

# Design of iterative methods with memory for solving nonlinear systems

Alicia Cordero<sup>1</sup>, Neus Garrido<sup>2</sup>, Juan Ramon Torregrosa<sup>1</sup>, and Paula Triguero-Navarro<sup>1</sup>

<sup>1</sup>Universidad Politécnica de Valencia

<sup>2</sup>Universitat Politecnica de Valencia

September 5, 2022

## Abstract

In this paper, we design two parametric classes of iterative methods without memory to solve nonlinear systems, whose convergence order is four and seven, respectively. From their error equations and to increase the convergence order without performing new functional evaluations, memory is introduced in these families of different forms. That allows us to increase from four to seven the convergence order in the first family and from seven to eleven in the second one. We perform some numerical experiments with big size systems for confirming the theoretical results and comparing the proposed methods along other known schemes.

**ARTICLE TYPE****CMMSE: Design of iterative methods with memory for solving nonlinear systems**

Alicia Cordero | Neus Garrido | Juan R. Torregrosa\* | Paula Triguero-Navarro

<sup>1</sup>Instituto de Matemáticas Multidisciplinar,  
Universitat Politècnica de València,  
Valencia, Spain

**Correspondence**

\*J.R. Torregrosa. Email: jr Torre@mat.upv.es

**Abstract**

In this paper, we design two parametric classes of iterative methods without memory to solve nonlinear systems, whose convergence order is four and seven, respectively. From their error equations and to increase the convergence order without performing new functional evaluations, memory is introduced in these families of different forms. That allows us to increase from four to seven the convergence order in the first family and from seven to eleven in the second one. We perform some numerical experiments with big size systems for confirming the theoretical results and comparing the proposed methods along other known schemes.

**KEYWORDS:**

Iterative schemes; Nonlinear systems; Memory schemes; Basins of attraction; Dynamical planes.

**1 | INTRODUCTION**

It is very common that, when solving real problems in Engineering or Science, we need to solve  $F(x) = 0$ , being  $F : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ . In general, it is difficult to solve these systems in exact way, so often an attempt is made to obtain an approximate solution rather than the exact solution. One way to achieve this is by iterative methods. These methods obtain a sequence of approximations,  $\{x^{(k)}\}$ , which, under certain conditions, converges to the solution of the system. One of the best known schemes is Newton's procedure, which is expressed as

$$x^{(k+1)} = x^{(k)} - [F'(x^{(k)})]^{-1} F(x^{(k)}), \quad k = 0, 1, \dots, \quad (1)$$

denoting as  $F'(x)$  the Jacobian matrix related to  $F$ .

Newton's scheme is well-known for its efficiency and simplicity, as well as for its quadratic convergence. When the derivative in (1) is replaced by the divided difference  $[x^{(k)} + F(x^{(k)}), x^{(k)}; F]$  we obtain the Steffensen's scheme<sup>1</sup>, which is a derivative-free and also has quadratic convergence. The calculation of an inverse of  $F'$  in the iterative expression of a method can be a drawback when the function to be studied cannot be derived, its derivative is too expensive to calculate or the Jacobian matrix is singular.

Different techniques have been used to construct Newton-like procedures, as weight functions, direct composition, estimations of the Jacobian matrices by using divided difference operators, etc. Therefore, some iterative methods for estimating the solutions of  $F(x) = 0$  have been analyzed with different order of convergence. The aim of these proposals is to accelerate the convergence or to improve the computational efficiency.

Recently, new parametric families of iterative methods were proposed in<sup>2,3,4</sup>, including a fast procedure for solving this kind of problems. Also iterative methods avoiding the Jacobian matrix have been proposed in<sup>5,6,7</sup>, with good orders of convergence. In them, Jacobian matrix  $F'(\cdot)$  is replaced by  $[\cdot, \cdot; F]$ , the divided difference operator. Matrix weight function procedure plays also a key role in the design of iterative schemes, as can be seen in<sup>8,9</sup>.

