-1)\bar{u}_{2,k}}{a^* \epsilon_k^{N-b}} \int_{\mathbb{R}^N} \bar{\xi}_k |x|^{-b} \bar{D}_k^p \mathrm{d}x, \end{aligned} \tag{2.13}$$

where the following equality is used,

$$\begin{aligned} \mu_{2,k} - \mu_{1,k} &= -\frac{p-1}{p+1} M_k^{\frac{p-1}{2}} \int_{\mathbb{R}^N} |x|^{-b} (u_{2,k}^{p+1} - u_{1,k}^{p+1}) \\ &= -\frac{p-1}{p+1} M_k^{\frac{p-1}{2}} (a^*)^{-\frac{p+1}{2}} \epsilon_k^{-\frac{N(p+1)}{2}} \int_{\mathbb{R}^N} |x|^{-b} (\bar{u}_{2,k}^{p+1} - \bar{u}_{1,k}^{p+1}) \\ &= -\frac{(p-1)\|\bar{u}_{2,k} - \bar{u}_{1,k}\|_{L^\infty(\mathbb{R}^N)}}{a^* \epsilon_k^{2+N-b}} \int_{\mathbb{R}^N} \bar{\xi}_k |x|^{-b} \bar{D}_k^p \mathrm{d}x. \end{aligned} \tag{2.14}$$

Multiplying (2.11) by $\bar{\xi}_k$ and integrating over $\mathbb{R}^N$, we then obtain that

$$\begin{aligned} &\epsilon_k^2 \int_{\mathbb{R}^N} |\nabla \bar{\xi}_k|^2 - \mu_{1,k}\epsilon_k^2 \int_{\mathbb{R}^N} |\bar{\xi}_k|^2 + \epsilon_k^2 \int_{\mathbb{R}^N} V(x)|\bar{\xi}_k|^2 \\ &= p\epsilon_k^b \int_{\mathbb{R}^N} |x|^{-b} \bar{D}_k^{p-1} |\bar{\xi}_k|^2 - \frac{(p-1)}{a^* \epsilon_k^{N-b}} \int_{\mathbb{R}^N} \bar{u}_{2,k} \bar{\xi}_k \int_{\mathbb{R}^N} \bar{\xi}_k |x|^{-b} \bar{D}_k^p \\ &\le p\epsilon_k^b \int_{\mathbb{R}^N} |x|^{-b} \bar{D}_k^{p-1} + \frac{(p-1)}{a^* \epsilon_k^{N-b}} \int_{\mathbb{R}^N} \bar{u}_{2,k} \int_{\mathbb{R}^N} |x|^{-b} \bar{D}_k^p \\ &\le C\epsilon_k^N \ \text{ as } \ k \to \infty, \end{aligned}$$

where we use the fact that $|\bar{\xi}_k|$ and $\bar{u}_{i,k}(\epsilon_k x)$ are uniformly bounded with respect to $k$, and $\bar{u}_{i,k}(\epsilon_k x)$ satisfies (2.6), where $i = 1, 2$. Recall from (1.9) that $\mu_{i,k}\epsilon_k^2 \to -1$ as $k \to \infty$, the above estimate further implies that there exists a constant $C_1 > 0$ such that

$$I := \epsilon_k^2 \int_{\mathbb{R}^N} |\nabla \bar{\xi}_k|^2 + \frac{1}{2} \int_{\mathbb{R}^N} |\bar{\xi}_k|^2 + \epsilon_k^2 \int_{\mathbb{R}^N} V(x)|\bar{\xi}_k|^2 < C_1 \epsilon_k^N \ \text{ as } \ k \to \infty. \tag{2.15}$$

Following [5, Lemma 4.5], we then conclude that for any $x_0 \in \mathbb{R}^N$, there exist a small constant $\delta > 0$ and $C_2 > 0$ such that

$$\int_{\partial B_\delta(x_0)} \Big[\epsilon_k^2 |\nabla \bar{\xi}_k|^2 + \frac{1}{2}|\bar{\xi}_k|^2 + \epsilon_k^2 V(x)|\bar{\xi}_k|^2\Big] \mathrm{d}S \le C_2 I \le C_1 C_2 \epsilon_K^N \ \text{ as } \ k \to \infty,$$

and thus the claim (2.9) is proved.

