B a cklund Transformation of (3+1)-Dimensional Variable Coefficient Potential-YTSF Equation and Related Problems

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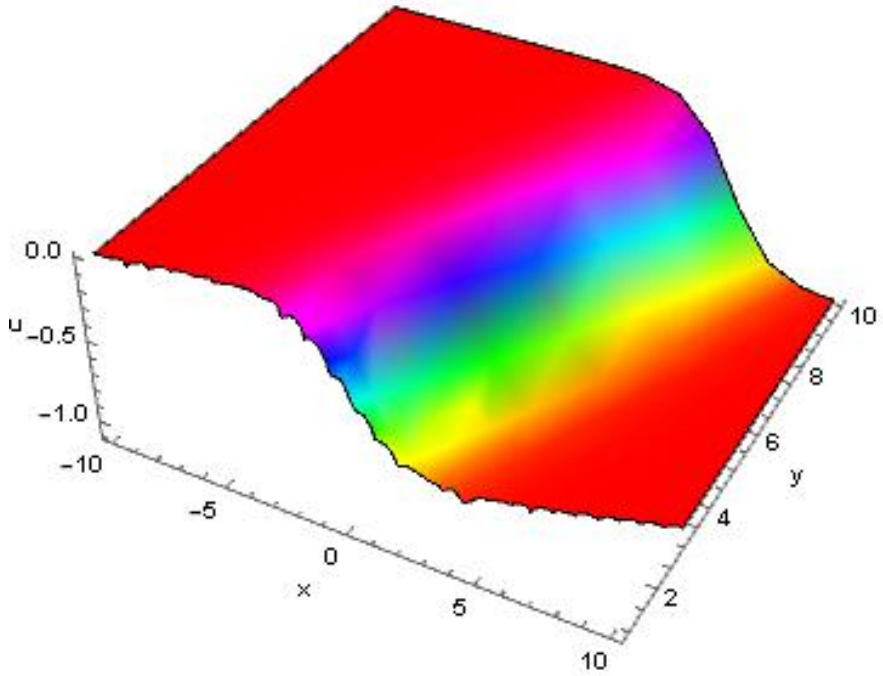
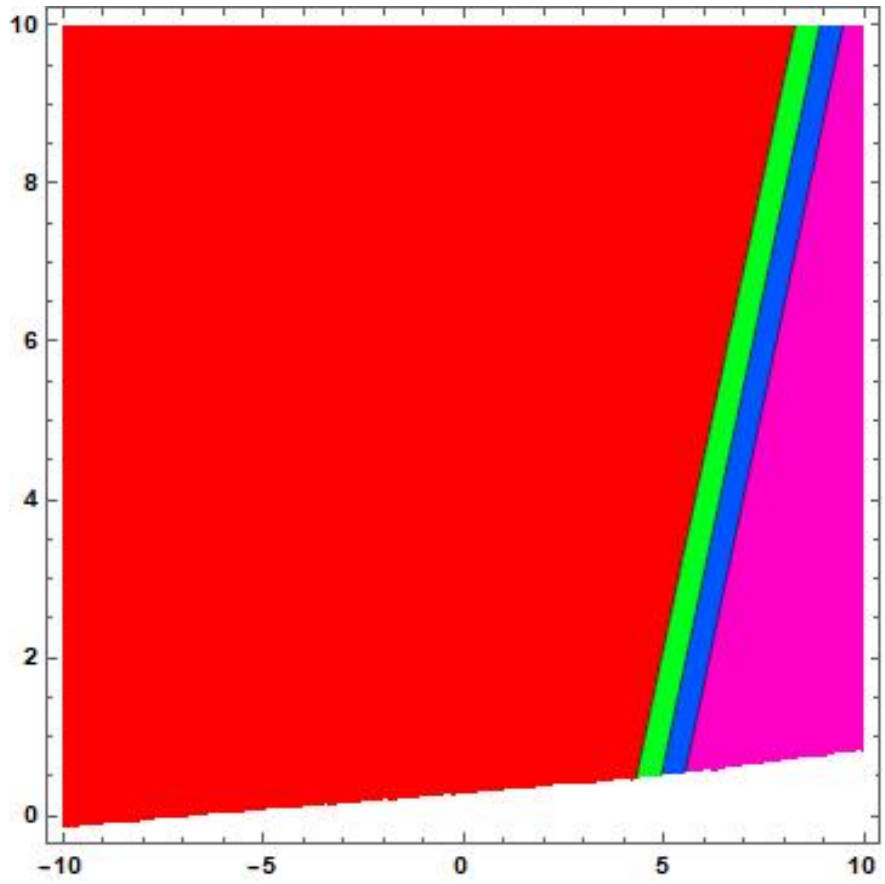
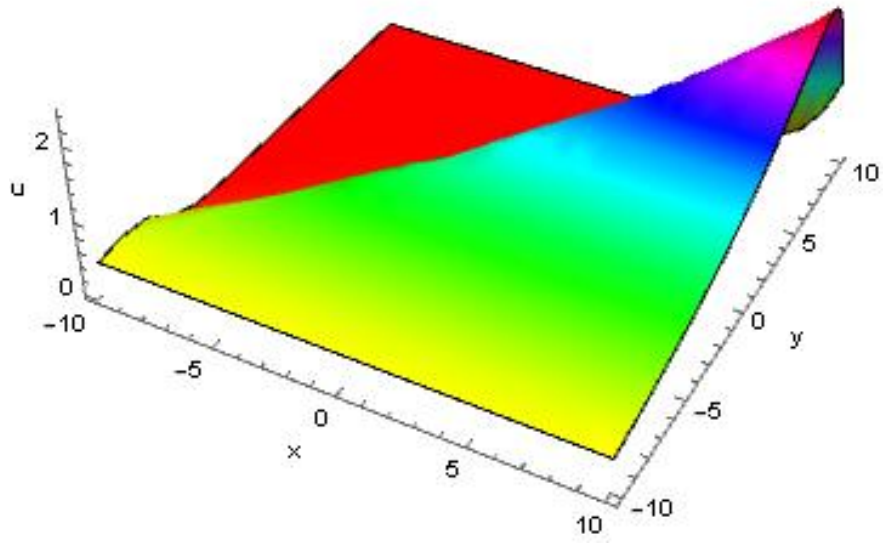
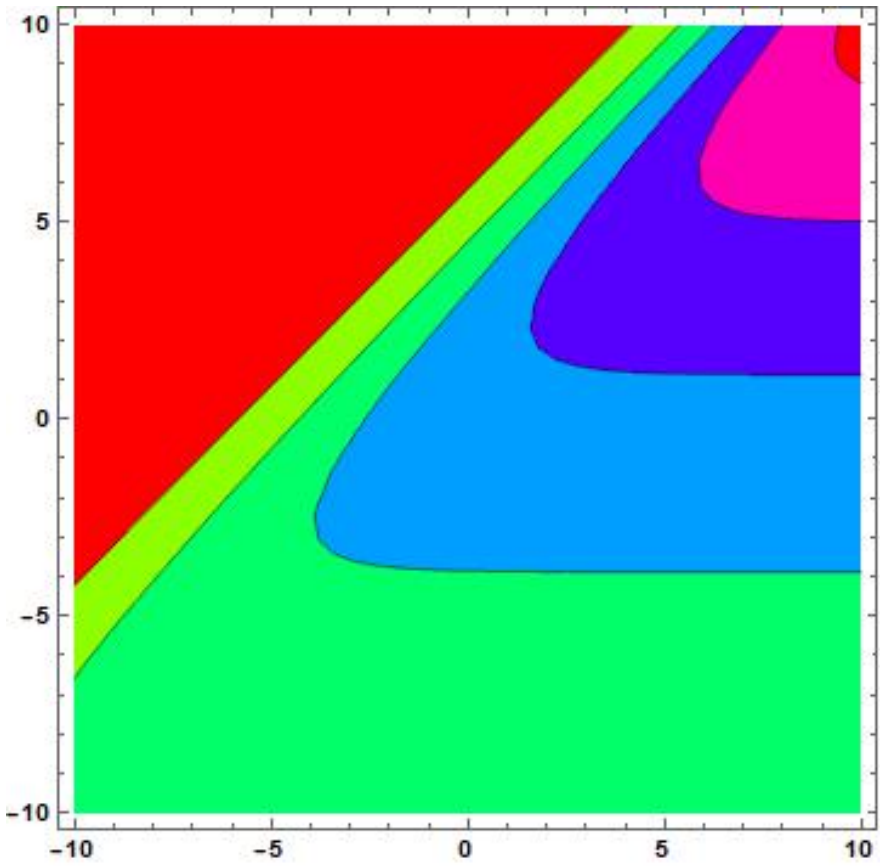
Abstract

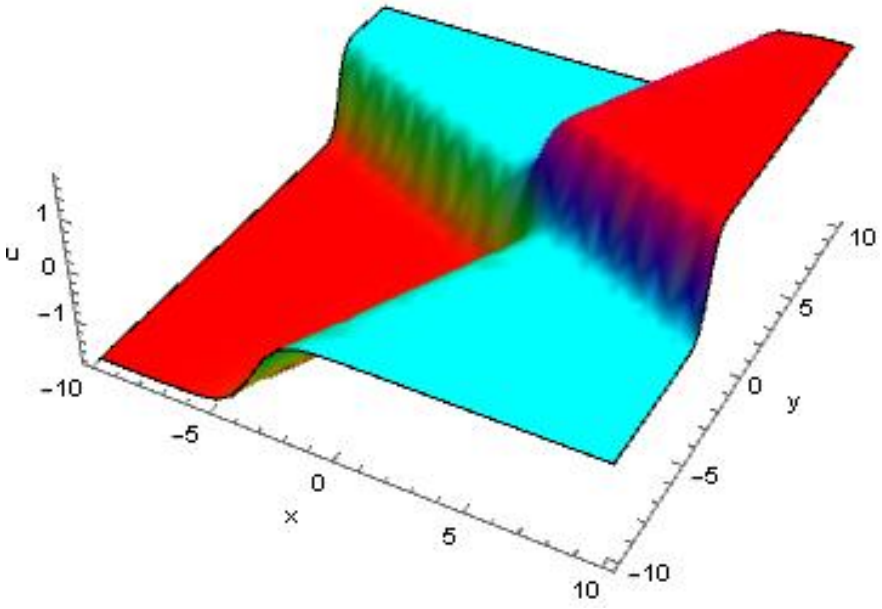
Based on the Bell polynomial method, the (3+1)-dimensional variable coefficient Potential-YTSF equation is transformed into bilinear form, and the double Bell polynomial B a cklund transformation, bilinear B a cklund transformation, Lax pair and infinite conservation law of this equation are constructed. Firstly, the Lax integrability of the equation is proved by the B a cklund transformation of double Bell polynomials, and the infinite conservation law is constructed. Secondly, the exact solution of the equation is obtained by bilinear B a cklund transformation and symbolic computation system Mathematica. Finally, we illustrate their properties by making some graphs of soliton solutions.