All the papers cited, and many others that appear in the literature, present iterative methods with high order of convergence but considerably increasing the computational cost. To avoid this increase, we have to resort to memory methods, that is, iterative schemes in which one iteration is obtained from several of the previous ones. Iterative schemes for solving systems by means of memory have been very recently constructed in some publications. In them, a new iterate is obtained from using at least two previous ones. In this way, it is proven that the convergence order increases but no functional evaluation is added,<sup>10,11,12,13</sup>.

In order to increase the quadratic convergence of Newton's method, Traub<sup>14</sup> proposed the following scheme

$$y^{(k)} = x^{(k)} - F'(x^{(k)})^{-1} F(x^{(k)}), \quad (2)$$

$$x^{(k+1)} = y^{(k)} - F'(x^{(k)})^{-1} F(y^{(k)}), \quad k = 0, 1, \dots, \quad (3)$$

which has order of convergence three.

In this paper, we design a derivative-free variant of Traub's method by replacing the derivatives by a divided difference with a parameter and a weight function. This yields the following parametric family, which as we shall see below is a class of iterative methods of fourth-order, that we denote by  $M_{4,\gamma}$ .

$$y^{(k)} = x^{(k)} - [w^{(k)}, x^{(k)}; F]^{-1} F(x^{(k)}),$$

$$x^{(k+1)} = y^{(k)} - H(\mu^{(k)})[y^{(k)}, x^{(k)}; F]^{-1} F(y^{(k)}),$$

where  $w^{(k)} = x^{(k)} + \gamma F(x^{(k)})$ ,  $\gamma \neq 0$ ,  $\gamma \in \mathbb{R}$  and the variable of the weight function is  $\mu^{(k)} = I - [w^{(k)}, x^{(k)}; F]^{-1}[y^{(k)}, w^{(k)}; F]$ . The first step of this method corresponds to Steffensen's scheme when  $\gamma = 1$ .

For increasing the order of convergence we introduce one more step to the parametric family  $M_{4,\gamma}$ , obtaining the following class which, as we will see, has order 7, and it is denoted by  $M_{7,\gamma}$ .

$$y^{(k)} = x^{(k)} - [w^{(k)}, x^{(k)}; F]^{-1} F(x^{(k)}),$$

$$z^{(k)} = y^{(k)} - H(\mu^{(k)})[y^{(k)}, x^{(k)}; F]^{-1} F(y^{(k)}),$$

$$x^{(k+1)} = z^{(k)} - G(\mu^{(k)}, \nu^{(k)})[z^{(k)}, y^{(k)}; F]^{-1} F(z^{(k)}),$$

where  $\nu^{(k)} = I - [w^{(k)}, x^{(k)}; F]^{-1}[z^{(k)}, y^{(k)}; F]H(\mu^{(k)})$ .

In addition, to design these families of iterative methods, we analyze several ways to introduce memory into their iterative formulas, replacing the parameter with an expression that uses the previous iterates and their functional evaluations. In this way, the order of the schemes is increased without adding new evaluations.

To prove the convergence order of the methods with memory we use the Ortega-Rheinboldt's Theorem, which appears in<sup>15</sup>.

**Theorem 1.** Let us consider a with memory iterative procedure  $\phi$ , generating the sequence  $\{x^{(k)}\}$  that converges to its root  $\alpha$ . Wether there is a constant  $\eta$  nonzero and positive  $t_i$ ,  $i = 0, \dots, m$  such that

$$|e_{k+1}| \leq \eta \prod_{i=0}^m |e_{k-i}|^{t_i},$$

is held, then the R-order of convergence of  $\phi$  is at least  $p$ , being  $p$  the unique positive root of

$$p^{m+1} - \sum_{i=0}^m t_i p^{m-i} = 0.$$

In Section 2, we study the order of convergence of the new classes of iterative methods to solve nonlinear systems. We also study in Section 2 how to introduce memory to these parametric families in order to increase the order of convergence without performing new functional evaluations. In Section 3 we perform some numerical experiments for confirming the theoretical results. The paper finishes with some conclusions and the references used in it.

