A Multiplicative calculus approach to solve applied nonlinear models

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Abstract

Problems like population growth, continuous stirred tank reactor (CSTR) and ideal gas are studied from the last four decades in the field medicine science, Engineering and applied science, respectively. One of the main motivation was to understand the pattern of such issues and how to fix them. With the help of applied Mathematics, such problems can be converted or modeled by nonlinear expressions with similar properties and the required solution can be obtained by iterative techniques. In this manuscript, we proposed a new iterative scheme for multiple roots (without prior knowledge of multiplicity m) by adopting multiplicative calculus rather than the standard calculus. The base of our scheme is on the well-known Schröder method and we retain the same second-order of convergence. In addition, we extend the order of convergence from second to fourth by constructing a two-step joint Schröder scheme with hybrid approach of ordinary and multiplicative calculus. Some numerical examples are tested to find the roots of nonlinear equations and results are found to be competent as compared to ordinary derivative methods. Finally, the convergence of schemes is also analyzed by basin of attractions that also support the theoretical aspects.

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Abstract

Problems like population growth, continuous stirred tank reactor (CSTR) and ideal gas are studied from the last four decades in the field medicine science, Engineering and applied science, respectively. One of the main motivation was to understand the pattern of such issues and how to fix them. With the help of applied Mathematics, such problems can be converted or modeled by nonlinear expressions with similar properties and the required solution can be obtained by iterative techniques. In this manuscript, we proposed a new iterative scheme for multiple roots (without prior knowledge of multiplicity m) by adopting multiplicative calculus rather than the standard calculus. The base of our scheme is on the well-known Schröder method and we retain the same second-order of convergence. In addition, we extend the order of convergence from second to fourth by constructing a two-step joint Schröder scheme with hybrid approach of ordinary and multiplicative calculus. Some numerical examples are tested to find the roots of nonlinear equations and results are found to be competent as compared to ordinary derivative methods. Finally, the convergence of schemes is also analyzed by basin of attractions that also support the theoretical aspects.

Mathematics subject classification: 65H05,65G99.
Keywords: Multiplicative derivative, Nonlinear equations, Schrörder method, Order of convergence.

1 Introduction

The solution of population growth, continuous stirred tank reactor (CSTR) and ideal gas are a big challenge for medical scientist, Engineer and Mathematician. We study the solution of such problems with the help of multiplicative calculus and hybrid approach rather than of the standard approaches. Multiplicative Calculus is also an important part of applied Mathematics.

In seventeenth century, Newton and Leibnitz created the differential and integral calculus concept based on subtraction and addition operation. Later on 1970's, Grossman and Katz [1] developed a different definition of differential and integral calculus that utilize the multiplication and division operation instead of addition and subtraction. This definition of differential and integral calculus named as Multiplicative Calculus. In 2008, Bashirov et al. [2], contributed on multiplicative calculus and its applications. After this, some authors worked on some applications of multiplicative calculus in different areas like in biology [3], in science and finance [4], in the biomedical sciences [5], in economic growth [6] etc.

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Very recently, Özyzpici et al. [7] and Ali Özyzpici [8] in the years of 2016 and 2020, suggested a new way to solve nonlinear equations with the help of multiplicative calculus approach (MCA). The earlier schemes [7, 8] with the help of MCA perform better numerical results as compared to the standard calculus approach. In these studies [7,8], researchers focused only on simple root problems. They didn't speak anything about the multiple roots. Because, finding the multiple roots of nonlinear expressions are more complicated and challenging as compared to simple root and multiple root with prior knowledge of multiplicity m. Some of the main reasons are to develop the convergence criteria/order and lengthy and complicated calculations for such problems.

With this motivation, we focus on new iterative techniques with the help of MCA. According to our best knowledge, there is not a single iterative with MCA that can provide the multiple roots of nonlinear equations. Another advantage of our scheme is that it does not require the prior knowledge of multiplicity m. The base of our scheme is on the well-known Schröder method. In addition to this, we also suggested a hybrid technique whose first substep is from the MCA and second substep from standard calculus approach. This new hybrid joint Schröder method enhance the convergence order from two to four. We compare our schemes with the existing methods on the basis of absolute error difference between two consecutive iterations, order of convergence, number of iterations, CPU timing, the graphs of absolute errors and bar graphs. We found that our methods perform much better in all ways of comparisons. Further, we also study the basin of attraction which also support the numerical results.

