Generating functions for series involving higher powers of inverse binomial coefficients and their applications

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Abstract

The purpose of this paper is to construct generating functions in terms of hypergeometric function and logarithm function for finite and infinite sums involving higher powers of inverse binomial coefficients. These generating functions provide a novel way of examining higher powers of inverse binomial coefficients from the perspective of these sums, assessing how several of these sums and these coefficients related to each other. A unique relation between the Euler-Frobenius polynomial and B-spline associated with exponential Euler spline is reported. Moreover, with the aid of derivative operator and functional equations for generating functions, many new computational formulas involving the special finite sums of higher powers of (inverse) binomial coefficients, the Bernoulli polynomials and numbers, Euler polynomials and numbers, the Stirling numbers, the harmonic numbers, and finite sums are derived. Moreover, A few recurrence relations containing these particular finite sums are given. Using these recurence relations, we give a solution of the problem which was given by Charalambides7, Exercise 30, p. 273. We give calculations algorithms for these finite sums. Applying these algorithms and Wolfram Mathematica 12.0, we give some plots and many values of these polynomials and finite sums.

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Generating functions for series involving higher powers of inverse binomial coefficients and their applications

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Abstract

The purpose of this paper is to construct generating functions in terms of hypergeometric function and logarithm function for finite and infinite sums involving higher powers of inverse binomial coefficients. These generating functions provide a novel way of examining higher powers of inverse binomial coefficients from the perspective of these sums, assessing how several of these sums and these coefficients related to each other. A unique relation between the Euler-Frobenius polynomial and B-spline associated with exponential Euler spline is reported. Moreover, with the aid of derivative operator and functional equations for generating functions, many new computational formulas involving the special finite sums of higher powers of (inverse) binomial coefficients, the Bernoulli polynomials and numbers, Euler polynomials and numbers, the Stirling numbers, the harmonic numbers, and finite sums are derived. Moreover, A few recurrence relations containing these particular finite sums are given. Using these recurence relations, we give a solution of the problem which was given by Charalambides[7] Exercise 30, p. 273. We give calculations algorithms for these finite sums. Applying these algorithms and Wolfram Mathematica 12.0, we give some plots and many values of these polynomials and finite sums.

KEYWORDS:
Generating function, finite sums, special functions and special numbers, derivative operator, spline, inverse binomial coefficients, recurrence relations, algorithm.
Mathematical Subject Classifications: 05A15, 11B68, 11B37, 35A23, 65Qxx, 03Dxx.

1 | INTRODUCTION

Finite sums and infinite sums, together with special functions, generating functions, special numbers and polynomials, provide a powerful tool in a wide variety of all branches of mathematics and other sciences. Finite sums involving powers of binomial binomial coefficients have been investigated to many years ago. It is known that the history of these sums have been traced back for long period of time. Finite sums have always attracted considerable attention of many researchers especially mathematicians, and other researchers involving statisticians, physicists, engineers, and social sciences. Finite sums involving binomial coefficients are an integral part of most scientific disciplines, and a basic understanding of their usage and limitations is essential for not only mathematics, but also for any scientist such as engineers. Similarly, computation of generating functions for special numbers and polynomials, finite and infinite sums, PDEs, QDEs, algorithms, moments, polynomial time algorithms, initial value problem, Subgraph generating functions in chemistry, biological molecules, etc have also been very useful applications.

Therefore, various manuscripts, research books, and monographs related to recurrence relations for alternating sums of powers of (inverse) binomial coefficients, the finite sums with their generating functions along with various applications have been published throughout the years (*cf.* [2]-[39]).

We [28] introduced many computational formulas and generating functions for the finite sums of powers of binomial coefficients. These formulas have many applications in mathematical models, the Franel numbers, Brownian paths, graph theory, and other real world problems that are associated watermelons with more than two chains (*cf.* [3]-[36]). Moreover, we [28] also presented various examples involving applications of sums of powers of binomial with their generating functions. The binomial coefficient can also be used to construct mathematical model for the cell population dynamics and computation the growth of number of cells of multicellular organism (*cf.* [10]).

With the aid of derivative operator, hyperbolic functions, hypergeometric functions, and functional equations for generating functions, this paper provides not only novel fundamental properties of finite sums for powers of the (inverse) binomial coefficients with their new generating functions, but also computational formulas and relations. Our aim in this paper is to present many novel computational formulas for finite sums involving higher powers of (inverse) binomial coefficients with their generating functions involving hypergeometric function and special functions.

We [28] defined following the sum:

$$y_6(j,b;\omega,p)=\sum_{k=0}^{b}\binom{b}{k}^{p}\frac{\omega^k k^j}{b!}, \tag{1}$$

where $b,j,p\in\mathbb{N}_0$ ($\mathbb{N}_0=\mathbb{N}\cup\{0\}=\{0,1,2,\dots\}$), $\omega\in\mathbb{R}$, set of real numbers or $\omega\in\mathbb{C}$, set of complex numbers, set of real numbers and $0^j=1$ for $j=0$.

Generating function for the sum $y_6(j,b;\omega,p)$ is given by

$$F_{y6}(u,b;\omega,p)={}_kF_m\left(-b,-b,...,-b;1,1,...,1;(-1)^p\omega e^u\right)=\sum_{j=0}^{\infty}b!y_6(j,b;\omega,p)\frac{u^j}{j!}, \tag{2}$$

where ${}_kF_m\left(x_1,...,x_k;w_1,...,w_m;u\right)$ denotes the following generalized hypergeometric series:

$${}_kF_m\left(x_1,...,x_k;w_1,...,w_m;u\right)=\sum_{n=0}^{\infty}\left(\frac{\prod\limits_{j=1}^{k}\left(x_j\right)_n}{\prod\limits_{j=1}^{m}\left(w_j\right)_n}\right)\frac{u^n}{n!}, \tag{3}$$

where the above series converges for all u if $k<m+1$, and for $|u|<1$ if $k=m+1$. Assuming that all parameters have general values, real or complex, except for the w_j, $j=1,2,...,m$ none of which is equal to zero or a negative integer. $(w)_v$ denotes the Pochhammer's symbol, defined by

$$(w)_v=\prod_{j=0}^{v-1}(w+j),$$

and $(w)_0=1$ for $w\neq 1$, $v\in\mathbb{N}$, and $w\in\mathbb{C}$ (*cf.* [9,14,28,34]).

Using (1) and (2), we also solved Srivastava's [33] p. 416, open problem 1.

Substituting $\omega=-1$, $p=3$, and $b=2v$ into (2), we have

$$F_{y6}(u,2v;-1,3)={}_3F_2\left(-2v,-2v,-2v;1,1;e^u\right)=\sum_{j=0}^{\infty}(2v)!y_6(j,2v;-1,3)\frac{u^j}{j!},$$

where

$$y_6(0,2v;-1,3)=\frac{(-1)^v}{(v!)^2}\binom{3v}{2v}.$$

Formulas similar to the previous one were studied by Dixon in 1891. Using the following well-known formula for the binomial coefficient $\binom{v}{k}$:

$$2\pi i\binom{v}{k}=\oint_{\delta}\frac{(1+w)^v}{w^{k+1}}dw,$$

where $\delta = \left\{ w \in \mathbb{C} : |w| < \frac{1}{2} \right\}$, Bostan et al. (see also[3]) showed that

$$(2\pi i)^3 \binom{v}{k}^3 = \oint_{\delta\times\delta\times\delta} \frac{\left(1+u_1\right)^{2v}\left(1+u_2\right)^{2v}\left(1+u_3\right)^{2v}}{u_1^{k+1}u_2^{k+1}u_3^{k+1}} du_1 du_2 du_3.$$

With the help of the previous formula, we obtain the following generating function for the sum $y_6(0, 2v; -1, 3)$:

$$\begin{aligned} F(t) &= \sum_{v=0}^{\infty} (2v)! y_6(0, 2v; -1, 3) t^v \\ &= \frac{1}{(2\pi i)^2} \oint_{\delta\times\delta} \frac{z_1 z_2 dz_1 dz_2}{z_1^2 z_2^2 - t\left(1+z_1\right)^2\left(1+z_2\right)^2\left(1-z_1 z_2\right)^2}. \end{aligned}$$

The generating function $F(t)$ satisfies the following differential equation:

$$t(27t+1)\frac{d^2F(t)}{dt^2} + (54t+1)\frac{dF(t)}{dt} + 6F(t) = 0$$

(*cf.*[3]).

Using the above differential equation, we get the following recurrence relation for the sum $y_6(0, 2v; -1, 3)$:

$$3(3v+2)(3v+1)(2v)! y_6(0, 2v; -1, 3) + (v+1)^2 (2v+2)! y_6(0, 2v+2; -1, 3) = 0.$$

Since $y_6(0, 0; -1, 3) = 1$, due to work of Bostan et al.[3], this recurrence relation leads to a proof of Dixon's identity with the aid of the mathematical induction method.

Substituting $k = p+1$, $m = p$ and $x_1 = x_2 = \cdots = x_{p+1} = 1, w_1 = w_2 = \cdots = w_p = -\beta$, $(-\beta \notin \{0, -1, -2, -3, \ldots\})$, and $u = (-1)^p \lambda e^z$ into (3), we construct the following generating function involving higher powers of inverse binomial coefficients:

Definition 1. Let $v, p \in \mathbb{N}_0$, $-\beta \notin \{0, -1, -2, -3, \ldots\}$ and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) with $|\lambda| < 1$. We define the sum $B_v(\beta; \lambda, p)$ by the following generating function:

$$\begin{aligned} F(z, v; \beta, \lambda, p) &= {}_{p+1}F_p\left(1, 1..., 1; -\beta, ..., -\beta; (-1)^p \lambda e^z\right) \\ &= \sum_{v=0}^{\infty} B_v(\beta; \lambda, p)\frac{z^v}{v!}. \end{aligned} \tag{4}$$

By using (4), we obtain explicit formula for $B_v(\beta; \lambda, p)$ by the following theorem:

Theorem 1. Let $v, p \in \mathbb{N}_0$ and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) with $|\lambda| < 1$. We have

$$B_v(\beta; \lambda, p) = \sum_{m=0}^{\infty} \frac{m^v \lambda^m}{\binom{\beta}{m}^p}, \tag{5}$$

where $\binom{\beta}{0} = 1$ and

$$\binom{\beta}{m} = \frac{\prod_{j=0}^{m-1}(\beta - j)}{m!}.$$

Many applications and numerical values of the sum $B_v(\beta; \lambda, p)$ are given throughout this paper. This sum is related to the certain classes of the infinite series involving Feynman diagrams. For example, Kalmykov et al.[16] gave the following multiple (inverse) binomial type sums, which are connection with the hypergeometric functions and the Feynman diagrams:

$$\sum_{m=0}^{\infty} \frac{w^m}{\binom{2m}{m}^d m^f} S_{a_1}(m-1)...S_{a_q}(m-1),$$

where $d = \pm 1$, f is any integer number, and $S_a(m)$ denotes the harmonic numbers of order a:

$$\sum_{j=1}^{m} \frac{1}{j^a}.$$

For $d = -1$, $d = 1$, and $a_j = 0$ with $j = 1, 2, ..., q$, a relation between the above sum and the sum $B_v(\beta; \lambda, p)$ is obtained.
The main motivation of this paper is not only to define the following finite sum involving higher powers of inverse binomial

coefficients:

$$\mathcal{S}_v(n;\lambda,p) := \sum_{j=0}^{n} \frac{j^v}{\binom{n}{j}^p}\lambda^j, \tag{6}$$

where $n, v, p \in \mathbb{N}_0$ and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$), but also to investigate fundamental properties of the generating function which is given by (4). Numerous properties of the sums $\mathcal{S}_v(n;\lambda,p)$ and $B_v(n;\lambda,p)$ are going to be given in the next sections. We will also introduce other alternative forms of the generating functions for the sum $\mathcal{S}_v(n;\lambda,p)$ with aid of special functions and certain special finite sums.