## 2 | CONVERGENCE ANALYSIS

The tools we are going to use, in demonstrating the convergence of these schemes, were defined and introduced by the authors in<sup>16</sup>.

Let us consider  $F : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$  as a sufficiently differentiable function in  $D$ , a neighbourhood of a root  $\alpha$  of  $F(x) = 0$ , we consider the operator

$$[x + h, x; F] = \int_0^1 F'(x + th) dt, \quad (4)$$

defined by Genochi-Hermite on<sup>15</sup>. Using the Taylor development of  $F'(x + th)$  at  $x$  and by direct integration, we get

$$[x + h, x; F] = F'(x) + \frac{1}{2}F''(x)h + \frac{1}{6}F'''(x)h^2 + O(h^3). \quad (5)$$

Let us also consider  $X = \mathbb{R}^{n \times n}$  be the Banach space of size  $n \times n$  real square matrixes, and  $H : X \rightarrow X$  a function defined as :

- $H'(u)(v) = H_1uv$ , where  $H' : X \rightarrow \mathcal{L}(X)$  and  $H_1 \in \mathbb{R}$ ,
- $H''(u, v)(w) = H_2uvw$ , being  $H'' : X \times X \rightarrow \mathcal{L}(X)$  and  $H_2 \in \mathbb{R}$ ,

where it is denoted by  $\mathcal{L}(X)$  the set of linear operators in  $X$ . When  $k \rightarrow \infty$ ,  $\mu^{(k)} \rightarrow 0$ , being 0 the zero matrix. Therefore, there exist real numbers  $H_0, H_1, H_2$  such that  $H$  can be expanded around 0 as:

$$H(\mu^{(k)}) = H_0I + H_1\mu^{(k)} + \frac{1}{2}H_2(\mu^{(k)})^2 + O((\mu^{(k)})^3),$$

being  $I$  is the  $n \times n$  identity matrix.

In the same way, we define a multivariable matrix function  $G(\mu^{(k)}, v^{(k)})$  so, there exist real numbers  $G_0, G_{11}, G_{12}, G_{2i}$  for  $i = 1, 2, 3$  and  $G_{3j}$  for  $j = 1, 2, 3, 4$  such that  $G$  can be expanded around  $(0, 0)$  as:

$$\begin{aligned} G(\mu^{(k)}, v^{(k)}) &= G_0I + G_{11}\mu^{(k)} + G_{12}v^{(k)} + \frac{1}{2}(G_{21}(\mu^{(k)})^2 + G_{22}\mu^{(k)}v^{(k)} \\ &+ G_{23}(v^{(k)})^2) + \frac{1}{6}(G_{31}(\mu^{(k)})^3 + G_{32}(\mu^{(k)})^2v^{(k)} \\ &+ G_{33}\mu^{(k)}(v^{(k)})^2 + G_{34}(v^{(k)})^3) + O_4(\mu^{(k)}, v^{(k)}), \end{aligned}$$

where  $O_4(\mu^{(k)}, v^{(k)})$  denotes all terms in which the sum of exponents of  $\mu^{(k)}$  and  $v^{(k)}$  is at least 4.

## 2.1 | Convergence analysis of $M_{4,\gamma}$

Next, we prove the order of convergence of parametric class  $M_{4,\gamma}$ .

**Theorem 2.** Let us consider  $F : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$  a differentiable enough function defined in a neighbourhood  $D$  of  $\alpha$ , such that  $F(\alpha) = 0$ . Let us also assume that  $F'(\alpha)$  is nonsingular. Let  $H(\mu)$  be a real matrix function satisfying  $H_0 = 1, H_1 = 1$  and  $|H_2| < \infty$ , being  $I$  the  $n \times n$  identity matrix. Therefore, being  $x^{(0)}$  an initial guess sufficiently near to  $\alpha$ , sequence  $\{x^{(k)}\}$  defined by  $M_{4,\gamma}$  converges to  $\alpha$  with order 4, for any non-zero value of parameter  $\gamma$ , and its error equation is:

$$e_{k+1} = \left( -C_3(I + \gamma F'(\alpha)) + C_2 \left( \left( 3I - \frac{H_2}{2} I + \gamma F'(\alpha) \right) C_2 + \gamma C_2 F'(\alpha) \right) \right) \quad (6)$$

$$C_2 (I + \gamma F'(\alpha)) e_k^4 + O(e_k^5). \quad (7)$$

being  $C_i = \frac{1}{i!} [F'(\alpha)]^{-1} F^{(i)}(\alpha)$ ,  $i = 2, 3, \dots$ , and  $e_k = x^{(k)} - \alpha$ .