The details of the paper are: Section 2 states the proposed multiplicative methods, Section 3 represents the convergence analysis of suggested methods, Section 4 demonstrates the experimental work of newly constructed schemes, Section 5 devotes to graphical analysis of new methods and finally, Section 6 depicts the outcomes of the findings.

1.1 Some basic terminologies

Definition 1.1 *The nonlinear function $g : \Omega \subset \mathbb{R} \to \mathbb{R}$ is multiplicative differentiable (g^*) at x or on Ω if it is positive and differentiable at x or on Ω, and defined as*

$$\begin{aligned} g^*(x) &= \frac{d^* g}{dx} = \lim_{h\to 0}\left(\frac{g(x+h)}{g(x)}\right)^{\frac{1}{h}}, \\ g^*(x) &= \lim_{h\to 0}\left(\triangle^* g\right)^{\frac{1}{h}} = e^{\frac{g'(x)}{g(x)}}, \\ &= e^{(lnOg)'(x)}. \end{aligned} \tag{1.1}$$

In the similar pattern, higher-order multiplicative derivative is defined as:

$$g^{**}(x) = e^{(lnOg^*)'(x)} = e^{(lnOg)''(x)}, \tag{1.2}$$

and in more general

$$g^{*(n)}(x) = e^{(lnOg)^{(n)}(x)}, n = 0, 1, 2, \ldots \tag{1.3}$$

where $(lnOg) = ln\,(g(x))$. Note that in equation (1.3) when $n = 0$ means no multiplicative derivative and it depicts the original function $g(x) = 1$.

1.2 Some results on multiplicative differentiation are

Consider g and h be multiplicative differentiable and ψ be ordinary differentiable functions. Let c be a positive constant then, we have

(i) $(c)^* = 1$

(ii) $(cg)^*(x) = g^*(x)$

(iii) $(gh)^*(x) = g^*(x)h^*(x)$

(iv) $(\frac{g}{h})^*(x) = \frac{g^*(x)}{h^*(x)}$

(v) $(g^{\Psi})^*(x) = g^*(x)^{\Psi(x)}.g(x)^{\Psi'(x)}$

(vi) $(gO\Psi)^*(x) = g^*\Psi(x)^{\Psi'(x)}$

Definition 1.2 *Suppose $g : \Omega \subset \mathbb{R} \to \mathbb{R}^+$ be a positive nonlinear equation. Then the multiplicative nonlinear equation is defined as*

$$g(x) = 1. \tag{1.4}$$

Theorem 1.3 *Let $g : \Omega \to \mathbb{R}$ be $(n+1)$ times multiplicative differential in an open interval Ω. Therefore for any $x, x+a \in \Omega$, $\exists$ a number $\eta \in (0,1)$ so that*

$$g(x+a) = \prod_{l=0}^{n} \left(g^{*(l)}(x)\right)^{\frac{a^l}{l!}} \left(g^{*(n+1)}(x+\eta a)\right)^{\frac{a^{n+1}}{(n+1)!}}. \tag{1.5}$$

2 Proposed Schemes with some basic terminologies

Here, we developed the following iterative methods to solve the nonlinear equation in context to multiplicative derivative.
Multiplicative Schröder Method (MSM)

$$x_{k+1} = x_k - \frac{ln\,(g(x_k))\,ln\,(g^*(x_k))}{\left(ln\,(g^*(x_k))\right)^2 - ln\,(g(x_k))\,ln\,(g^{**}(x_k))}, \quad \forall k = 0, 1, 2, \cdots. \tag{2.1}$$