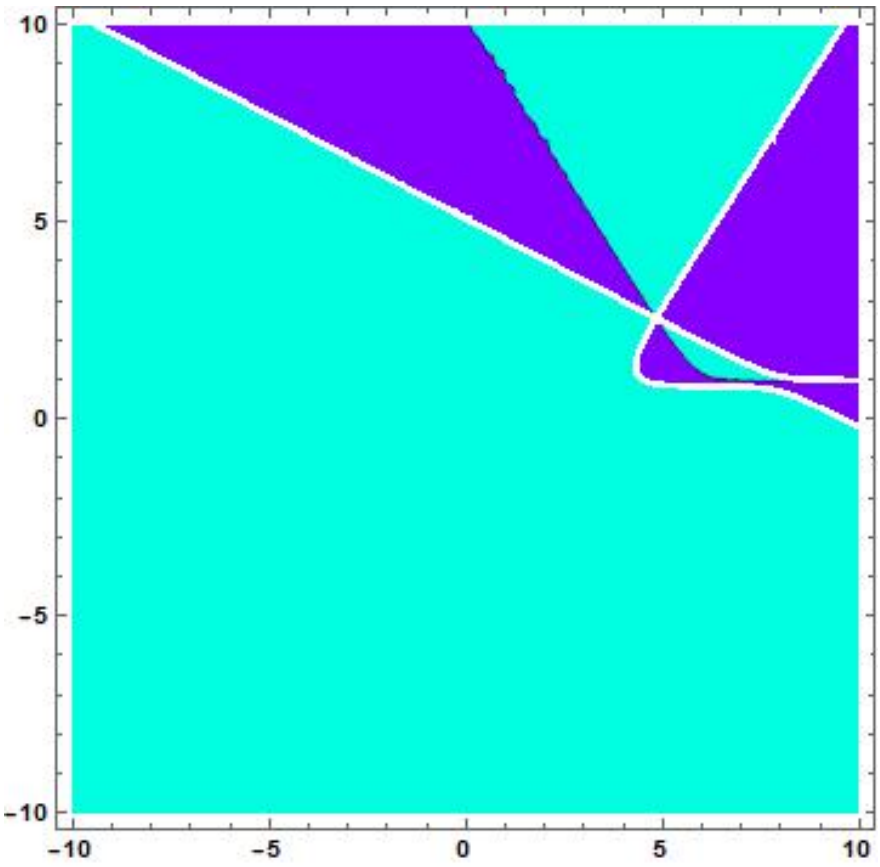


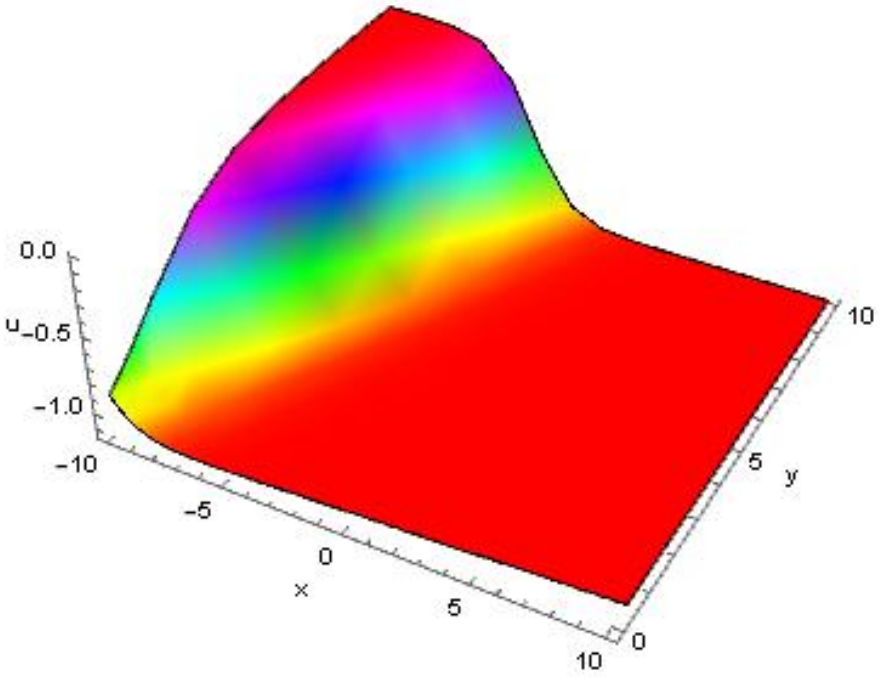


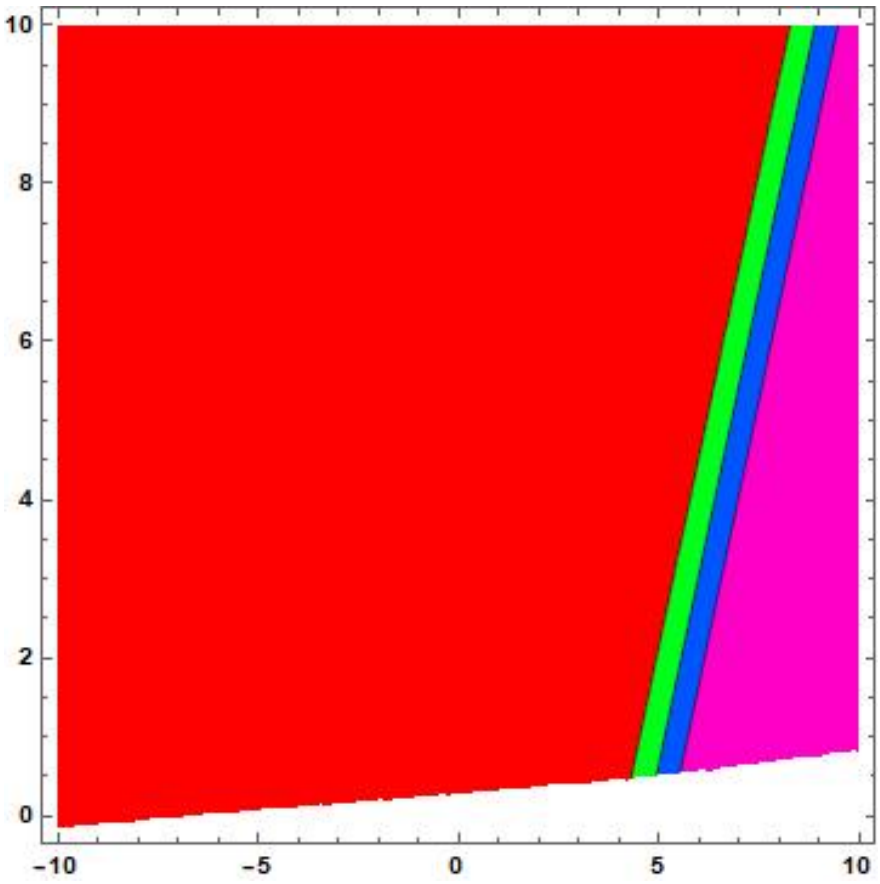


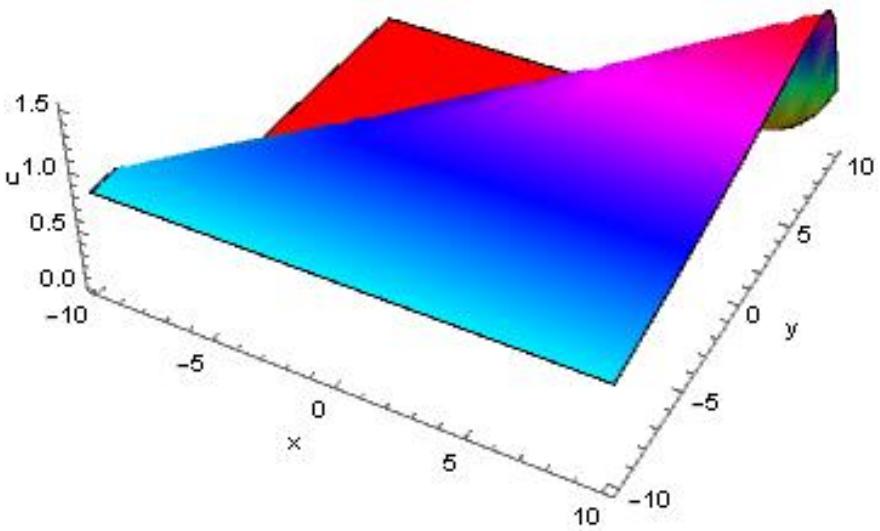


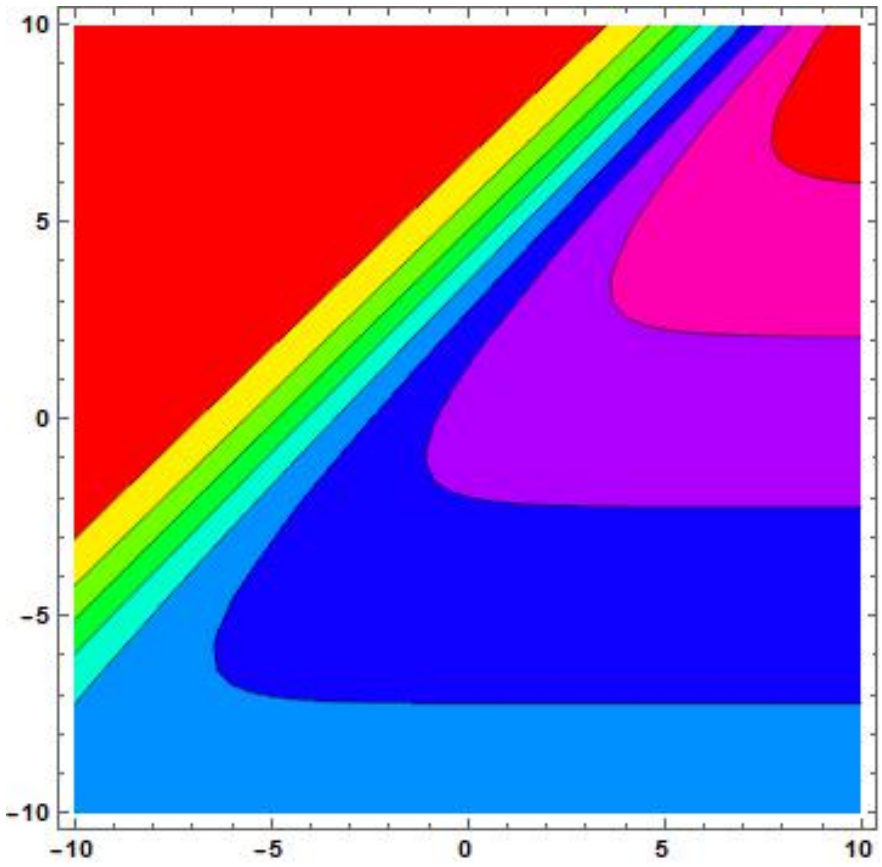


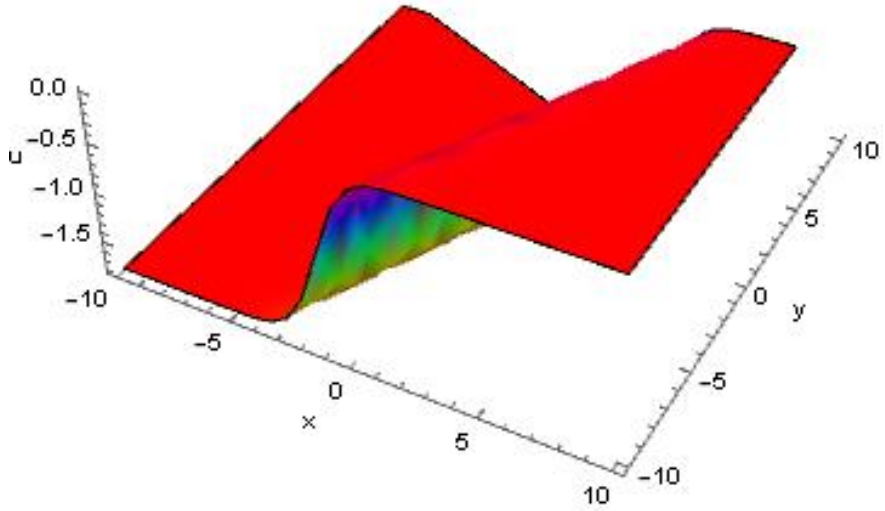


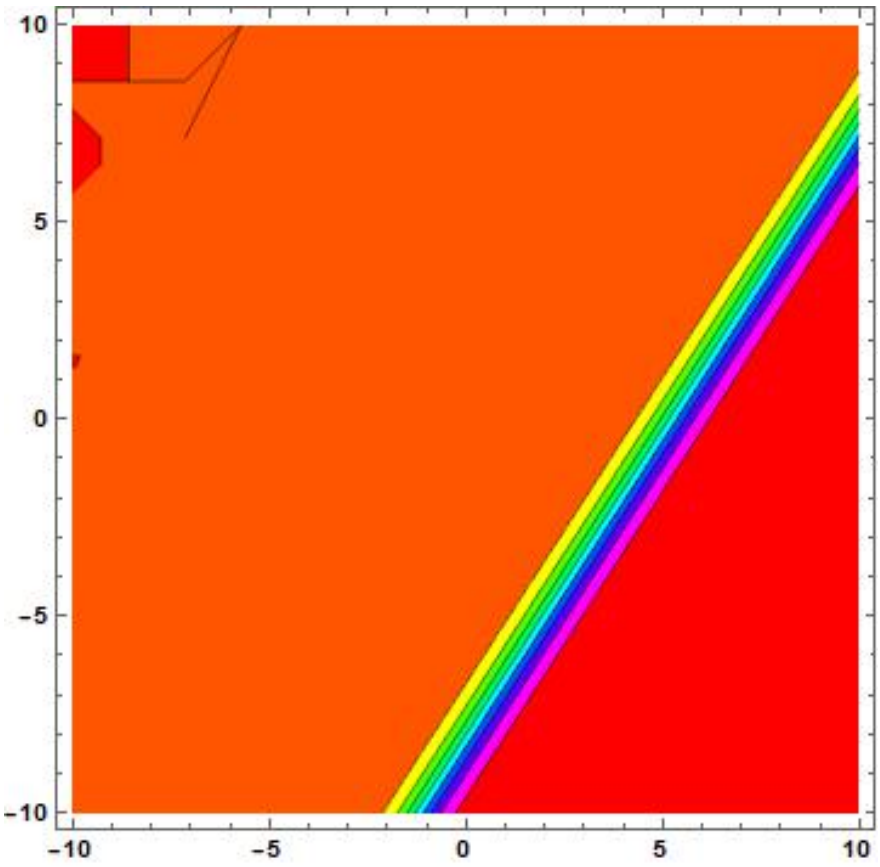


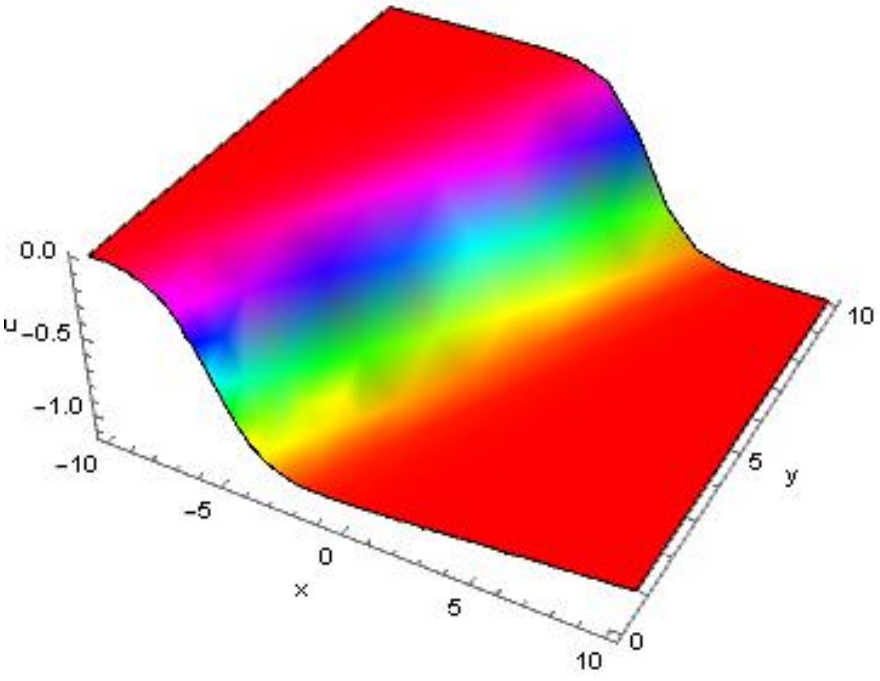


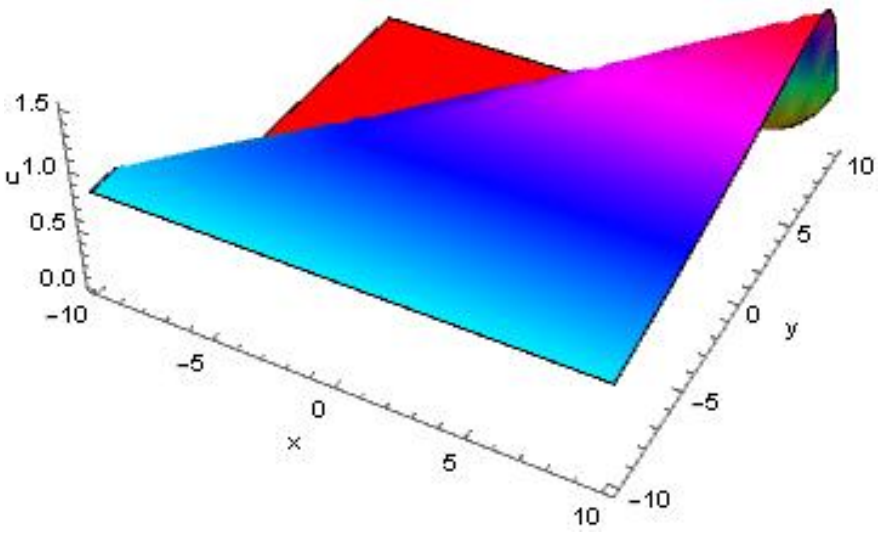


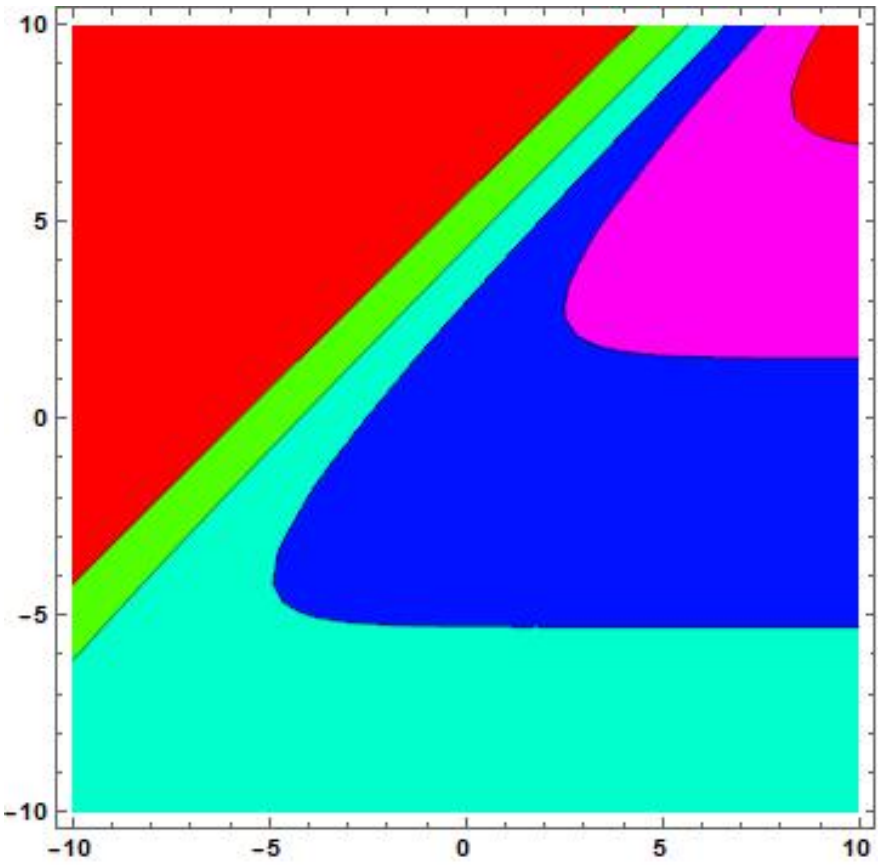


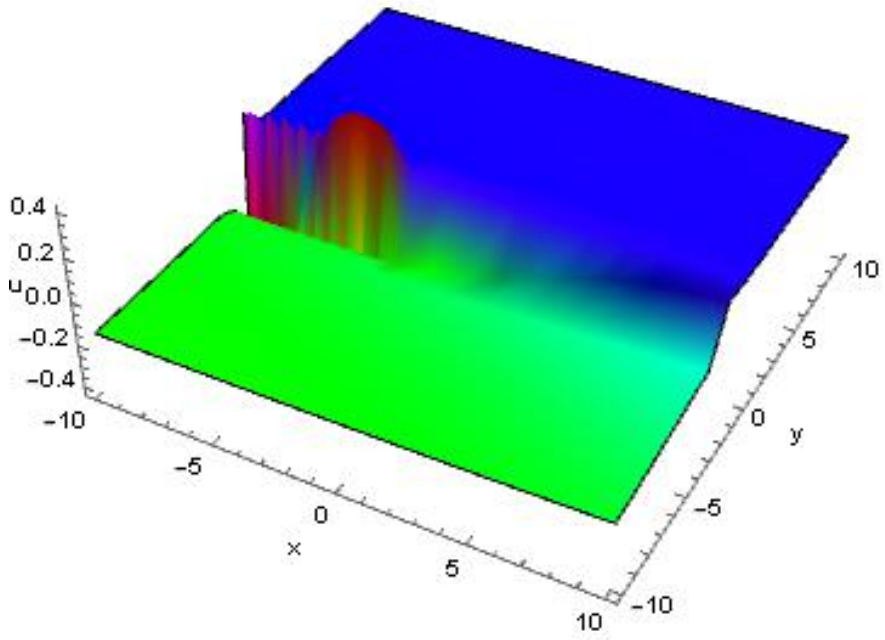


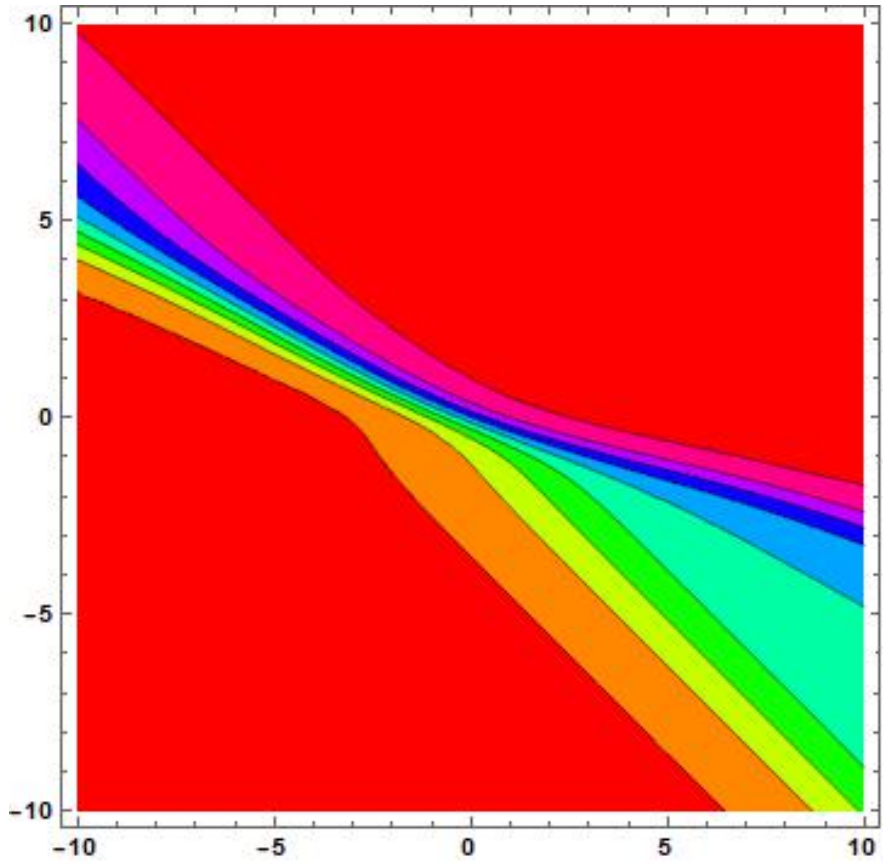


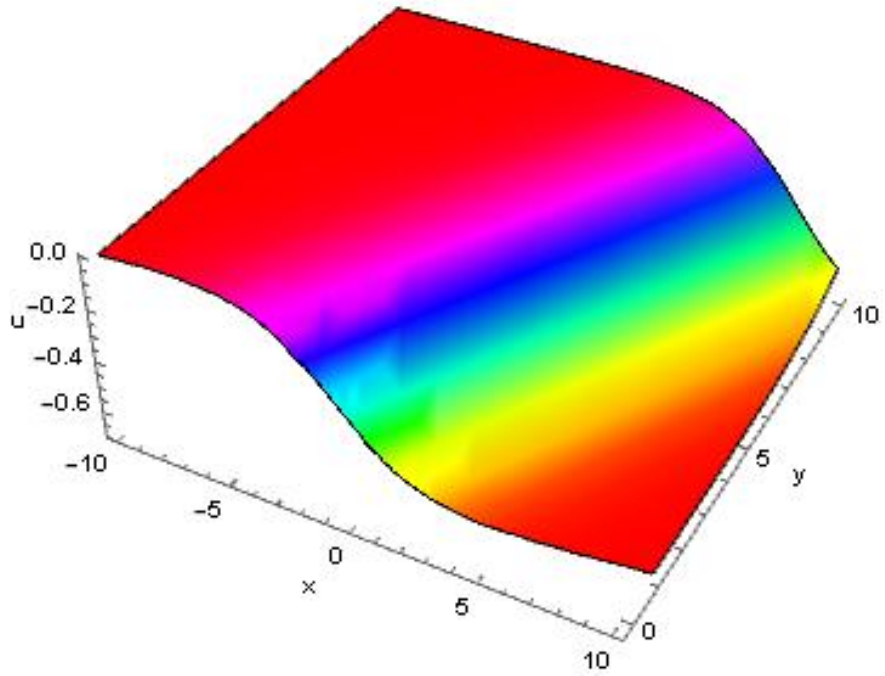


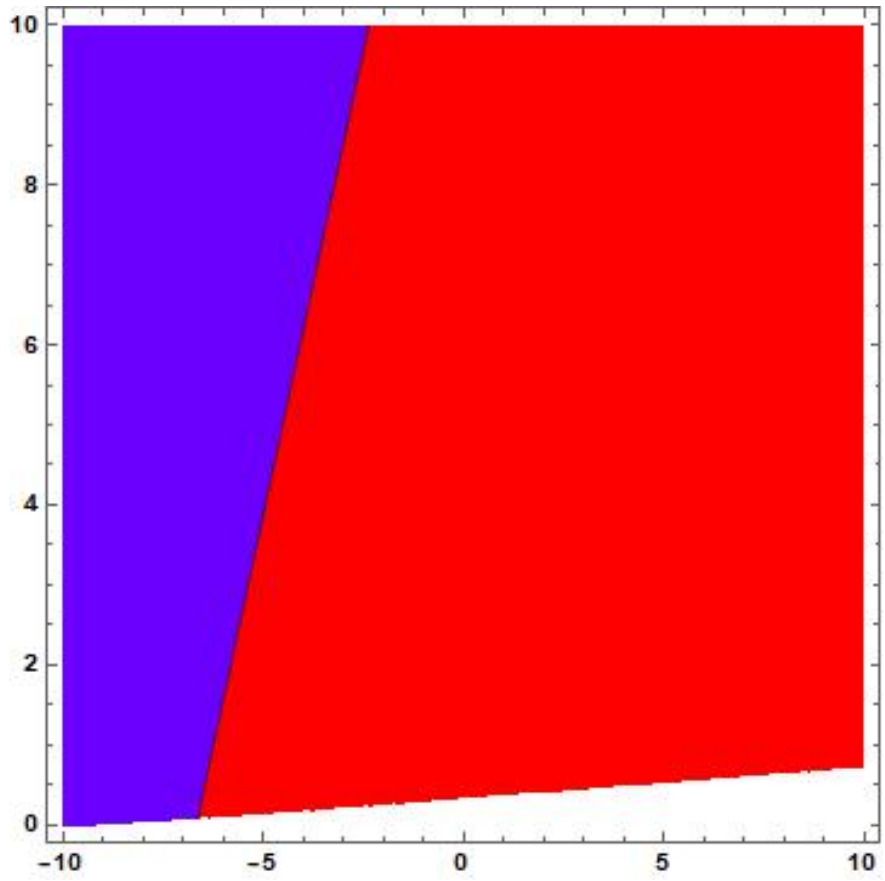


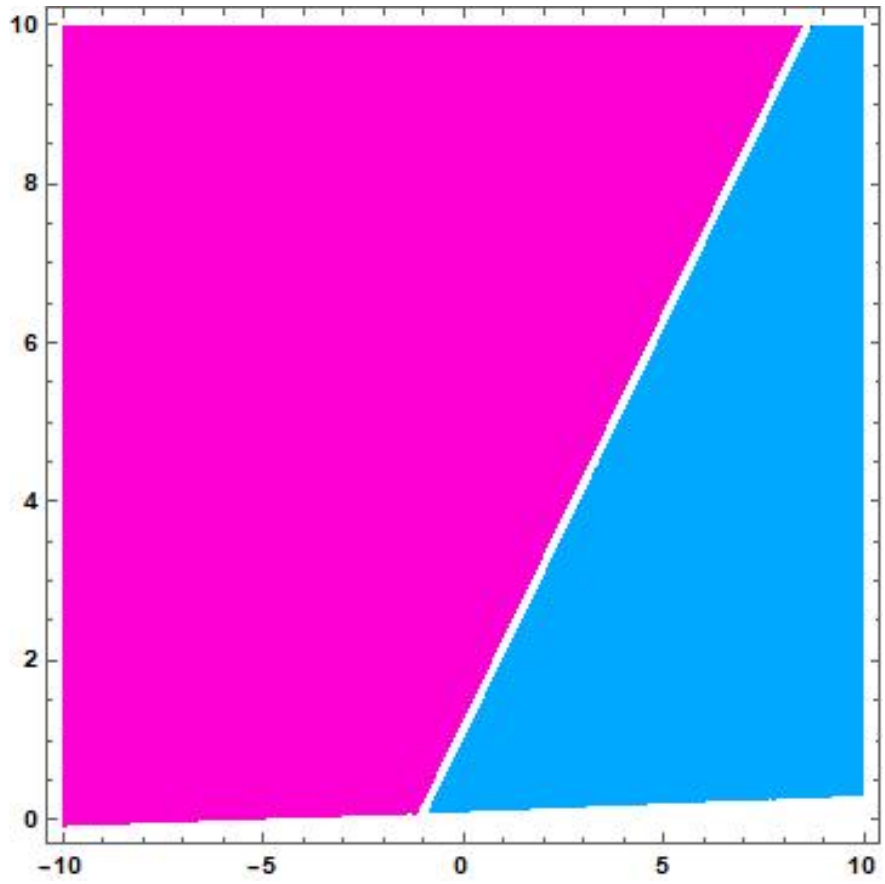


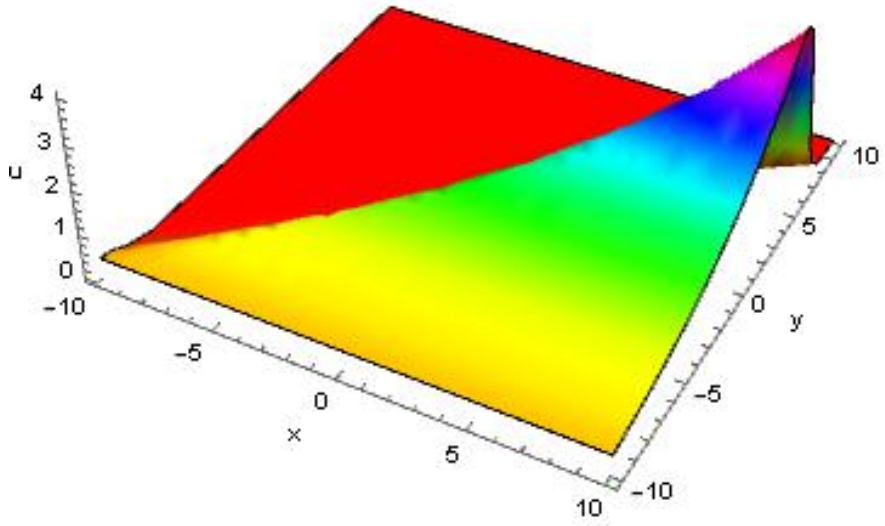


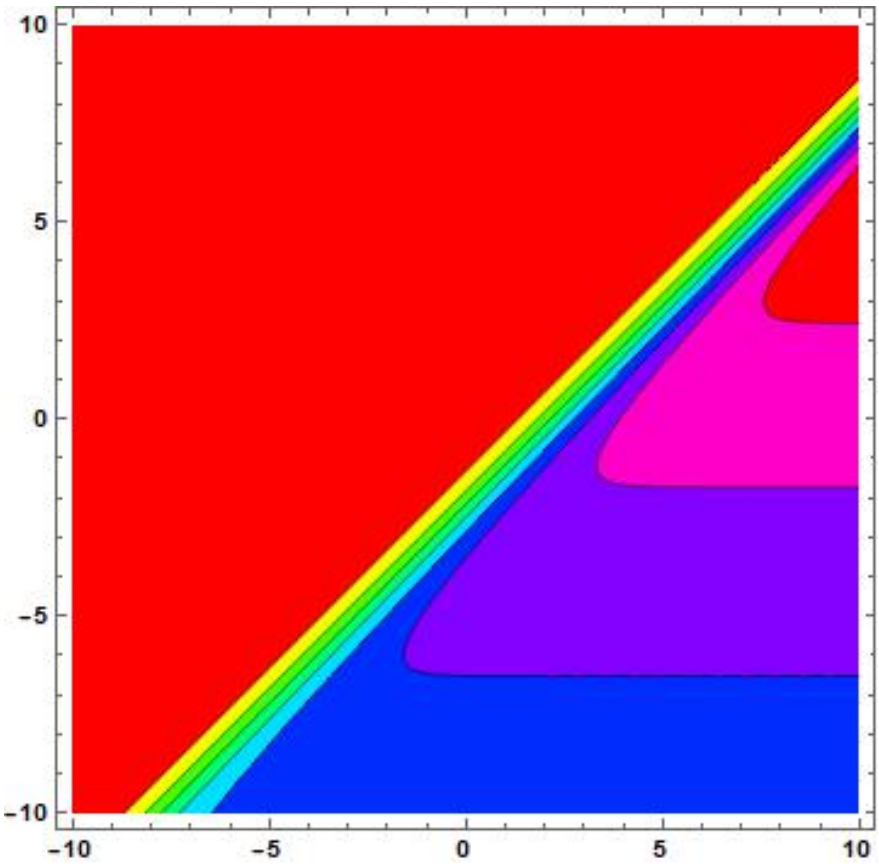


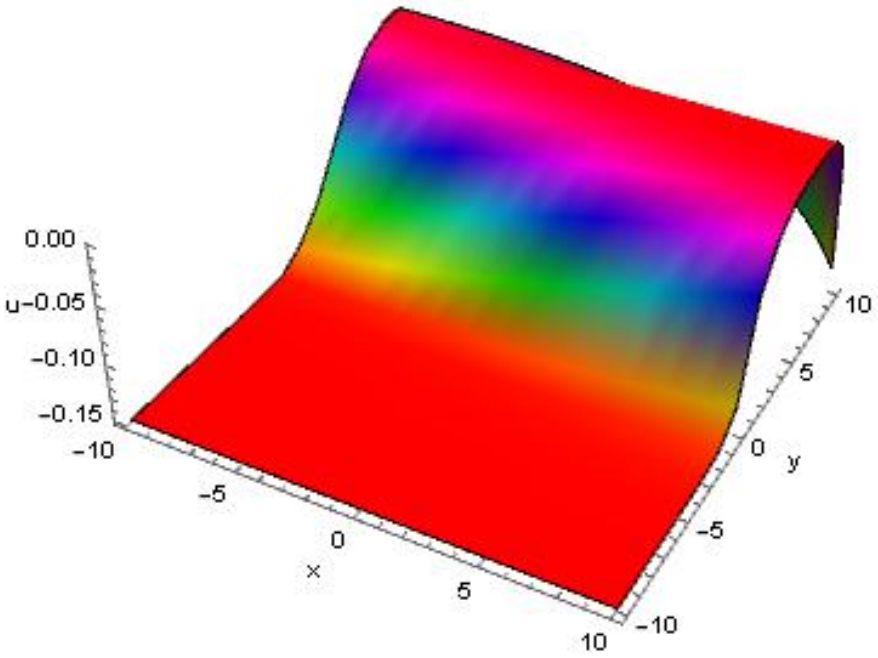


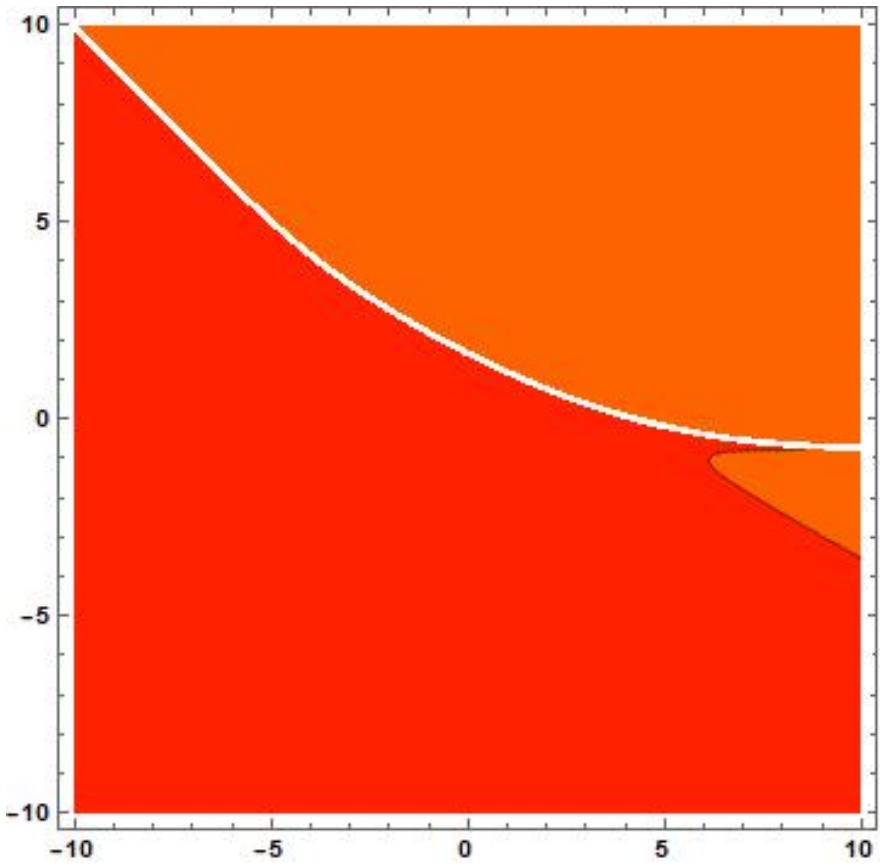


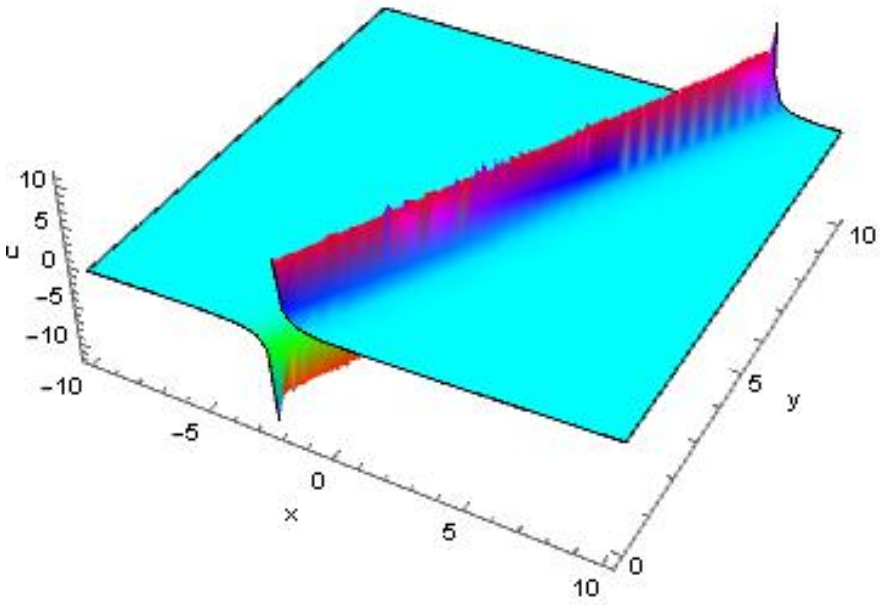


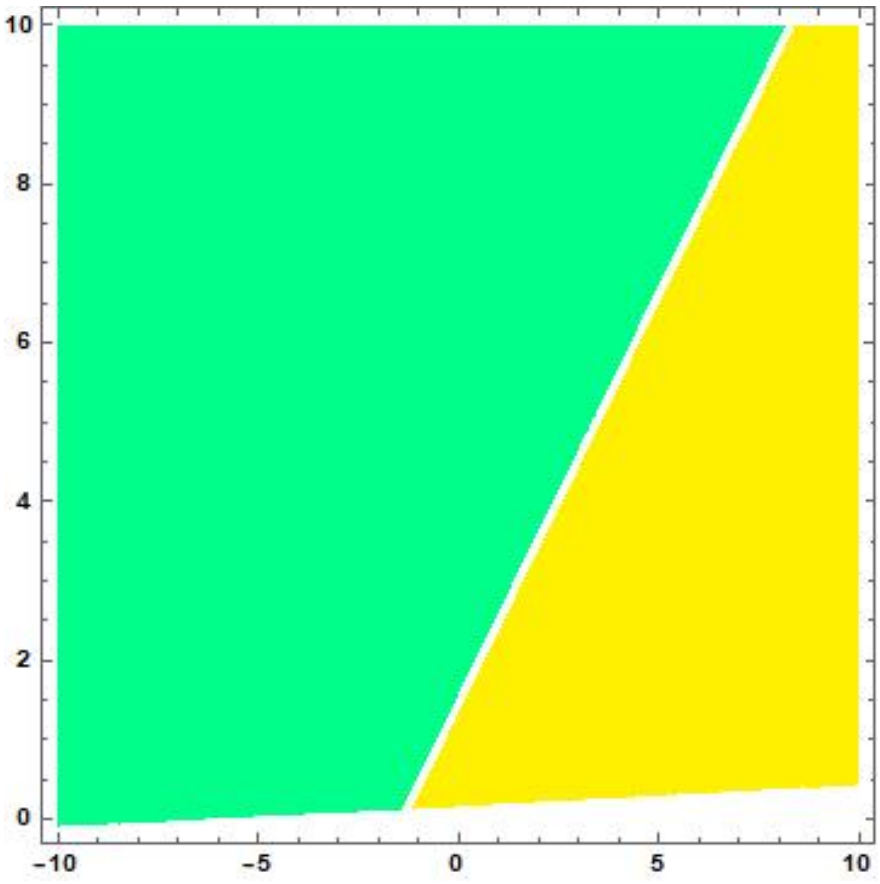


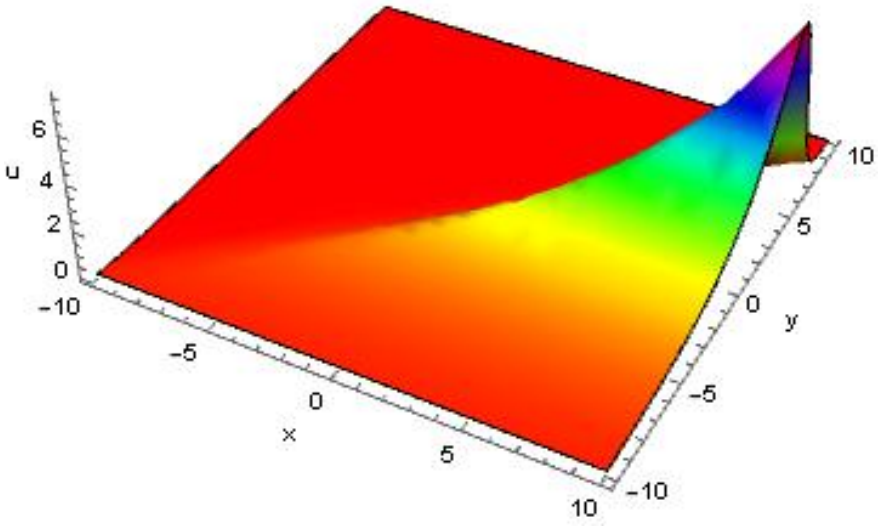


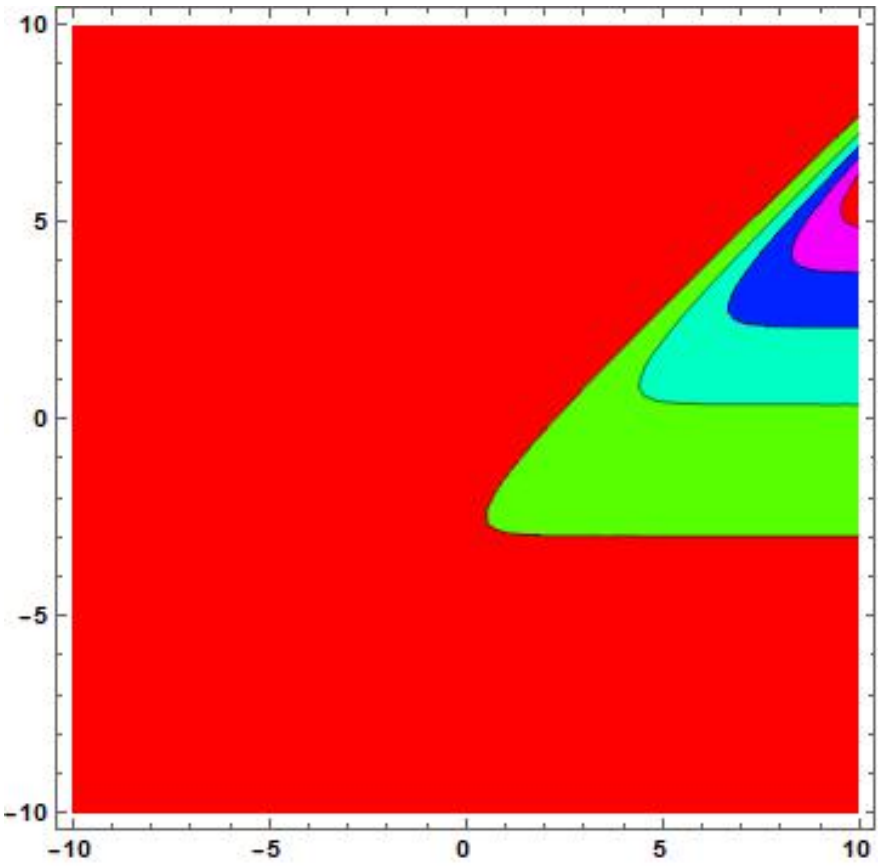


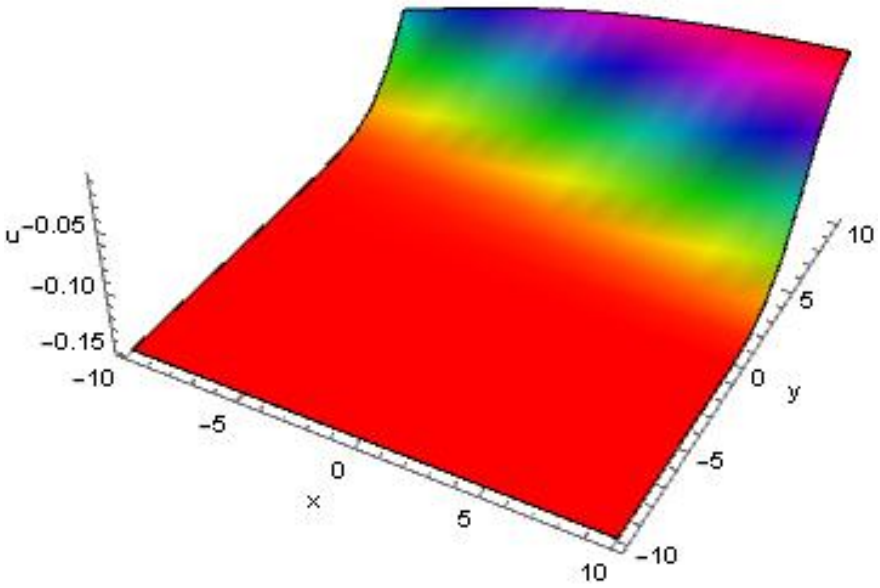


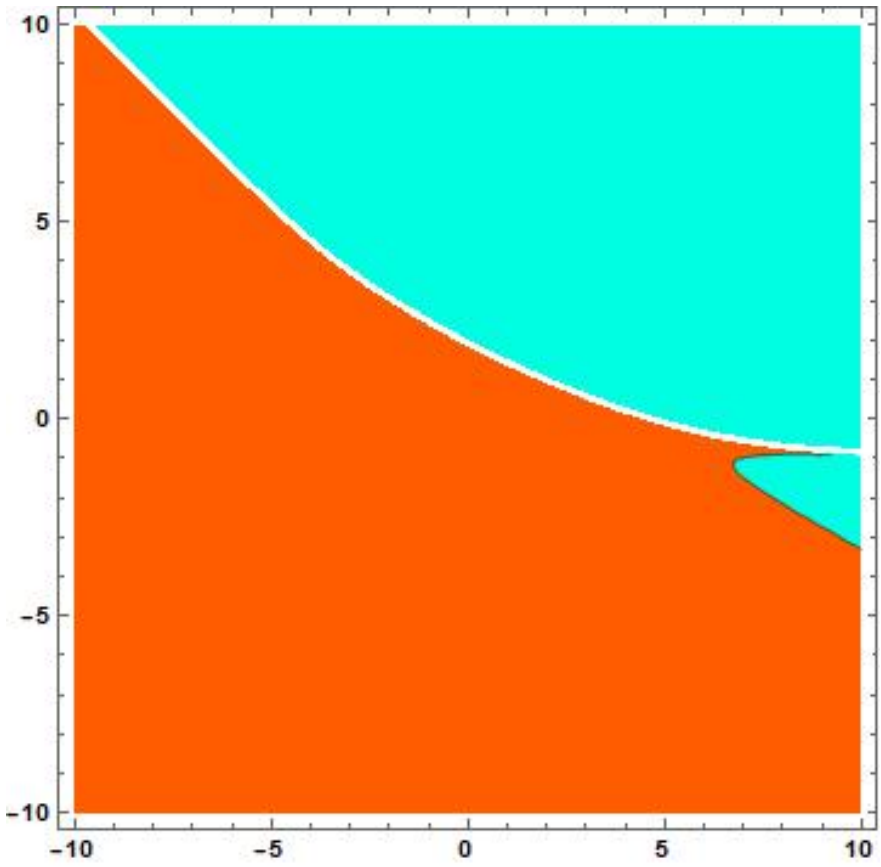


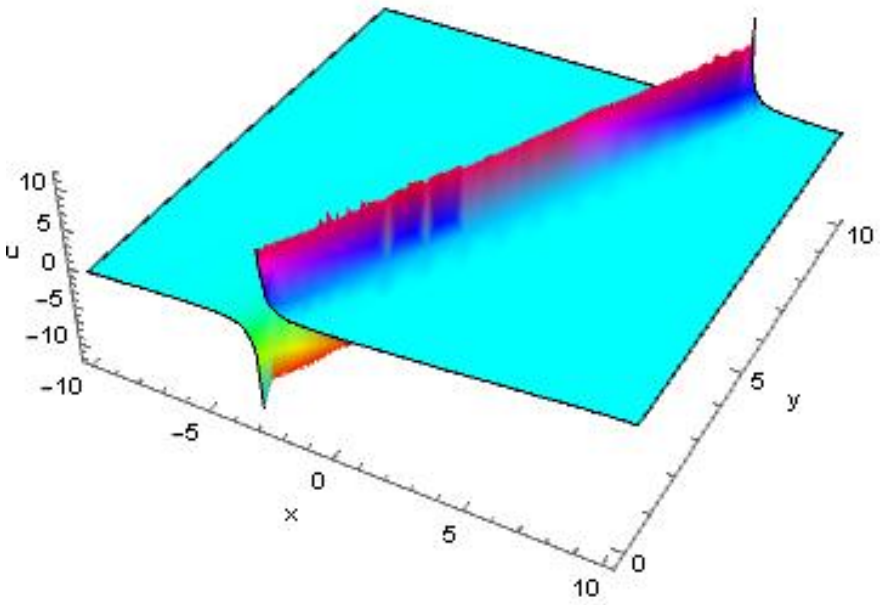


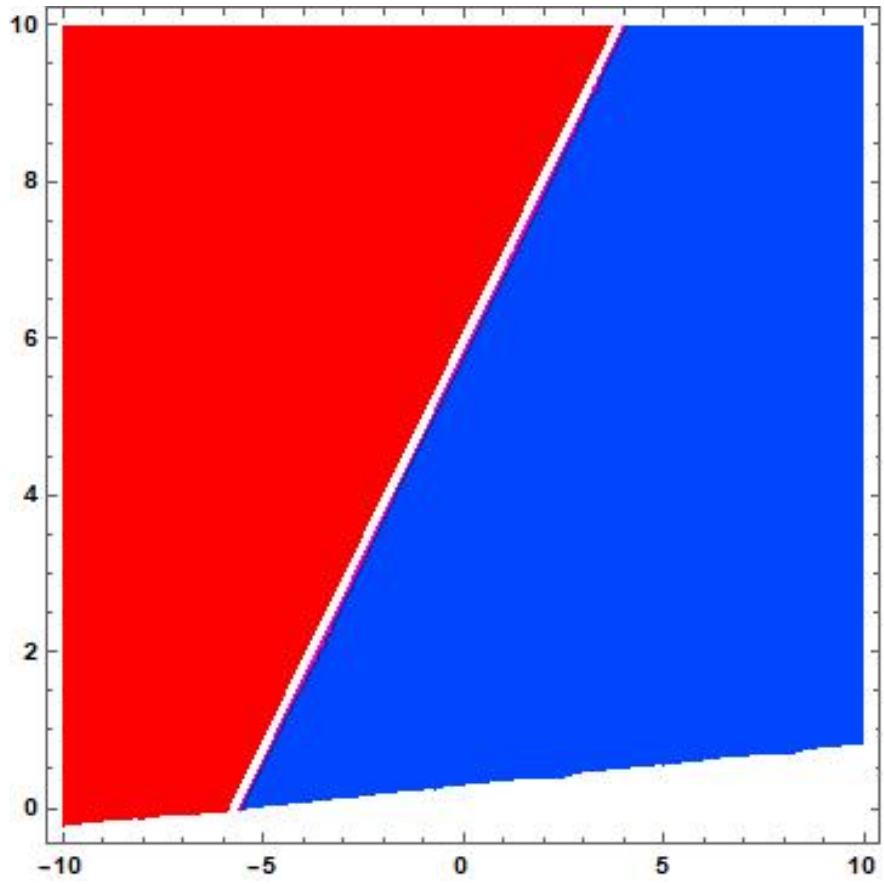


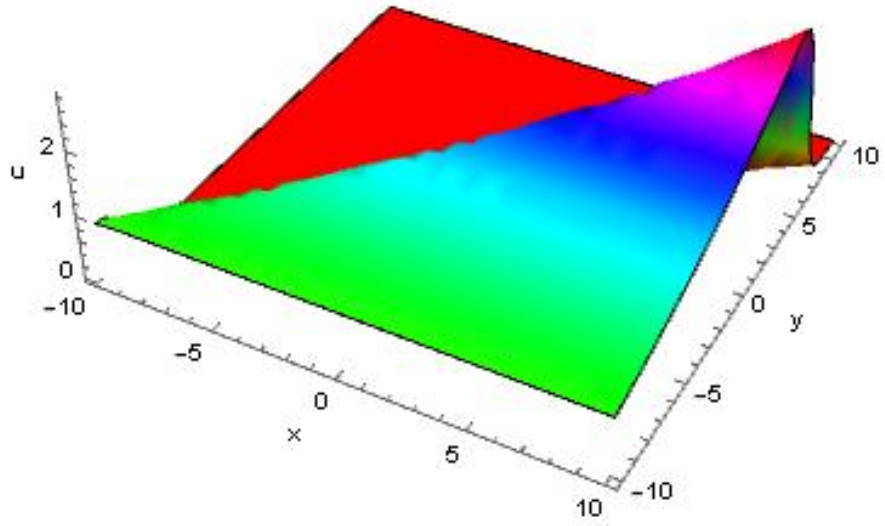


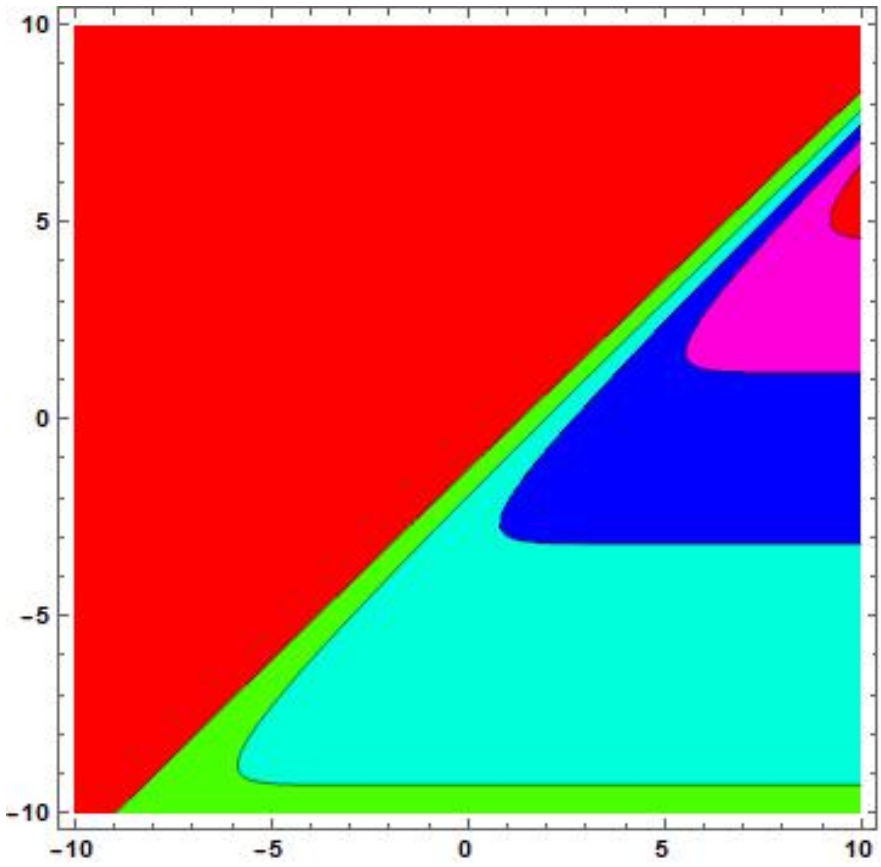


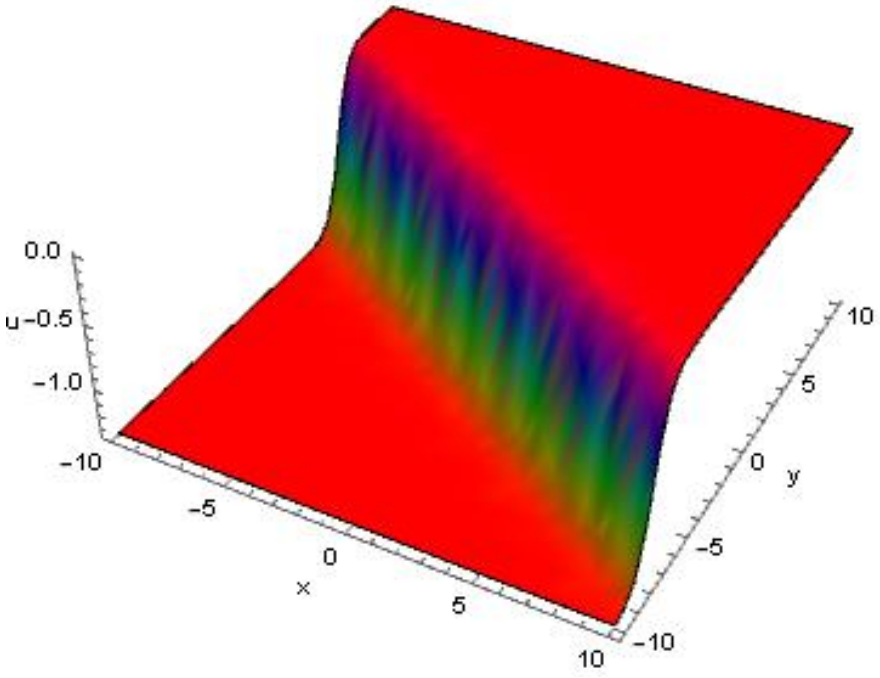


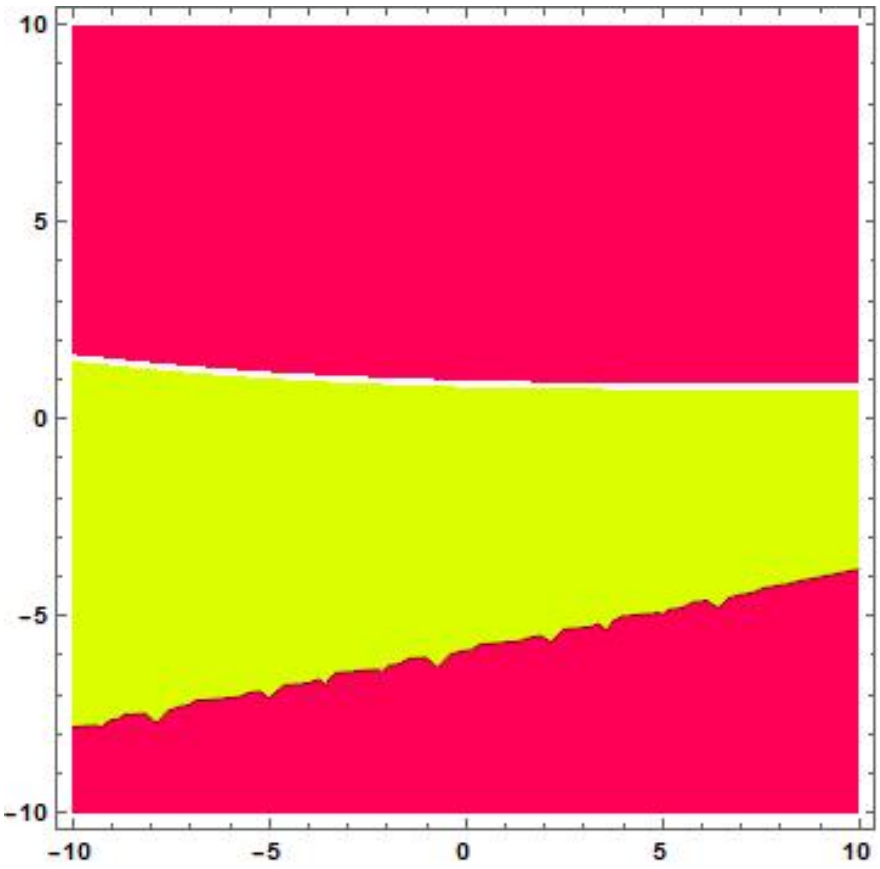


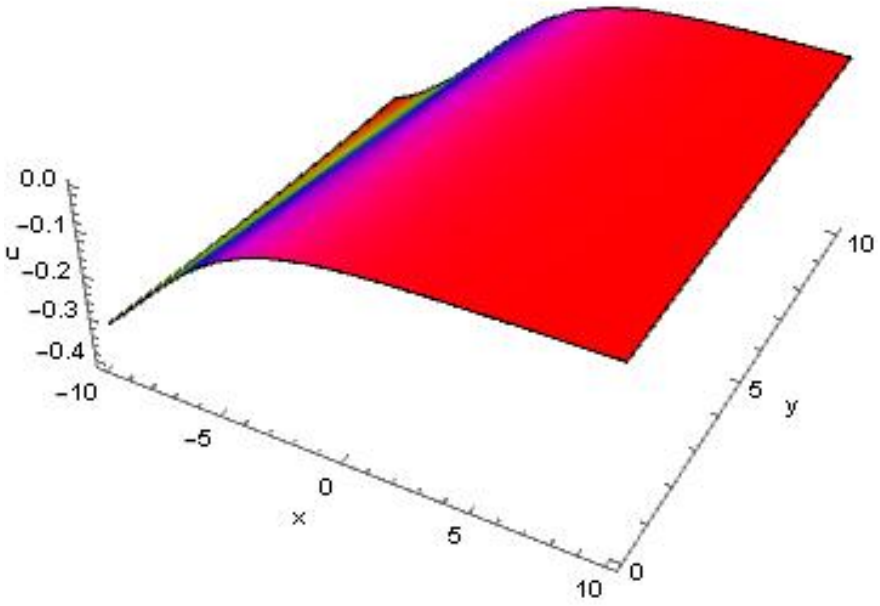


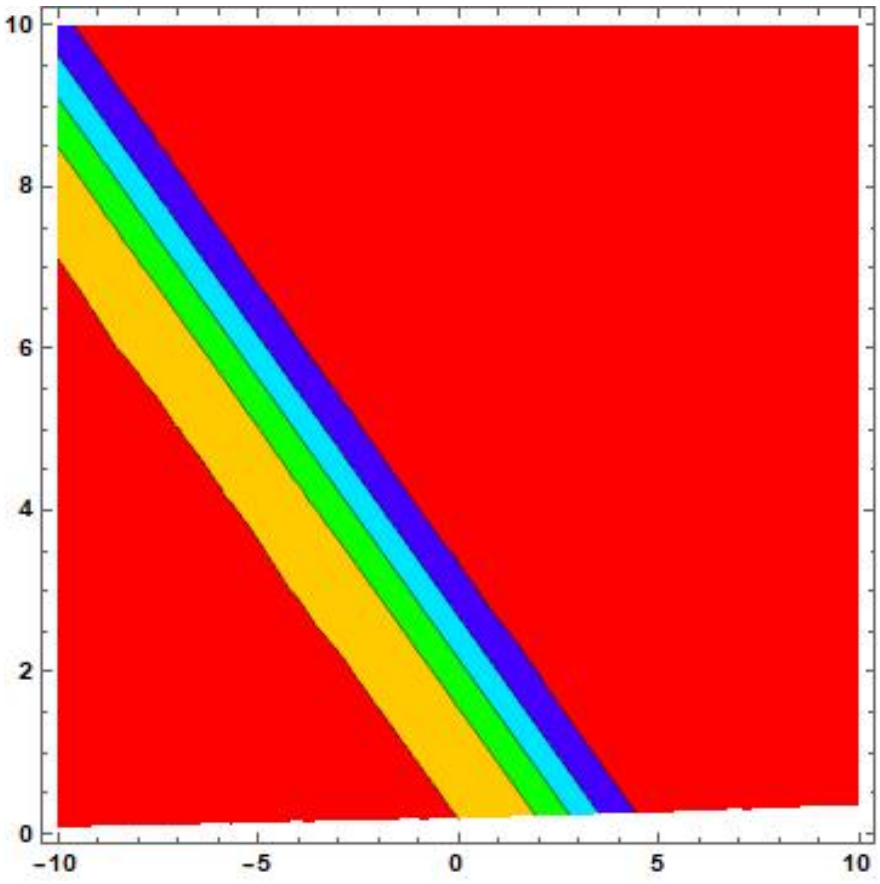


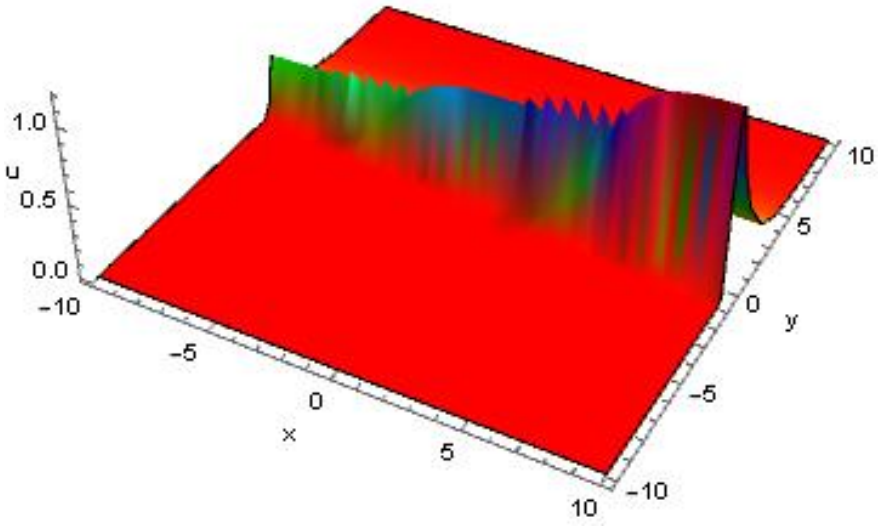


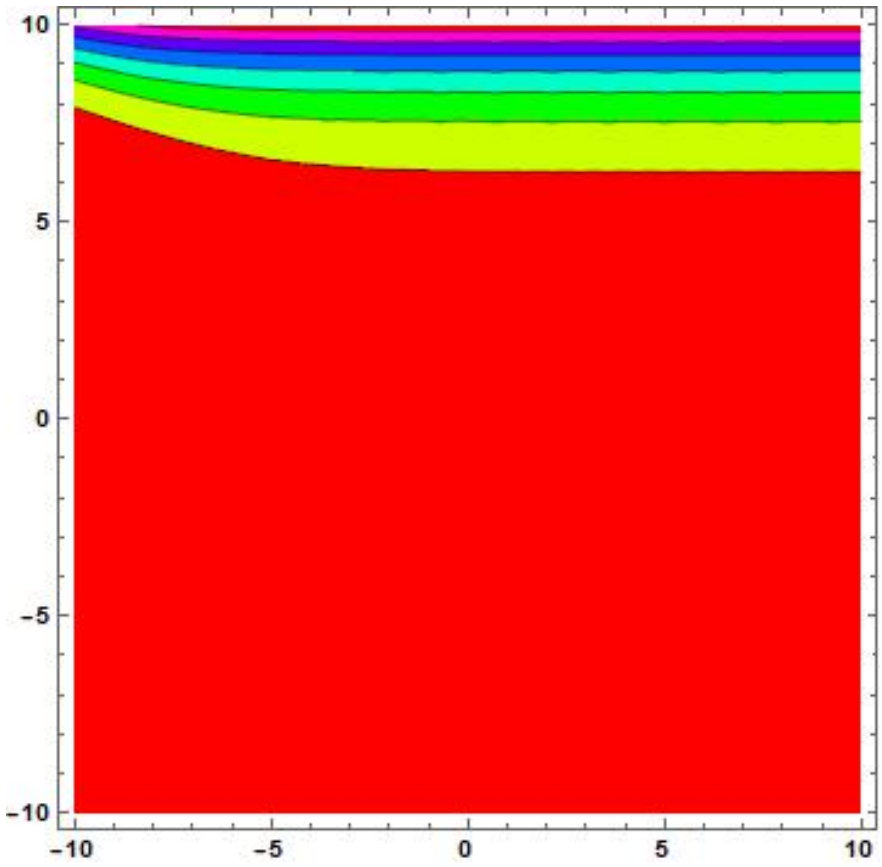


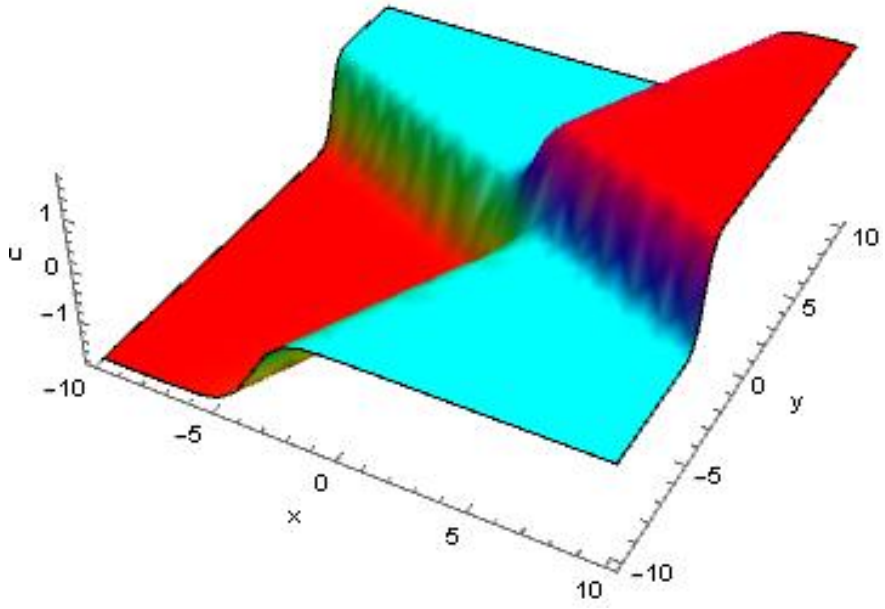


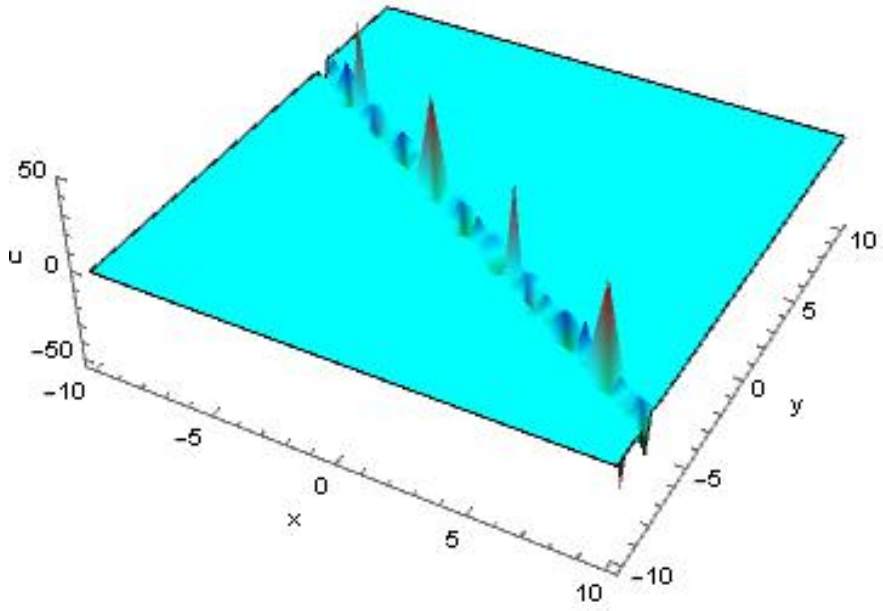


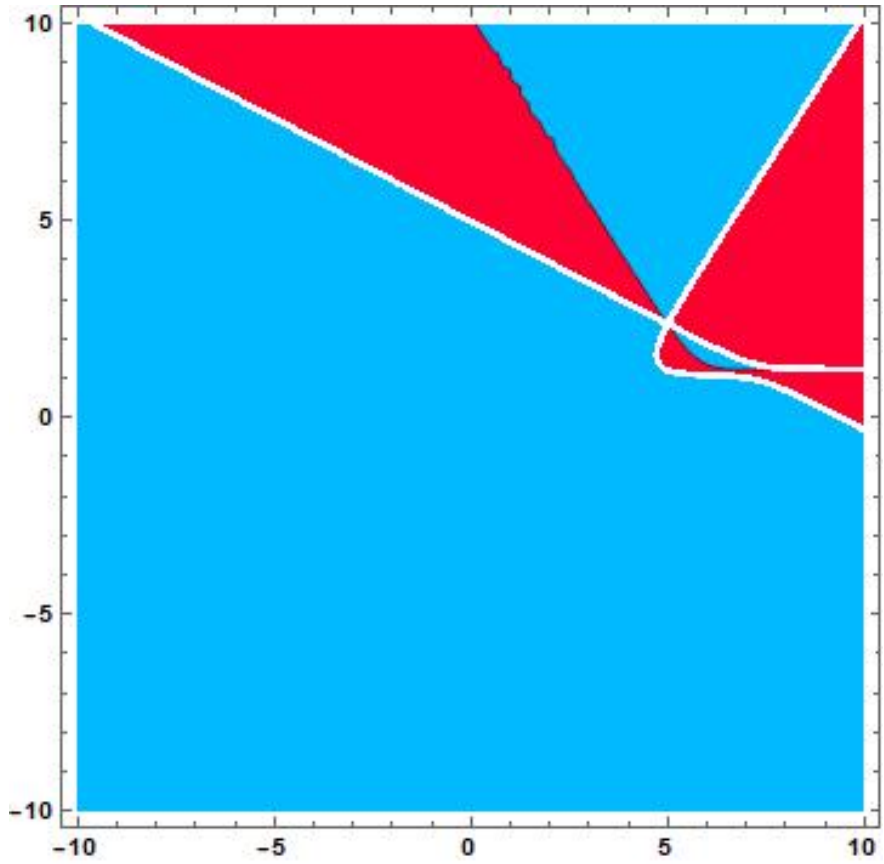


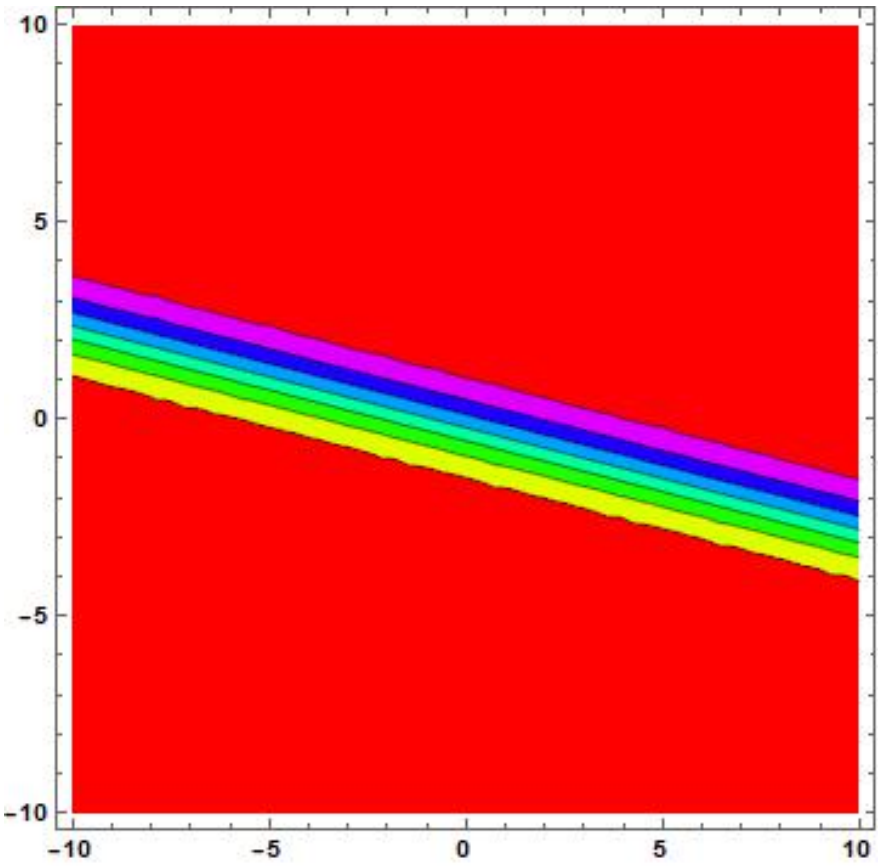


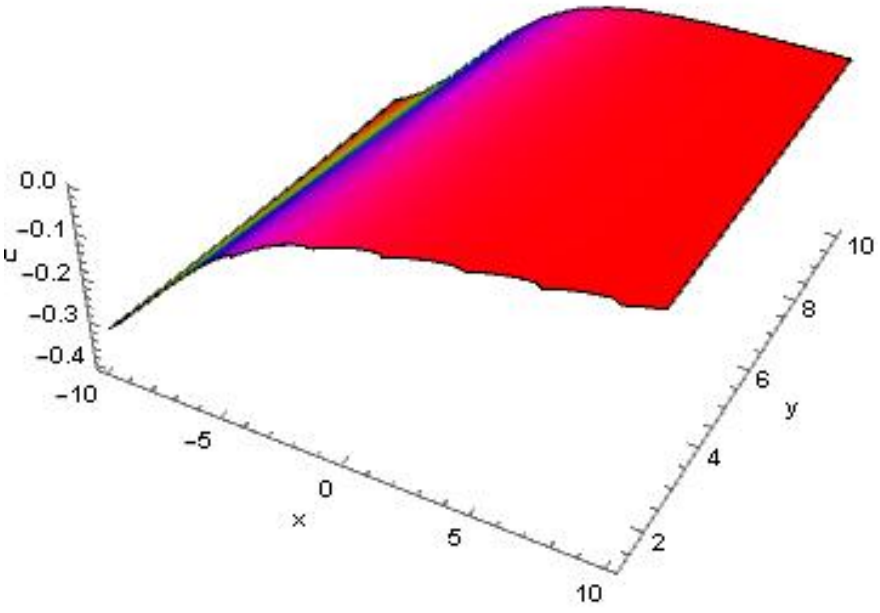


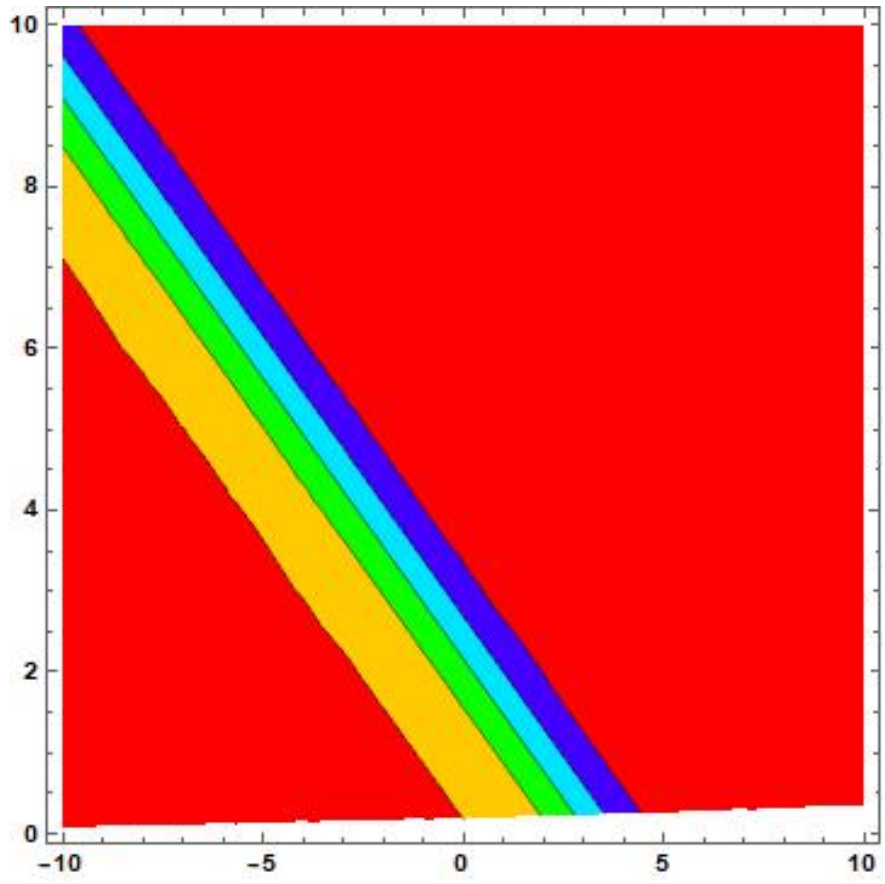


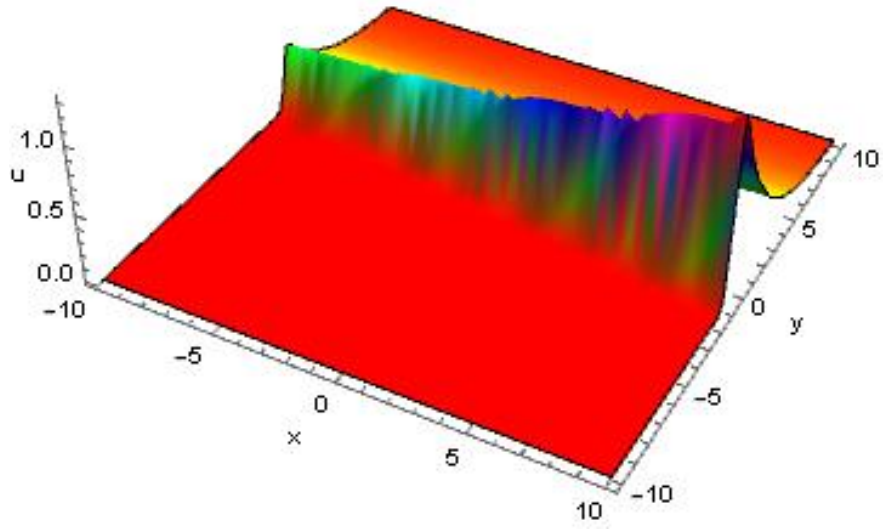


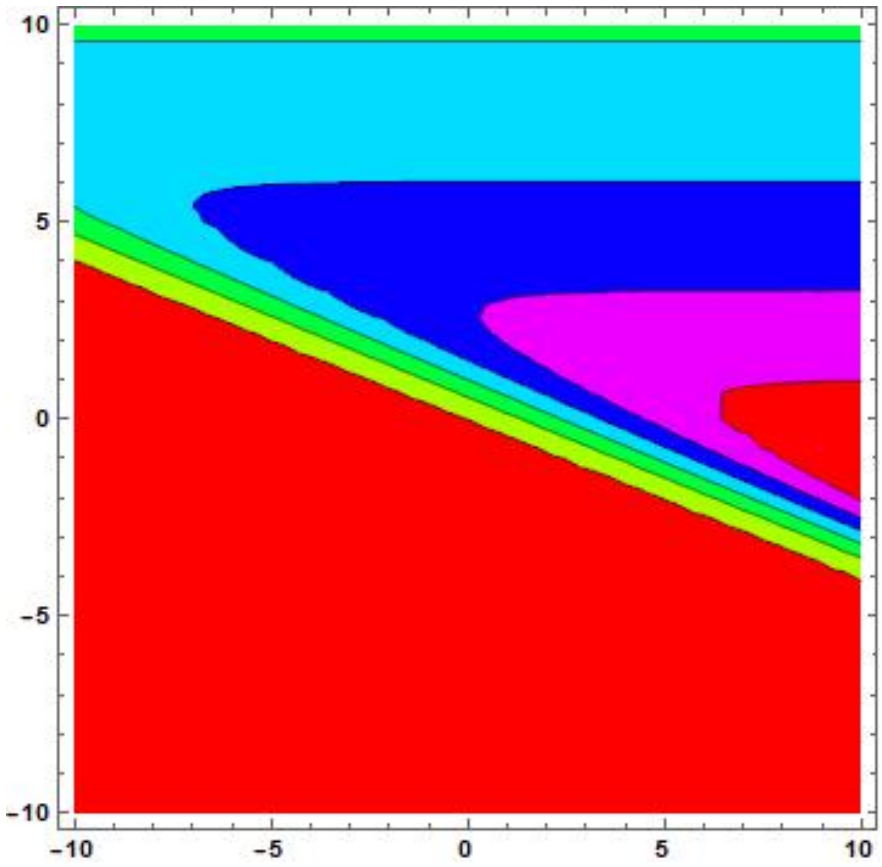


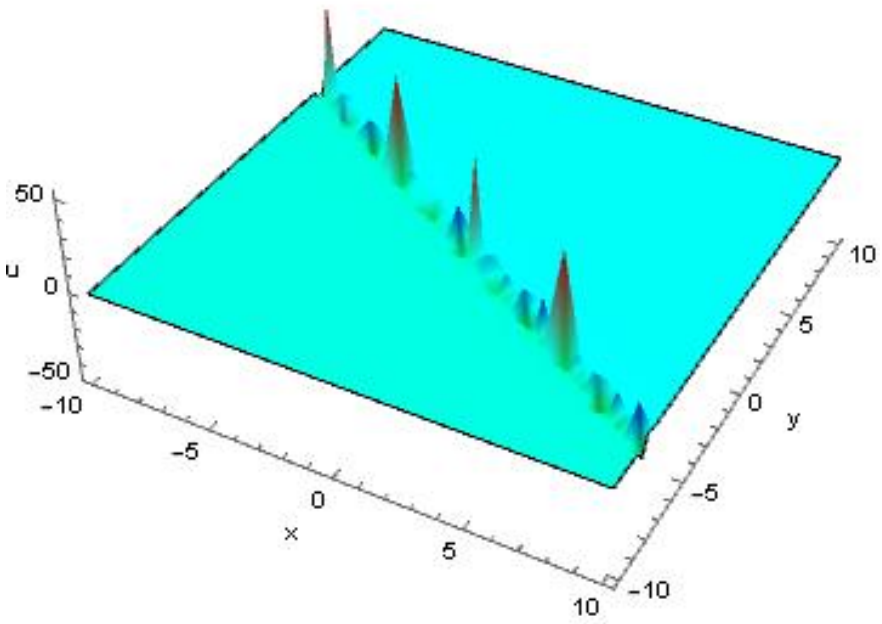


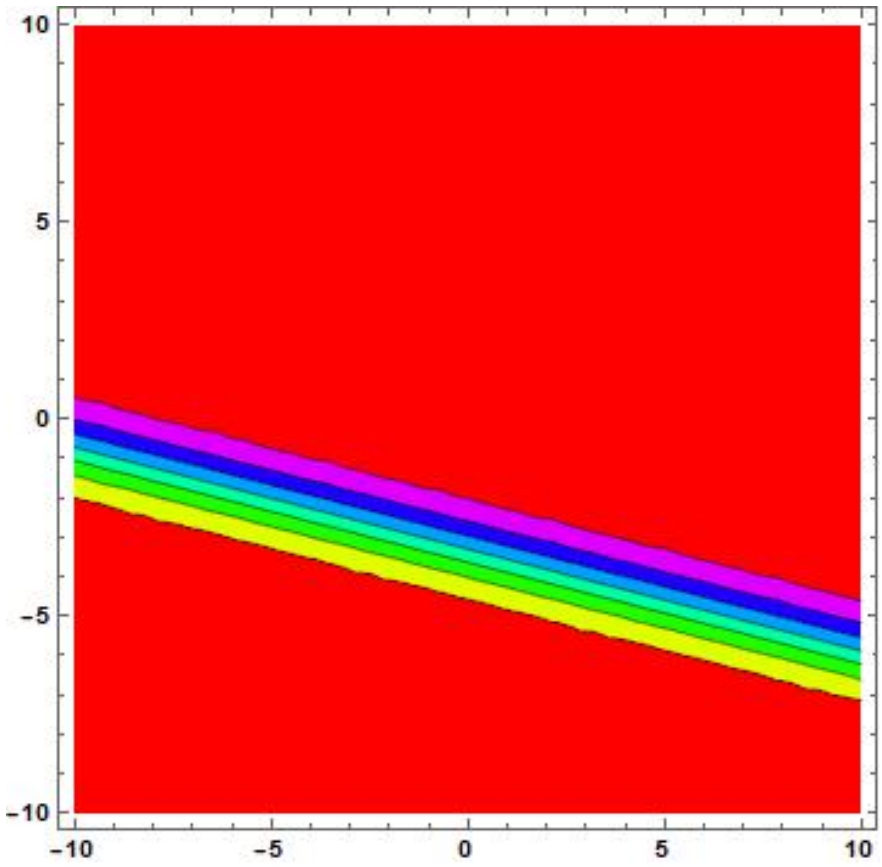


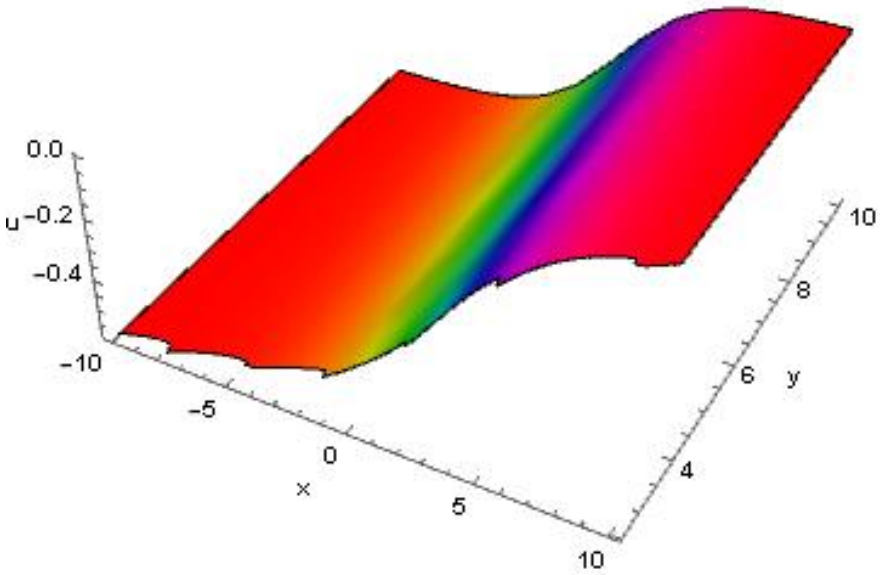


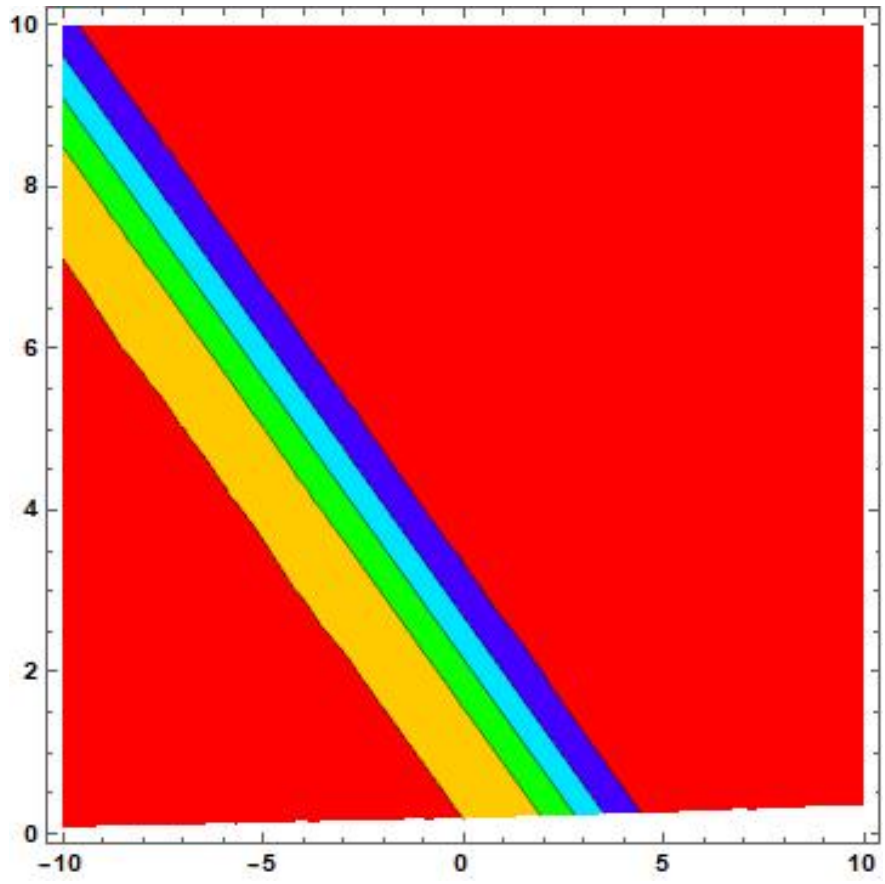


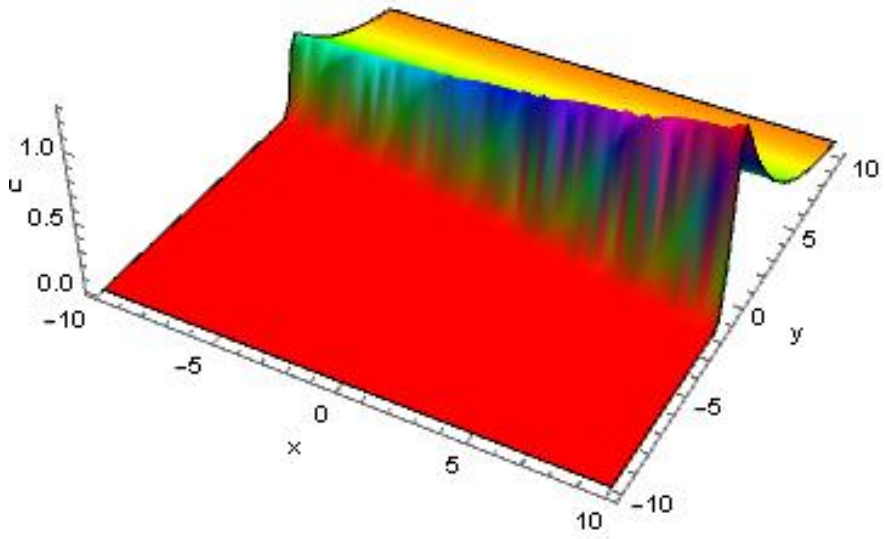


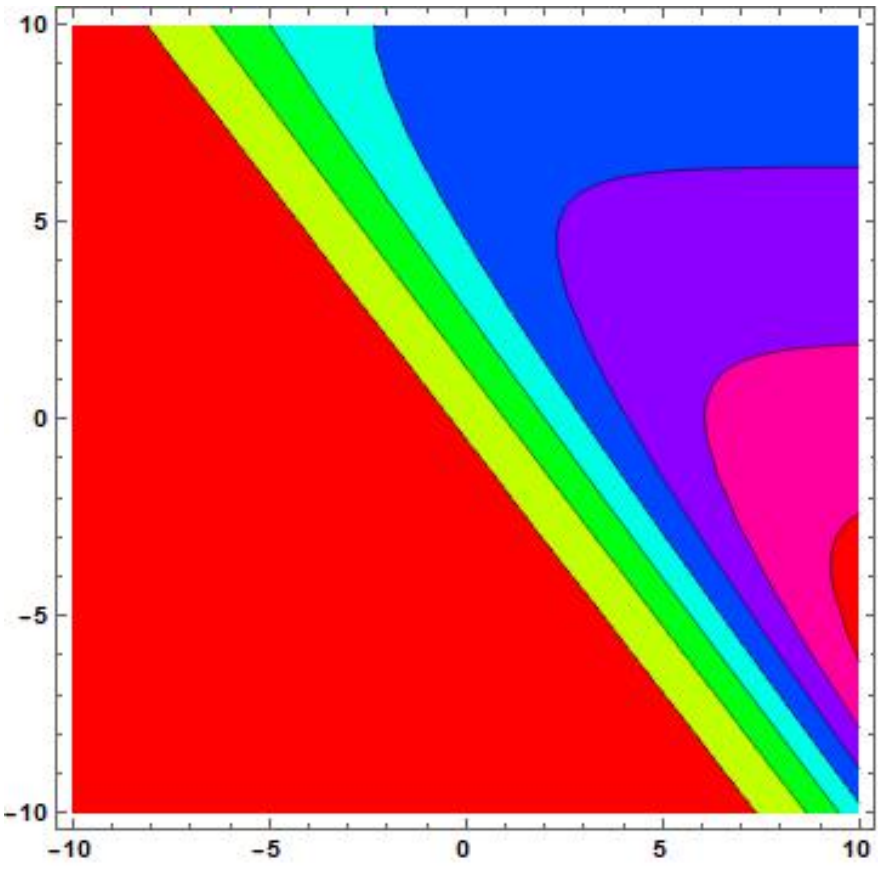












Bäcklund Transformation of (3+1)-Dimensional Variable Coefficient Potential-YTSF Equation and Related Problems

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Abstract: Based on the Bell polynomial method, the (3+1)-dimensional variable coefficient Potential-YTSF equation is transformed into bilinear form, and the double Bell polynomial Bäcklund transformation, bilinear Bäcklund transformation, Lax pair and infinite conservation law of this equation are constructed. Firstly, the Lax integrability of the equation is proved by the Bäcklund transformation of double Bell polynomials, and the infinite conservation law is constructed. Secondly, the exact solution of the equation is obtained by bilinear Bäcklund transformation and symbolic computation system Mathematica. Finally, we illustrate their properties by making some graphs of soliton solutions.