Define

$$\xi_k(x) = \bar{\xi}_k(\epsilon_k x). \tag{2.16}$$

We shall prove Theorem 1.1 by deriving a contradiction through the following three steps.

*Step 1.* There exist a subsequence of $k$, still denoted by $k$, and a constant $b_0 \in \mathbb{R}$ such that $\xi_k(x) \to \xi_0(x)$ in $C_{loc}(\mathbb{R}^N)$ as $k \to \infty$, where

$$\xi_0 = b_0\Big(\frac{p-1}{2}(x \cdot \nabla w) + \frac{2-b}{2}w\Big). \tag{2.17}$$

Actually, note that $\widehat{u}_{i,k}(x)$ satisfies the following equation

$$-\Delta \widehat{u}_{i,k}(x) + \epsilon_k^2 V(\epsilon_k x)\widehat{u}_{i,k}(x) - \frac{\widehat{u}_{i,k}^p(x)}{|x|^b} = \mu_{i,k}\epsilon_k^2 \widehat{u}_{i,k}(x) \ \text{ in } \mathbb{R}^N. \tag{2.18}$$

Denote

$$\widehat{D}_k^{s-1} := \int_0^1 \big[t\widehat{u}_{2,k} + 1 - t\widehat{u}_{1,k}\big]^{s-1} \mathrm{d}t, \tag{2.19}$$

so that $\xi_k$ satisfies

$$-\Delta \xi_k + C_k(x)\xi_k = g_k(x) \ \text{ in } \mathbb{R}^N, \tag{2.20}$$

where the cofficient $C_k(x)$ satisfies

$$C_k(x) = \epsilon_k^2 V(\epsilon_k x) - p|x|^{-b}\widehat{D}_k^{p-1} - \epsilon_k^2 \mu_{1,k}, \tag{2.21}$$

and the inhomogeneous term $g_k(x)$ satisfies

$$g_k(x) := -\frac{(p-1)\widehat{u}_{2,k}}{a^*}\int_{\mathbb{R}^N} \xi_k |x|^{-b}\widehat{D}_k^p \mathrm{d}x. \tag{2.22}$$

Here (1.5) and (2.4) are used. Recalling from (2.5) that $\widehat{u}_{i,k}$ is uniformly bounded in $L^\infty(\mathbb{R}^N)$, we obtain that $|x|^{-b}\widehat{D}^{p-1}$ is uniformly bounded in $L^r_{loc}(\mathbb{R}^N)$, where $r \in (\frac{N}{2}, \frac{N}{b})$. Since $\|\xi_k\|_{L^\infty(\mathbb{R}^N)} \le 1$, the standard elliptic regularity then implies (cf[19]) that $\|\xi_k\|_{W^{2,r}_{loc}(\mathbb{R}^N)} \le C$ and thus $\|\xi_k\|_{C^\alpha_{loc}} \le C$ for some $\alpha \in (0, 2-b)$, where the constant $C > 0$ is independent of $k$. Therefore, there exists a subsequence of $\{k\}$, still denoted by $\{k\}$, and a function $\xi_0 = \xi_0(x)$ such that $\xi_a(x) \to \xi_0(x)$ in $C_{loc}(\mathbb{R}^N)$ as $k \to \infty$. Applying (1.7), (1.9) and dircet calculations yield from (2.21) and (2.22) that

$$C_k(x) \to 1 - p|x|^{-b}w^{p-1} \ \text{ uniformly in } C_{loc}(\mathbb{R}^N) \text{ as } k \to \infty,$$

and

$$g_k(x) \to -\frac{(p-1)w}{a^*}\int_{\mathbb{R}^N} \xi_0|x|^{-b}w^p \ \text{ uniformly in } C_{loc}(\mathbb{R}^N) \text{ as } k \to \infty,$$

which implies from (2.20) that $\xi_0$ solves

$$\mathcal{L}\xi_0 = -\Delta\xi_0 + \xi_0 - \frac{pw^{p-1}}{|x|^b} = -\frac{(p-1)w}{a^*}\int_{\mathbb{R}^N} \xi_0|x|^{-b}w^p. \tag{2.23}$$