Joint Multiplicative Schröder Method (JMSM)

$$\begin{aligned} y_k =& x_k - \frac{ln\,(g(x_k))\,ln\,(g^*(x_k))}{\left(ln\,(g^*(x_k))\right)^2 - ln\,(g(x_k))\,ln\,(g^{**}(x_k))}, \\ x_{k+1} =& y_k - \frac{g(y_k)g'(y_k)}{\left(g'(y_k)\right)^2 - g'(y_k)g''(y_k)}, \quad \forall k = 0, 1, 2... \end{aligned} \tag{2.2}$$

3 Convergence analysis

Theorem 3.1 *Assume the sufficiently multiplicative differential function $g : \Omega \subseteq \mathbb{R} \to \mathbb{R}^+$ with r_1 multiplicative zero in an open interval Ω. Whenever x_0 is sufficiently close to r_1, then multiplicative Schröder scheme* (2.1) *has quadratic convergence.*

Proof Let r_1 is a multiplicative zero of function $g(x)$ so that $g(r_1) = 1$. Since the function $g(x)$ is sufficiently multiplicative differentiable, therefore by using equation (1.5), and the error equation $e_k = r_1 - x_k$, we have

$$g(r_1) = 1 = g(x_k) g^*(x_k)^{e_k} g^{**}(x_k)^{\frac{e_k^2}{2}} g^{***}(c_1)^{\frac{e_k^3}{6}}, \tag{3.1}$$

$$g(r_1) = 1 = g(x_k) g^*(x_k)^{e_k} g^{**}(c_2)^{\frac{e_k^2}{2}}, \tag{3.2}$$

here c_1, c_2 are between r_1 and x_k. Now Raise the power of (3.1) by $ln\,(g^*(x_k))$ gives

$$1 = g(x_k)^{ln(g^*(x_k))} g^*(x_k)^{ln(g^*(x_k))e_k} g^{**}(x_k)^{\frac{ln(g^*(x_k))}{2} e_k^2} g^{***}(c_1)^{\frac{ln(g^*(x_k))}{6} e_k^3}, \tag{3.3}$$

and raising the power of (3.2) by $e_k\ ln\,(g^{**}(x_k))$ gives

$$1 = g(x_k)^{e_k ln(g^{**}(x_k))} g^*(x_k)^{ln(g^{**}(x_k))e_k^2} g^{**}(c_2)^{ln(g^{**}(x_k))\frac{e_k^3}{2}}. \tag{3.4}$$

Dividing (3.3) with (3.4) gives

$$g(x_k)^{ln(g^*(x_k))} \left(\frac{g^*(x_k)^{ln(g^*(x_k))}}{g(x_k)^{ln(g^{**}(x_k))}} \right)^{e_k} \left(\frac{g^{**}(x_k)^{\frac{ln(g^*(x_k))}{2}}}{g^*(x_k)^{ln(g^{**}(x_k))}} \right)^{e_k^2} \left(\frac{g^{***}(c_1)^{\frac{ln(g^*(x_k))}{6}}}{g(c_2)^{\frac{ln(g^{**}(x_k))}{2}}} \right)^{e_k^3} = 1. \tag{3.5}$$

After using natural log on both sides of (3.5), one can have

$$ln\,(g^*(x_k))\, ln\,(g(x_k)) + ln \left(\frac{g^*(x_k)^{ln(g^*(x_k))}}{g(x_k)^{ln(g^{**}(x_k))}} \right) e_k + ln \left(\frac{g^{**}(x_k)^{\frac{ln(g^*(x_k))}{2}}}{g^*(x_k)^{ln(g^{**}(x_k))}} \right) e_k^2 + O(e_k^3) = 0. \tag{3.6}$$