Keywords: Bell polynomial method;bilinear Bäcklund transformation;Lax pair;infinite conservation law

1 Introduction

In nonlinear science, nonlinear partial differential equation is a differential equation whose degree is higher than one. It is an important branch of modern mathematics. Whether in theory or in practical applications, nonlinear partial differential equations are used to describe problems in the fields of mechanics, control engineering, ecological and economic systems, chemical circulation systems, and epidemiology. Nonlinear partial differential equations, also known as nonlinear mathematical physics equations, nonlinear evolution equations. It is a mathematical model of nonlinear phenomena in many modern science and engineering fields such as physical chemistry, biology, atmospheric space science, nonlinear optics and astrophysics. With the development of symbolic computation, scholars have proposed many methods for solving nonlinear problems, such

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as Hirota bilinear method [1], Bäcklund transformation method [2], homogeneous balance method [3], F-expansion method [4], inverse scattering method [5] and so on.

$$u_{xxxz} + 4u_x u_{xz} + 2u_z u_{xx} - 4u_{xt} + 3u_{yy} = 0 \tag{1}$$

Equation (1) is often used to describe the dynamics of solitons and nonlinear waves in a domain.In reference [6],the self-Bäcklund transformation of equation (1) was constructed by using strong symmetry, and the separated variable solution of the equation was obtained. In reference [7],a new multi-periodic soliton solution of the (3+1)-dimensional Potential-YTSF equation was constructed by using Hirota bilinear form and generalized three-wave test method. Based on the Bell polynomial method,this paper will study the (3+1)-dimensional Potential-Yu-Toda-Sasa-Fukuyama (Potential-YTSF) equation with variable coefficients.

$$h_1(t)u_{xxxz} + h_2(t)u_x u_{xz} + h_3(t)u_z u_{xx} + h_4(t)u_{xt} + h_5(t)u_{yy} = 0 \tag{2}$$

The exact solutions of the double Bell polynomials under Bäcklund transformation,bilinear Bäcklund transformation and bilinear Bäcklund transformation.Where $u = u(x, y, z, t)$,when $h_1(t) = 1, h_2(t) = 4, h_3(t) = 2, h_4(t) = -4, h_5(t) = 3$,equation (2) becomes equation (1).Therefore,it is meaningful to study equation (2).

2 P-Polynomial and Bilinear Form of (3+1)-Dimensional Variable Coefficient Potential YTSF Equation

We introduce the definition of Bell polynomials and related results [8]~[11].We use Bell polynomials to construct the bilinear form of equation (2),and let

$$u = q_x, \tag{3}$$

Where $q = q(x, y, z, t)$.Substituting (3) into equation (2) to obtain

$$h_1(t)q_{xxxxz} + h_2(t)q_{xx}q_{xxz} + h_3(t)q_{xz}u_{xxx} + h_4(t)q_{xxt} + h_5(t)q_{xyy} = 0, \tag{4}$$

When $h_2(t) = h_3(t) = 3h_1(t)$,integrating equation (4) once for x yields