Note that $\mathcal{S}_v(n;\lambda,p)$ can also be considered as a polynomial with respect to the variable λ.

By using (2) and (4), we will give some new novel formulas involving $B_v(n;\omega,p)$, $\mathcal{S}_v(n;\lambda,p)$, $y_6(m,n;\lambda,p)$, and other special numbers and polynomials.

Another important motivation for this paper is to give relations between the sum $\mathcal{S}_v(m;\lambda,p)$ and the following rational sum (or the numbers) $y(m,\phi)$:

$$y(m,\phi) = (-1)^m \sum_{k=0}^{m} \frac{1}{(\phi-1)^{m+1-k}\,\vartheta\left\{\phi^{k+1}\right\}}, \tag{7}$$

where ϑ denotes the following Euler operator, which is given in[24] p. 80, Eq. (2),

$$\vartheta = \phi\frac{d}{d\phi},$$

$\phi \neq 0,1$, $\phi \in \mathbb{R}$ and $m \in \mathbb{N}_0$. We gave relations among the rational sum $y(m,\phi)$, special numbers and polynomials, special integrals, hypergeometric series, Dirichlet series, and also zeta type functions (*cf.*[30,31]). For $\phi \neq 0,1$, $\phi \in \mathbb{R}$, $w \in \mathbb{C}$, $0 < |w| < 1$ and $\left|\frac{(\phi-1)w}{\phi}\right| < 1$, this rational sum is given by the following generating function:

$$G(w,\phi) = \frac{\ln\left(1-\frac{\phi-1}{\phi}w\right)}{(w^2-w)(1-\phi)^2} = \sum_{v=0}^{\infty} y(v,\phi)\left((1-\phi)w\right)^v, \tag{8}$$

where the principal branch of the logarithm is assumed to be taken (*cf.*[32]).

1.1 | Preliminaries

In order to give main results of this paper in the next sections, we need the following generating functions for well-known special numbers and polynomials.

The Frobenius-Euler polynomials $H_v(s;u)$ are defined by

$$\frac{e^{su}}{e^u-\eta} = \sum_{v=0}^{\infty} \frac{H_v(s;\eta)}{(1-\eta)\,v!}u^v, \tag{9}$$

for $s=0$, $H_v(\eta) := H_v(0;\eta)$ denotes the Frobenius-Euler numbers (or Eulerian numbers, or Euler Frobenius numbers), and also $H_v(-1) := E_v$, which denotes the Euler numbers (*cf.*[29,34]).

The Apostol-Bernoulli polynomials $\mathcal{B}_v(s;\eta)$ are defined by

$$\frac{ue^{su}}{\eta e^u-1} = \sum_{v=0}^{\infty} \frac{\mathcal{B}_v(s;\eta)}{v!}u^v, \tag{10}$$

for $s=0$, $\mathcal{B}_v(\eta) := \mathcal{B}_v(0;\eta)$ denotes the Apostol-Bernoulli numbers and for $\eta=1$, $\mathcal{B}_v(s,\eta) := B_v(s)$ denotes the Bernoulli polynomials (*cf.*[2,29,34]).

The harmonic numbers H_v are defined by

$$F_1(u) = \ln(1-u) = (u-1)\sum_{v=0}^{\infty} H_v u^v, \tag{11}$$

where $H_0 = 0$ and $|u| < 1$ (*cf.*[2,7,31,33,34]).

Let $j \in \mathbb{N}_0$. The Stirling numbers of the second kind $S(c,j)$ are defined by

$$F_{S_2}(w,j) = (e^w-1)^j = \sum_{c=0}^{\infty} S(c,j)\frac{j!w^c}{c!}, \tag{12}$$

(*cf.*[2]-[34]). Using (12), for $c, j \in \mathbb{N}_0$, the following results are easy to achieve: $S(c,c) = 1$. If $j > c$, then $S(c,j) = 0$.

The numbers $Y_{\upsilon}(\theta)$ are defined by

$$g(u;\theta)=\frac{2}{\theta^{2}u+\theta-1}=\sum_{\upsilon=0}^{\infty}Y_{\upsilon}(\theta)\frac{u^{\upsilon}}{\upsilon!},\tag{13}$$

where

$$Y_{\upsilon}(\theta)=-\frac{2\theta^{2\upsilon}\upsilon!}{(1-\theta)^{\upsilon+1}}$$

(*cf.* [25] Eq. (2.13)).

In [27], we defined the generating function for the combinatorial numbers $y_1(n,k;\lambda)$ as follows:

$$F_{y_1}(u,k;\lambda)=\frac{1}{k!}\left(\lambda e^{u}+1\right)^{k}=\sum_{n=0}^{\infty}y_1(n,k;\lambda)\frac{u^{n}}{n!},\tag{14}$$

where $k\in\mathbb{N}_0$, $\lambda\in\mathbb{C}$ and

$$y_1(n,k;\lambda)=\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}j^{n}\lambda^{j}.$$

The polynomials $C_m(a,b)$ and $S_m(a,b)$ are defined respectively by

$$e^{az}\cos(bz)=\sum_{m=0}^{\infty}C_m(a,b)\frac{z^{m}}{m!}\tag{15}$$

and

$$e^{az}\sin(bz)=\sum_{m=0}^{\infty}S_m(a,b)\frac{z^{m}}{m!}\tag{16}$$

(*cf.* [17,35]).

Next, we outline the results of the article as follows:

In Section 2, using hypergeometric series and special functions, we construct generating functions for infinite and finite sums of powers of the inverse binomial coefficients. Using this generating function, we obtain many new formulas and relations for the sums of powers of the (inverse) binomial coefficients. We also derive a relation between the Euler-Frobenius polynomial and B-spline associated with exponential Euler spline.

In Section 3, using derivative operator and Euler operator, we present many formulas for the sums $y_1(\upsilon,d;\lambda)$, $y_6(m,n;\lambda,p)$, $\mathcal{S}_{\upsilon}(n;\lambda,p)$, and $\boldsymbol{B}_{\upsilon}(\beta;\lambda,p)$, and the Stirling numbers.

In Section 4, with the aid of generating functions with their functional equations, we give many new relations among the sums $y_6(m,n;\lambda,p)$, $\mathcal{S}_{\upsilon}(m;\lambda,q)$, $y_1(\upsilon,k;\lambda)$, and the Stirling numbers of the second kind. We also derive many recurrence relations for the numbers $\mathcal{S}_{\upsilon}(n;\lambda,p)$ with the aid of the rational sum $y(n,\lambda)$.

In Section 5, we give calculations algorithms for equations (6), (39), and (48). Using these algorithms, we give not only some 2D plots and 3D surfaces graphs for the polynomial $\mathcal{S}_{\upsilon}(n;x,p)$ with respect to x, but also some numerical values of these polynomials with the tables. Using (5), some the values for some convergence intervals for the $\boldsymbol{B}_{\upsilon}(\beta;\lambda,p)$ are given.

In Section 6 is the Conclusion section.

2 | GENERATING FUNCTIONS FOR FINITE SUMS INVOLVING HIGHER POWERS OF INVERSES OF BINOMIAL COEFFICIENTS

The aim of this section is to construct generating functions for finite sums of powers of the inverse binomial coefficients with the aid of hypergeometric series and special functions. We obtain some novel formulas and relations for the finite sums of powers of the (inverse) binomial coefficients. We also present some alternating generating functions for special values of the sum $\mathcal{S}_{\upsilon}(n;\lambda,p)$ and $\boldsymbol{B}_{j}(\beta;\lambda,p)$.

We define a new class of polynomials $\mathcal{P}_{\upsilon}(x,\beta;\lambda,p)$ whose coefficients are the sum $\boldsymbol{B}_{j}(\beta;\lambda,p)$ by the following generating functions:

$$G(z;\beta,\lambda,p)=F(z;\beta,\lambda,p)e^{xz}=\sum_{\upsilon=0}^{\infty}\mathcal{P}_{\upsilon}(x,\beta;\lambda,p)\frac{z^{\upsilon}}{\upsilon!}\tag{17}$$

where $-\beta\notin\{0,-1,-2,-3,\dots\}$, $\upsilon,p\in\mathbb{N}_0$ and $x,\lambda\in\mathbb{R}$ (or $\mathbb{C}$) with $|\lambda|<1$.

By using (17), we get

$$\sum_{v=0}^{\infty}\mathcal{P}_v(x,\beta;\lambda,p)\frac{z^v}{v!}=\sum_{v=0}^{\infty}\frac{(xz)^v}{v!}\sum_{v=0}^{\infty}B_v(\beta;\lambda,p)\frac{z^v}{v!}$$

Therefore

$$\sum_{v=0}^{\infty}\mathcal{P}_v(x,\beta;\lambda,p)\frac{z^v}{v!}=\sum_{v=0}^{\infty}\sum_{j=0}^{v}\binom{v}{j}x^{v-j}B_j(\beta;\lambda,p)\frac{z^v}{v!}.$$

Equating the coefficients of $\frac{z^v}{v!}$ on both sides of the in the previous equation, we arrive at the computation formula for the polynomials $\mathcal{P}_v(x,\beta;\lambda,p)$:

Theorem 2. Let $v, p\in\mathbb{N}_0$ and $\lambda,\beta\in\mathbb{R}$ (or $\mathbb{C}$) with $|\lambda|<1$. Then we have

$$\mathcal{P}_v(x,\beta;\lambda,p)=\sum_{j=0}^{v}\binom{v}{j}x^{v-j}B_j(\beta;\lambda,p). \tag{18}$$

With the aid of (6), we also define the following polynomials:

$$\mathcal{Q}_v(x,n;\lambda,p)=\sum_{j=0}^{v}\binom{v}{j}x^{v-j}\mathcal{S}_j(n;\lambda,p). \tag{19}$$

Applying the kth derivative, with respect to x, to equation (17), we get the following higher-order partial differential equation:

$$\frac{\partial^k}{\partial x^k}\{G(z;\beta,\lambda,p)\}=t^kG(z;\beta,\lambda,p).$$

From the above equation, we arrive at the following theorem, which gives us higher-order partial differential equation for the polynomial $\mathcal{P}_v(x,\beta;\lambda,p)$.

Theorem 3. Let $k\in\mathbb{N}_0$ and $v\in\mathbb{N}$. Then we have

$$\frac{\partial^k}{\partial x^k}\left\{\mathcal{P}_v(x,\beta;\lambda,p)\right\}=(v)^{\underline{k}}\mathcal{P}_{v-k}(x,\beta;\lambda,p),$$

where $(v)^{\underline{k}}=\prod_{j=0}^{k-1}(v-j)$ and $(v)^{\underline{0}}=1$.

Equation (4) can also be represented as follows:

$$F(z;\beta,\lambda,p)=\sum_{j=0}^{\infty}\frac{(\lambda e^z)^j}{\binom{\beta}{j}^p}=\sum_{v=0}^{\infty}B_v(\beta;\lambda,p)\frac{z^v}{v!}, \tag{20}$$

where $|\lambda|<1$.

Combining (17) with (20), we obtain the following alternative form of the generating function in (17) as follows:

$$G(z;\beta,\lambda,p)=\sum_{j=0}^{\infty}\frac{\lambda^j}{\binom{\beta}{j}^p}e^{(j+x)z}=\sum_{v=0}^{\infty}\mathcal{P}_v(x,\beta;\lambda,p)\frac{z^v}{v!}. \tag{21}$$

By using (21), we obtain the following alternative form of the polynomials $\mathcal{P}_v(x,\beta;\lambda,p)$ in (18) as follows:

Theorem 4. Let $v\in\mathbb{N}$. Then we have

$$\mathcal{P}_v(x,\beta;\lambda,p)=\sum_{j=0}^{\infty}\frac{\lambda^j}{\binom{\beta}{j}^p}(j+x)^v,$$

where $|\lambda|<1$.