*Proof.* Let us consider the Taylor expansion of  $F(x^{(k)})$  around  $\alpha$

$$F(x^{(k)}) = F'(\alpha) (e_k + C_2 e_k^2 + C_3 e_k^3 + C_4 e_k^4 + O(e_k^5)). \quad (8)$$

We also consider, equivalently to the previous case, the expansion of  $F(w^{(k)})$  around  $\alpha$ , where  $e_w = w^{(k)} - \alpha$ .

Now, from the previous expressions we calculate the Taylor expansion of  $[w^{(k)}, x^{(k)}; F]$ .

$$\begin{aligned} [w^{(k)}, x^{(k)}; F] &= F'(\alpha) (I + C_2(2I + \gamma F'(\alpha))e_k \\ &+ (\gamma C_2 F'(\alpha)C_2 + C_3(3I + 3\gamma F'(\alpha) + \gamma^2 F'(\alpha)F'(\alpha))) e_k^2) + O(e_k^3). \end{aligned}$$

Next, for the inverse of the divided difference operator  $[w^{(k)}, x^{(k)}; F]$ , we obtain

$$[w^{(k)}, x^{(k)}; F]^{-1} = (I + X_2 e_k + X_3 e_k^2 + O(e_k^3)) F'(\alpha)^{-1},$$

where

$$\begin{aligned} X_2 &= -C_2(2I + \gamma F'(\alpha)), \\ X_3 &= 4C_2^2 + \gamma C_2 F'(\alpha) C_2 + 2\gamma C_2^2 F'(\alpha) + \gamma^2 (C_2 F'(\alpha))^2 \\ &\quad - C_3(3I + 3\gamma F'(\alpha) + \gamma^2 F'(\alpha) F'(\alpha)). \end{aligned}$$

Then,

$$\begin{aligned} y^{(k)} - \alpha &= e_k - [w^{(k)}, x^{(k)}; F]^{-1} F(x^{(k)}) \\ &= C_2(I + \gamma F'(\alpha)) e_k^2 \\ &\quad - (2C_2^2 + 2\gamma C_2^2 F'(\alpha) + \gamma^2 (C_2 F'(\alpha))^2 - C_3(2I + 3\gamma F'(\alpha) + \gamma^2 F'(\alpha) F'(\alpha))) e_k^3 \\ &\quad + O(e_k^4). \end{aligned}$$

In order to obtain the Taylor expansion of  $\mu^{(k)} = I - [w^{(k)}, x^{(k)}; F]^{-1} [y^{(k)}, w^{(k)}; F]$ , we have

$$\begin{aligned} [y^{(k)}, w^{(k)}; F] &= F'(\alpha)(I + C_2(I + \gamma F'(\alpha))e_k \\ &\quad + (\gamma C_2^2 F'(\alpha) + \gamma C_2 F'(\alpha) C_2 + C_2^2 + C_3(I + 2\gamma F'(\alpha) + \gamma^2 F'(\alpha)^2)) e_k^2) + O(e_k^3). \end{aligned}$$

So,

$$\mu^{(k)} = C_2 e_k + (-C_2(C_2(3I + \gamma F'(\alpha)) + \gamma F'(\alpha) C_2) + C_3(2I + \gamma F'(\alpha))) e_k^2 + O(e_k^3).$$