Since $-\Delta w + w - w^{p-1}w|x|^{-b} = 0$, we have

$$-\Delta(\partial_i w) + \partial_i w - pw^{p-1}\partial_i w|x|^{-b} + bw^{p-1}|x|^{-(b+2)}x_i = 0,$$

and hence $\mathcal{L}(\partial_i w) = -bw^{p-1}|x|^{-(b+2)}x_i$. Moreover, by direct calculations, we get

$$\begin{aligned}\mathcal{L}(x_i\partial_i w) &= -\Delta(x_i\partial_i w) + x_i\partial_i w - pw^{p-1}|x|^{-b}x_i\partial_i w\\ &= x_i\mathcal{L}(\partial_i w) - 2\partial_i(\partial_i w).\end{aligned}$$

We then deduce from above two facts that

$$\begin{aligned}\mathcal{L}(x\cdot\nabla w)&=-bw^{p-1}w|x|^{-b}-2\Delta w\\&=-bw^{p-1}w|x|^{-b}-2(w-w^{p-1}w|x|^{-b})\\&=(2-b)w^{p-1}w|x|^{-b}-2w.\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathcal{L}(w)&=-\Delta w+w-pw^{p-1}|x|^{-b}w\\&=-\Delta w+w-w^{p-1}w|x|^{-b}-(p-1)w^{p-1}w|x|^{-b}\\&=-(p-1)w^{p-1}w|x|^{-b}.\end{aligned}$$

Therefore, we conclude that

$$\mathcal{L}\Big(\frac{p-1}{2}(x\cdot\nabla w)+\frac{2-b}{2}w\Big)=-(p-1)w,$$

which then implies that (2.17) holds true in view of the non-degeneracy of $\mathcal{L}$.

*Step 2.* The constant $b_0=0$. Recalling from (2.5) that $\widehat{u}_{i,k}$ is uniformly bounded in $L^\infty(\mathbb{R}^N)$, we obtain that $|x|^{-b}\widehat{u}^p_{i,k}$ is uniformly bounded in $L^r_{loc}(\mathbb{R}^N)$, where $r\in(\frac{N}{2},\frac{N}{b})$. By [19, Theorem 9.11] and (2.18), one can deduce that $\widehat{u}_{i,k}$ is uniformly bounded in $W^{2,r}_{loc}(\mathbb{R}^N)$. Since $0<b<\min\{2,\frac{N}{2}\}$, $\widehat{u}_{i,k}$ is uniformly bounded in $H^2_{loc}(\mathbb{R}^N)$. Applying Cauchy-Schwarz inequaty, one can deduce from (2.4) that

$$\begin{aligned}\epsilon_k^2\int_{B_\delta(0)}\big(x\cdot\nabla\bar{u}_{i,k}(x)\big)\Delta\bar{u}_{i,k}(x)&=\epsilon_k^2\int_{B_\delta(0)}\Big(x\cdot\nabla\widehat{u}_{i,k}\big(\frac{x}{\epsilon_k}\big)\Big)\Delta\widehat{u}_{i,k}\big(\frac{x}{\epsilon_k}\big)\\&=\epsilon_k^N\int_{B_{\frac{\delta}{\epsilon_k}}(0)}\Big(x\cdot\nabla\widehat{u}_{i,k}(x)\Big)\Delta\widehat{u}_{i,k}(x)\\&\le\epsilon_k^{N-1}\delta\int_{B_{\frac{\delta}{\epsilon_k}}(0)}|\nabla\widehat{u}_{i,k}(x)||\Delta\widehat{u}_{i,k}(x)|\\&\le\frac{\epsilon_k^{N-1}\delta}{2}\Big(\int_{B_{\frac{\delta}{\epsilon_k}}(0)}|\nabla\widehat{u}_{i,k}(x)|^2+\int_{B_{\frac{\delta}{\epsilon_k}}(0)}|\Delta\widehat{u}_{i,k}(x)|\Big)\\&\le\frac{\epsilon_k^{N-1}\delta}{2}\|\widehat{u}_{i,k}\|^2_{H^2_{B_{\frac{\delta}{\epsilon_k}}(0)}}<\infty.\end{aligned}$$