2.1 | Some alternative forms of the generating functions for the sums $B_v(\beta;\lambda,p)$

Here, we give some alternative forms of the generating functions in the equation (4) for the sums $B_v(\beta;\lambda,p)$.

A generating function for $B_v(\beta;\lambda,p)$ including $\cosh(z)$ and $\sinh(z)$ is given by

$$\begin{aligned}F(z,v;n,p) &= \sum_{j=0}^{\infty}\frac{\lambda^j}{\binom{\beta}{j}^p}(\cosh(jz)+\sinh(jz)) \\ &=\sum_{v=0}^{\infty}B_v(\beta;\lambda,p)\frac{z^v}{v!},\end{aligned} \tag{22}$$

where $|\lambda| < 1$.

Substituting $z = x + iy$ into (22), we get

$$\sum_{j=0}^{\infty} \frac{\lambda^j}{\binom{\beta}{j}^p} e^{jx} \cos(jy) = \sum_{v=0}^{\infty} \frac{B_v(\beta; \lambda, p)}{v!} \sum_{k=0}^{\left[\frac{v}{2}\right]} (-1)^k \binom{v}{2k} x^{v-2k} y^{2k}, \tag{23}$$

and

$$\sum_{j=0}^{\infty} \frac{\lambda^j}{\binom{\beta}{j}^p} e^{jx} \sin(jy) = \sum_{v=0}^{\infty} \frac{B_v(\beta; \lambda, p)}{v!} \sum_{k=1}^{\left[\frac{v+1}{2}\right]} (-1)^{k+1} \binom{v}{2k-1} x^{v-2k+1} y^{2k-1}. \tag{24}$$

Substituting $x = y$ into (23) and (24), we get the following results, respectively:

$$\sum_{j=0}^{\infty} \frac{\lambda^j}{\binom{\beta}{j}^p} e^{jx} \cos(jx) = \sum_{v=0}^{\infty} \sum_{k=0}^{\left[\frac{v}{2}\right]} (-1)^k \binom{v}{2k} B_v(\beta; \lambda, p) \frac{x^v}{v!} \tag{25}$$

and

$$\sum_{j=0}^{\infty} \frac{\lambda^j}{\binom{\beta}{j}^p} e^{jx} \sin(jx) = \sum_{v=0}^{\infty} \sum_{k=1}^{\left[\frac{v+1}{2}\right]} (-1)^{k+1} \binom{v}{2k-1} B_v(\beta; \lambda, p) \frac{x^v}{v!}. \tag{26}$$

Combining (25) and (26) with the following certain finite sums

$$2 \sum_{k=0}^{\left[\frac{v}{2}\right]} (-1)^k \binom{v}{2k} = (1-i)^v + (1+i)^v$$

and

$$2i \sum_{k=1}^{\left[\frac{v+1}{2}\right]} (-1)^{k+1} \binom{v}{2k-1} = (1+i)^v - (1-i)^v$$

(*cf.*[13] Eq. (2.26) and Eq. (2.30)), we get the following generating functions:

$$\sum_{j=0}^{\infty} \frac{\lambda^j}{\binom{\beta}{j}^p} e^{jx} \cos(jx) = \sum_{v=0}^{\infty} B_v(\beta; \lambda, p) \frac{\left((1-i)^v + (1+i)^v\right) x^v}{2v!}$$

and

$$\sum_{j=0}^{\infty} \frac{\lambda^j}{\binom{\beta}{j}^p} e^{jx} \sin(jx) = \sum_{v=0}^{\infty} B_v(\beta; \lambda, p) \frac{\left((1+i)^v - (1-i)^v\right) x^v}{2iv!},$$

where $|\lambda| < 1$.

By making use of the Taylor series of the function e^{mx} in (20), we get

$$\sum_{v=0}^{\infty} B_v(\beta; \lambda, p) \frac{z^v}{v!} = \sum_{j=0}^{\infty} \frac{\lambda^j}{\binom{\beta}{j}^p} \sum_{v=0}^{\infty} \frac{(jz)^v}{v!}.$$

Equating the coefficients of $\frac{z^v}{v!}$ on both sides of the in the previous equation, we arrive at the computation formula for the numbers $B_v(\beta; \lambda, p)$, given by the equation (5).

Combining (25) with (15) yields

$$\sum_{v=0}^{\infty} \sum_{j=0}^{\infty} \frac{\lambda^j}{\binom{\beta}{j}^p} C_v(j, j) \frac{x^v}{v!} = \sum_{v=0}^{\infty} \frac{B_v(\beta; \lambda, p)}{v!} \sum_{k=0}^{\left[\frac{v}{2}\right]} (-1)^k \binom{v}{2k} x^v.$$

Combining the above equation with the following known formula

$$\sum_{k=0}^{\left[\frac{v}{2}\right]} (-1)^k \binom{v}{2k} = \sqrt{2^v} \cos\left(\frac{v\pi}{4}\right),$$

($cf.$[13] Eq. (2.27)), we get

$$\sum_{v=0}^{\infty}\sum_{j=0}^{\infty}\frac{\lambda^j}{\binom{\beta}{j}^p}C_v(j,j)\frac{x^v}{v!}=\sum_{v=0}^{\infty}B_v(\beta;\lambda,p)\frac{\sqrt{2^v}\cos\left(\frac{v\pi}{4}\right)x^v}{v!}.$$

By equalizing the coefficients of $\frac{x^v}{v!}$ on both sides of the previous equation, we arrive at the following theorem after the necessary calculations.

Theorem 5. Let $v, p \in \mathbb{N}_0$ and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) with $|\lambda| < 1$. Then we have

$$\sum_{j=0}^{\infty}\frac{\lambda^j}{\binom{\beta}{j}^p}C_v(j,j)=\sqrt{2^v}B_v(\beta;\lambda,p)\cos\left(\frac{v\pi}{4}\right). \tag{27}$$

Substituting $v = 1$ into (27), we get the following result:

Corollary 1. Let $p \in \mathbb{N}_0$ and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) with $|\lambda| < 1$. Then we have

$$B_1(\beta;\lambda,p)=2\sum_{j=0}^{\infty}\frac{\lambda^j}{\binom{\beta}{j}^p}C_1(j,j).$$

Substituting $v = 2$ into (27), we get the following result:

Corollary 2. Let $p \in \mathbb{N}_0$ and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) with $|\lambda| < 1$. Then we have

$$\sum_{j=0}^{\infty}\frac{\lambda^j}{\binom{\beta}{j}^p}C_2(j,j)=0.$$

Substituting $v = 4m$, $m \in \mathbb{N}_0$ into (27), we get the following result:

Corollary 3. Let $n, m, p \in \mathbb{N}_0$ and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$). Then we have

$$B_{4m}(\beta;\lambda,p)=\frac{1}{(-1)^m 4^m}\sum_{j=0}^{\infty}\frac{\lambda^j}{\binom{\beta}{j}^p}C_{4m}(j,j).$$

Substituting $v = 8m$, $m \in \mathbb{N}_0$ into (27), we get the following result:

Corollary 4. Let $n, m, p \in \mathbb{N}_0$ and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) with $|\lambda| < 1$. Then we have

$$B_{8m}(\beta;\lambda,p)=2^{-4m}\sum_{j=0}^{\infty}\frac{\lambda^j}{\binom{\beta}{j}^p}C_{8m}(j,j).$$

Combining (26) with (16) yields

$$\sum_{v=0}^{\infty}\sum_{j=0}^{\infty}\frac{\lambda^j}{\binom{\beta}{j}^p}S_v(j,j)\frac{x^v}{v!}=\sum_{v=0}^{\infty}\sum_{k=0}^{\left[\frac{v-1}{2}\right]}(-1)^k\binom{v}{2k+1}B_v(\beta;\lambda,p)\frac{x^v}{v!}.$$

Combining the above equation with the following known formula

$$\sum_{k=0}^{\left[\frac{v-1}{2}\right]}(-1)^k\binom{v}{2k+1}=\sqrt{2^v}\sin\left(\frac{v\pi}{4}\right)$$

($cf.$[13] Eq. (2.31)), we get

$$\sum_{v=0}^{\infty}\sum_{j=0}^{\infty}\frac{\lambda^j}{\binom{\beta}{j}^p}S_v(j,j)\frac{x^v}{v!}=\sum_{v=0}^{\infty}B_v(\beta;\lambda,p)\frac{\left(\sqrt{2}x\right)^v\sin\left(\frac{v\pi}{4}\right)}{v!}.$$

By equalizing the coefficients of $\frac{x^v}{v!}$ on both sides of the previous equation, we arrive at the following theorem after the necessary calculations.

Theorem 6. Let $n, v, p \in \mathbb{N}_0$ and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) with $|\lambda| < 1$. Then we have

$$\sum_{j=0}^{\infty} \frac{\lambda^j}{\binom{\beta}{j}^p} S_v(j,j) = \sqrt{2^v} B_v(\beta;\lambda,p) \sin\left(\frac{v\pi}{4}\right). \tag{28}$$

Substituting $v = 1$ into (27), we get the following result:

Corollary 5. Let $n, p \in \mathbb{N}_0$ and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) with $|\lambda| < 1$. Then we have

$$B_1(n;\lambda,p) = 2\sum_{j=0}^{\infty} \frac{\lambda^j}{\binom{\beta}{j}^p} S_1(j,j).$$

Substituting $v = 4m$, $m \in \mathbb{N}_0$ into (27), we get the following result:

Corollary 6. Let $n, m, p \in \mathbb{N}_0$ and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) with $|\lambda| < 1$. Then we have

$$\sum_{j=0}^{\infty} \frac{\lambda^j}{\binom{\beta}{j}^p} S_{4m}(j,j) = 0.$$

Substituting $v = 4m+2$, $m \in \mathbb{N}_0$ into (27), we get the following result:

Corollary 7. Let $n, m, p \in \mathbb{N}_0$ and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) with $|\lambda| < 1$. Then we have

$$B_{4m+2}(\beta;\lambda,p) = \frac{1}{(-1)^m 2^{2m+1}} \sum_{j=0}^{\infty} \frac{\lambda^j}{\binom{\beta}{j}^p} S_{4m+2}(j,j).$$

Here, using (20), we get

$$\sum_{j=0}^{\infty} \frac{\lambda^j}{\binom{\beta}{j}^p} \sum_{k=0}^{j} \binom{j}{k} (e^z-1)^k = \sum_{v=0}^{\infty} B_v(\beta;\lambda,p)\frac{z^v}{v!}.$$

Combining the previous equation with (12) yields the following functional equation:

$$\sum_{j=0}^{\infty} \frac{\lambda^j}{\binom{\beta}{j}^p} \sum_{k=0}^{j} \binom{j}{k} k! F_{S_2}(z,k) = \sum_{v=0}^{\infty} B_v(\beta;\lambda,p)\frac{z^v}{v!}.$$

Using the above functional equation, we get

$$\sum_{j=0}^{\infty} \frac{\lambda^j}{\binom{\beta}{j}^p} \sum_{k=0}^{j} \binom{j}{k} k! \sum_{v=0}^{\infty} S(v,k)\frac{z^v}{v!} = \sum_{v=0}^{\infty} B_v(\beta;\lambda,p)\frac{z^v}{v!}.$$

By equalizing the coefficients of $\frac{z^v}{v!}$ on both sides of the previous equation, we arrive at the following theorem after the necessary calculations.