Therefore, if we denote  $M_3 = -C_2(C_2(3I + \gamma F'(\alpha)) + \gamma F'(\alpha) C_2) + C_3(2I + \gamma F'(\alpha))$ , then  $\mu^{(k)} = C_2 e_k + M_3 e_k^2 + O(e_k^3)$  and

$$\begin{aligned} H(\mu^{(k)}) &= H_0 + H_1 \mu^{(k)} + \frac{1}{2} H_2 (\mu^{(k)})^2 + O_3(\mu^{(k)}) = I + \mu^{(k)} + \frac{H_2}{2} (\mu^{(k)})^2 + O_3(\mu^{(k)}) \\ &= I + C_2 e_k + \left( M_3 + \frac{H_2}{2} C_2^2 \right) e_k^2 + O(e_k^3). \end{aligned}$$

Applying the Genochi-Hermite formula,

$$[y^{(k)}, x^{(k)}; F] = F'(\alpha) (I + C_2 e_k + (C_3 + C_2^2(I + \gamma F'(\alpha))) e_k^2) + O(e_k^3).$$

Then,  $[y^{(k)}, x^{(k)}; F]^{-1} = (I + R_2 e_k + R_3 e_k^2 + O(e_k^3)) F'(\alpha)^{-1}$ , where

$$\begin{aligned} R_2 &= -C_2, \\ R_3 &= -C_3 + C_2(I + \gamma F'(\alpha) C_2 - \gamma C_2 F'(\alpha)). \end{aligned}$$

If we denote by  $e_y = y^{(k)} - \alpha$ , then

$$\begin{aligned} e_{k+1} &= e_y - \left( I + C_2 e_k + \left( M_3 + \frac{H_2}{2} C_2^2 \right) e_k^2 + O(e_k^3) \right) \\ &\quad \left( (I + R_2 e_k + R_3 e_k^2 + O(e_k^3))(e_y + C_2 e_y^2 + O(e_y^3)) \right) \\ &= \left( -C_3(I + \gamma F'(\alpha)) + C_2 \left( (I + \gamma F'(\alpha)) C_2 + C_2 \left( 2I - \frac{H_2}{2} I + \gamma F'(\alpha) \right) \right) \right) \\ &\quad C_2 (I + \gamma F'(\alpha)) e_k^4 + O(e_k^5). \end{aligned}$$

It is proven that family  $M_{4,\gamma}$  has order 4 under these standard conditions. Moreover, if we assume that  $H_2 = 2I$ , then the expression of the error equation is

$$\begin{aligned} e_{k+1} &= (-C_3(I + \gamma F'(\alpha)) + C_2 ((I + \gamma F'(\alpha)) C_2 + C_2(I + \gamma F'(\alpha)))) \\ &\quad C_2 (I + \gamma F'(\alpha)) e_k^4 + O(e_k^5). \end{aligned}$$

□

## 2.2 | Introducing memory to $M_{4,\gamma}$

We have shown that parametric family  $M_{4,\gamma}$  has order 4. Now, we are going to design other higher order methods from this one by introducing memory.

From the error equation, if parameter  $\gamma$  is  $\gamma = -[F'(\alpha)]^{-1}$ , then we increase the order, but as we do not know the solution  $\alpha$ , we must estimate this expression.

A known way to approximate this expression is by using divided difference operators, whereby, we approximate the parameter as  $\gamma_k = -[x^{(k)}, x^{(k-1)}, F]^{-1}$ . By replacing this on the family  $M_{4,\gamma}$ , we get a method with memory, denoted by  $M_4D$ .

Another way to approximate this parameter is by using Kurchatov's divided difference operator, that is,  $\gamma_k = -[2x^{(k)} - x^{(k-1)}, x^{(k-1)}, F]^{-1}$ . If we replace the parameter of the family  $M_{4,\gamma}$  by this approximation, we obtain a method with memory that we denote by  $M_4K$ . Let us now calculate the order of these two methods with memory. In the proof of these results we use Theorem 1 and the following notation. If  $\lim_{k \rightarrow \infty} \frac{f(x_k)}{g(x_k)} = C$ , being  $C$  a nonzero constant, we denote it by  $f \sim Cg$ .