Therefore, we use integration by parts to get that

$$
\begin{aligned}
&-\epsilon_k^2\int_{B_\delta(0)}(x\cdot\nabla\bar u_{i,k})\Delta\bar u_{i,k}\\
=&-\epsilon_k^2\int_{\partial B_\delta(0)}\frac{\partial\bar u_{i,k}}{\partial\nu}(x\cdot\nabla\bar u_{i,k})\mathrm{d}S+\epsilon_k^2\int_{B_\delta(0)}\nabla\bar u_{i,k}\cdot\nabla(x\cdot\nabla\bar u_{i,k})\\
=&-\epsilon_k^2\int_{\partial B_\delta(0)}\frac{\partial\bar u_{i,k}}{\partial\nu}(x\cdot\nabla\bar u_{i,k})\mathrm{d}S+\epsilon_k^2\sum_{j=1}^N\int_{B_\delta(0)}(\frac{\partial\bar u_{i,k}}{\partial x_j})^2+\frac12 x\cdot\nabla(\frac{\partial\bar u_{i,k}}{\partial x_j})^2\\
=&-\epsilon_k^2\int_{\partial B_\delta(0)}\frac{\partial\bar u_{i,k}}{\partial\nu}(x\cdot\nabla\bar u_{i,k})\mathrm{d}S\\
&+\epsilon_k^2\Big[\int_{B_\delta(0)}|\nabla\bar u_{i,k}|^2-\frac N2|\nabla\bar u_{i,k}|^2+\frac12\int_{\partial B_\delta(0)}|\nabla\bar u_{i,k}|^2(x\cdot\nu)\Big]\mathrm{d}S\\
=&\,\epsilon_k^2\int_{\partial B_\delta(0)}-\frac{\partial\bar u_{i,k}}{\partial\nu}(x\cdot\nabla\bar u_{i,k})+\frac12(x\cdot\nu)|\nabla\bar u_{i,k}|^2\mathrm{d}S+\frac{(2-N)\epsilon_k^2}{2}\int_{B_\delta(0)}|\nabla\bar u_{i,k}|^2\\
=&\,\epsilon_k^2\int_{\partial B_\delta(0)}\Big[-\frac{\partial\bar u_{i,k}}{\partial\nu}(x\cdot\nabla\bar u_{i,k})+\frac12(x\cdot\nu)|\nabla\bar u_{i,k}|^2+\frac{2-N}{4}(\nabla\bar u_{i,k}^2\cdot\nu)\Big]\mathrm{d}S\\
&-\frac{2-N}{2}\int_{B_\delta(0)}\Big[\epsilon_k^2V(x)\bar u_{i,k}^2(x)-\epsilon_k^b\bar u_{i,k}^{p+1}|x|^{-b}-\mu_{i,k}\epsilon_k^2\bar u_{i,k}^2(x)\Big],
\end{aligned}
\tag{2.24}
$$

where the last equality follows from the following equation

$$
\begin{aligned}
\frac{(2-N)\epsilon_k^2}{2}\int_{B_\delta(0)}|\nabla\bar u_{i,k}|^2=&\frac{2-N}{4}\int_{\partial B_\delta(0)}(\nabla\bar u_{i,k}^2\cdot\nu)\mathrm{d}S\\
&-\frac{2-N}{2}\int_{B_\delta(0)}\Big[\epsilon_k^2V(x)\bar u_{i,k}^2(x)-\epsilon_k^b\bar u_{i,k}^{p+1}|x|^{-b}-\mu_{i,k}\epsilon_k^2\bar u_{i,k}^2(x)\Big].
\end{aligned}
\tag{2.25}
$$

On the other hand, multiplying (2.7) by $(x\cdot\nabla\bar u_{i,k})$, where $i=1.2$, and integrating it over $B_\delta(0)$, where $\delta>0$ is small enough, we deduce for $i=1,2$,