Theorem 7. Let $v \in \mathbb{N}_0$. Then we have

$$B_v(\beta;\lambda,p) = \sum_{j=0}^{\infty} \frac{\lambda^j}{\binom{\beta}{j}^p} \sum_{k=0}^{j} \binom{j}{k} k! S(v,k),$$

where $|\lambda| < 1$.

Substituting $\beta = -n$, $(n \in \mathbb{N})$ into (5), we get the following result:

Corollary 8. Let $v \in \mathbb{N}_0$ and $n \in \mathbb{N}$. Then we have

$$B_v(-n;\lambda,p) = \sum_{m=0}^{\infty} ((-1)^m m!)^p \frac{m^v \lambda^m}{\left((n)_m\right)^p}, \tag{29}$$

where $|\lambda| < 1$.

For special values of λ, p, and v, we give the following alternating generating functions for the sum $\mathcal{S}_v(n;\lambda,p)$:

Putting $v = 0$ and $\lambda = p = 1$ in (6), we have

$$\mathcal{S}_0(n;1,1) = \sum_{m=0}^{n} \frac{1}{\binom{n}{m}} \tag{30}$$

(*cf.*[20,18]).

Alternative forms of the generating functions for the sum $\mathcal{S}_0(n;1,1)$ are given as follows:

$$\frac{2\ln(1-z)}{z^2-2z}=\sum_{n=0}^{\infty}\mathcal{S}_0(n;1,1)z^n \tag{31}$$

(*cf.*[20], see also[18]),

$$F_{ib}(u)=\frac{2}{2-u}\frac{1}{1-u}-\frac{2\ln(1-u)}{(2-u)^2}=\sum_{n=0}^{\infty}\mathcal{S}_0(n;1,1)u^n \tag{32}$$

(*cf.*[7] Exercise 30*, p. 272),

$$-\frac{2z\ln(1-z)}{(2-z)^3}-\frac{z\,(3z-4)}{(2-z)^2\,(1-z)^2}=\sum_{n=0}^{\infty}\mathcal{S}_1(n;1,1)z^n$$

(*cf.*[4,18]), and

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathcal{S}_v(n;1,1)z^n-2\sum_{j=0}^{v}(-1)^{v+j+1}\frac{(j+1)}{(2-z)^{j+2}}W_{v,j}\ln(1-z)\\ &=\sum_{0\le j\le d\le v-1}^{v}(-1)^{v+j+1}\frac{(j+1)}{(2-z)^{d-j+1}}W_{v,v-j}\left(\frac{(d-j+1)\left((1-(1-z)^{d-v}\right)}{(d-v)(2-z)}+(1-z)^{d-v-1}\right)\\ &+\frac{2}{1-z}\sum_{j=0}^{v}(-1)^{v+j}\frac{(j+1)}{(2-z)^{j+1}}W_{v,j}\end{aligned}$$

where $W_{n,m}$ denotes the Worpitzky numbers (A028246), defined by

$$W_{n,m}=\sum_{d=0}^{m}(-1)^{m+d}\binom{m}{d}(d+1)^n=m!S(m,n)$$

(*cf.*[4]).

2.2 | Euler-Frobenius polynomial related to B-spline and exponential Euler spline

Here using (5), we give a relation between the Euler-Frobenius polynomial related to B-spline and exponential Euler spline.

Putting $n=1$ and $\lambda=-\mu$ in (29), we get

$$B_v(-1;-\mu,p)=\sum_{m=0}^{\infty}m^v\mu^m, \tag{33}$$

where $|\mu|<1$.

Substituting $p=0$ into (5), for $|\lambda|<1$, the sum $B_v(\beta;\lambda,p)$ reduces to the well-known interpolation function for the Euler-Frobenius polynomials $\Pi_v(\lambda)$ and the Eulerian polynomials (or Euler Frobenius polynomials) $\mathcal{A}_n(\lambda)$, which are related the B-spline and spline functions:

$$\sum_{m=0}^{\infty}(m+1)^v\lambda^m=\frac{\Pi_v(\lambda)}{(1-\lambda)^{v+1}},$$

where $|\lambda|<1$ and the polynomials $\Pi_v(\lambda)$ are defined by

$$\frac{1}{\lambda-e^u}=\sum_{v=0}^{\infty}\frac{\Pi_v(\lambda)}{(\lambda-1)^{v+1}}\frac{u^v}{v!}, \tag{34}$$

(*cf.*[23] p. 391, Lemma 7. 1.) and

$$\sum_{m=0}^{\infty}m^v\lambda^m=\frac{\mathcal{A}_n(\lambda)}{(1-\lambda)^{v+1}},$$

where $|\lambda|<1$ and the Eulerian polynomials $\mathcal{A}_n(\lambda)$, defined by

$$\frac{1-\lambda}{1-\lambda e^{w(1-\lambda)}}=\sum_{d=0}^{\infty}\mathcal{A}_d(\lambda)\frac{w^d}{d!},$$

where

$$\mathcal{A}_d(\lambda) = \sum_{e=1}^{d} \left\langle \begin{matrix} d \\ e \end{matrix} \right\rangle \lambda^e$$

and

$$\left\langle \begin{matrix} d \\ e \end{matrix} \right\rangle = \sum_{c=0}^{e} (-1)^c \binom{d+1}{c} (1+e-c)^d,$$

which are given the number of permutations of $\{1, 2, \ldots, d\}$ with permutation $e-1$ ascents (*cf.*[1,5,6,15 23] p. 391, Lemma 7.1).

Combining (34) with (10), we get

$$-\frac{1}{\lambda} \sum_{v=0}^{\infty} \frac{\mathcal{B}_v(0;\lambda)}{v!} u^v = \sum_{v=0}^{\infty} \frac{\Pi_v(\lambda)}{v!\,(\lambda-1)^{v+1}} u^{v+1}.$$

Therefore

$$\Pi_{v-1}(\lambda) = -\frac{(\lambda-1)^v}{\lambda} \mathcal{B}_v(0;\lambda),$$

where $v \in \mathbb{N}$.

We observe that in the light of the above considerations, it can easily be seen that the $B_v(\beta; \lambda, p)$ series is also a generalization of the interpolation functions for the Euler-Frobenius polynomials $\Pi_v(\lambda)$ and the Eulerian polynomials.

For $\lambda_+ := \{0, \lambda\}$ with $\lambda \in (-\infty, \infty)$, in terms of the B-spline

$$Q(\lambda) := Q_{k+1}(\lambda) = \frac{1}{k!}\lambda_+^k - \frac{1}{k!}\binom{k+1}{1}(\lambda-1)_+^k + \cdots + \frac{(-1)^{k+1}}{k!}(\lambda-1-k)_+^k,$$

the Euler-Frobenius polynomials $\Pi_v(\lambda)$ can be expressed by

$$\Pi_k(\lambda) = k! \sum_{j=0}^{k-1} Q_{k+1}(j+1)\lambda^j,$$

where $Q(\lambda) > 0$ if $0 < \lambda < k+1$, $Q(\lambda) = 0$ elsewhere (*cf.*[23]).

Using (9), a relation between the polynomials $\Pi_v(\lambda)$ and $H_v(s;\lambda)$ is given by

$$\sum_{v=0}^{\infty} \frac{H_v(s;\lambda)}{1-\lambda} \frac{u^v}{v!} = e^{su} \sum_{v=0}^{\infty} \frac{\Pi_v(\lambda)}{(\lambda-1)^{v+1}} \frac{u^v}{v!},$$

we get

$$\sum_{v=0}^{\infty} H_v(s;\lambda) \frac{u^v}{v!} = \sum_{v=0}^{\infty} \sum_{j=0}^{v} \frac{\Pi_j(\lambda) s^{v-j}}{(\lambda-1)^j} \frac{u^v}{v!}.$$

By equalizing the coefficients of $\frac{u^v}{v!}$ on both sides of the previous equation, we arrive at the following result after the necessary calculations.

$$H_v(s;\lambda) = \sum_{j=0}^{v} \frac{\Pi_j(\lambda) s^{v-j}}{(\lambda-1)^j}.$$

Remark 1. Polynomial spline functions and B-spline functions have been considered as broken polynomials with certain smoothness (*cf.*[15 23] p. 391, Lemma 7.1). There are many geometric interpretation of the Euler-Frobenius polynomial associated with the B-spline and the exponential Euler spline. B-spline functions can be used in many different mathematical model which are used to real world problems associated with computer geometric design, mathematical analysis, medicine, engineering etc.

2.3 | Formulas for special values of the sum $\mathcal{S}_v(n; \lambda, p)$

Here, we give some formulas derived from some special values of the sum $\mathcal{S}_v(n; \lambda, p)$. For the special values of v, λ, and p, some special values the sum $\mathcal{S}_v(n; \lambda, p)$ have been recently considered by Belbachir[4], Charalambides[7], Sury[36], Gould[13,12], Mansour[18], Riphon[21], Sprugnoli[37,38], and Rockett[22].

Substituting $\lambda = -1$ and $p = 1$ into (6), according to the work of Sprugnoli[38], we have the following recurrence relation:

Corollary 9. Let $n, v \in \mathbb{N}_0$. Then we have

$$\mathcal{S}_{v+1}(n;-1,1) = (n+1)\left(\mathcal{S}_v(n;-1,1) - \mathcal{S}_{v+1}(n+1;-1,1) + (-1)^{n+1}(n+1)^v\right).$$

Theorem 8. Let $n \in \mathbb{N}$ with $n > 1$ and $\lambda \neq 0$. Then we have

$$(-1)^{n-1} y(n-1,\lambda) + \frac{2}{\lambda^n(\lambda-1)} + (\lambda+1)\sum_{j=0}^{n-1}\frac{(\lambda-1)^{j-n-1}}{\lambda^{j+1}}$$
$$= 2\sum_{j=0}^{n}\frac{\mathcal{S}_0(n-j;1,1)}{\lambda^{n-j}(\lambda-1)^{j+1}} - 2\sum_{j=0}^{n-1}\frac{\mathcal{S}_0(n-1-j;1,1)}{\lambda^{n-j}(\lambda-1)^{j+1}} + \frac{1}{2}\sum_{j=0}^{n-2}\frac{\mathcal{S}_0(n-2-j;1,1)}{\lambda^{n-j}(\lambda-1)^{j+1}}.$$

Proof. Using (32), we get the following functional equation:

$$\ln(1-z) = -\frac{1}{2}(2-z)^2 F_{ib}(z) + \frac{2-z}{1-z}. \tag{35}$$

Substituting $z = \frac{\lambda-1}{\lambda}w$ into the equation (35) and combining with the equation (8), we get the following functional equation:

$$G(w,\lambda) = \frac{2-\frac{\lambda-1}{\lambda}w}{\left(w^2-w\right)\left(1-\frac{\lambda-1}{\lambda}w\right)} + 2\frac{\left(1-\frac{\lambda-1}{2\lambda}w\right)^2}{\left(w-w^2\right)}F_{ib}\left(\frac{\lambda-1}{\lambda}w\right).$$

Combining the above equation with (32) and (8), we get

$$\sum_{n=0}^{\infty}(-1)^n(\lambda-1)^{n+2}y(n,\lambda)w^{n+1} = \left(-2+\frac{\lambda-1}{\lambda}w\right)\sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{(\lambda-1)^j}{\lambda^j}w^n$$
$$+2\left(1-\frac{\lambda-1}{2\lambda}w\right)^2\sum_{n=0}^{\infty}\sum_{j=0}^{n}\left(\frac{\lambda-1}{\lambda}\right)^{n-j}\mathcal{S}_0(n-j;1,1)w^n.$$

After some elementary calculations and also by equalizing the coefficients of w^n on both sides of the previous equation, we arrive at the desired result after the necessary calculations. □

Substituting $v = 0$, $\lambda = -1$ and $p = 1$ into (6), we have the following result:

Corollary 10. If n is an even non-negative integer, then

$$\mathcal{S}_0(n;-1,1) = \frac{2(n+1)}{n+2}, \tag{36}$$

otherwise $\mathcal{S}_0(n;-1,1) = 0$.