Simplifying the equation (3.6) and substituting in (2.1) we have

$$\begin{aligned} \frac{ln\,(g(x_k))\, ln\,(g^*(x_k))}{(ln\,(g^*(x_k)))^2 - ln\,(g(x_k))\, ln\,(g^{**}(x_k))} &= -\, e_k + \frac{e_k^2}{2} \left(\frac{ln\,(g^*(x_k))\, ln\,(g^{**}(x_k))}{ln\,(g^*(x_k))^2 - ln\,(g(x_k))\, ln\,(g^{**}(x_k))} \right) + O(e_k^3). \\ x_{k+1} - r &= x_k - r - \frac{ln\,(g(x_k))\, ln\,(g^*(x_k))}{(ln\,(g^*(x_k)))^2 - ln\,(g(x_k))\, ln\,(g^*(x_k))}, \\ e_{k+1} &= e_k - e_k + e_k^2(B) + O(e_k^3), \\ e_{k+1} &= e_k^2(B) + O(e_k^3), \\ where \quad B = \frac{1}{2} & \left(\frac{ln\,(g^*(x_k))\, ln\,(g^{**}(x_k))}{ln\,(g^*(x_k))^2 - ln\,(g(x_k))\, ln\,(g^{**}(x_k))} \right). \end{aligned} \tag{3.7}$$

Hence, the technique (2.1) has quadratic convergence.

Theorem 3.2 *Let the function $g : \Omega \subset \mathbb{R} \rightarrow \mathbb{R}+$ has a root r in an open interval I. Assume that $g(x)$ is sufficiently ordinary, and multiplicative differentiable about the point r. Then the method JMSM given in (2.2) has fourth order of convergence.*

Proof Proof:Let the function $g(x) - 1 = 0$ has simple root r and $e_k = x_k - r$ be the error. Assume a function $H(x)$ defined as

$$H(x) = y(x) - \frac{g(y(x))g'(y(x))}{g'(y(x))^2 - g(y(x))g''(y(x))} \tag{3.8}$$

where

$$y(x) = x - \frac{(lng(x))(lng^*(x))}{(lng^*(x))^2 - (lng(x))(lng^{**}(x))} \tag{3.9}$$

With the help of software Mathematica, we have $y'(r) = 0,\ y''(r) = 0$, and the function $H(x)$ satisfies

$$H(r) = r \text{ and } H^{(n)}(r) = 0,\ n = 1, 2, 3. \tag{3.10}$$

Therefore, $H^4(r)$ can be written as

$$H^4(r) = -\frac{3g''(r)\left(g'(r)^2 - g''(r)\right)^2}{g'(r)^3} \tag{3.11}$$

By applying the Taylor's series of $H(x_k)$ about the point r with conditions (3.10), one gets

$$x_{k+1} = H(x_k) = H(r) + \frac{H^4(r)}{4!}e_k^4 + O(e_k^5) \tag{3.12}$$

$$e_{k+1} = \frac{H^4(r)}{4!}e_k^4 + O(e_k^5) \tag{3.13}$$

which proves the fourth order of convergence of JMSM.

4 Experimental work

In this section, some experiments have been performed on suggested iterative methods for solving nonlinear equations. All the numerical work has been done on $Mathematica$ 11 Software with the stopping criterion $|g(x_k)| < 10^{-50}$ in ordinary derivative case and in multiplicative derivative case $|g_1(x_k) - 1| < 10^{-50}$. The experimental work of proposed multiplicative Schröder method (MSM), and joint multiplicative Schröder method ($JMSM$) are compared with ordinary Schröder method (SM), and modified Newton's method (MNM) [9]. In every numerical work

the approximate computational order of convergence (ACOC) ρ, iteration index k, and consecutive iteration error $|x_{k+1} - x_k|$ presented in [10] are evaluated and showed in Tables 1–4.
Remark: The meaning of expression $m(\pm n)$ is $m \times 10^{\pm n}$ in all the tables.

Example 4.1 *Firstly, we consider the population growth model that formulate the following nonlinear equation*

$$g(x) = \frac{1000}{1564}e^x + \frac{435}{1564}(e^x - 1) - 1.$$

In this model we evaluate the birth rate denoted as x, if in a specific local area has 1000 thousand people at first and 435 thousand move into the local area in the first year. Likewise, assume 1564 thousand individuals toward the finish of one year. The computed results towards the root $x_r = 0.1009979\ldots$ are displayed in Table 1. Clearly, the methods MSM, and $JMSM$ shows better results in terms of consecutive error and number of iteration in comparison of existing ones.

Table 1: Convergence behavior of the methods MNM, SM, MSM, $JMSM$ at approximation $x_0 = 1$.