$$
\begin{aligned}
&-\epsilon_k^2\int_{B_\delta(0)}(x\cdot\nabla\bar u_{i,k})\Delta\bar u_{i,k}\\
=&\,\epsilon_k^2\int_{B_\delta(0)}\big[\mu_{i,k}-V(x)\big]\bar u_{i,k}(x\cdot\nabla\bar u_{i,k})+\epsilon_k^b\int_{B_\delta(0)}|x|^{-b}\bar u_{i,k}^p x(x\cdot\nabla\bar u_{i,k})\\
=&-\frac{\epsilon_k^2}{2}\int_{B_\delta(0)}\bar u_{i,k}^2\Big\{N\big[\mu_{i,k}-V(x)\big]-\big(x\cdot\nabla V(x)\big)\Big\}\\
&+\frac{\epsilon_k^2}{2}\int_{\partial B_\delta(0)}\bar u_{i,k}^2\big[\mu_{i,k}-V(x)\big](x\cdot\nu)\mathrm{d}S\\
&+\frac{\epsilon_k^b}{p+1}\Big[\int_{\partial B_\delta(0)}|x|^{-b}\bar u_{i,k}^{p+1}(x\cdot\nu)\mathrm{d}S-\int_{B_\delta(0)}\bar u_{i,k}^{p+1}\frac{N-b}{|x|^b}\Big].
\end{aligned}
\tag{2.26}
$$

Substituting (2.24) into (2.26) yields that

$$
\begin{aligned}
&\epsilon_k^2 \int_{\partial B_\delta(0)} \Big[ -\frac{\partial \bar{u}_{i,k}}{\partial \nu}(x \cdot \nabla \bar{u}_{i,k}) + \frac{1}{2}(x\cdot\nu)|\nabla \bar{u}_{i,k}|^2 + \frac{2-N}{4}(\nabla \bar{u}_{i,k}^2 \cdot \nu)\Big] \mathrm{d}S \\
&= \epsilon_k^2 \int_{B_\delta(0)} \bar{u}_{i,k}^2 \big(V(x) - \mu_{i,k} + \frac{1}{2} x \cdot \nabla V\big) \\
&\quad - \frac{2(p+1) - N(p-1) - 2b}{2(p+1)} \epsilon_k^b \int_{B_\delta(0)} \bar{u}_{i,k}^{p+1} |x|^{-b} + I_i,
\end{aligned} \tag{2.27}
$$

where the term satisfies

$$
I_i = \frac{\epsilon_k^2}{2} \int_{\partial B_\delta(0)} \bar{u}_{i,k}^2 \big[\mu_{i,k} - V(x)\big](x\cdot\nu)\mathrm{d}S + \frac{\epsilon_k^b}{p+1} \int_{\partial B_\delta(0)} |x|^{-b} \bar{u}_{i,k}^{p+1}(x \cdot \nu)\mathrm{d}S. \tag{2.28}
$$

Since it follows from (1.5) that $\mu_{i,k}\epsilon_k^2 \int_{\mathbb{R}^N} \bar{u}_{i,k}^2 + \frac{p-1}{p+1}\epsilon_k^b \int_{\mathbb{R}^N} \bar{u}_{i,k}^2 = \epsilon^{2+N} a^* I(M_k)$, we deduce from (2.24)–(2.28) that

$$
\begin{aligned}
&- \epsilon_k^2 \int_{B_\delta(0)} \Big[V(x) + \frac{1}{2}[x \cdot \nabla V(x)]\Big] \bar{u}_{i,k}^2 + \epsilon^{2+N} a^* I(M_k) \\
&= I_i + \epsilon_k^2 \int_{\partial B_\delta(0)} \Big[\frac{\partial \bar{u}_{i,k}}{\partial \nu}(x\cdot\nabla \bar{u}_{i,k}) - \frac{1}{2}(x\cdot\nu)|\nabla \bar{u}_{i,k}|^2 - \frac{2-N}{4}(\nabla \bar{u}_{i,k}^2 \cdot \nu)\Big]\mathrm{d}S \\
&\quad + \mu_{i,k}{\epsilon_k}^2 \int_{\mathbb{R}^N \setminus B_\delta(0)} \bar{u}_{i,k}^2 + \frac{N(p-1)+2b-4}{2(p+1)} \epsilon_k^b \int_{\mathbb{R}^N} |x|^{-b} \bar{u}_{i,k}^{p+1} \\
&\quad + \frac{2(p+1) - N(p-1) - 2b}{2(p+1)} \epsilon_k^b \int_{\mathbb{R}^N \setminus B_\delta(0)} \bar{u}_{i,k}^{p+1} |x|^{-b},
\end{aligned}
$$

which implies that

$$
\frac{4 - N(p-1) - 2b}{2} \epsilon_k^b \int_{\mathbb{R}^N} \bar{D}_k^p |x|^{-b} \bar{\xi}_k = T_1 + T_2 + T_3 + T_4 + T_5 + T_6, \tag{2.29}
$$