Noting that in the literature, there are many studies, which have been carried out on the equation (36) in recent years. See, for instance, related references: (*cf.*[4,18,26]); and the references cited in each of these earlier works.

Among other special results, we give the following further remarks and observation on the polynomial $\mathcal{S}_v(n;\lambda,p)$:

Remark 2. Putting $v = 0$ and $p = 1$ in (6), we have

$$\mathcal{S}_0(n;\lambda,1) = \frac{n+1}{(\lambda+1)\left(1+\frac{1}{\lambda}\right)^{n+1}}\sum_{j=1}^{n+1}\frac{(\lambda^j+1)\left(1+\frac{1}{\lambda}\right)^j}{j}$$

(*cf.*[18]). Substituting $\lambda = 1$ into (6), we have

$$\mathcal{S}_0(n;1,p) = (n+1)^p\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\frac{(-1)^j\binom{k}{j}}{n-k+j+1}\right)^p$$

(*cf.*[18]). Substituting $v = 0$, $\lambda = 1$ and $p = 2$ into (6), we have

$$\mathcal{S}_0(n;1,2) = 2(n+1)^2\sum_{k=0}^{n}\sum_{j=0}^{k}\frac{(-1)^j\binom{k}{j}}{(n+j+2)(n-k+1)} \tag{37}$$

(*cf.*[18]). Putting $v = \lambda = p = 1$ in (6), we have

$$\mathcal{S}_1(n;1,1) = \frac{(n+1)(2^n-1)}{2^n} + \sum_{j=0}^{n-2}\frac{(n-j)(n-j-1)2^{j-m-1}}{j+1}$$

(*cf.* [18]).

3 | FORMULAS FOR THE SUMS $y_1(v,d;\lambda)$, $y_6(m,n;\lambda,p)$, $\mathcal{S}_v(n;\lambda,p)$ AND $B_v(\beta;\lambda,p)$ DERIVE FROM DERIVATIVE OPERATOR

In this section, using the Euler operator $\vartheta = \lambda\frac{d}{d\lambda}$ and $D_z = \frac{d}{dz}$, we give some formulas for finite sums involving the sums $y_1(v,d;\lambda)$, $y_6(m,n;\lambda,p)$, $\mathcal{S}_v(n;\lambda,p)$, and $B_v(\beta;\lambda,p)$.

Let $v \in \mathbb{N}$. Setting

$$\vartheta^v := \left(\lambda\frac{d}{d\lambda}\right)^v$$

denote v operations $\lambda\frac{d}{d\lambda}$ each on the analytic function $f(\lambda)$, λ and $\frac{d}{d\lambda}$ not being permutable. That is, the operator ϑ^v is given by the following formula:

$$\vartheta^v\{f(\lambda)\} = \sum_{j=1}^{v}(-1)^j\frac{1}{j!}\sum_{l=1}^{j}(-1)^l\binom{j}{l}j^v\lambda^j\frac{d^j}{d\lambda^j}\{f(\lambda)\} \tag{38}$$

(*cf.* [24] p. 80, Eq. (2)).

Combining (38) with (12), we also have the following well-known formula:

$$\vartheta^v\{f(\lambda)\} = \sum_{j=0}^{v}S(v,j)\lambda^j\frac{d^j}{d\lambda^j}\{f(\lambda)\},$$

(*cf.* [24] p. 80, Eq. (2)).

Applying the operator ϑ to equation (1), we get the following result:

Corollary 11. Let $v, m \in \mathbb{N}_0$. We have

$$y_6(m,n;\lambda,p) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}^p\vartheta^m\left\{\lambda^k\right\}.$$

Combining the above equation with (38), we get

$$y_6(m,n;\lambda,p) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}^p\sum_{j=1}^{m}(-1)^j\frac{1}{j!}\sum_{l=1}^{j}(-1)^l\binom{j}{l}j^m\lambda^j\frac{d^j}{d\lambda^j}\left\{\lambda^k\right\}.$$

Substituting (12) into the above equation, we arrive at the following result:

Theorem 9.

$$y_6(m,n;\lambda,p) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}^p\sum_{j=1}^{m}S(m,j)\lambda^j\frac{d^j}{d\lambda^j}\left\{\lambda^k\right\}.$$

Noting that $\mathcal{S}_v(n;\lambda,p)$ is a λ variable polynomial of degree n.

Applying the operator ϑ to the equation (6), we also get the following result:

$$\mathcal{S}_v(n;\lambda,p) = \sum_{j=0}^{n}\frac{1}{\binom{n}{j}^p}\vartheta^v\left\{\lambda^j\right\}.$$

Combining the above equation with (38), we arrive at the following theorem:

Theorem 10. Let $n \in \mathbb{N}_0$. We have

$$\mathcal{S}_v(n;\lambda,p) = \sum_{j=0}^{n}\frac{1}{\binom{n}{j}^p}\sum_{k=0}^{v}S(v,k)\lambda^k\frac{d^k}{d\lambda^k}\left\{\lambda^j\right\}. \tag{39}$$

Another different explicit formula for the numbers $\mathcal{S}_v(n;\lambda,p)$ is also given by the following formula:

Corollary 12. Let $v \in \mathbb{N}_0$. We have

$$B_v(\beta;\lambda,p) = \frac{\partial^v}{\partial z^v}\{F(z;\beta,\lambda,p)\}\big|_{z=0},$$

where $|\lambda| < 1$.

Theorem 11. Let $v \in \mathbb{N}_0$. We have

$$\sum_{m=0}^{\infty} \binom{-b}{m} \vartheta^{v} \left\{ \lambda^{m} \right\} = b \binom{b+v}{v} \sum_{d=0}^{v} (-1)^{d} \binom{v}{d} \frac{d! y_1 (v, d; \lambda)}{(b+d) (\lambda+1)^{b+d}}. \tag{40}$$

Proof. For the proof of assertion of the theorem, we can use the following derivative operator $D_z^v \left\{ x^{-b} \right\}$ (*cf.* [12] Eq. (8)):

$$D_z^v \left\{ x^{-b} \right\} = b \binom{b+v}{v} \sum_{d=0}^{v} (-1)^{d} \binom{v}{d} \frac{x^{-b-d}}{b+d} D_z^v \left\{ x^{d} \right\}, \tag{41}$$

where b is a real number, x is a function of z. Substituting

$$x = \lambda e^{t} + 1$$

into (41), we get

$$\frac{\partial^{v}}{\partial t^{v}} \left\{ \left(\lambda e^{t} + 1 \right)^{-b} \right\} |_{t=0} = b \binom{b+v}{v} \sum_{d=0}^{v} (-1)^{d} \binom{v}{d} \frac{D_z^v \left\{ \left(\lambda e^{t} + 1 \right)^{d} \right\} |_{t=0}}{(b+d) (\lambda+1)^{b+d}}. \tag{42}$$

Assuming that $\left| \lambda e^{t} \right| < 1$. Applying the binomial theorem to the previous equation and using (14), we obtain

$$\frac{\partial^{v}}{\partial t^{v}} \left\{ \sum_{m=0}^{\infty} \binom{-b}{m} \left(\lambda e^{t} \right)^{m} \right\} |_{t=0} = b \binom{b+v}{v} \sum_{d=0}^{v} (-1)^{d} \binom{v}{d} \frac{1}{(b+d) (\lambda+1)^{b+d}} \frac{\partial^{v}}{\partial t^{v}} \left\{ \sum_{m=0}^{\infty} y_1 (m, d; \lambda) \frac{t^{m}}{m!} \right\} |_{t=0}.$$

After the necessary calculations are done in the above equation, we arrive at the assertion of the theorem. □

Putting the following formula in (40)

$$d! y_1 (v, d; \lambda) = \frac{d^{v}}{du^{v}} \left\{ (\lambda e^{u} + 1)^{d} \right\} |_{u=0}$$

(*cf.* [39]), then we have the following corollary:

Corollary 13. Let $v \in \mathbb{N}_0$. We have

$$\sum_{m=0}^{\infty} \binom{-b}{m} \vartheta^{v} \left\{ \lambda^{m} \right\} = \sum_{d=0}^{v} \frac{(-1)^{d} b \binom{b+v}{v} \binom{v}{d}}{(b+d) (\lambda+1)^{b+d}} \frac{d^{v}}{du^{v}} \left\{ (\lambda e^{u} + 1)^{d} \right\} |_{u=0} .$$

For $p = 0$, combining (20) with (42), after similar calculations in the proof of the previous theorem, the following result is also obtained:

Corollary 14. Let $v \in \mathbb{N}_0$. We have

$$\sum_{m=0}^{\infty} \binom{-b}{m} \vartheta^{v} \left\{ \lambda^{m} \right\} = b \binom{b+v}{v} \sum_{d=0}^{v} (-1)^{d} \binom{v}{d} \frac{\mathcal{S}_v(d; \lambda, 0)}{D_\lambda \left\{ (\lambda+1)^{b+d+1} \right\}}. \tag{43}$$

Substituting $b = v = 1$ into (40), and using $y_1 (1, 0; \lambda) = 0$ and $y_1 (1, 1; \lambda) = \lambda$, we easily have the following known infinite series representation:

$$\sum_{m=0}^{\infty} \binom{-1}{m} m \lambda^{m-1} = -\frac{1}{(\lambda+1)^{2}}.$$

Combining (40) with (43), we arrive at the following theorem:

Theorem 12. Let $v \in \mathbb{N}_0$. We have

$$\sum_{d=0}^{v} (-1)^{d} \binom{v}{d} \frac{1}{D_\lambda \left\{ (\lambda+1)^{b+d+1} \right\}} \sum_{j=0}^{v} \left(\frac{1}{j!(v-j)!} - 1 \right) \vartheta^{v} \left\{ \lambda^{j} \right\} = 0.$$

4 | RELATIONS AMONG THE FINITE SUMS $y_6(m, n; \lambda, p)$, $\mathcal{S}_v(m; \mu, q)$, $y_1(v, k; \lambda)$, AND THE STIRLING NUMBERS

In this section, using generating functions with their functional equations, we give some relations among the finite sums $y_6(m, n; \lambda, p)$, $\mathcal{S}_v(m; \mu, q)$, $y_1(v, k; \lambda)$, and the Stirling numbers of the second kind.

Using (2) and (4), we get

$$F_{y6}(z,n;\lambda,p)F(z,v;\mu,q)=\sum_{m=0}^{\infty}\sum_{j=0}^{m}\binom{m}{j}y_6(j,n;\lambda,p)B_{m-j}(\beta;\mu,q)\frac{z^m}{m!}$$

and

$$F_{y6}(z,n;\lambda,p)F(z,v;\mu,q)=\sum_{m=0}^{\infty}\frac{1}{n!}\sum_{j=0}^{\infty}\sum_{l=0}^{v}\frac{\binom{v}{l}^q}{\binom{\beta}{j}^p}\lambda^j\mu^l\frac{(l+j)^m z^m}{m!},$$

where $|\lambda|<1$. Therefore, we arrive at the following theorem:

Theorem 13. Let $n\in\mathbb{N}$. We have

$$\sum_{j=0}^{m}\binom{m}{j}y_6(j,n;\lambda,p)B_{m-j}(\beta;\mu,q)=\sum_{j=0}^{\infty}\sum_{l=0}^{v}\frac{\binom{v}{l}^q}{n!\binom{\beta}{j}^p}\lambda^j\mu^l(l+j)^m, \tag{44}$$

where $|\lambda|<1$.

