INDEFINITE HALMOS, EGERVARY AND Sz.-NAGY DILATIONS

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Abstract: Let $\mathcal{M}$ be an indefinite inner product module over a *-ring of characteristic 2. We show that every self-adjoint operator on $\mathcal{M}$ admits Halmos, Egervary and Sz.-Nagy dilations.
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1. Introduction

In 1950, Halmos [22] made a deep insight into structure theory of operators on Hilbert space by exhibiting any contraction as a part of a unitary. In 1953, Sz.-Nagy [39] showed that Halmos result can be extended to powers of contractions using a unitary operator. In 1963, T. Ando [5] showed that there is a version of Sz.-Nagy dilation for commuting contractions. Combined with spectral theory and theory of (several) complex variables, today, dilation theory of contractions is a rapidly evolving area of research and for a comprehensive look, we refer [1, 4–7, 9–16, 19–21, 27, 28, 31–37, 40–43]. Started in 1970's, dilations of contractions acting on Lebesgue spaces and Banach spaces followed Hilbert space developments [2, 3, 17, 18, 24, 30, 38].

In 2021, by identifying essential mechanisms of dilation theory, Bhat, De and Rakshit [8] obtained surprising results in the set theory context and vector spaces. In 2022, further study in the context of vector spaces was carried by Krishna and Johnson [26]. We note that another vector space variant is also studied by Han, Larson, Liu and Liu [23]. Recently Krishna introduced the notion of magic contractions and derived Sz.-Nagy dilation for p-adic Hilbert spaces and modules [25].

In this paper, we derive indefinite inner product module versions of Halmos dilation (Theorem 2.2), Egervary N-dilation (Theorem 2.3), Sz.-Nagy dilation (Theorem 2.4). Our article is highly motivated from the paper of Halmos [22], Egervary [16], Schaffer [36], Sz.-Nagy [39], Bhat, De and Rakshit [8], Krishna and Johnson [26] and Krishna [25].

2. Indefinite Halmos, Egervary and Sz.-Nagy Dilations

We are going to use the following notions. A ring $\mathcal{R}$ with an automorphism $*$ which is either identity or of order 2 is called as an *-ring. Throughout the paper we assume that characteristic of ring is 2.

Definition 2.1. *[29] Let $\mathcal{V}$ be a module over $\mathcal{R}$. We say that $\mathcal{V}$ is an indefinite inner product module (we write IIPM) if there is a map (called as indefinite inner product) $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathcal{R}$ satisfying following.*

(i) *If $x \in \mathcal{V}$ is such that $\langle x, y \rangle = 0$ for all $y \in \mathcal{V}$, then $x = 0$.*
(ii) *$\langle x, y \rangle = \langle y, x \rangle^*$ for all $x, y \in \mathcal{V}$.*
(iii) *$\langle \alpha x + y, z \rangle = a\langle x, z \rangle + \langle y, z \rangle$ for all $a \in \mathcal{R}$, for all $x, y, z \in \mathcal{V}$.*

Let $\mathcal{V}$ be a IIPM and $T : \mathcal{V} \to \mathcal{V}$ be a morphism. We say that T is adjointable if there is a morphism, denoted by $T^* : \mathcal{V} \to \mathcal{V}$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$, $\forall x, y \in \mathcal{V}$. Note that (i) in Definition 2.1 says that adjoint, if exists, is unique. An adjointable morphism U is said to be a unitary if $UU^* = U^*U = I_{\mathcal{V}}$, the identity operator on $\mathcal{V}$. An adjointable morphism P is said to be projection if $P^2 = P^* = P$. An adjointable morphism T is said to be an isometry if $T^*T = I_{\mathcal{V}}$. An adjointable morphism T is said to be self-adjoint if $T^* = T$. We denote the identity operator on $\mathcal{V}$ by $I_{\mathcal{V}}$.
Our first result is the indefinite Halmos dilation.

Theorem 2.2. ***(Indefinite Halmos dilation)*** *Let $\mathcal{V}$ be a IIPM over a *-ring of characteristic 2 and $T : \mathcal{V} \to \mathcal{V}$ be a self-adjoint morphism. Then the morphism*

$$U \coloneqq \begin{pmatrix} T & I_{\mathcal{V}} + T \\ I_{\mathcal{V}} + T & T \end{pmatrix}$$

is unitary on $\mathcal{V} \oplus \mathcal{V}$. In other words,

$$T = P_{\mathcal{V}} U|_{\mathcal{V}}, \quad T^* = P_{\mathcal{V}} U^*|_{\mathcal{V}},$$

where $P_{\mathcal{V}} : \mathcal{V} \oplus \mathcal{V} \ni (x, y) \mapsto x \in \mathcal{V}$.

Proof. A direct calculation says that

$$V \coloneqq \begin{pmatrix} T & I_{\mathcal{V}} + T \\ I_{\mathcal{V}} + T & T \end{pmatrix}$$

is the inverse and adjoint of U. □

Our second result is the indefinite Egervary N-dilation.

