A NEW PROOF OF CONTINUOUS WELCH BOUNDS

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Abstract: We give a new proof of continuous Welch bounds obtained by M. Krishna *[arXiv:2109.09296]*. Our proof is motivated from the proof of Welch *[IEEE Transactions on Information Theory, 1974]*.
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1. A new proof of continuous Welch bounds

In [1], following continuous version of Welch bounds are proved using (continuous) frames/spectral theory.

Theorem 1.1. *[1] Let (Ω, μ) be a measure space and $\{\tau_\alpha\}_{\alpha\in\Omega}$ be a normalized continuous Bessel family for a d-dimensional Hilbert space $\mathcal{H}$. If the diagonal $\Delta := \{(\alpha, \alpha) : \alpha \in \Omega\}$ is measurable in the measure space $\Omega \times \Omega$, then*

$$\int_{\Omega\times\Omega} |\langle \tau_\alpha, \tau_\beta\rangle|^{2m}\, d(\mu\times\mu)(\alpha,\beta) = \int_\Omega\int_\Omega |\langle \tau_\alpha, \tau_\beta\rangle|^{2m}\, d\mu(\alpha)\, d\mu(\beta) \geq \frac{\mu(\Omega)^2}{\binom{d+m-1}{m}}, \quad \forall m \in \mathbb{N}. \tag{1}$$

Equality holds in Inequality (1) if and only if $\{\tau_\alpha\}_{\alpha\in\Omega}$ is a tight continuous frame. Further, we have the ***higher order continuous Welch bounds***

$$\sup_{\alpha,\beta\in\Omega,\alpha\neq\beta} |\langle \tau_\alpha, \tau_\beta\rangle|^{2m} \geq \frac{1}{(\mu\times\mu)((\Omega\times\Omega)\setminus\Delta)}\left[\frac{\mu(\Omega)^2}{\binom{d+m-1}{m}} - (\mu\times\mu)(\Delta)\right], \quad \forall m \in \mathbb{N}. \tag{2}$$

In particular, we have the ***first order continuous Welch bound***

$$\sup_{\alpha,\beta\in\Omega,\alpha\neq\beta} |\langle \tau_\alpha, \tau_\beta\rangle|^{2} \geq \frac{1}{(\mu\times\mu)((\Omega\times\Omega)\setminus\Delta)}\left[\frac{\mu(\Omega)^2}{d} - (\mu\times\mu)(\Delta)\right].$$

Proof. We give a new proof. Due to unitary equivalence of any two same dimensional Hilbert spaces, it suffices to prove the theorem for the standard d-dimensional Hilbert space $\mathbb{C}^d$. Our proof is motivated from the proof of Welch [2]. Let $\tau_\alpha := (a_1^{(\alpha)}, a_2^{(\alpha)}, \dots, a_d^{(\alpha)})$ for all $\alpha \in \Omega$. Define

$$B_m := \int_{\Omega\times\Omega} |\langle \tau_\alpha, \tau_\beta\rangle|^{2m}\, d(\mu\times\mu)(\alpha,\beta) = \int_\Omega\int_\Omega |\langle \tau_\alpha, \tau_\beta\rangle|^{2m}\, d\mu(\alpha)\, d\mu(\beta).$$

Then

$$\begin{aligned} B_m &= \int_\Delta |\langle \tau_\alpha, \tau_\beta\rangle|^{2m}\, d(\mu\times\mu)(\alpha,\beta) + \int_{(\Omega\times\Omega)\setminus\Delta} |\langle \tau_\alpha, \tau_\beta\rangle|^{2m}\, d(\mu\times\mu)(\alpha,\beta) \\ &= (\mu\times\mu)(\Delta) + \int_{(\Omega\times\Omega)\setminus\Delta} |\langle \tau_\alpha, \tau_\beta\rangle|^{2m}\, d(\mu\times\mu)(\alpha,\beta) \\ &\leq (\mu\times\mu)(\Delta) + \sup_{\alpha,\beta\in\Omega,\alpha\neq\beta} |\langle \tau_\alpha, \tau_\beta\rangle|^{2m} (\mu\times\mu)((\Omega\times\Omega)\setminus\Delta). \end{aligned}$$