By using (4), we get

$$\sum_{j=0}^{\infty}\frac{\lambda^j}{\binom{\beta}{j}^p}\sum_{k=0}^{j}\binom{j}{k}(e^z-1)^k=\sum_{v=0}^{\infty}B_v(\beta;\lambda,p)\frac{z^v}{v!}.$$

Combining the above equation with (12), we obtain

$$\sum_{j=0}^{\infty}\frac{\lambda^j}{\binom{\beta}{j}^p}\sum_{k=0}^{j}\binom{j}{k}k!\sum_{v=0}^{\infty}S(v,k)\frac{z^v}{v!}=\sum_{v=0}^{\infty}B_v(\beta;\lambda,p)\frac{z^v}{v!}.$$

By equalizing the coefficients of $\frac{z^v}{v!}$ on both sides of the previous equation, we arrive at the following theorem after the necessary calculations:

Theorem 14. Let $p,v\in\mathbb{N}_0$. Then we have

$$B_v(\beta;\lambda,p)=\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{\binom{j}{k}}{\binom{\beta}{j}^p}k!\lambda^j S(v,k), \tag{45}$$

where $|\lambda|<1$.

By using (4), we get

$$\sum_{j=0}^{\infty}\frac{\lambda^j}{\binom{\beta}{j}^p}\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}(e^z+1)^k=\sum_{v=0}^{\infty}B_v(\beta;\lambda,p)\frac{z^v}{v!}.$$

Combining the above equation with (14), we obtain

$$\sum_{j=0}^{n}\frac{\lambda^j}{\binom{n}{j}^p}\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}k!\sum_{v=0}^{\infty}y_1(v,k;1)\frac{z^v}{v!}=\sum_{v=0}^{\infty}\mathcal{S}_v(n;\lambda,p)\frac{z^v}{v!}.$$

By equalizing the coefficients of $\frac{z^v}{v!}$ on both sides of the previous equation, we arrive at the following result after the necessary calculations:

Theorem 15. Let $p,v\in\mathbb{N}_0$. Then we have

$$B_v(\beta;\lambda,p)=\sum_{j=0}^{\infty}\sum_{k=0}^{j}(-1)^{j-k}\frac{\binom{j}{k}}{\binom{\beta}{j}^p}k!\lambda^j y_1(v,k;1), \tag{46}$$

where $|\lambda|<1$.

Since

$$k!y_1(v,k;1)=\frac{d^v}{dt^v}\left\{\left(e^t+1\right)^k\right\}|_{t=0}$$

(*cf.*[11,27]), we modify equation (46) by the following corollary:

Corollary 15. Let $p, v \in \mathbb{N}_0$. Then we have

$$B_v(\beta;\lambda,p)=\sum_{j=0}^{\infty}\sum_{k=0}^{j}(-1)^{j-k}\frac{\binom{j}{k}}{\binom{\beta}{j}^{p}}\lambda^{j}\frac{d^{v}}{dt^{v}}\left\{\left(e^{t}+1\right)^{k}\right\}|_{t=0},$$

where $|\lambda|<1$.

Theorem 16. Let $m \in \mathbb{N}_0$. Then we have

$$\int_{0}^{\lambda}\frac{(1+x)^{m+1}-1}{x-1}dx=\frac{\left(S_0(m;\lambda,1)+\lambda^{m+1}(m+1)\,y(m,-\lambda)\right)(1+\lambda)^{m+2}}{(m+1)\,\lambda^{m+1}}.$$

Proof. Combining the following known equation

$$\sum_{j=0}^{m}\frac{1}{\binom{m}{j}}\lambda^{j}=(m+1)\sum_{j=0}^{m}\frac{\lambda^{m+1}+\lambda^{m-j}}{(j+1)(1+\lambda)^{m+1-j}}$$

(*cf.*[36]) with (6), we get

$$S_0(m;\lambda,1)=(m+1)\sum_{j=0}^{m}\frac{\lambda^{m+1}}{(j+1)(1+\lambda)^{m+1-j}}+(m+1)\lambda^{m+1}\sum_{j=0}^{m}\frac{1}{(j+1)\lambda^{j+1}(1+\lambda)^{m+1-j}}.$$

Combining the right hand side of the above equation with (7), after some elementary calculations, we obtain

$$\sum_{j=0}^{m}\frac{1}{(j+1)(1+\lambda)^{m+1-j}}=\frac{S_0(m;\lambda,1)+\lambda^{m+1}(m+1)\,y(m,-\lambda)}{(m+1)\,\lambda^{m+1}}. \tag{47}$$

Combining the left side of the equation (47) with the following equation

$$\begin{aligned}\sum_{j=0}^{m}\frac{1}{(j+1)(1+\lambda)^{m+1-j}}&=\frac{1}{(1+\lambda)^{m+2}}\sum_{j=0}^{m}\int_{0}^{\lambda}(1+x)^{j}\,dx\\&=\frac{1}{(1+\lambda)^{m+2}}\int_{0}^{\lambda}\frac{(1+x)^{m+1}-1}{x-1}dx,\end{aligned}$$

we arrive at the desired result. □

Some special values of $S_v(n;\lambda,p)$ are given as follows:

Combining the following well-known formula (*cf.*[8] Eq. (2.15),[36]):

$$\frac{1}{\binom{n}{j}}=(n+1)\sum_{l=0}^{j}(-1)^{j-l}\binom{j}{l}\frac{1}{n-l+1}$$

with (6), for $p\geq 1$, we arrive at the following corollary:

Corollary 16. Let $n \in \mathbb{N}_0$. Then we have

$$S_v(n;\lambda,p)=(n+1)\sum_{j=0}^{n}\sum_{l=0}^{j}(-1)^{j-l}\frac{\binom{j}{l}}{\binom{n}{j}^{p-1}}\frac{j^{v}\lambda^{j}}{n-l+1}. \tag{48}$$

Substituting $v=0$ and $p=1$ and $\lambda=-1$ into (6), we have

$$S_0(2n;-1,1)=2+\sum_{j=1}^{2n-1}\frac{(-1)^{j}}{\binom{2n}{j}}.$$

Combining the above equation with the following well-known formula, for $n\in\mathbb{N}$,

$$\sum_{j=1}^{2n-1}\frac{(-1)^{j-1}}{\binom{2n}{j}}=\frac{1}{n+1}$$

(*cf.*[37]), we have the following result:

Corollary 17. Let $n \in \mathbb{N}_0$. Then we have

$$\mathcal{S}_0(2n; -1, 1) = \frac{2n+1}{n+1}.$$

Substituting $v = p = 1$ and $\lambda = -1$ into (6), we have

$$\mathcal{S}_1(2n; -1, 1) = 2n - \sum_{j=1}^{2n-1} \frac{(-1)^j j}{\binom{2n}{j}}.$$

Combining the above equation with the following well-known formula, for $n \in \mathbb{N}$,

$$\sum_{j=1}^{2n-1} \frac{(-1)^{j-1} j}{\binom{2n}{j}} = \frac{n}{n+1}$$

(*cf.*[37]), we arrive at the following result:

Corollary 18. Let $n \in \mathbb{N}_0$. Then we have

$$\mathcal{S}_1(2n; -1, 1) = \frac{2n^2+n}{n+1}.$$

Remark 3. Substituting $v = 0$, $p = 1$ and $\lambda = -1$ into (6), we have

$$\mathcal{S}_0(n; -1, 1) = \sum_{j=0}^{n} \frac{(-1)^v}{\binom{n}{j}} = (1 + (-1)^n)\frac{n+1}{n+2},$$

(*cf.*[37]). This sum is also derived from the following well-known formula:

$$\sum_{j=0}^{n} \frac{(-1)^v}{\binom{x}{j}} = \left(1 + \frac{(-1)^n}{\binom{x+1}{n+1}}\right)\frac{x+1}{x+2},$$

where $x \neq -1, -2$ (*cf.*[37]).

Remark 4. The finite and infinite sums involving $y_6(0, n; \lambda, p)$, inverse binomial summation formulas involving the hypergeometric transformation formulas were given[21] and[28].

Substituting the following well-known formula, which is related to (30),

$$\mathcal{S}_0(n; 1, 1) = 2(n+1)y(n, -1) \tag{49}$$

(*cf.*[30] Eq. (6.5)) into the following equation

$$y(n-1, -1) = 2y(n, -1) - \frac{2}{n+1} \tag{50}$$

(*cf.*[30] Corollary 12), after some calculations, we find the following recurrence relation for the sums $\mathcal{S}_0(n; 1, 1)$:

Theorem 17. Let $n \in \mathbb{N}$. Then we have

$$2n\mathcal{S}_0(n; 1, 1) - (n+1)\,\mathcal{S}_0(n-1; 1, 1) = 2n. \tag{51}$$

Remark 5. Putting $v = 0$, $\lambda = p = 1$ in (6), we have (49), which is also given by generating function (32). By using (32), Charalambides[7] Exercise 30*, p. 272 also gave the following formulas:

$$\mathcal{S}_0(n; 1, 1) = \sum_{v=0}^{n} \frac{n+1}{v+1} 2^{v-n}, \tag{52}$$

where $n \in \mathbb{N}_0$ (*cf.*[7] p. 273). Here we also note that there are many other different proof of the equation (52), See, for instance, (*cf.*[4,22,36,26,18]); and the references cited in each of these earlier works. With the aid of (51), we also have solution of Exercise 30* in[7] p. 273.

4.1 | Formulas for the sums of inverses of binomial coefficients derived from the numbers $y(m, \lambda)$

Here, using generating functions with their functional equations, we obtain some novel formulas for the sums of inverses of binomial coefficients derived from the numbers $y(m, \lambda)$.

By combining (32) with (11) and (13) yields the following functional equation:

$$F_{ib}(z) = g(z; -1)\left(\frac{1}{z-1} + \frac{F_1(z)}{2-z}\right). \tag{53}$$

By using (53), we get

$$\sum_{n=0}^{\infty} \mathcal{S}_0(n; 1, 1)z^n = -\sum_{n=0}^{\infty} Y_n(-1)\frac{z^n}{n!}\sum_{n=0}^{\infty} z^n + \frac{z-1}{2}\sum_{n=0}^{\infty} Y_n(-1)\frac{z^n}{n!}\sum_{n=0}^{\infty}\sum_{j=0}^{n} 2^{-j}H_{n-j}z^n.$$

Therefore

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{S}_0(n; 1, 1)z^n &= -\sum_{n=0}^{\infty}\sum_{j=0}^{n} Y_j(-1)\frac{z^n}{j!} + \frac{1}{2}\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\sum_{j=0}^{k}\frac{2^{-j}Y_{n-k-1}(-1)H_{k-j}}{(n-k-1)!}z^n \\ &\quad -\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}\frac{Y_{n-k}(-1)H_{k-j}}{2^{j+1}(n-k)!}z^n.\end{aligned}$$

Now equating the coefficients of z^n on both sides of the above equation, we arrive at the following theorem:

Theorem 18. Let $n \in \mathbb{N}$. Then we have

$$\mathcal{S}_0(n; 1, 1) = -\sum_{j=0}^{n}\frac{Y_j(-1)}{j!} + \sum_{k=0}^{n-1}\sum_{j=0}^{k}\frac{Y_{n-k-1}(-1)H_{k-j}}{2^{j+1}(n-k-1)!} - \sum_{k=0}^{n}\sum_{j=0}^{k}\frac{Y_{n-k}(-1)H_{k-j}}{2^{j+1}(n-k)!}.$$

Substituting $\lambda = -1$ into (7), we obtain, respectively

$$y(m, -1) = \sum_{j=0}^{m}\frac{2^{j-m-1}}{j+1} \tag{54}$$

(*cf.*[30]) and

$$y(m, -1) = \sum_{v=0}^{m}\sum_{n=0}^{v}(-1)^v\frac{2^{v-m-1}B_n S_1(v, n)}{v!}, \tag{55}$$

Combining (54) with (52), we arrive at (49).