where $T_i$ satisfies

$$
\begin{aligned}
T_1 &:= \frac{I_2 - I_1}{\|\bar{u}_{2,k} - \bar{u}_{1,k}\|_{L^\infty}} \\
&= \epsilon_k^b \int_{\partial B_\delta(0)} |x|^{-b} \bar{D}_k^p \bar{\xi}_k (x \cdot \nu) \mathrm{d}S - \frac{\epsilon_k^2}{2} \int_{\partial B_\delta(0)} (\bar{u}_{2,k} + \bar{u}_{1,k}) \bar{\xi}_k V(x)(x \cdot \nu) \mathrm{d}S \\
&+ \frac{\epsilon_k^2 \mu_{2,k}}{2} \int_{\partial B_\delta(0)} (\bar{u}_{2,k} + \bar{u}_{1,k}) \bar{\xi}_k (x \cdot \nu) \mathrm{d}S + \frac{(\mu_{2,k} - \mu_{1,k})\epsilon_k^2}{2\|\bar{u}_{2,k} - \bar{u}_{1,k}\|_{L^\infty}} \int_{\partial B_\delta(0)} \bar{u}_{1,k}^2 (x \cdot \nu) \mathrm{d}S, \\
T_2 &= -\frac{2-N}{4} \epsilon_k^2 \int_{\partial B_\delta(0)} \nabla[(\bar{u}_{2,k} + \bar{u}_{1,k})\bar{\xi}_k] \cdot \nu, \\
T_3 &= -\frac{\epsilon_k^2}{2} \int_{\partial B_\delta(0)} (x \cdot \nu)(\nabla \bar{u}_{2,k} + \nabla u_{1,k}) \cdot \nabla \bar{\xi}_k \\
&\quad + \epsilon_k^2 \int_{\partial B_\delta(0)} \Big[ (x \cdot \nabla \bar{u}_{2,k})(\nu \cdot \nabla \bar{\xi}_k) + (\nu \cdot \nabla \bar{u}_{1,k})(x \cdot \nabla \xi_k) \Big], \\
T_4 &= \mu_{2,k} \epsilon_k^2 \int_{\mathbb{R}^N \setminus B_\delta(0)} (\bar{u}_{2,k} + \bar{u}_{1,k}) \bar{\xi}_k + \frac{(\mu_{2,k} - \mu_{1,k})\epsilon_k^2}{\|\bar{u}_{2,k} - \bar{u}_{1,k}\|_{L^\infty}} \int_{\mathbb{R}^N \setminus B_\delta(0)} \bar{u}_{1,k}^2, \\
T_5 &= \frac{2(p+1) - N(p-1) - 2b}{2} \epsilon_k^b \int_{\mathbb{R}^N \setminus B_\delta(0)} \bar{D}_k^p \bar{\xi}_k |x|^{-b}, \\
T_6 &= \epsilon_k^2 \int_{B_\delta(0)} \Big[ V(x) + \frac{1}{2}[x \cdot \nabla V(x)] \Big] (\bar{u}_{1,k} + \bar{u}_{2,k}) \bar{\xi}_k.
\end{aligned}
$$

Here $\bar{D}_k^{s-1} = \int_0^1 [t\bar{u}_{2,k} + (1-t)\bar{u}_{1,k}]^{s-1} \mathrm{d}t$.

We now estimate the right hand side of (2.29). We first consider the term $T_1$. Using Hölder inequatility, we derive from (2.5) and (2.6) that

$$
\begin{aligned}
&\epsilon_k^b \int_{\partial B_\delta(0)} |x|^{-b} \bar{D}_k^p \bar{\xi}_k (x \cdot \nu) \mathrm{d}S - \frac{\epsilon_k^2}{2} \int_{\partial B_\delta(0)} (\bar{u}_{2,k} + \bar{u}_{1,k}) \bar{\xi}_k V(x)(x \cdot \nu) \mathrm{d}S \\
&+ \frac{\epsilon_k^2 \mu_{2,k}}{2} \int_{\partial B_\delta(0)} (\bar{u}_{2,k} + \bar{u}_{1,k}) \bar{\xi}_k (x \cdot \nu) \mathrm{d}S = o(e^{-\frac{C\delta}{\epsilon_k}}) \ \text{ as } \ k \to \infty.
\end{aligned} \tag{2.30}
$$