**Theorem 3.** Let us consider  $F : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$  a differentiable enough function defined in an neighbourhood  $D$  of the root of  $F$ ,  $\alpha$ . We assume that  $F'(\alpha)$  is nonsingular. Let  $H(\mu)$  be a real matrix function that satisfies  $H_0 = 1$ ,  $H_1 = 1$  and  $H_2 = 2$ , being  $I$  the  $n \times n$  identity matrix. Therefore, being the initial guess  $x^{(0)}$  sufficiently close to  $\alpha$ , sequence  $\{x^{(k)}\}$  obtained by  $M_4D$  converges to  $\alpha$  with order  $p = 2 + \sqrt{6} \approx 4.449$  and method  $M_4K$  converges to the root  $\alpha$  with convergence order  $p = 2 + 2\sqrt{2} \approx 4.8284$ .

*Proof.* Let us now consider the Taylor expansion of  $F(x^{(k-1)})$ ,  $F'(x^{(k-1)})$  and  $F''(x^{(k-1)})$  around  $\alpha$ , in the same way as in Theorem 2. Let us now calculate  $[x^{(k)}, x^{(k-1)}; F]$  using the Genochi-Hermite formula with  $h = e_k - e_{k-1}$ ,

$$[x^{(k)}, x^{(k-1)}; F] = F'(\alpha) (I + C_2(e_k + e_{k-1})) + O_2(e_k, e_{k-1}).$$

Then,

$$[x^{(k)}, x^{(k-1)}; F]^{-1} = (I - C_2(e_k + e_{k-1}))F'(\alpha)^{-1} + O_2(e_k, e_{k-1}).$$

Therefore,  $\gamma_k = -(I - C_2(e_k + e_{k-1}))F'(\alpha)^{-1} + O_2(e_k, e_{k-1})$  and

$$I + \gamma_k F'(\alpha) = C_2(e_k + e_{k-1}) + O_2(e_{k-1}, e_k).$$

Thus,  $I + \gamma_k F'(\alpha) \sim e_{k-1}$ . By the error equation (6) and the above relation we have

$$e_{k+1} \sim e_{k-1}^2 e_k^4. \quad (9)$$

In addition, suppose that the R-order of the method is at least  $p$ . Therefore, it is satisfied

$$e_{k+1} \sim D_{k,p} e_k^p,$$

where  $D_{k,p}$  tends to the asymptotic error constant,  $D_p$ , when  $k \rightarrow \infty$ . Then,

$$e_{k+1} \sim D_{k,p} (D_{k-1,p} e_{k-1}^p)^p = D_{k,p} D_{k-1,p}^p e_{k-1}^{p^2}. \quad (10)$$

In the same way that relation (9) is obtained, it follows that

$$e_{k+1} \sim e_{k-1}^2 (D_{k-1,p} e_{k-1}^p)^4 = D_{k-1,p}^4 e_{k-1}^{4p+2}. \quad (11)$$

Then, by equaling the exponents of  $e_{k-1}$  in (10) and (11), we get that

$$p^2 = 4p + 2,$$

whose only positive solution defines convergence order of  $M_4D$ , being  $p = 2 + \sqrt{6} \approx 4.449$ .

On the other hand, applying the Genochi-Hermite formula,

$$[2x^{(k)} - x^{(k-1)}, x^{(k-1)}; F] = F'(\alpha) (I + 2C_2 e_k - 2C_3 e_{k-1} e_k + C_3 e_{k-1}^2 + 4C_3 e_k^2) + O_3(e_k, e_{k-1}).$$

Then,

$$[2x^{(k)} - x^{(k-1)}, x^{(k-1)}; F]^{-1} = (I - 2C_2 e_k - C_3 e_{k-1}^2 + 2C_3 e_{k-1} e_k + 4(C_2^2 - C_3) e_k^2) F'(\alpha)^{-1} + O_3(e_k, e_{k-1}).$$