Combining (55) with (49) gives directly the following theorem:

Theorem 19. Let $m \in \mathbb{N}$. Then we have

$$\mathcal{S}_0(m; 1, 1) = (m+1)\sum_{v=0}^{m}\sum_{n=0}^{v}(-1)^v\frac{2^{v-m}B_n S_1(v, n)}{v!}.$$

5 | GRAPHS AND NUMERIC VALUES FOR $\mathcal{S}_v(n; \lambda, p)$ WITH THE HELP OF ALGORITHMS AND SOME CONVERGENCE TABLES FOR $B_v(\beta; \lambda, p)$

In this section, we use the equations from the previous sections to give some algorithms for computing the values of the polynomial $\mathcal{S}_v(n; x, p)$ with their some 2D graphs and 3D surfaces graphs. These algorithms are constructed by the help of equations (6), (39) and (48). By the aid of these algorithms, some numerical values of these polynomials are also given by the tables. Using (5), the values for some convergence intervals for the $B_v(\beta; \lambda, p)$ are calculated.

By (6), we give Algorithm 1.

Algorithm 1 Let $v, n, p \in \mathbb{N}_0$, and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$). Then, this algorithm returns the sum $S_v(n; \lambda, p)$.

---

procedure `SUM_S`(v: nonnegative integer, n: nonnegative integer, λ: real or complex number, p: nonnegative integer)
  Begin
  Local variable j : nonnegative integer
  return `Sum`$\Big($`Power`$[j, v] *$ `Power`$[\lambda, j]$ /`Power`[`Binomial_Coef`$[n, j], p], j, 0, n\Big)$
end procedure

By (5) and (6), we give Algorithm 2.

Algorithm 2 Let $v, p \in \mathbb{N}_0$ and $\beta, \lambda \in \mathbb{R}$ (or $\mathbb{C}$). Then, this algorithm returns the sum $B_v(\beta; \lambda, p)$.

---

procedure `SUM_B`(v: nonnegative integer, β: real or complex number, λ: real or complex number, p: nonnegative integer)
  Begin
  Local variable m : nonnegative integer
  if $\beta \notin \mathbb{N}_0$ & $|\lambda| < 1$ **then**
    return `Sum`$\Big($`Power`$[m, v] *$ `Power`$[\lambda, m]$ /`Power`[`Binomial_Coef`$[\beta, m], p], m, 0,$ `Infinity`$\Big)$
  else if $\beta \in \mathbb{N}_0$ & $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) **then**
    return `SUM_S`$\Big(v, \beta, \lambda, p\Big)$
  else
    return 0
  end if
end procedure

By implementing Algorithm 1 in Wolfram Mathematica 12.0, we give some values of $S_v(n; \lambda, p)$ in Table 1 and 2D plots of $S_v(n; \lambda, p)$ in Figure 1 for some special cases.

	$n = 1$	$n = 2$	$n = 3$	$n = 4$
$v = 1$	λ	$\frac{\lambda}{1024} + 2\lambda^2$	$\frac{\lambda}{59049} + \frac{2\lambda^2}{59049} + 3\lambda^3$	$\frac{\lambda}{1048576} + \frac{\lambda^2}{30233088} + \frac{3\lambda^3}{1048576} + 4\lambda^4$
$v = 2$	λ	$\frac{\lambda}{1024} + 4\lambda^2$	$\frac{\lambda}{59049} + \frac{4\lambda^2}{59049} + 9\lambda^3$	$\frac{\lambda}{1048576} + \frac{\lambda^2}{15116544} + \frac{9\lambda^3}{1048576} + 16\lambda^4$
$v = 3$	λ	$\frac{\lambda}{1024} + 8\lambda^2$	$\frac{\lambda}{59049} + \frac{8\lambda^2}{59049} + 27\lambda^3$	$\frac{\lambda}{1048576} + \frac{\lambda^2}{7558272} + \frac{27\lambda^3}{1048576} + 64\lambda^4$
$v = 4$	λ	$\frac{\lambda}{1024} + 16\lambda^2$	$\frac{\lambda}{59049} + \frac{16\lambda^2}{59049} + 81\lambda^3$	$\frac{\lambda}{1048576} + \frac{\lambda^2}{3779136} + \frac{81\lambda^3}{1048576} + 256\lambda^4$

TABLE 1 Table of the sum $S_v(n; \lambda, p)$ for $v \in \{1, 2, 3, 4\}$, $n \in \{1, 2, 3, 4\}$ and $p = 10$.

Note that the graphs of the polynomials in the 3rd row of the Table 1 are as in in Figure 1 .

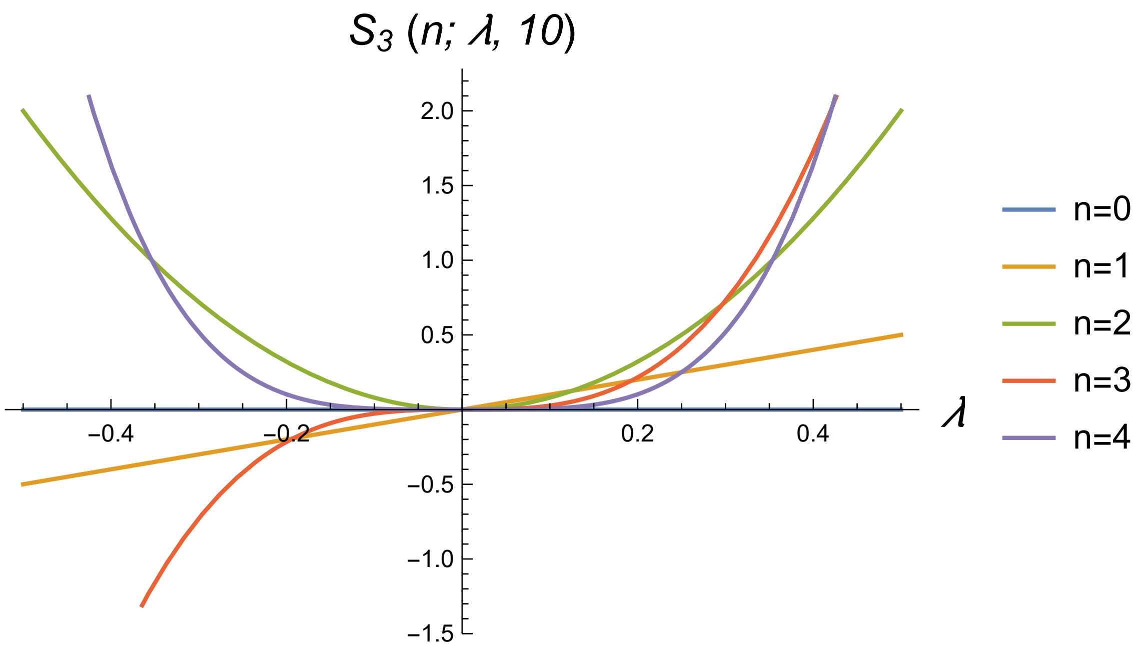

FIGURE 1 Plots of the sum $S_{\upsilon}(n;\lambda,p)$ for $\upsilon=3$, $n\in\{0,1,2,3,4\}$, $\lambda\in\left[-\frac{1}{2},\frac{1}{2}\right]$, $p=10$.

By implementing (19) in Wolfram Mathematica 12.0, we give some 2D plots of the polynomials $Q_{\upsilon}(x,n;\lambda,p)$ in Figure 2 and Figure 3 for their special cases.

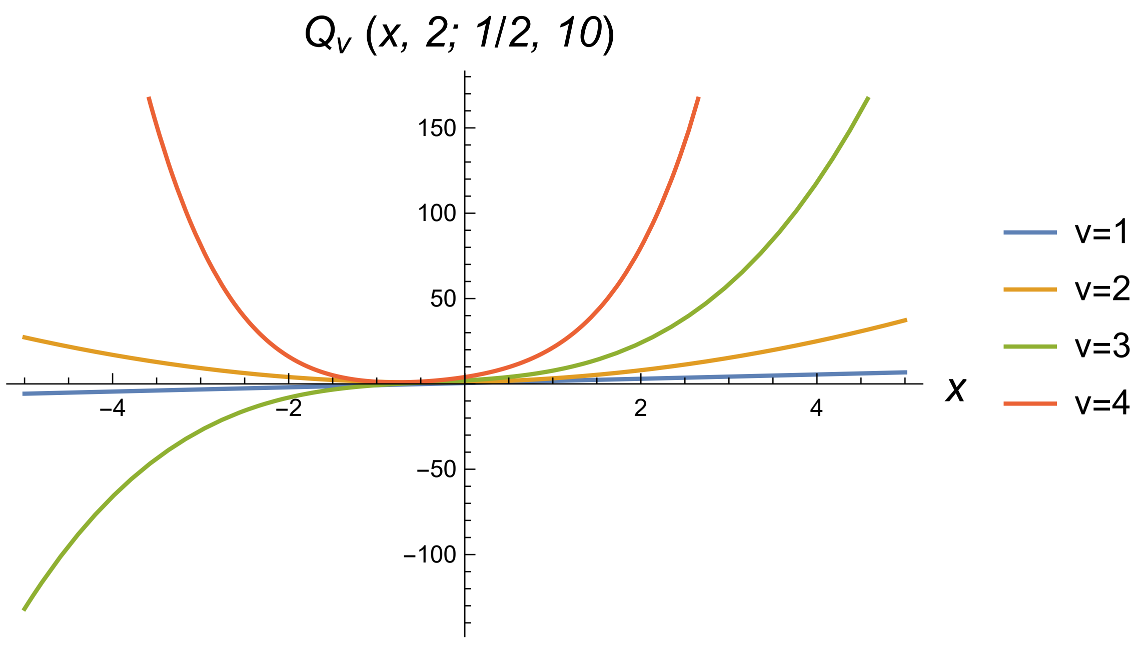

FIGURE 2 Plots of the polynomials $Q_{\upsilon}(x,n;\lambda,p)$ for $\upsilon\in\{1,2,3,4\}$, $x\in\left[-\frac{1}{5},\frac{1}{5}\right]$, $n=2$, $\lambda=\frac{1}{2}$, $p=10$.

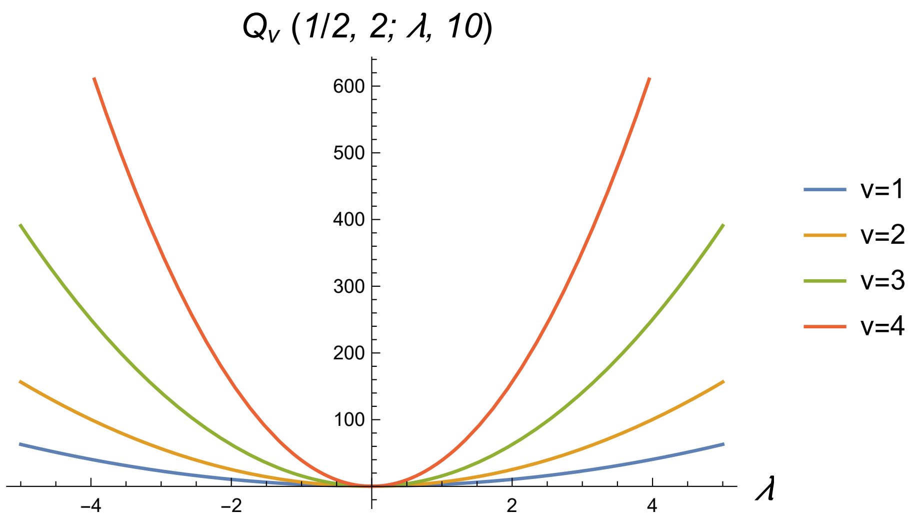

FIGURE 3 Plots of the polynomials $\mathcal{Q}_v(x, n; \lambda, p)$ for $v \in \{1, 2, 3, 4\}$, $x = \frac{1}{2}$, $n = 2$, $\lambda \in [-5, 5]$, $p = 10$.