Schemes	k	$\lvert x_{(k+1)} - x_{(k)} \rvert$	ρ	*Total number of iterations*	*CPU time(Seconds)*
	2	3.7(−2)	2.000	5	0.281
MNM	3	6.7(−4)			
	4	2.1(−7)			
SM	2	4.2(−1)	2.006	6	0.329
	3	1.1(−1)			
	4	5.8(−3)			
	2	1.7(−2)	2.001	4	0.313
MSM	3	1.2(−5)			
	4	6.4(−12)			
	2	1.3(−4)	4.000	4	0.469
$JMSM$	3	2.5(−19)			
	4	3.3(−78)			

Example 4.2 *Here, we study the nonlinear problem $g(x) = xe^{x^2} - sin^2(x) + 3cox(x) - 4$ having the root $x_r = 1.06513\ldots$. The evaluated results are demonstrated in Table 2 which shows that proposed multiplicative schemes converges to the root in only four to five iterations.*

Example 4.3 *Now, we tested the methods on continuous stirred tank reactor problem which was converted into the following mathematical expression by Douglas [11].*

$$\kappa \frac{2.98(s + 2.25)}{(s + 1.45)(s + 2.85)^2(s + 4.35)} = -1. \tag{4.1}$$

Here, κ denotes gain of the proportional controller. For the values of κ control system is stable but when $\kappa = 0$, we have the poles of the open-loop transferred function as the solutions of following nonlinear equation:

$$g(x) = x^4 + 1.5x^3 + 47.49x^2 + 83.06325x + 5.123266875. \tag{4.2}$$

The function $g(x)$ has root -2.85 with multiplicity $m = 2$. The outcomes of suggested methods are demonstrated in Table 3 and results are equally competent as compared to methods MNM, and SM.

Table 2: Convergence behavior of the methods MNM, SM, MSM, $JMSM$ at approximation $x_0 = 0.75$.

Schemes	k	$\lvert x_{(k+1)} - x_{(k)}\rvert$	ρ	*Total number of iterations*	*CPU time(Seconds)*
MNM	2	2.1(−1)	2.938	8	0.265
	3	2.1(−1)			
	4	1.9(−1)			
SM	2	9.9(−2)	2.000	7	0.469
	3	9.1(−2)			
	4	4.3(−2)			
MSM	2	8.1(−2)	1.998	4	0.328
	3	3.7(−3)			
	4	1.1(−5)			
$JMSM$	2	1.9(−2)	4.016	4	0.469
	3	2.5(−7)			
	4	5.8(−27)			

Table 3: Convergence behavior of the methods MNM, SM, MSM, $JMSM$ at approximation $x_0 = -2.5$.

Schemes	k	$\lvert x_{(k+1)} - x_{(k)}\rvert$	ρ	*Total number of iterations*	*CPU time(Seconds)*
MNM	2	8.0(−6)	1.903	4	0.375
	3	1.5(−12)			
	4	5.6(−26)			
SM	2	1.6(−5)	2.187	4	0.266
	3	5.8(−10)			
	4	8.1(−21)			
MSM	2	3.4(−5)	2.266	4	0.251
	3	2.7(−9)			
	4	1.7(−19)			
$JMSM$	2	3.5(−1)	4.790	3	0.454
	3	1.2(−4)			
	4	2.7(−21)			

Example 4.4 *Lastly, we worked on van der waal equation of ideal gas [10] andwhich describes the characteristics of real gas and formed into the following mathematical expression*