Theorem 2.3. ***(Indefinite Egervary N-dilation)*** *Let $\mathcal{V}$ be a IIPM over a *-ring of characteristic 2 and $T : \mathcal{V} \to \mathcal{V}$ be a self-adjoint morphism. Let N be a natural number. Then the morphism*

$$U \coloneqq \begin{pmatrix} T & 0 & 0 & \cdots & 0 & 0 & I_{\mathcal{V}} + T \\ I_{\mathcal{V}} + T & 0 & 0 & \cdots & 0 & 0 & T \\ 0 & I_{\mathcal{V}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & I_{\mathcal{V}} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & I_{\mathcal{V}} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & I_{\mathcal{V}} & 0 \end{pmatrix}_{(N+1)\times(N+1)}$$

is unitary on $\oplus_{k=1}^{N+1} \mathcal{V}$ and

$$(1) \qquad T^k = P_{\mathcal{V}} U^k|_{\mathcal{V}}, \quad \forall k = 1, \dots, N, \quad (T^*)^k = P_{\mathcal{V}} (U^*)^k|_{\mathcal{V}}, \quad \forall k = 1, \dots, N,$$

where $P_{\mathcal{V}} : \oplus_{k=1}^{N+1} \mathcal{V} \ni (x_k)_{k=1}^{N+1} \mapsto x_1 \in \mathcal{V}$.

Proof. A direct calculation of power of U gives Equation (1). To complete the proof, now we need show that U is unitary. Define

$$V := \begin{pmatrix} T & I_{\mathcal{V}}+T & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & I_{\mathcal{V}} & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I_{\mathcal{V}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & I_{\mathcal{V}} \\ I_{\mathcal{V}}+T & T & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}_{(N+1)\times(N+1)}.$$

Then $UV = VU = I_{\oplus_{k=1}^{N+1}\mathcal{V}}$ and $U^* = V$. □

Note that the Equation (1) holds only upto N and not for $N+1$ and higher natural numbers. In the following theorem, given a IIPM $\mathcal{V}$, $\oplus_{n=-\infty}^{\infty}\mathcal{V}$ is the IIPM defined by

$$\oplus_{n=-\infty}^{\infty}\mathcal{V} := \{\{x_n\}_{n=-\infty}^{\infty}, x_n \in \mathcal{V}, \forall n \in \mathbb{Z}, x_n \neq 0 \text{ only for finitely many } n'\text{s}\}$$

equipped with inner product

$$\langle \{x_n\}_{n=-\infty}^{\infty}, \{y_n\}_{n=-\infty}^{\infty} \rangle := \sum_{n=-\infty}^{\infty} \langle x_n, y_n \rangle, \quad \forall \{x_n\}_{n=-\infty}^{\infty}, \{y_n\}_{n=-\infty}^{\infty} \in \oplus_{n=-\infty}^{\infty}\mathcal{V}.$$

Our third result is the indefinite Sz.-Nagy dilation.

Theorem 2.4. ***(Indefinite Sz.-Nagy dilation)*** *Let $\mathcal{V}$ be a IIPM over a *-ring of characteristic 2 and $T : \mathcal{V} \to \mathcal{V}$ be a self-adjoint morphism. Let $U := (u_{n,m})_{-\infty \leq n,m \leq \infty}$ be the morphism defined on $\oplus_{n=-\infty}^{\infty}\mathcal{V}$ given by the infinite matrix defined as follows:*

$$u_{0,0} := T, \quad u_{0,1} := I_{\mathcal{V}}+T, \quad u_{-1,0} := I_{\mathcal{V}}+T, \quad u_{-1,1} := T,$$
$$u_{n,n+1} := I_{\mathcal{V}}, \quad \forall n \in \mathbb{Z}, n \neq 0, 1, \quad u_{n,m} := 0 \quad \textit{otherwise},$$

i.e.,

$$U = \begin{pmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & I_{\mathcal{V}} & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & I_{\mathcal{V}} & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & I_{\mathcal{V}}+T & T & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \boxed{T} & I_{\mathcal{V}}+T & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & I_{\mathcal{V}} & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & I_{\mathcal{V}} & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}_{\infty\times\infty}$$

where T is in the $(0,0)$ position (which is boxed), is unitary on $\oplus_{n=-\infty}^{\infty}\mathcal{V}$ and

(2) $$T^n = P_{\mathcal{V}}U^n|_{\mathcal{V}}, \quad \forall n \in \mathbb{N}, \quad (T^*)^n = P_{\mathcal{V}}(U^*)^n|_{\mathcal{V}}, \quad \forall n \in \mathbb{N},$$

where $P_{\mathcal{V}} : \oplus_{n=-\infty}^{\infty}\mathcal{V} \ni (x_n)_{n=-\infty}^{\infty} \mapsto x_0 \in \mathcal{V}$.