Some surface plots of the two variable polynomials $\mathcal{Q}_v(x, n; \lambda, p)$ with respect to the variables x and λ are given as follows:

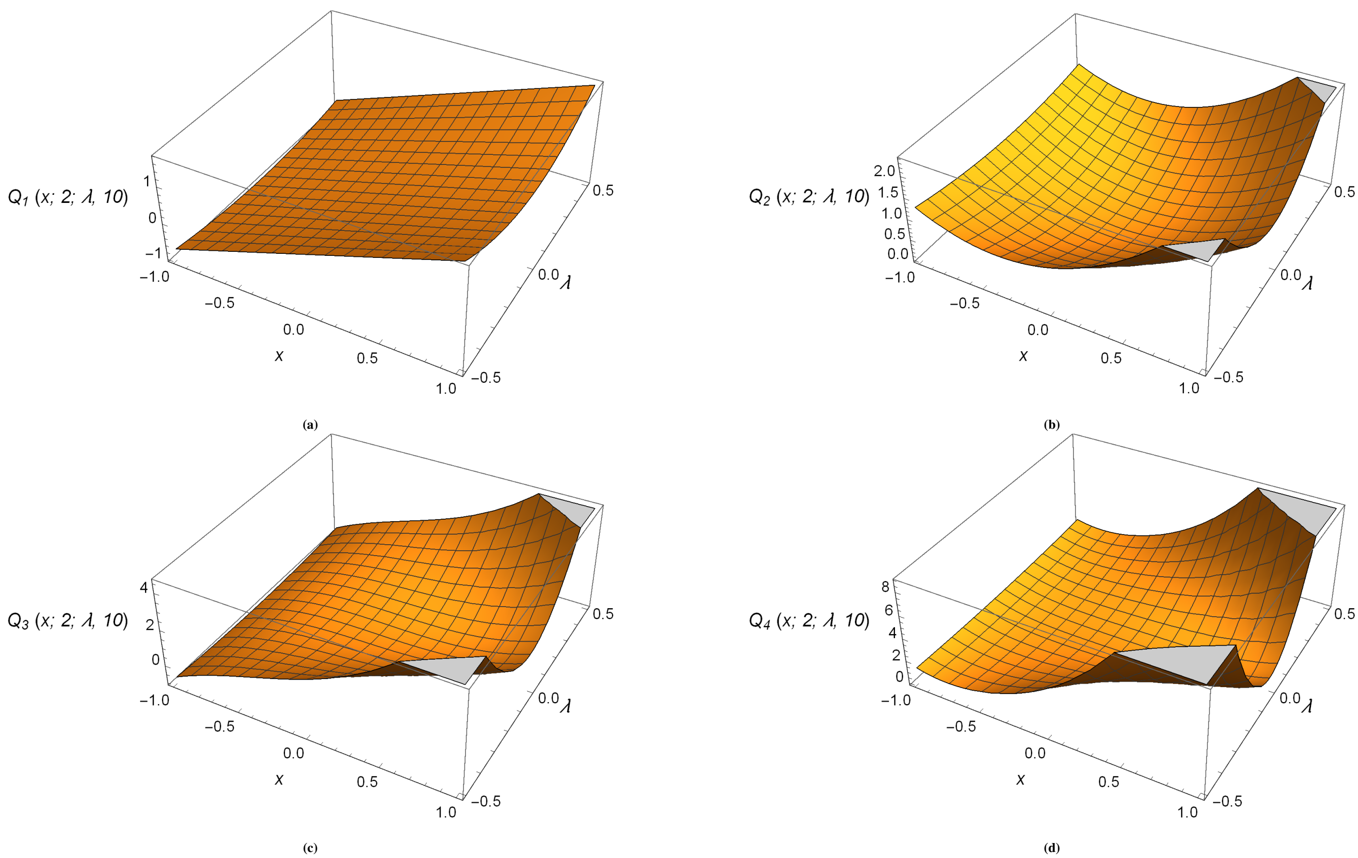

FIGURE 4 Surface plots of the polynomials $\mathcal{Q}_v(x, n; \lambda, p)$ for $x \in [-1, 1]$, $n = 2$, $\lambda \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, $p = 10$ with cases: **(a)** $v = 1$; **(b)** $v = 2$; **(c)** $v = 3$; **(d)** $v = 4$.

Note that different values of the v which is a degree of the polynomial $\mathcal{Q}_v(x, n; \lambda, p)$ are effected the surface plots.

5.1 | Hypergeometric geometric representation and numerical values of the sum $B_v(\beta;\lambda,p)$

Here, with aid of Algorithm 2, we give hypergeometric geometric representation and some numerical values of the sum $B_v(\beta;\lambda,p)$.

By using (3) and (5) with mathematical induction, we arrive at the following theorem:

Theorem 20. Let $v, p \in \mathbb{N}_0$ and $\lambda \in \mathbb{R} \setminus \{0\}$ (or $\mathbb{C} \setminus \{0\}$) with $|\lambda| < 1$. We have

$$ {}_{p+v}F_{p+v-1}(\underbrace{2,\dots,2}_{(p+v)-\text{times}};\underbrace{1,\dots,1}_{p-\text{times}},\underbrace{1-\beta,\dots,1-\beta}_{(v-1)-\text{times}};(-1)^p\lambda) = \frac{\beta^p B_v(\beta;\lambda,p)}{\lambda}. \tag{56} $$

By implementing (56) with Algorithm 2 in Wolfram Mathematica 12.0, we give some numerical values of the sum $B_v(\beta;\lambda,p)$ in Table 2 in which the notation HypergeometricPFQ[$\{a_1,\dots,a_p\}$, $\{b_1,\dots,b_q\}$, z] denotes the Hypergeometric function ${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z)$.

	$p=0$	$p=1$	$p=2$
$v=0$	$\frac{1}{1-\lambda}$	$\mathtt{Hypergeometric2F1}[1,1,-\beta,-\lambda]$	$\mathtt{HypergeometricPFQ}[\{1,1,1\},\{-\beta,-\beta\},\lambda]$
$v=1$	$\frac{\lambda}{(-1+\lambda)^2}$	$\frac{\lambda\mathtt{Hypergeometric2F1}[2,2,1-\beta,-\lambda]}{\beta}$	$\frac{\lambda\mathtt{HypergeometricPFQ}[\{2,2,2\},\{1-\beta,1-\beta\},\lambda]}{\beta^2}$
$v=2$	$-\frac{\lambda(1+\lambda)}{(-1+\lambda)^3}$	$\frac{\lambda\mathtt{HypergeometricPFQ}[\{2,2,2\},\{1,1-\beta\},-\lambda]}{\beta}$	$\frac{\lambda\mathtt{HypergeometricPFQ}[\{2,2,2,2\},\{1,1-\beta,1-\beta\},\lambda]}{\beta^2}$
$v=3$	$\frac{\lambda(1+4\lambda+\lambda^2)}{(-1+\lambda)^4}$	$\frac{\lambda\mathtt{HypergeometricPFQ}[\{2,2,2,2\},\{1,1,1-\beta\},-\lambda]}{\beta}$	$\frac{\lambda\mathtt{HypergeometricPFQ}[\{2,2,2,2,2\},\{1,1,1-\beta,1-\beta\},\lambda]}{\beta^2}$
$v=4$	$-\frac{\lambda(1+11\lambda+11\lambda^2+\lambda^3)}{(-1+\lambda)^5}$	$\frac{\lambda\mathtt{HypergeometricPFQ}[\{2,2,2,2,2\},\{1,1,1,1-\beta\},-\lambda]}{\beta}$	$\frac{\lambda\mathtt{HypergeometricPFQ}[\{2,2,2,2,2,2\},\{1,1,1,1-\beta,1-\beta\},\lambda]}{\beta^2}$

TABLE 2 Table of the sum $B_v(\beta;\lambda,p)$ for $v \in \{0,1,2,3,4\}$ and $p \in \{0,1,2\}$.

With the aid of the Table 2 , some values of the sum $B_v(\beta;\lambda,p)$ are given explicitly as follows:

$$ \begin{aligned} B_1(\beta;\lambda,1) &= \frac{\lambda\,{}_2F_1(2,2;1-\beta;-\lambda)}{\beta}, \\ B_1(\beta;\lambda,2) &= \frac{\lambda\,{}_3F_2(2,2,2;1-\beta,1-\beta;\lambda)}{\beta^2}, \\ B_2(\beta;\lambda,1) &= \frac{\lambda\,{}_3F_2(2,2,2;1,1-\beta;-\lambda)}{\beta}, \\ B_2(\beta;\lambda,2) &= \frac{\lambda\,{}_4F_3(2,2,2,2;1,1-\beta,1-\beta;\lambda)}{\beta^2}, \end{aligned} $$

and also

$$ \begin{aligned} B_3(\beta;\lambda,1) &= \frac{\lambda\,{}_4F_3(2,2,2,2;1,1,1-\beta;-\lambda)}{\beta}, \\ B_3(\beta;\lambda,2) &= \frac{\lambda\,{}_5F_4(2,2,2,2,2;1,1,1-\beta,1-\beta;\lambda)}{\beta^2}, \\ B_3(\beta;\lambda,3) &= \frac{\lambda\,{}_6F_5(2,2,2,2,2,2;1,1,1-\beta,1-\beta,1-\beta;-\lambda)}{\beta^3}, \\ B_3(\beta;\lambda,4) &= \frac{\lambda\,{}_7F_6(2,2,2,2,2,2,2;1,1,1-\beta,1-\beta,1-\beta,1-\beta;\lambda)}{\beta^4}, \end{aligned} $$

and so on.

6 | CONCLUSION

Many formulas and identities were derived from generating functions for the rational sum $y(n, \lambda)$ and their applications, blended by the methods and techniques used in mathematical analysis and analytic number theory (*cf.*[30,31,32]). We then constructed generating functions associated with hypergeometric function and logarithm function for the finite sum $\mathcal{S}_v(n; \lambda, p)$ and infinite sum $B_v(\beta; \lambda, p)$ involving higher powers of inverse binomial coefficients. Using functional equations of these generating functions, we presented various formulas and relations involving sums of higher powers of (inverse) binomial coefficients, the Bernoulli polynomials and numbers, Euler polynomials and numbers, the Stirling numbers, the (alternating) harmonic numbers, the Leibnitz numbers and polynomials, and combinatorial sums.

We showed that the infinite sum $B_v(\beta; \lambda, p)$ was a generalization of the interpolation functions for the Euler-Frobenius polynomials and the Eulerian polynomials. It is well-known that the finite sums and infinite sums have subsequently been studied in analytic number theory, in combinatorics analysis, in analysis of other sciences. The results of this paper may also serve as a reference for researchers in various fields such as mathematics, computational engineering, probability and statistics, and in other sciences.

In future studies, it is planned to investigate other relationsships among the certain sums involving higher powers of (inverse) binomial coefficients with their generating functions and the family of zeta functions and also exponential Euler type spline.

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CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

AUTHOR CONTRIBUTIONS

This paper has only one author.

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