On the other hand, using the definition of standard inner product in $\mathbb{C}^d$ and Cauchy-Schwarz inequality,

$$\begin{aligned}
B_m &= \int_\Omega \int_\Omega \left|\sum_{r=1}^d a_r^{(\alpha)}\overline{a_r^{(\beta)}}\right|^{2m} d\mu(\alpha)\,d\mu(\beta)\\
&= \int_\Omega \int_\Omega \sum_{u_1\cdots u_m=1, v_1\cdots v_m=1}^d \prod_{p=1}^m a_{u_p}^{(\alpha)}\overline{a_{v_p}^{(\alpha)} a_{u_p}^{(\beta)}} a_{v_p}^{(\beta)}\,d\mu(\alpha)\,d\mu(\beta)\\
&= \sum_{x_1\cdots x_d, y_1\cdots y_d, \sum_{r=1}^d x_r=\sum_{s=1}^d y_s=m} \binom{m}{x_1,\dots,x_d}\binom{m}{y_1,\dots,y_d}\left|\int_\Omega \prod_{r=1}^d \left(a_r^{(\alpha)}\right)^{x_r}\left(\overline{a_r^{(\alpha)}}\right)^{y_r} d\mu(\alpha)\right|^2\\
&\geq \sum_{x_1\cdots x_d, \sum_{r=1}^d x_r=m} \binom{m}{x_1,\dots,x_d}^2\left|\int_\Omega \prod_{r=1}^d \left|a_r^{(\alpha)}\right|^{2x_r} d\mu(\alpha)\right|^2\\
&\geq \frac{1}{\left\|\sum_{x_1\cdots x_d, \sum_{r=1}^d x_r=m}\right\|}\left|\sum_{x_1\cdots x_d, \sum_{r=1}^d x_r=m} \binom{m}{x_1,\dots,x_d}\int_\Omega \prod_{r=1}^d |a_r^{(\alpha)}|^{2x_r} d\mu(\alpha)\right|^2\\
&= \frac{1}{\binom{d+m-1}{m}}\left|\sum_{x_1\cdots x_d, \sum_{r=1}^d x_r=m} \binom{m}{x_1,\dots,x_d}\int_\Omega \prod_{r=1}^d |a_r^{(\alpha)}|^{2x_r} d\mu(\alpha)\right|^2\\
&= \frac{1}{\binom{d+m-1}{m}}\left|\int_\Omega \sum_{x_1\cdots x_d, \sum_{r=1}^d x_r=m} \binom{m}{x_1,\dots,x_d}\prod_{r=1}^d |a_r^{(\alpha)}|^{2x_r} d\mu(\alpha)\right|^2\\
&= \frac{1}{\binom{d+m-1}{m}}\left|\int_\Omega \left(\sum_{p=1}^d a_p^{(\alpha)}\overline{a_p^{(\alpha)}}\right)^m d\mu(\alpha)\right|^2\\
&= \frac{1}{\binom{d+m-1}{m}}\left|\int_\Omega (1)^m\, d\mu(\alpha)\right|^2 = \frac{\mu(\Omega)^2}{\binom{d+m-1}{m}}.
\end{aligned}$$

Hence

$$\begin{aligned}
(\mu\times\mu)(\Delta) + \sup_{\alpha,\beta\in\Omega,\alpha\neq\beta} |\langle\tau_\alpha,\tau_\beta\rangle|^{2m}(\mu\times\mu)((\Omega\times\Omega)\setminus\Delta) &\geq \int_{\Omega\times\Omega} |\langle\tau_\alpha,\tau_\beta\rangle|^{2m}\,d(\mu\times\mu)(\alpha,\beta)\\
&\geq \frac{\mu(\Omega)^2}{\binom{d+m-1}{m}}.
\end{aligned}$$

$\square$

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