Moreover, we deduce from (2.14) that

$$
\frac{|\mu_{2,k} - \mu_{1,k}|\epsilon_k^2}{\|\bar{u}_{2,k} - \bar{u}_{1,k}\|_{L^\infty}} \leq \frac{p-1}{a^* \epsilon_k^{N-b}} \int_{\mathbb{R}^N} \bar{\xi}_k |x|^{-b} \bar{D}_k^p \leq C \ \text{ as } \ k \to \infty \tag{2.31}
$$

which and (2.30) yield that

$$
T_1 = \frac{I_2 - I_1}{\|\bar{u}_{2,k} - \bar{u}_{1,k}\|_{L^\infty}} = o(e^{-\frac{C\delta}{\epsilon_k}}) \ \text{ as } \ k \to \infty, . \tag{2.32}
$$

Moreover, if $\delta > 0$ is small, then it follows from (2.5), (2.6) and (2.15) that

$$\begin{aligned}
|T_2| &= \Big| -\frac{2-N}{4}\epsilon_k^2 \int_{\partial B_\delta(0)} \nabla[(\bar{u}_{2,k}+\bar{u}_{1,k})\bar{\xi}_k]\cdot\nu \Big| \\
&\leq C\epsilon_k^2 \Big[ \Big( \int_{\partial B_\delta(0)} |\nabla(\bar{u}_{2,k}+\bar{u}_{1,k})|^2 \Big)^{\frac{1}{2}} \Big( \int_{\partial B_\delta(0)} |\bar{\xi}_k|^2 \Big)^{\frac{1}{2}} \\
&\quad + \Big( \int_{\partial B_\delta(0)} |\nabla\bar{\xi}_k|^2 \Big)^{\frac{1}{2}} \Big( \int_{\partial B_\delta(0)} |\bar{u}_{2,k}+\bar{u}_{1,k}|^2 \Big)^{\frac{1}{2}} \Big] \\
&\leq C\epsilon_k^2 \big( e^{-\frac{C\delta}{\epsilon_k}} \epsilon_k^{\frac{N}{2}} + \epsilon_k^{\frac{N-2}{2}} e^{-\frac{C\delta}{\epsilon_k}} \big) \\
&\leq C\epsilon_k^{\frac{N+2}{2}} e^{-\frac{C\delta}{\epsilon_k}} \ \text{ as } \ k\to\infty,
\end{aligned} \tag{2.33}$$

where $C > 0$ is independent of $k$. Similar to the estimates (2.30)–(2.33), one can deduce that

$$|T_3|, |T_4|, |T_5| = o(e^{-\frac{C\delta}{\epsilon_k}}) \ \text{ as } \ k\to\infty. \tag{2.34}$$

As for $T_6$, one has

$$\begin{aligned}
&\epsilon_k^2 \int_{B_\delta(0)} V(x)(\bar{u}_{1,k}+\bar{u}_{2,k})\bar{\xi}_k \\
&= \epsilon_k^{2+N} \int_{B_{\frac{\delta}{\epsilon_k}}(0)} V(\epsilon_k x)(\widehat{u}_{1,k}+\widehat{u}_{2,k})\xi_k \\
&= \epsilon_k^{2+N+d} \int_{B_{\frac{\delta}{\epsilon_k}}(0)} \big(1+o(1)\big) h(x)(\widehat{u}_{1,k}+\widehat{u}_{2,k})\xi_k \\
&= \big(2+o(1)\big)\epsilon_k^{2+N+d} \int_{\mathbb{R}^N} h(x) w \xi_0 \ \text{ as } \ k\to\infty.
\end{aligned}$$