$$h_1(t)q_{xxxz} + 3h_1(t)q_{xx}q_{xz} + h_4(t)q_{xt} + h_5(t)q_{yy} = 0, \tag{5}$$

Then the equation has the following P-polynomial representation

$$E(q) = h_1(t)P_{xxxz}(q) + h_4(t)P_{xt}(q) + h_5(t)P_{yy}(q) = 0. \tag{6}$$

Let $q = 2\ln f$,equation (6) be in bilinear form

$$(h_1(t)D_x^3D_z + h_4(t)D_xD_t + h_5(t)D_y^2)f \cdot f = 0. \tag{7}$$

Where f is an undetermined function about $x, y, z, t.D_x^3D_z, D_xD_t, D_y^2$ satisfies the D-operator. The definition of D-operator is:

$$D_x^m D_z^n(f \cdot g) = (\frac{\partial}{\partial x} - \frac{\partial}{\partial x'})^m(\frac{\partial}{\partial z} - \frac{\partial}{\partial z'})^n f(x,z)g(x,z) \mid_{x=x',z=z'} \tag{8}$$

$$D_x^m D_t^n(f \cdot g) = (\frac{\partial}{\partial x} - \frac{\partial}{\partial x'})^m(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'})^n f(x,t)g(x,t) \mid_{x=x',t=t'} \tag{9}$$

3 Bi-Bell polynomial B$\ddot{a}$cklund transformation, bilinear B$\ddot{a}$cklund transformation, Lax pair and infinite conservation law for (3+1)-dimensional Potential-YTSF equation with variable coefficients

3.1 Bi-Bell polynomial B$\ddot{a}$cklund transformation for (3+1)-dimensional Potential-YTSF equation with variable coefficients

We use Bell polynomials to construct the bilinear B$\ddot{a}$cklund transformation of the (3+1)-dimensional Potential-YTSF equation.Let q, q' be the solution of equation (5),then

$$E(q) = h_1(t)q_{xxxz} + 3h_1(t)q_{xx}q_{xz} + h_4(t)q_{xt} + h_5(t)q_{yy} = 0, \tag{10}$$

$$E(q') = h_1(t)q'_{xxxz} + 3h_1(t)q'_{xx}q'_{xz} + h_4(t)q'_{xt} + h_5(t)q'_{yy} = 0, \tag{11}$$

Subtracting and transforming the two forms

$$q = w - v, q' = w + v, \tag{12}$$

we get

$$E(q') - E(q) = h_1(t)(q'-q)_{xxxz} + \frac{3}{2}h_1(t)[(q'-q)_{xz}(q'+q)_{xx} + (q'+q)_{xz}(q'-q)_{xx}] \tag{13}$$

$$+h_4(t)(q'-q)_{xt} + h_5(t)(q'-q)_{yy}$$

$$= 2h_1(t)v_{xxxz} + 2h_1(t)[3v_{xz}w_{xx} + 3w_{xz}v_{xx}] + 2h_4(t)v_{xt} + 2h_5(t)v_{yy} = 0.$$

Selecting constraint conditions

$$\mathcal{Y}_{xz} - M\mathcal{Y}_x = \lambda, \tag{14}$$

By differentiating both sides of equation (14) with respect to x,we have

$$w_{xxz} + v_{xx}v_z + v_x v_{xz} - Mv_{xx} = 0. \tag{15}$$

Substituting equations (14) and (15) into equation (13),we get

$$\begin{aligned} E(q') - E(q) &= h_1(t)v_{xxxz} + h_1(t)[3v_{xz}w_{xx} + 3w_{xz}v_{xx}] + h_4(t)v_{xt} + h_5(t)v_{yy} \\ &= [h_1(t)\mathcal{Y}_{xxx} + \alpha\mathcal{Y}_y + \beta\mathcal{Y}_x]_z + [h_4(t)\mathcal{Y}_t - \alpha\mathcal{Y}_z - 3h_1(t)\lambda\mathcal{Y}_x]_x \\ &+ [h_5(t)\mathcal{Y}_y - \beta\mathcal{Y}_z + \mu]_y + 3h_1(t)v_{xx}w_{xz} - 3h_1(t)v_x w_{2xz} - 3h_1(t)v_x^2 v_{xz} = 0. \end{aligned} \tag{16}$$

Therefore,the (3+1)-dimensional variable coefficient Potential-YTSF equation has a double Bell polynomial B$\ddot{a}$cklund transformation