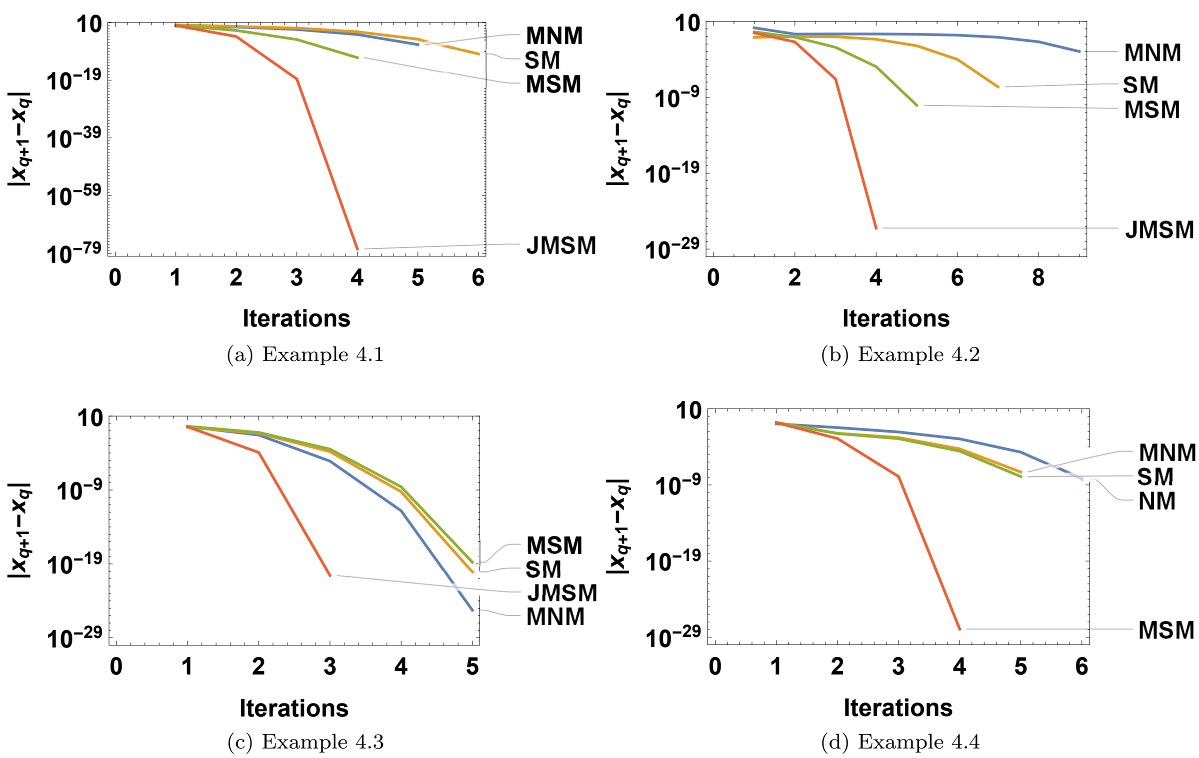
$$g(x) = x^3 - 5.22x^2 + 9.0825x - 5.2675.$$

One of its root $x_r = 1.75$ has multiplicity $m = 2$. The performance of different iterative schemes have been shown in Table 4, and one can easily conclude that the proposed methods MSM and $JMSM$ converges much faster to the root than the other methods MNM and SM.

Remark: *Further, the error analysis of numerical 4.1 to 4.4 has been shown in the Figure 1. It is clear from all sub figures of Figure 1 that proposed method error reduction is more faster than existing methods. In similar way, iteration comparisons of different existing methods with proposed methods is depicted in the Figure 2. Clearly, the proposed method converges to root in less number of iterations as compared with other schemes.*

Table 4: Convergence behavior of the methods MNM, SM, MSM, $JMSM$ at approximation $x_0 = 1.9$.

Schemes	k	$\lvert x_{(k+1)} - x_{(k)} \rvert$	ρ	*Total number of iterations*	*CPU time(Seconds)*
MNM	2	9.0(−3)	1.995	6	0.297
	3	1.1(−3)			
	4	2.0(−5)			
SM	2	1.7(−3)	2.050	5	0.359
	3	5.4(−5)			
	4	5.0(−8)			
MSM	2	1.2(−3)	2.033	5	0.281
	3	2.7(−5)			
	4	1.3(−8)			
$JMSM$	2	1.2(−3)	4.014	4	0.391
	3	1.3(−8)			
	4	1.2(−28)			