Moreover, since $\nabla h(x)\cdot x = dh(x)$, one can derive that

$$\begin{aligned}
&\frac{\epsilon_k^2}{2} \int_{B_\delta(0)} \big[x\cdot\nabla V(x)\big](\bar{u}_{1,k}+\bar{u}_{2,k})\bar{\xi}_k \\
&= \big(1+o(1)\big)\frac{\epsilon_k^2}{2} \int_{B_\delta(0)} [\nabla h(x)\cdot x](\bar{u}_{1,k}+\bar{u}_{2,k})\bar{\xi}_k \\
&= \big(1+o(1)\big) d\frac{\epsilon_k^2}{2} \int_{B_\delta(0)} h(x)(\bar{u}_{1,k}+\bar{u}_{2,k})\bar{\xi}_k \\
&= \big(1+o(1)\big) d\frac{\epsilon_k^{2+N+d}}{2} \int_{B_{\frac{\delta}{\epsilon_k}}(0)} h(x)(\widehat{u}_{1,k}+\widehat{u}_{2,k})\xi_k \\
&= \big(1+o(1)\big) d\epsilon_k^{2+N+d} \int_{\mathbb{R}^N} h(x) w \xi_0 \ \text{ as } \ k\to\infty.
\end{aligned}$$

It then follows from the above estimates that

$$T_6 = O(\epsilon_k^{2+N+d}) \ \text{ as } \ k\to\infty. \tag{2.35}$$

As for the left hand side of (2.29), one can deduce from (2.34) and (2.35) that

$$\begin{aligned} O(\epsilon_k^{2+N+d}) &= \frac{4-N(p-1)-2b}{2}\epsilon_k^b \int_{\mathbb{R}^N} \bar{D}_k^p |x|^{-b}\bar{\xi}_k \\ &= \frac{4-N(p-1)-2b}{2}\epsilon_k^b \int_{\mathbb{R}^N} \bar{D}_k^p(\epsilon_k x)|\epsilon_k x|^{-b}\bar{\xi}_k(\epsilon_k x) \\ &= \frac{4-N(p-1)-2b}{2}\epsilon_k^N \int_{\mathbb{R}^N} \widehat{D}_k^p |x|^{-b}\xi_k \\ &= \frac{4-N(p-1)-2b}{2}\big(1+o(1)\big)\epsilon_k^N \int_{\mathbb{R}^N} w^p|x|^{-b}\xi_0 \ \text{ as } \ k\to\infty, \end{aligned} \tag{2.36}$$

where $\bar{D}_k^p$ and $\widehat{D}_k^p$ are defined as in (2.10) and (2.19), respectively, and the last identity holds due to the fact that $\widehat{D}_k^p \to w^p$ uniformly in $\mathbb{R}^N$ as $k\to\infty$. Using (2.17), it then follows from (2.36) that

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} w^p|x|^{-b}\xi_0 \\ &= \int_{\mathbb{R}^N} w^p|x|^{-b} b_0\Big(\frac{p-1}{2}(x\cdot\nabla w) + \frac{2-b}{2}w\Big) \\ &= \frac{2-b}{2}b_0\int_{\mathbb{R}^N} w^{p+1}|x|^{-b} + \frac{p-1}{2(p+1)}b_0\int_{\mathbb{R}^N}|x|^{-b}(x\cdot\nabla w^{p+1}) \\ &= \frac{2-b}{2}b_0\int_{\mathbb{R}^N} w^{p+1}|x|^{-b} + \frac{p-1}{2(p+1)}b_0(b-N)\int_{\mathbb{R}^N} w^{p+1}|x|^{-b} \\ &= \Big[\frac{2-b}{2} + \frac{p-1}{2(p+1)}(b-N)\Big]b_0\int_{\mathbb{R}^N} w^{p+1}|x|^{-b}. \end{aligned}$$

Since $(2-b)(p+1)+(p-1)(b-N) > 4-2b-(4-2b) = 0$, under the assumption (1.3), we have $b_0 = 0$, i.e., $\xi_0 = 0$ in $\mathbb{R}^N$.

*Step 3.* $\xi_0 \equiv 0$ cannot occur. Let $y_k$ be the point satisfying $|\xi_k(y_k)| = \|\xi_k\|_{L^\infty(\mathbb{R}^N)} = 1$. Since $\widehat{u}_{i,k}$ decays exponentially uniformly for $k\to\infty$, applying the maximum principle to (2.20) yields that $|y_k| \le C$ uniformly in $k$. Therefore, taking a subsequence if necessary, we assume $y_k \to y_0$ as $k\to\infty$ and thus $|\xi_0(y_0)| = \lim_{k\to\infty}|\xi_k(y_k)| = 1$, which contradicts to the fact that $\xi_0 \not\equiv 0$. The proof of Theorem 1.1 is therefore complete. □

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