$$\begin{cases} \mathcal{Y}_{xz} - M\mathcal{Y}_x = \lambda, \\ h_1(t)\mathcal{Y}_{xxx} + \alpha\mathcal{Y}_y + \beta\mathcal{Y}_x = 0, \\ h_4(t)\mathcal{Y}_t - \alpha\mathcal{Y}_z - 3h_1(t)\lambda\mathcal{Y}_x = 0, \\ h_5(t)\mathcal{Y}_y - \beta\mathcal{Y}_z + \mu = 0. \end{cases} \tag{17}$$

Where $\lambda, \alpha = \beta, \mu$ is an arbitrary constant.

3.2 Bilinear B$\ddot{a}$cklund Transformation of (3+1)-Dimensional Variable Coefficient Potential-YTSF Equation

In the transformation

$$v = ln\frac{f}{g}, w = lnfg \tag{18}$$

After that,we can get the bilinear form of the B$\ddot{a}$cklund transformation

$$\begin{cases} (D_x D_z - MD_x - \lambda)f \cdot g = 0, \\ (h_1(t)D_x^3 + \alpha D_y + \beta D_x)f \cdot g = 0, \\ (h_4(t)D_t - \alpha D_z - 3h_1(t)\lambda D_x)f \cdot g = 0, \\ (h_5(t)D_y - \beta D_z + \mu)f \cdot g = 0. \end{cases} \tag{19}$$

3.3 Lax Integrability of (3+1)-dimensional Potential YTSF Equation

Here we consider the following transformation

$$v = \ln\varphi, w = q + \ln\varphi \tag{20}$$

Next,according to the double Bell polynomial Bäcklund transformation,the linear system of the equation is obtained

$$\begin{aligned}
&h_1(t)\varphi_{xxx} + 3h_1(t)u_x\varphi_x + \alpha\varphi_y + \beta\varphi_x = 0,\\
&\varphi q_{xz} + \varphi_{xz} - M\varphi_x - \lambda\varphi = 0,\\
&h_4(t)\varphi_t - \alpha\varphi_z + 3h_1(t)\lambda\varphi_x = 0,\\
&h_5(t)\varphi_y - \beta\varphi_z + \mu\varphi = 0.
\end{aligned} \tag{21}$$

The corresponding Lax Operator from equation (21)

$$\begin{aligned}
L_1(\varphi) &= M\varphi_x + q_{xz}\varphi + \varphi_x\varphi_z - \lambda\varphi,\\
L_2(\varphi) &= \varphi_t - \frac{h_1(t)}{h_4(t)}\varphi_{xxx} - \frac{3h_1(t)}{h_4(t)}u\varphi_x - \frac{h_5(t)}{h_4(t)}\partial_z^{-1}\varphi_{yy}.
\end{aligned} \tag{22}$$

It satisfies the compatibility condition $\varphi_{txz} = \varphi_{xzt}$.So equation (22) is the Lax pair of the equation.

3.4 The infinite conservation law of (3+1)-dimensional Potential YTSF equation

Take transformation

$$2\eta = q'_x - q_x \tag{23}$$

Based on equations (23) and (12),we get

$$v_x = \eta, w_x = q_x + \eta \tag{24}$$

Substituting equation (24) into (17) to obtain

$$\begin{aligned}
&q_{xz} + \eta_z + \eta\partial_x^{-1}\eta_z - M\eta = \lambda,\\
&\partial_z[h_1(t)\eta_{xx} + 3h_1(t)\eta(q_x+\eta)_x + h_1(t)\eta^3 + \alpha\partial_x^{-1}\eta_y + \alpha\eta]\\
&+ \partial_x[h_4(t)\partial_x^{-1}\eta_t - \alpha\partial_x^{-1}\eta_z - 3h_1(t)\lambda\eta] + \partial_y[h_5(t)\partial_x^{-1}\eta_y - \alpha\partial_x^{-1}\eta_z] = 0,
\end{aligned} \tag{25}$$

Where

$$M = -\varepsilon, \lambda = \varepsilon^2, \eta = \varepsilon + \sum_{n=1}^{\infty}\mathcal{F}_n(u, u_x, u_{xx}, \cdots)\varepsilon^{-n} \tag{26}$$

Substituting equation (26) into equation (25),we get

$$\mathcal{F}_1 = q_{xz} = u_z, \mathcal{F}_2 = \mathcal{F}_{1,z} = q_{xzz} = u_{zz} \tag{27}$$

$$\mathcal{F}_{n+1} = \mathcal{F}_{n,z} + \partial_x^{-1}\mathcal{F}_{n,z} + \sum_{k=1}^{n}\mathcal{F}_k(\mathcal{F}_{n-k} + \partial_x^{-1}\mathcal{F}_{n-k,z}) \tag{28}$$