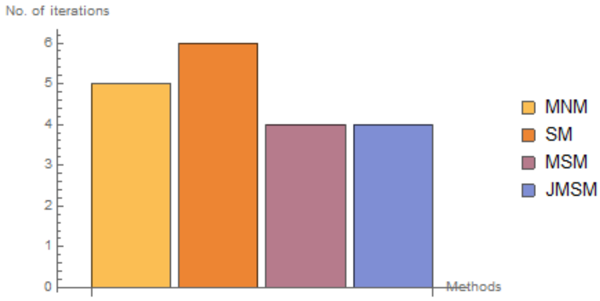
(a) Example 4.1

(b) Example 4.2

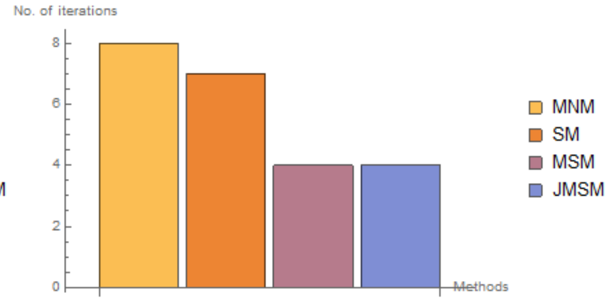
(c) Example 4.3

(d) Example 4.4

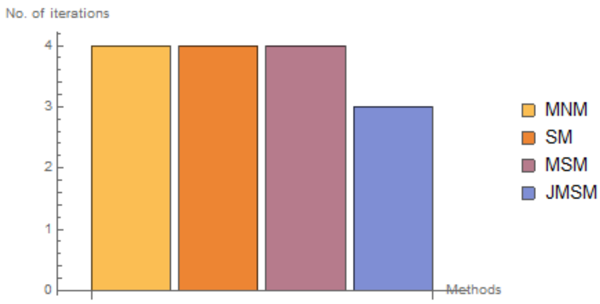
Figure 1: Graphical Error Analysis



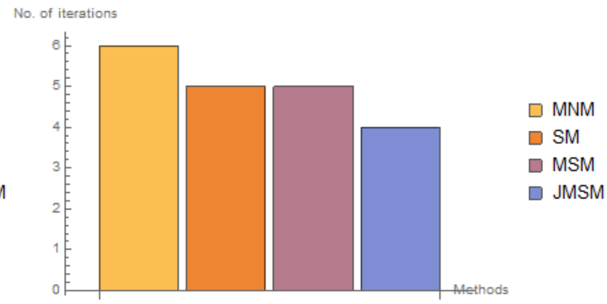
(a) Example 4.1



(b) Example 4.2



(c) Example 4.3



(d) Example 4.4

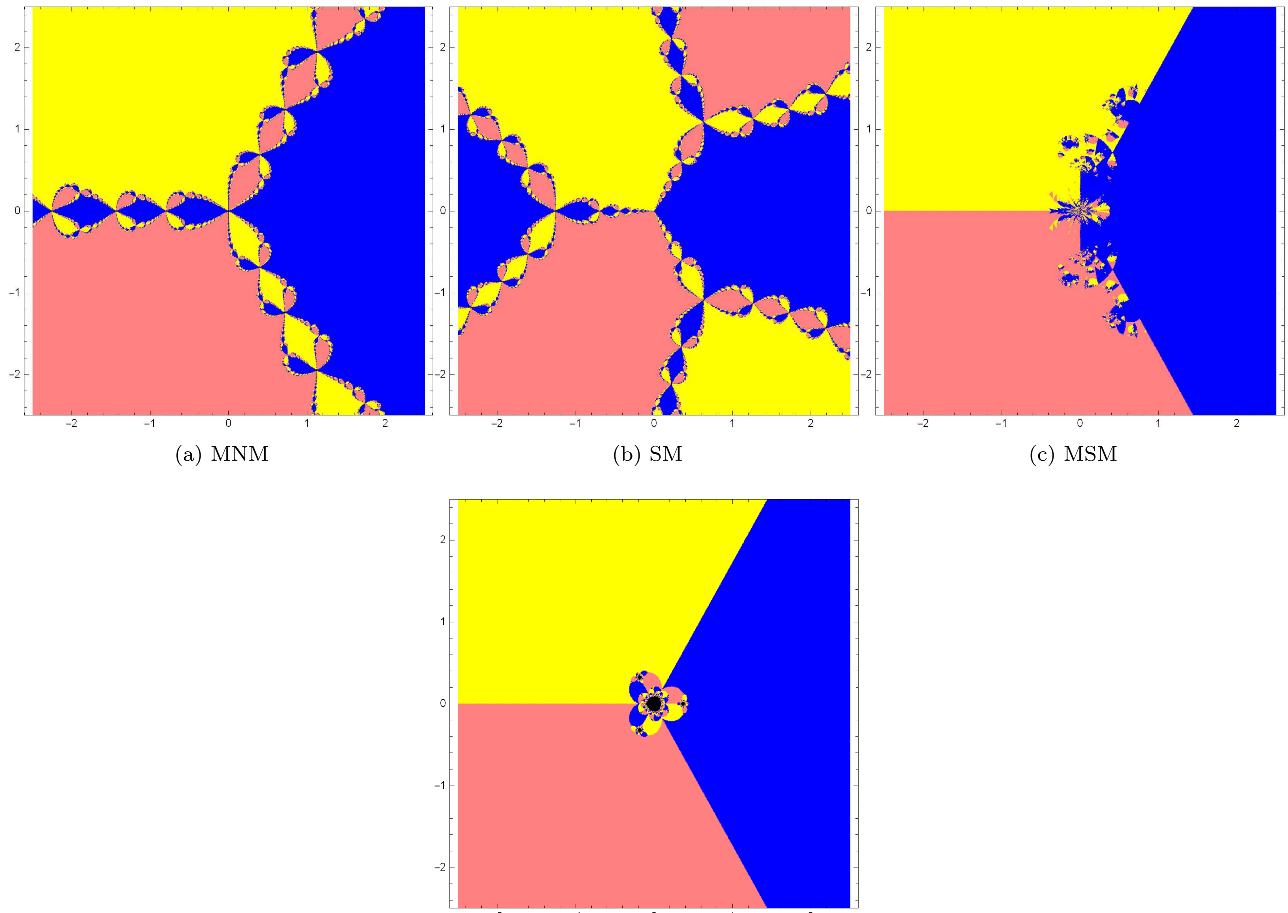
Figure 2: Iteration Analysis

5 Basin of Attraction

The concept of basin of attraction confirms the convergence of all the possible roots of the nonlinear equation within specified rectangular region. So, here we presents the convergence of modified Newton's method (MNM), ordinary Schröder method (SM), multiplicative Schröder method (MSM), and joint multiplicative Schröder method (JMSM) on different initial values in the rectangular region $[-2, 2] \times [-2, 2]$ *by dynamical planes explained in [12]. The basin of attractions are shown in Figure 3 for the scaler equation* $z^3 - 1$ *and each image is plotted by an initial guess as an ordered pair of* 256 *complex points of abscissa and coordinate axis. If an initial point does not converge to the root then it is plotted with black color otherwise different colors are used to represent different roots* $1, -i, i$ *with a tolerance of* 10^{-3}*. We observe that very less divergence area is depicted by proposed schemes MSM and JMSM.*

6 Conclusions

This paper presented two new iterative methods with the help of MCA. The basis of our new Schröder and joint Schröder iterative methods stand on multiplicative derivatives. We studied the convergence analysis of newly con-



(a) MNM (b) SM (c) MSM (d) JMSM

Figure 3: Dynamical planes of new and existing methods for function $z^3 - 1 = 0$

structed schemes. Our new schemes didn't required the prior knowledge of multiplicity m. In addition, we also provide the more efficient solution to the population growth, continuous stirred tank reactor (CSTR), ideal gas and academic problems as compared to the existing solutions. We compared our techniques on the basis of (i) absolute error difference between two consecutive iterations (ii) order of convergence (iii) number of iterations (iv) CPU timing (v) the graphs of absolute errors and (vi) bar graphs. In all the six different ways, we found that our methods perform much better in comparison of the existing methods. Finally, we study the basin of attraction which also support the numerical results. In future work, we will focus on the multi-point iterative methods for multiple roots as well as for system of nonlinear equations. This area will open the new pandora of iterative methods.

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