Proof. We get Equation (2) by calculation of powers of U. The matrix $V \coloneqq (v_{n,m})_{-\infty\leq n,m\leq \infty}$ defined by

$$v_{0,0} \coloneqq T, \quad v_{0,-1} \coloneqq I_{\mathcal{V}}+T, \quad v_{1,0} \coloneqq I_{\mathcal{V}}+T, \quad v_{1,-1} \coloneqq T,$$
$$v_{n,n-1} \coloneqq I_{\mathcal{V}}, \quad \forall n \in \mathbb{Z}, n\neq 0,1, \quad v_{n,m} \coloneqq 0 \quad \text{otherwise},$$

i.e.,

$$V = \begin{pmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & I_{\mathcal{V}} & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & I_{\mathcal{V}} & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & I_{\mathcal{V}}+T & \boxed{T} & 0 & 0 & \cdots \\ \cdots & 0 & 0 & T & I_{\mathcal{V}}+T & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & I_{\mathcal{V}} & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & I_{\mathcal{V}} & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}_{\infty\times\infty}$$

where T is in the (0.0) position (which is boxed), satisfies $UV = VU = I_{\oplus_{n=-\infty}^{\infty}\mathcal{V}}$ and $U^* = V$. □

We note that explicit sequential form of U is

$$U(x_n)_{n=-\infty}^{\infty} = (\dots, x_{-2}, x_{-1}, (I_{\mathcal{V}}+T)x_0 + Tx_1, \boxed{Tx_0 + (I_{\mathcal{V}}+T)x_1}, x_2, x_2, \dots)$$

where $Tx_0 + (I_{\mathcal{V}}+T)x_1$ is in the 0 position (which is boxed) and U^* is

$$U^*(x_n)_{n=-\infty}^{\infty} = (\dots, x_{-3}, x_{-2}, \boxed{(I_{\mathcal{V}}+T)x_{-1} + Tx_0}, Tx_{-1} + (I_{\mathcal{V}}+T)x_0, x_1, \dots),$$

where $(I_{\mathcal{V}}+T)x_{-1}+Tx_0$ is in the 0 position (which is boxed). We next wish to derive indefinite isometric Sz.-Nagy dilation.

Theorem 2.5. ***(Indefinite isometric Sz.-Nagy dilation)*** *Let $\mathcal{V}$ be a IIPM over a *-ring of characteristic 2 and $T : \mathcal{V} \to \mathcal{V}$ be a self-adjoint morphism. Let $U \coloneqq (u_{n,m})_{0\leq n,m\leq\infty}$ be the morphism defined on $\oplus_{n=0}^{\infty}\mathcal{V}$ given by the infinite matrix defined as follows:*

$$u_{0,0} \coloneqq T, \quad u_{2,1} \coloneqq I_{\mathcal{V}}+T, \quad u_{n+1,n} \coloneqq I_{\mathcal{V}}, \quad \forall n \geq 2, \quad u_{n,m} \coloneqq 0 \quad \textit{otherwise},$$

i.e.,

$$U = \begin{pmatrix} \boxed{T} & 0 & 0 & 0 & 0 & 0 & \cdots \\ I_{\mathcal{V}}+T & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & I_{\mathcal{V}} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & I_{\mathcal{V}} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & I_{\mathcal{V}} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & I_{\mathcal{V}} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}_{\infty\times\infty}$$

where T is in the $(0,0)$ position (which is boxed), is isometry on $\oplus_{n=0}^{\infty}\mathcal{V}$ and

$$T^n = P_{\mathcal{V}}U^n|_{\mathcal{V}}, \quad \forall n \in \mathbb{N}, \quad (T^*)^n = P_{\mathcal{V}}(U^*)^n|_{\mathcal{V}}, \quad \forall n \in \mathbb{N}, \tag{3}$$

where $P_{\mathcal{V}} : \oplus_{n=0}^{\infty}\mathcal{V} \ni (x_n)_{n=0}^{\infty} \mapsto x_0 \in \mathcal{V}$.

Proof. It suffices to note the adjoint of U is

$$U^* = \begin{pmatrix} \boxed{T} & I_{\mathcal{V}} + T & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & I_{\mathcal{V}} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & I_{\mathcal{V}} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & I_{\mathcal{V}} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & I_{\mathcal{V}} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}_{\infty \times \infty}$$

where T is in the $(0,0)$ position (which is boxed). □

We now formulate following problems.

Problem 2.6.

(i) ***Whether there is an indefinite Ando dilation? If yes, whether one can dilate commuting three, four, ... commuting self-adjoint morphisms to commuting unitaries?***
(ii) ***Whether there is (a kind of) uniqueness of indefinite Halmos dilation?***
(iii) ***Whether there is a indefinite intertwining-lifting theorem (commutant lifting theorem)?***

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