Substituting equation (26) into the second equation in equation (25),the infinite conservation law of equation (2) can be expressed as

$$\mathcal{L}_{n,z} + \mathcal{M}_{n,x} + \mathcal{G}_{n,y} + \mathcal{P}_{n,t} = 0(n = 1, 2, 3, \cdots) \tag{29}$$

$\mathcal{L}_n, \mathcal{M}_n, \mathcal{G}_n, \mathcal{P}_n$ can be expressed as

$$\begin{aligned}\mathcal{L}_n = h_1(t)(&\mathcal{F}_{n,xx} + 3\mathcal{F}_{n+1,x} + 3q_{xx}\mathcal{F}_n + 3\mathcal{F}_{n+2} + 3\sum_{k=1}^{n}\mathcal{F}_k\mathcal{F}_{n-k,x} \\ &+ 3\sum_{k=1}^{n+1}\mathcal{F}_k\mathcal{F}_{n+1-k} + \sum_{i+j+k=n}\mathcal{F}_i\mathcal{F}_j\mathcal{F}_k) + \alpha\partial_x^{-1}\mathcal{F}_{n,y} + \alpha\mathcal{F}_n.\end{aligned} \tag{30}$$

$$\mathcal{M}_n = -3h_1(t)\lambda\mathcal{F}_n - \alpha\partial_x^{-1}\mathcal{F}_{n,z}, \mathcal{G}_n = h_5(t)\partial_x^{-1}\mathcal{F}_{n,y} - \alpha\partial_x^{-1}\mathcal{F}_{n,z}, \mathcal{P}_n = h_4(t)\mathcal{F}_n \tag{31}$$

4 Soliton Solutions and Properties of (3+1)-dimensional Potential YTSF Equation

In order to obtain the soliton solution of equation (2),we set

$$\begin{aligned}&f = f(x, y, z, t) = 1 + f_1\varepsilon + f_2\varepsilon^2 + f_3\varepsilon^3 + \cdots, \\ &g = g(x, y, z, t) = 1 + g_1\varepsilon + g_2\varepsilon^2 + g_3\varepsilon^3 + \cdots, \\ &f_i = f_i(x, y, z, t), g_i = g_i(x, y, z, t)(i = 1, 2, 3, \cdots).\end{aligned} \tag{32}$$

Substituting equation (32) into the bilinear equation (19),and let the coefficient of each power of ε be 0.The coefficients of ε is

$$\begin{aligned}&-\lambda f_1 - \lambda g_1 - Mf_{1x} + Mg_{1x} + f_{1xz} + g_{1xz} = 0, \\ &\alpha f_{1y} - \alpha g_{1y} + \alpha f_{1x} - \alpha g_{1x} + h_1(t)f_{1xxx} - h_1(t)g_{1xxx} = 0, \\ &h_4(t)f_{1t} - h_4(t)g_{1t} - \alpha f_{1z} + \alpha g_{1z} - 3\lambda h_1(t)f_{1x} + 3\lambda h_1(t)g_{1x} = 0, \\ &\mu f_1 + \mu g_1 - \alpha f_{1z} + \alpha g_{1z} + h_5(t)f_{1y} - h_5(t)g_{1y} = 0.\end{aligned} \tag{33}$$

The coefficients of ε^2 is

$$\begin{aligned}
&-\lambda f_2-\lambda g_1 f_1-\lambda g_2-M g_1 f_{1x}-g_{1z} f_{1x}-M f_{2x}+M f_1 g_{1x}-f_{1z} g_{1x}\\
&+M g_{2x}+g_1 f_{1xz}+f_{2xz}+f_1 g_{1xz}+g_{2xz}=0,\\
&\alpha g_1 f_{1y}+\alpha f_{2y}-\alpha f_1 g_{1y}-\alpha g_{2y}+\alpha g_1 f_{1x}+\alpha f_{2x}-\alpha f_1 g_{1x}-\alpha g_{2x}-3h_1(t) g_{1x} f_{1xx}\\
&+3h_1(t) f_{1x} g_{1xx}+h_1(t) g_1 f_{1xxx}+h_1(t) f_{2xxx}-h_1(t) f_1 g_{1xxx}-h_1 g_{2xxx}=0,\\
&h_4(t) g_1 f_{1t}+h_4(t) f_{2t}-h_4(t) f_1 g_{1t}-h_4(t) g_{2t}-\alpha g_1 f_{1z}-\alpha f_{2z}+\alpha f_1 g_{1z}+\alpha g_{2z}\\
&-3\lambda h_1(t) g_1 f_{1x}-3\lambda h_1(t) f_{2x}+3\lambda h_1(t) f_1 g_{1x}+3\lambda h_1(t) g_{2x}=0,\\
&\mu f_2+\mu f_1 g_1+\mu g_2-\alpha g_1 f_{1z}-\alpha f_{2z}+\alpha f_1 g_{1z}+\alpha g_{2z}+h_5(t) g_1 f_{1y}+h_5(t) f_{2y}\\
&-h_5(t) f_1 g_{1y}-h_5(t) g_{2y}=0.
\end{aligned} \tag{34}$$

The coefficients of ε^3 is

$$\begin{aligned}
&-\lambda f_3-\lambda f_2 g_1-\lambda f_1 g_2-\lambda g_3-M g_2 f_{1x}-g_{2z} f_{1x}-M g_1 f_{2x}-g_{1z} f_{2x}\\
&-M f_{3x}+M f_2 g_{1x}-f_{2z} g_{1x}+M f_1 g_{2x}-f_{1z} g_{2x}+M g_{3x}+g_2 f_{1xz}+g_1 f_{2xz}\\
&+f_{3xz}+f_2 g_{1xz}+f_1 g_{2xz}+g_{3xz}=0,\\
&\alpha g_2 f_{1y}+\alpha g_1 f_{2y}+\alpha f_{3y}-\alpha f_2 g_{1y}-\alpha f_1 g_{2y}-\alpha g_{3y}+\alpha g_2 f_{1x}+\alpha g_1 f_{2x}+\alpha f_{3x}\\
&-\alpha f_2 g_{1x}-\alpha f_1 g_{2x}-\alpha g_{3x}-3h_1(t) g_{2x} f_{1xx}-3h_1(t) g_{1x} f_{2xx}+3h_1(t) f_{2x} g_{1xx}\\
&+3h_1(t) f_{1x} g_{2xx}+h_1(t) g_2 f_{1xxx}+h_1(t) g_1 f_{2xxx}+h_1(t) f_{3xxx}-h_1(t) f_2 g_{1xxx}\\
&-h_1(t) f_1 g_{2xxx}-h_1(t) g_{3xxx}=0,\\
&h_4(t) g_2 f_{1t}+h_4(t) g_1 f_{2t}+h_4(t) f_{3t}-h_4(t) f_2 g_{1t}-h_4(t) f_1 g_{2t}-h_4(t) g_{3t}-\alpha g_2 f_{1z}\\
&-\alpha g_1 f_{2z}-\alpha f_{3z}+\alpha f_2 g_{1z}+\alpha f_1 g_{2z}+\alpha g_{3z}-3\lambda h_1(t) g_2 f_{1x}-3\lambda h_1(t) g_1 f_{2x}\\
&-3\lambda h_1(t) f_{3x}+3\lambda h_1(t) f_2 g_{1x}+3\lambda h_1(t) f_1 g_{2x}+3\lambda h_1(t) g_{3x}=0,\\
&\mu f_3+\mu f_2 g_1+\mu f_1 g_2+\mu g_3-\alpha g_2 f_{1z}-\alpha g_1 f_{2z}-\alpha f_{3z}+\alpha f_2 g_{1z}+\alpha f_1 g_{2z}+\alpha g_{3z}\\
&+h_5(t) g_2 f_{1y}+h_5(t) g_1 f_{2y}+h_5(t) f_{3y}-h_5(t) f_2 g_{1y}-h_5(t) f_1 g_{2y}-h_5(t) g_{3y}=0.
\end{aligned} \tag{35}$$

First calculate the single solitary wave solution of the equation,we let

$$f_1=m(t)\exp(\eta_1)+\gamma(t), g_1=n(t)\exp(\theta_1)+\rho(t) \tag{36}$$

where $\eta_1=k_1x+l_1y+s_1z+w_1(t), \theta_1=p_1x+q_1y+r_1z+\varpi_1(t)$,substituteing it into equation (33) to get

$$k_1=\frac{1}{2}\sqrt{-\frac{2\alpha}{3h_1(t)}+\frac{\sqrt[3]{2}\alpha^2(12\lambda h_1(t)+h_5(t))}{3h_1(t)\Theta_1^{\frac{1}{3}}}+\frac{\Theta_1}{3\sqrt[3]{2}h_1(t)h_5(t)}} \tag{37}$$

$$-\frac{1}{2}\sqrt{-\frac{4\alpha}{3h_1(t)}-\frac{\sqrt[3]{2}\alpha^2(12\lambda h_1(t)+h_5(t))}{3h_1(t)\Theta_1^{\frac{1}{3}}}-\frac{\Theta_1}{3\sqrt[3]{2}h_1(t)h_5(t)}-\frac{2\Theta_3}{h_1(t)h_5(t)\sqrt{\Theta_4}}}$$

$$p_1=\frac{1}{2}\sqrt{-\frac{2\alpha}{3h_1(t)}+\frac{\sqrt[3]{2}\alpha^2(12\lambda h_1(t)+h_5(t))}{3h_1(t)\Theta_1^{\frac{1}{3}}}+\frac{\Theta_1^{\frac{1}{3}}}{3\sqrt[3]{2}h_1(t)h_5(t)}}$$

$$-\frac{1}{2}\sqrt{-\frac{4\alpha}{3h_1(t)}-\frac{\sqrt[3]{2}\alpha^2(12\lambda h_1(t)+h_5(t))}{3h_1(t)\Theta_1^{\frac{1}{3}}}-\frac{\Theta_1}{3\sqrt[3]{2}h_1(t)h_5(t)}+\frac{2\Theta_3}{h_1(t)h_5(t)\sqrt{\Theta_4}}}$$

$$r_1=\frac{1}{\alpha^2}[-\frac{\alpha(3M\alpha+\mu)}{4}+\frac{3\alpha^2\lambda h_1(t)h_5(t)\sqrt{\Theta_4}}{\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{\alpha^2h_5^2(t)\sqrt{\Theta_4}}{4\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{1}{8\sqrt[3]{2}}\Theta_1^{\frac{1}{3}}\sqrt{\Theta_4}$$
$$-\frac{1}{8}h_1(t)h_5(t)\Theta_4^{\frac{3}{2}}+\frac{1}{4}\alpha h_5(t)\sqrt{\Theta_5}+\frac{3\alpha^2\lambda h_1(t)h_5(t)\sqrt{\Theta_5}}{\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}$$
$$+\frac{\alpha^2h_5(t)\sqrt{\Theta_5}}{4\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{1}{8\sqrt[3]{2}}\Theta_1^{\frac{1}{3}}\sqrt{\Theta_5}+\frac{1}{8}h_1(t)h_5(t)\Theta_5^{\frac{3}{2}}]$$

$$s_1=\frac{1}{\alpha^2}[\frac{\alpha(3M\alpha+\mu)}{4}+\frac{3\alpha^2\lambda h_1(t)h_5(t)\sqrt{\Theta_4}}{\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{\alpha^2h_5^2(t)\sqrt{\Theta_4}}{4\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{1}{8\sqrt[3]{2}}\Theta_1^{\frac{1}{3}}\sqrt{\Theta_4}$$
$$-\frac{1}{8}h_1(t)h_5(t)\Theta_4^{\frac{3}{2}}+\frac{1}{4}\alpha h_5(t)\sqrt{\Theta_5}+\frac{3\alpha^2\lambda h_1(t)h_5(t)\sqrt{\Theta_5}}{\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}$$
$$+\frac{\alpha^2h_5(t)\sqrt{\Theta_5}}{4\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{1}{8\sqrt[3]{2}}\Theta_1^{\frac{1}{3}}\sqrt{\Theta_5}+\frac{1}{8}h_1(t)h_5(t)\Theta_5^{\frac{3}{2}}]$$

$$l_1=\frac{1}{\alpha^2}[\frac{3\alpha(M\alpha-\mu)}{4h_5(t)}+\frac{3\alpha^2\lambda h_1(t)\sqrt{\Theta_4}}{\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{\alpha^2h_5(t)\sqrt{\Theta_4}}{4\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{1}{8\sqrt[3]{2}h_5(t)}\Theta_1^{\frac{1}{3}}\sqrt{\Theta_4}$$
$$-\frac{1}{8}h_1(t)\Theta_4^{\frac{3}{2}}+\frac{1}{4}\alpha\sqrt{\Theta_5}+\frac{3\alpha^2\lambda h_1(t)\sqrt{\Theta_5}}{\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}$$
$$+\frac{\alpha^2h_5(t)\sqrt{\Theta_5}}{4\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{1}{8\sqrt[3]{2}h_5(t)}\Theta_1^{\frac{1}{3}}\sqrt{\Theta_5}+\frac{1}{8}h_1(t)\Theta_5^{\frac{3}{2}}]$$

$$q_1=\frac{1}{\alpha^2}[\frac{3\alpha(\mu-M\alpha)}{4h_5(t)}+\frac{3\alpha^2\lambda h_1(t)\sqrt{\Theta_4}}{\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{\alpha^2h_5(t)\sqrt{\Theta_4}}{4\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{1}{8\sqrt[3]{2}h_5(t)}\Theta_1^{\frac{1}{3}}\sqrt{\Theta_4}$$
$$-\frac{1}{8}h_1(t)\Theta_4^{\frac{3}{2}}+\frac{1}{4}\alpha\sqrt{\Theta_5}+\frac{3\alpha^2\lambda h_1(t)\sqrt{\Theta_5}}{\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}$$
$$+\frac{\alpha^2h_5(t)\sqrt{\Theta_5}}{4\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{1}{8\sqrt[3]{2}h_5(t)}\Theta_1^{\frac{1}{3}}\sqrt{\Theta_5}+\frac{1}{8}h_1(t)\Theta_5^{\frac{3}{2}}]$$

$$w_1'(t)=\frac{1}{\alpha h_4(t)m(t)}[\frac{\alpha m(t)(3M\alpha+\mu)}{4}+\frac{3}{2}\alpha\lambda h_1(t)m(t)\sqrt{\Theta_4}+\frac{3\alpha^2\lambda h_1(t)h_5(t)\sqrt{\Theta_4}m(t)}{\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}$$

$$+\frac{\alpha^2 h_5^2(t)\sqrt{\Theta_4}m(t)}{4\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{1}{8\sqrt[3]{2}}\Theta_1^{\frac{1}{3}}\sqrt{\Theta_4}m(t)-\frac{1}{8}h_1(t)h_5(t)\Theta_4^{\frac{3}{2}}m(t)$$

$$+\frac{1}{4}\alpha h_5(t)\sqrt{\Theta_5}m(t)+\frac{3\alpha^2\lambda h_1(t)h_5(t)\sqrt{\Theta_5}m(t)}{\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{\alpha^2 h_5^2(t)\sqrt{\Theta_5}m(t)}{4\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}$$

$$+\frac{1}{8\sqrt[3]{2}}\Theta_1^{\frac{1}{3}}\sqrt{\Theta_5}m(t)+\frac{1}{8}h_1(t)h_5(t)\Theta_5^{\frac{3}{2}}m(t)-\alpha h_4(t)m'(t)]$$

$$\varpi_1'(t)=\frac{1}{\alpha h_4(t)n(t)}[-\frac{\alpha n(t)(3M\alpha+\mu)}{4}+\frac{3}{2}\alpha\lambda h_1(t)n(t)\sqrt{\Theta_4}+\frac{3\alpha^2\lambda h_1(t)h_5(t)\sqrt{\Theta_4}n(t)}{\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}$$

$$+\frac{\alpha^2 h_5^2(t)\sqrt{\Theta_4}n(t)}{4\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{1}{8\sqrt[3]{2}}\Theta_1^{\frac{1}{3}}\sqrt{\Theta_4}n(t)-\frac{1}{8}h_1(t)h_5(t)\Theta_4^{\frac{3}{2}}n(t)$$

$$+\frac{1}{4}\alpha h_5(t)\sqrt{\Theta_5}n(t)+\frac{3\alpha^2\lambda h_1(t)h_5(t)\sqrt{\Theta_5}n(t)}{\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{\alpha^2 h_5^2(t)\sqrt{\Theta_5}n(t)}{4\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}$$

$$-\frac{3}{2}\alpha\lambda h_1(t)n(t)\sqrt{\Theta_5}+\frac{1}{8\sqrt[3]{2}}\Theta_1^{\frac{1}{3}}\sqrt{\Theta_5}n(t)+\frac{1}{8}h_1(t)h_5(t)\Theta_5^{\frac{3}{2}}n(t)-\alpha h_4(t)n'(t)]$$

$$\Theta_1=27\Theta_3^2h_1(t)h_5(t)-72\alpha^3\lambda h_1h_5^2+2\alpha^3h_5^3(t)+\sqrt{\Theta_2},\Theta_3=M\alpha^2-\alpha\mu \tag{38}$$

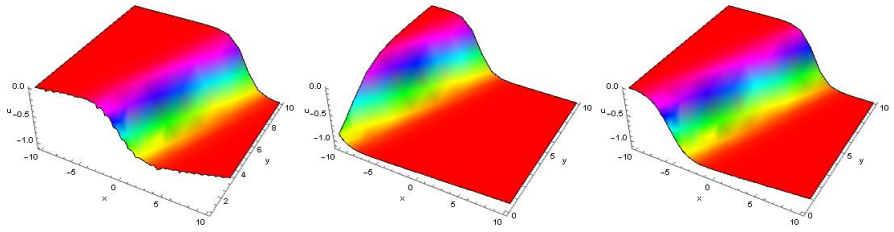
$$\Theta_2=-4h_5^3(t)[12\alpha^2\lambda h_1(t)+\alpha^2h_5(t)]^3+h_5^2(t)[27\Theta_3^2h_1(t)-72\alpha^3\lambda h_1(t)h_5(t)+2\alpha^3h_5^2(t)]^2 \tag{39}$$

$$\Theta_4=-\frac{2\alpha}{3h_1(t)}+\frac{\sqrt[3]{2}\alpha^2(12\lambda h_1(t)+h_5(t))}{3h_1(t)\Theta_1^{\frac{1}{3}}}+\frac{\Theta_1^{\frac{1}{3}}}{3\sqrt[3]{2}h_1(t)h_5(t)} \tag{40}$$

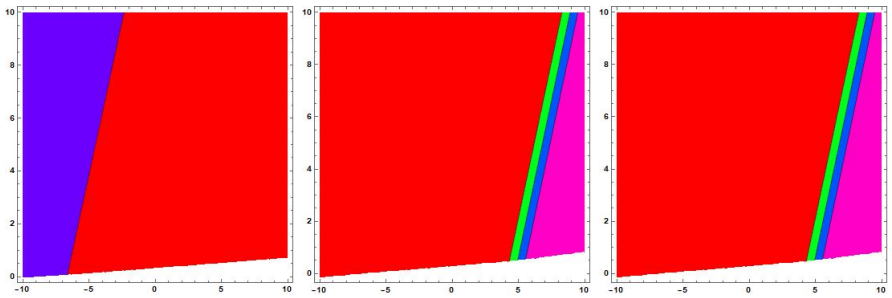
$$\Theta_5=-\frac{4\alpha}{3h_1(t)}-\frac{\sqrt[3]{2}\alpha^2(12\lambda h_1(t)+h_5(t))}{3h_1(t)\Theta_1^{\frac{1}{3}}}-\frac{\Theta_1}{3\sqrt[3]{2}h_1(t)h_5(t)}+\frac{2\Theta_3}{h_1(t)h_5(t)\sqrt{\Theta_4}} \tag{41}$$

Supposing that $f_2=g_2=0,f_i=g_i=0(i=3,4,\cdots)$,the single soliton solution of the variable coefficient (3+1)-dimensional Potential-YTSF equation is

$$u=[\ln(\frac{1+f_1}{1+g_1})]_x=\frac{m(t)\exp(\eta_1)[k_1-\exp(\theta_1)n(t)(p_1-k_1)]-p_1n(t)\exp(\theta_1)(1+\gamma(t))}{(1+n(t)\exp(\theta_1))(1+m(t)\exp(\eta_1)+\gamma(t))} \tag{42}$$

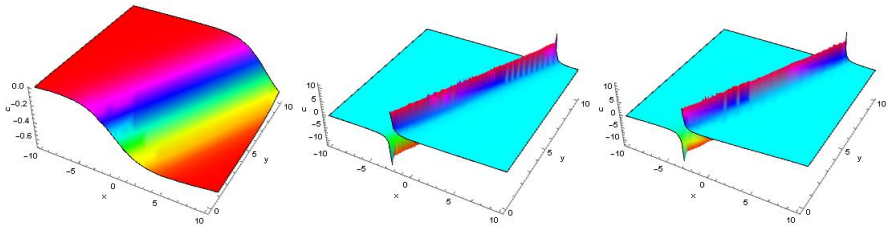


(a)t=0 (b)t=0.5 (c)t=-0.5

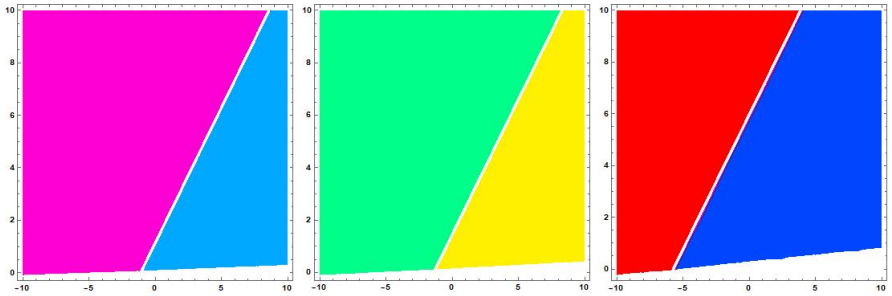


(d) (e) (f)

Fig.1$h_1(t) = \arctan(t), n(t) = t, m(t) = t^2, \gamma(t) = \sin(t), h_5(t) = \cos(t), r_1 = -1, k_1 = 3$

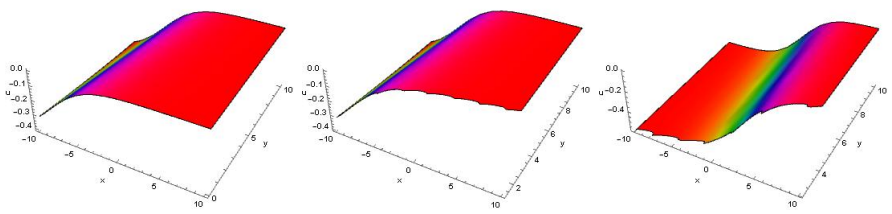


(a)t=0 (b)t=0.5 (c)t=-0.5

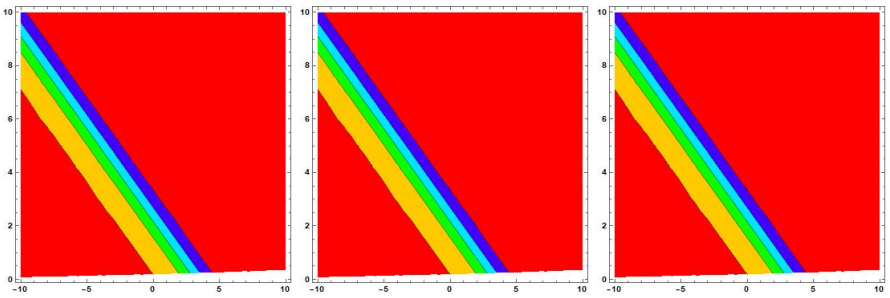


(d) (e) (f)

Fig.2$h_1(t) = \arctan(t), n(t) = \exp(t)\arctan(t), m(t) = t^2\cos(t), \gamma(t) = \sin(t)\cosh(t), h_5(t) = \cos(t), r_1 = -1, k_1 = 3,$



(a)t=0 (b)t=0.5 (c)t=-0.5



(d) (e) (f)

Fig.3$h_4(t) = t^2, n(t) = \sin(t) + \arctan(t), m(t) = t^2 + \cos(t), \gamma(t) = t + \sin(t), h_5(t) = \cos(t), r_1 = -1, k_1 = 3,$

Next we calculate the double solitary wave solution of the equation,let

$$f_1 = m_1(t)\exp(\eta_1) + m_2(t)\exp(\eta_2) + \gamma(t), f_2 = m_{12}(t)\exp(\eta_1 + \eta_2) \tag{43}$$

$$g_1 = n_1(t)\exp(\theta_1) + n_2(t)\exp(\theta_2) + \rho(t), g_2 = n_{12}(t)\exp(\theta_1 + \theta_2) \tag{44}$$

where $\eta_i = k_i x + l_i y + s_i z + w_i(t), \theta_i = p_i x + q_i y + r_i z + \varpi_i(t)(i = 1, 2)$,substituting equation (45) and (46) into equation (36)

$$q_1 = 0, q_2 = 0, s_1 = M, s_2 = 0, r_1 = 0, r_2 = 0, k_2 = 0, p_1 = 0, p_2 = 0, l_2 = 0, \tag{45}$$

$$\mu = 0, \lambda = 0, l_1 = -k_1 - \frac{k_1^3 M h_1(t) m_1(t)\gamma(t)}{h_4(t)[\gamma(t)(m_1'(t) + m_1(t)w_1'(t)) - m_1(t)\gamma'(t)]},$$

$$\rho(t) = \frac{\gamma(t)\rho'(t)}{\gamma'(t)}, w_2(t) = \int[-\frac{m_2'(t)}{m_2(t)} + \frac{\gamma'(t)}{\gamma(t)}]dt, \varpi_2(t) = \int[-\frac{n_2'(t)}{n_2(t)} + \frac{\gamma'(t)}{\gamma(t)}]dt$$

$$n_1(t) = -\frac{\gamma(t)n_1'(t)}{\gamma(t)\varpi_1'(t) - \gamma'(t)}, n_{12}(t) = \frac{n_2(t)n_{12}'(t)\gamma(t)}{\gamma(t)n_2'(t) - n_2(t)\gamma(t)z_1'(t) - n_2(t)\gamma'(t)}$$

$$m_{12}(t) = \frac{m_1(t)m_2(t)\gamma(t)m_{12}'(t)}{m_2(t)\gamma(t)m_1'(t) + m_1(t)\gamma(t)m_2'(t) - 2m_1(t)m_2(t)\gamma'(t)},$$

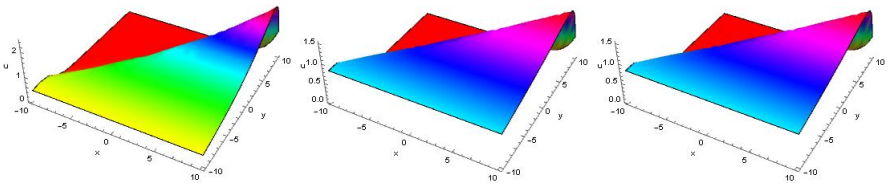
$$h_5(t) = -\frac{h_4^2(t)[\gamma(t)(m_1'(t) + m_1(t)w_1'(t)) - m_1(t)\gamma'(t)]^2}{k_1 m_1(t)\gamma(t)[k_1^2 M m_1(t)h_1(t)\gamma(t) + h_4(t)[\Theta_6]]}$$

$$\Theta_6 = \gamma(t)(m_1'(t) + m_1(t)w_1'(t)) - m_1(t)\gamma'(t) \tag{46}$$

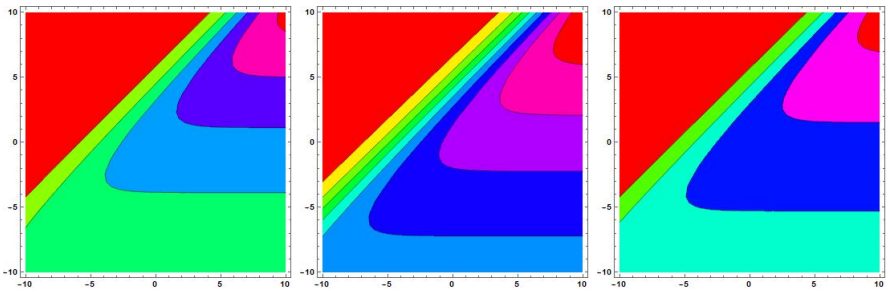
Supposing that $f_3 = g_3 = 0, f_i = g_i = 0(i = 4, 5, \cdots), \varepsilon = 1$,the double soliton solution of the (3+1)-dimensional variable coefficient Potential-YTSF equation is

$$u = [\ln(\frac{1 + f_1 + f_2}{1 + g_1 + g_2})]_x \tag{47}$$

$$= \frac{k_1 m_1(t)\exp(\eta_1) + \exp(\eta_2)[(k_1+k_2)m_{12}(t)\exp(\eta_1) + k_2 m_2(t)]}{1 + m_1(t)\exp(\eta_1) + m_{12}(t)\exp(\eta_1+\eta_2) + m_2(t)\exp(\eta_2) + \gamma(t)}$$
$$+ \frac{p_1 n_1(t)\exp(\theta_1) + \exp(\theta_2)[\exp(\theta_1)(p_1+p_2)n_{12}(t) + p_2 n_2(t)]}{1 + n_1(t)\exp(\theta_1) + n_{12}(t)\exp(\theta_1+\theta_2) + n_2(t)\exp(\theta_2) + \rho(t)}$$

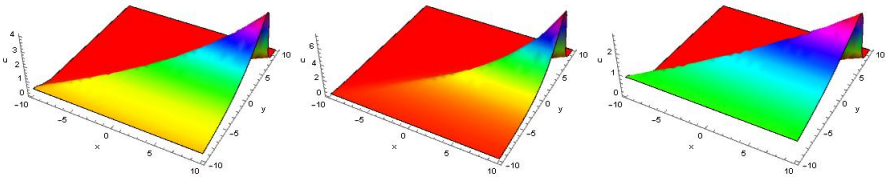


(a)t=5 (b)t=5.5 (c)t=6

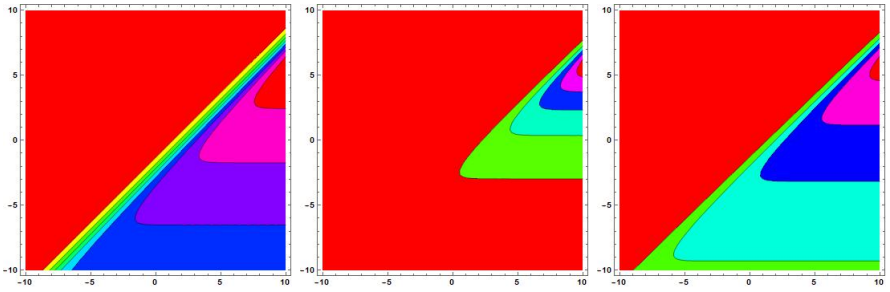


(d) (e) (f)

Fig.4$w_1(t) = \cos(t), h_1(t) = \sin(t), \gamma(t) = \arctan(t), h_4(t) = t^2, m_2(t) = \exp(t), m_{12}(t) = \cosh(t), m_1(t) = t, z = 0, k_1 = 2, M = 4$

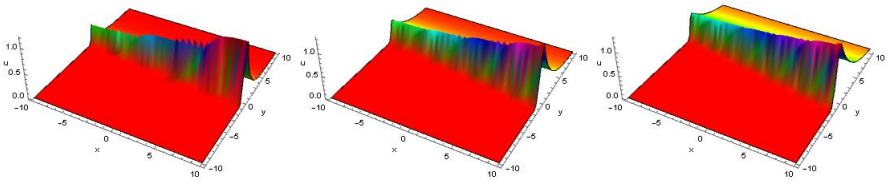


(a)t=2 (b)t=2.3 (c)t=2.8

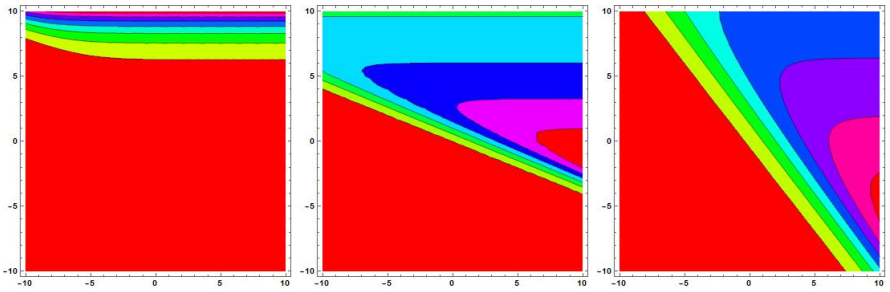


(d) (e) (f)

Fig.5$w_1(t) = t\cos(t), h_1(t) = \sin(t)\cosh(t), \gamma(t) = \arctan(t), h_4(t) = t^2\arctan(t), m_2(t) = \exp(t)\cos(t), m_{12}(t) = t\cosh(t), m_1(t) = t, z = 0, k_1 = 2, M = 4$



(a)t=0 (b)t=0.6 (c)t=1



(d) (e) (f)

Fig.6$w_1(t) = t + \cos(t), h_1(t) = \sin(t) + \cosh(t), \gamma(t) = \arctan(t), h_4(t) = t^2 + \arctan(t), m_2(t) = \exp(t) + \cos(t), m_{12}(t) = t + \cosh(t), m_1(t) = t, z = 0$

Next,we calculate the three-soliton solution of the equation.We let

$$\begin{aligned} f_1 &= m_1(t)\exp(\eta_1) + m_2(t)\exp(\eta_2) + m_3(t)\exp(\eta_3) + \gamma(t), \\ f_2 &= m_{12}(t)\exp(\eta_1+\eta_2) + m_{13}(t)\exp(\eta_1+\eta_3) + m_{23}(t)\exp(\eta_2+\eta_3), \\ f_3 &= m_{123}(t)\exp(\eta_1+\eta_2+\eta_3), \end{aligned} \tag{48}$$

$$\begin{aligned} g_1 &= n_1(t)\exp(\theta_1) + n_2(t)\exp(\theta_2) + n_3(t)\exp(\theta_3) + \rho(t), \\ g_2 &= n_{12}(t)\exp(\theta_1+\theta_2) + n_{13}(t)\exp(\theta_1+\theta_3) + n_{23}(t)\exp(\theta_2+\theta_3), \\ g_3 &= n_{123}(t)\exp(\theta_1+\theta_2+\theta_3) \end{aligned} \tag{49}$$

where $\eta_i = k_i x + l_i y + s_i z + w_i(t), \theta_i = p_i x + q_i y + r_i z + \varpi_i(t)(i = 1, 2, 3)$,substituting equations (51) and (52) into equation (38)

$$k_2 = 0, k_3 = -k_1, p_1 = 0, p_2 = 0, p_3 = 0, \tag{50}$$

$$m_2(t) = -\frac{m_1(t)m_{12}(t)\gamma(t)\rho(t)m_2'(t)}{-m_1(t)\gamma(t)\rho(t)m_{12}'(t) + m_{12}(t)[-2m_1(t)\rho(t)\gamma'(t) + \gamma(t)(\Theta_7)]}$$

$$n_2(t) = \frac{n_{123}(t)n_{13}(t)\gamma(t)\rho(t)n_2'(t)}{-n_{123}(t)\gamma(t)\rho(t)n_{13}'(t) + n_{13}(t)[n_{123}(t)\rho(t)\gamma'(t) + \gamma(t)(\Theta_8)]},$$

$$m_3(t) = \frac{m_1(t)m_{12}(t)m_{23}(t)m_3'(t)}{m_{12}(t)m_{23}(t)m_1'(t) - m_1(t)m_{23}(t)m_{12}'(t) + m_1(t)m_{12}(t)m_{23}'(t)}$$

$$n_3(t) = \frac{n_1(t)n_{13}(t)\gamma(t)\rho(t)n_3'(t)}{n_1(t)\gamma(t)\rho(t)n_{13}'(t) - n_{13}(t)[n_1(t)\rho(t)\gamma'(t) + \gamma(t)(\rho(t)n_1'(t) - 2n_1(t)\rho'(t))]}$$

$$n_{12}(t) = -\frac{n_1(t)n_{123}(t)n_{13}(t)\gamma(t)\rho(t)n_{12}'(t)}{-n_1(t)n_{13}(t)\gamma(t)\rho(t)n_{123}'(t) + n_{123}(t)[n_1(t)\gamma(t)\rho(t)n_{13}'(t) + \Theta_9]}$$

$$m_{13}(t) = \frac{m_1(t)m_{12}(t)m_{23}(t)\gamma(t)\rho(t)m_{13}'(t)}{-m_1(t)m_{23}(t\gamma(t)\rho(t)m_{12}'(t) + m_{12}(t)[m_1(t)\gamma(t)\rho(t)m_{23}'(t) + \Theta_{10}])}$$

$$n_{23}(t) = \frac{n_1(t)n_{123}(t)\gamma(t)\rho(t)n_{23}'(t)}{n_1(t)\gamma(t)\rho(t)n_{123}'(t) + n_{123}(t)[n_1(t)\rho(t)\gamma'(t) + \gamma(t)(-\rho(t)n_1'(t) + n_1(t)\rho'(t))]}$$

$$m_{123}(t) = \frac{m_1(t)m_{23}(t)\gamma(t)\rho(t)m_{123}'(t)}{m_1(t)\gamma(t)\rho(t)m_{23}'(t) - m_{23}(t)[m_1(t)\rho(t)\gamma'(t) + \gamma(t)(-\rho(t)m_1'(t) + m_1(t)\rho(t))]}$$

$$\varpi_3'(t) = \frac{[p_1s_1 + p_3s_1 - k_1(r_1 + r_3)]m_1(t)n_{13}(t)\gamma(t)\rho(t)n_1'(t) + \Theta_{11}}{(p_1s_1 - k_1r_1)m_1(t)n_1(t)n_{13}(t)\gamma(t)\rho(t)}$$

$$\Theta_7 = \rho(t)m_1'(t) + m_1(t)\rho'(t), \Theta_8 = \rho(t)n_{123}'(t) + n_{123}(t)\rho'(t) \tag{51}$$

$$\Theta_9 = n_{13}(t)(-\gamma(t)\rho(t)n_1'(t) - 2n_1(t)\rho(t)\gamma'(t) + n_1(t)\gamma(t)\rho'(t)) \tag{52}$$

$$\Theta_{10} = m_{23}(t)(2\gamma(t)\rho(t)m_1'(t) - 2m_1(t)\rho(t)\gamma'(t) + m_1(t)\gamma(t)\rho'(t)) \tag{53}$$

$$\Theta_{11} = n_1(t)[(-p_1s_1 + k_1r_1)m_1(t)\gamma(t)\rho(t)n_{13}'(t) + \Theta_{12}], \tag{54}$$

$$\Theta_{12} = n_{13}(t)[[(-k_1 + p_3)r_1 + p_1(s_1 - r_3)]m_1(t)\rho(t)\gamma'(t) + \gamma(t)[\Theta_{13}]] \tag{55}$$

$$\Theta_{13} = \rho(t)[(-p_3r_1 + p_1r_3)m_1'(t) + m_1(t)[(-p_3r_1 + p_1r_3)w_1'(t) + (p_3s_1 - k_1r_3)\varpi_1'(t)]] + \Theta_{14} \tag{56}$$

$$\Theta_{14} = (-p_1s_1 - p_3s_1 + k_1(r_1 + r_3))m_1(t)\rho'(t) \tag{57}$$

Supposing that $f_4 = g_4 = 0, f_i = g_i = 0(i = 5, 6, \cdots), \varepsilon = 1$,the triple soliton solution of the variable coefficient (3+1)-dimensional Potential-YTSF equation is

$$u = [\ln(\frac{1 + f_1 + f_2 + f_3}{1 + g_1 + g_2 + g_3})]_x \tag{58}$$

$$= \frac{k_1m_1(t)\exp(\eta_1) + k_2m_2(t)\exp(\eta_2) + k_3m_3(t)\exp(\eta_3) + \Delta_1}{1 + f_1 + f_2 + f_3}$$

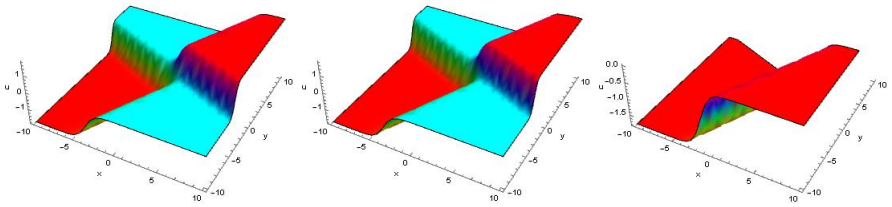
$$+\frac{p_1n_1(t)\exp(\theta_1) + p_2n_2(t)\exp(\theta_2) + p_3n_3(t)\exp(\theta_3) + \Delta_3}{1 + g_1 + g_2 + g_3}$$

$$\Delta_1 = (k_1 + k_2)m_{12}(t)\exp(\eta_1 + \eta_2) + (k_1 + k_3)m_{13}(t)\exp(\eta_1 + \eta_3) + \Delta_2$$

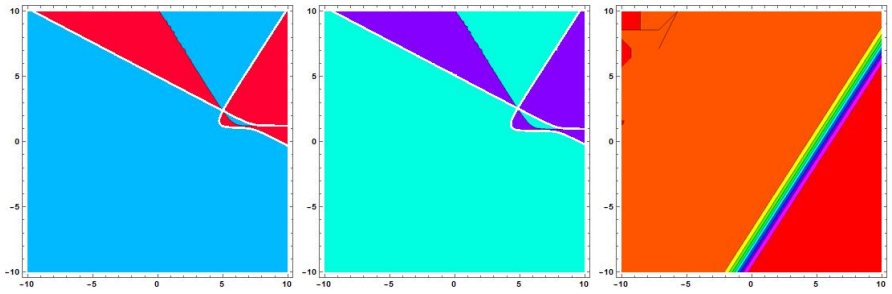
$$\Delta_2 = (k_2 + k_3)m_{23}(t)\exp(\eta_2 + \eta_3) + (k_1 + k_2 + k_3)m_{123}(t)\exp(\eta_1 + \eta_2 + \eta_3)$$

$$\Delta_3 = (p_1 + p_2)n_{12}(t)\exp(\theta_1 + \theta_2) + (p_1 + p_3)n_{13}(t)\exp(\theta_1 + \theta_3) + \Delta_4$$

$$\Delta_4 = (p_2 + p_3)n_{23}(t)\exp(\theta_2 + \theta_3) + (p_1 + p_2 + p_3)n_{123}(t)\exp(\theta_1 + \theta_2 + \theta_3)$$

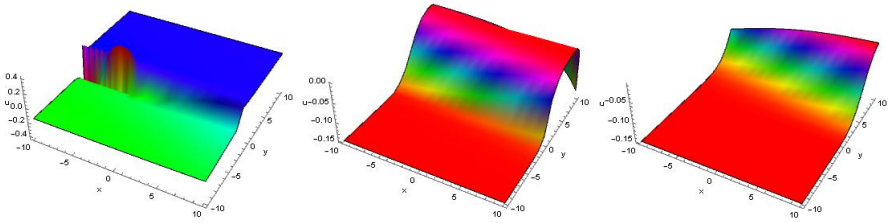


(a)t=0 (b)t=0.5 (c)t=2

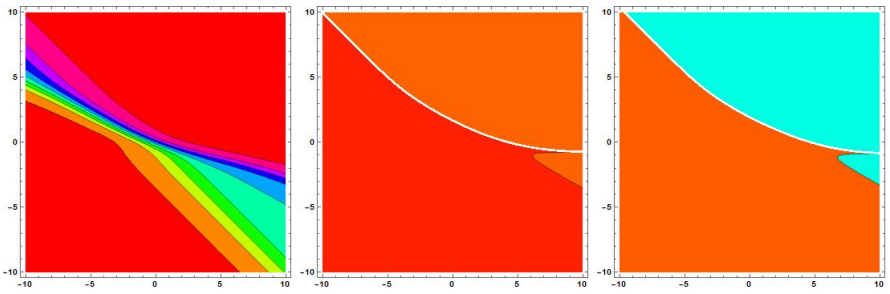


(d) (e) (f)

Fig.7$m_1(t) = \cos(t), w_1(t) = \tan(t), \rho(t) = \sin(t), \gamma(t) = \cos(t), m_{12}(t) = \exp(t), n_1(t) = t^2, m_{23}(t) = t^2, m_3(t) = \arctan(t), z_1(t) = \cosh(t), m_2(t) = t, m_{13}(t) = \sin(t), m_{123}(t) = \arccos(t)$

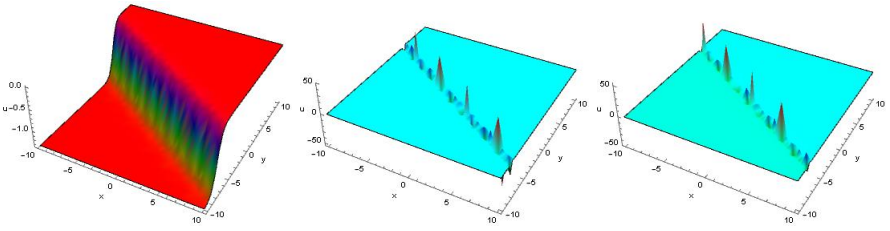


(a)t=0 (b)t=0.3 (c)t=0.5

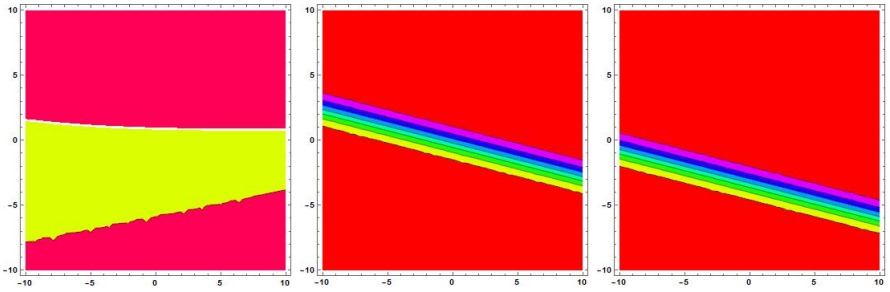


(d) (e) (f)

Fig.8$m_1(t) = \cos(t)\arctan(t), w_1(t) = t\tan(t), \rho(t) = t\sin(t), \gamma(t) = \cos(t), m_{12}(t) = t^2\exp(t), n_1(t) = t^2\sin(t), m_{23}(t) = t^2, m_3(t) = \arctan(t), z_1(t) = \exp(t)\cosh(t), m_2(t) = t\exp(t), m_{13}(t) = \sin(t), m_{123}(t) = \arccos(t)$



(a)t=6 (b)t=8 (c)t=8.3



(d) (e) (f)

Fig.9$m_1(t) = \cos(t) + \arctan(t), w_1(t) = t + \tan(t), \rho(t) = t + \sin(t), \gamma(t) = \cos(t), m_{12}(t) = t^2 + \exp(t), n_1(t) = t^2 + \sin(t), m_{23}(t) = t^2, m_3(t) = \arctan(t), \varpi_1(t) = \exp(t) + \cosh(t), m_2(t) = t + \exp(t), m_{13}(t) = \sin(t), m_{123}(t) = \arccos(t)$

5 Conclusion

Ref. [6] obtained the exact solution of equation (1) by using the self-Bäcklund transformation. In [7], the exact solution of (3+1)-dimensional constant coefficient equation (1) is obtained by traveling wave transformation method. Reference [13] obtained the exact solution of the constant coefficient equation (1) by the generalized projective Ricatti method. Reference [14] constructed a new form of solution of the constant coefficient equation (1) by using the KdV equation and its various solutions to generate various solutions of the nonlinear evolution equation. In this paper, based on the Bell polynomial method, the (3+1)-dimensional variable coefficient Potential-YTSF equation is transformed into P-polynomial, and the double Bell polynomial Bäcklund transformation and bilinear form Bäcklund transformation, Lax pair and infinite conservation law are obtained by P-polynomial.The exact solution of the equation is obtained by using the bilinear Bäcklund transformation and symbolic computation system Mathematica.By solving different parameters containing t, a part of the three-dimensional graph and contour map are obtained,and the physical meaning of the understanding is interpreted through these different graphs.

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Data Availability

All data generated or analyzed during this paper are included in this published article.

Declarations

Conflict of interest

The authors declare that there is no conflict of interests regarding the research effort and the publication of this study.

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