Therefore,

$$I + \gamma_k F'(\alpha) = (2C_2 e_k + C_3 e_{k-1}^2 - 2C_3 e_{k-1} e_k - 4(C_2^2 - C_3) e_k^2) F'(\alpha)^{-1} + O_3(e_k, e_{k-1}).$$

Thus  $I + \gamma_k F'(\alpha)$  can have the behaviour of  $e_k$ ,  $e_k e_{k-1}$ ,  $e_k^2$  or  $e_{k-1}^2$ . Obviously, the factors  $e_k e_{k-1}$  and  $e_k^2$  tend faster to 0 than  $e_k$ , so we have to see whether  $e_k$  or  $e_{k-1}^2$  converges faster. Let us assume now that the R-order of the scheme is, at least,  $p$ . Therefore, it is satisfied

$$e_{k+1} \sim D_{k,p} e_k^p,$$

where  $D_{k,p}$  tends to  $D_p$ , the asymptotic error constant, when  $k \rightarrow \infty$ . Then, we have

$$\frac{e_k}{e_{k-1}^2} \sim \frac{D_{k-1,p} e_{k-1}^p}{e_{k-1}^2}.$$

Then, if  $p > 2$  we obtain that  $\frac{D_{k-1,p} e_{k-1}^p}{e_{k-1}^2}$  converges to 0 when  $k \rightarrow \infty$ . Thus, if  $p > 2$ , then  $I + \gamma_k F'(\alpha) \sim e_{k-1}^2$ .

From error equation (6) and the above relation we obtain,

$$e_{k+1} \sim e_{k-1}^4 e_k^4. \quad (12)$$

In addition, by assuming that the R-order of the scheme is, at least,  $p$  we have the relation (10). In the same way that we obtain relation (12), we have

$$e_{k+1} \sim e_{k-1}^4 (D_{k-1,p} e_{k-1}^p)^4 = D_{k-1,p}^4 e_{k-1}^{4p+4}. \quad (13)$$

Then, equating the exponents of  $e_{k-1}$  in (10) and (13) it follows that

$$p^2 = 4p + 4,$$

whose only positive solution is the order of convergence of method  $M_4K$  (Theorem 1), being  $p = 2 + 2\sqrt{2} \approx 4.8284$ .  $\square$

The previous two methods with memory have been obtained using the variable  $x^{(k-1)}$ . Now, we are going to see what happens when the approximation  $y^{(k-1)}$  is used instead, that is, we choose  $\gamma_k = -[x^{(k)}, y^{(k-1)}, F]^{-1}$ , and  $M_4D_y$  is defined by substituting this into the parametric family and we also choose  $\gamma_k = -[2x^{(k)} - y^{(k-1)}, y^{(k-1)}, F]^{-1}$ , and  $M_4K_y$  is defined by substituting this into the parametric family.

Now, we establish the order of these two methods with memory,  $M_4D_y$  and  $M_4K_y$ , whose demonstration is similar to the previous one.

**Theorem 4.** Let us consider a sufficiently differentiable function  $F : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$  defined in an neighbourhood  $D$  of the root  $\alpha$  of  $F$ . We assume that  $F'(\alpha)$  is nonsingular. Let  $H(\mu)$  be a real matrix function satisfying  $H_0 = 1$ ,  $H_1 = 1$  and  $H_2 = 2$ , where  $I$  is the  $n \times n$  identity matrix. Then, taking an estimate  $x^{(0)}$  close enough to  $\alpha$ , sequence  $\{x^{(k)}\}$  generated by method  $M_4D_y$  converges to  $\alpha$  with order  $p = 5$  and the sequence generated by  $M_4K_y$  converges to the root  $\alpha$  with order  $p = 6$ .

### 2.3 | Convergence analysis of $M_{7,\gamma}$

In the nex result, we establish the convergence of parametric class  $M_{7,\gamma}$ , which is independent of the value of parameter  $\gamma$ .

**Theorem 5.** Let us consider a sufficiently differentiable function  $F : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$  defined in a neighbourhood  $D$  of the roof  $\alpha$  of  $F$ . We assume that  $F'(\alpha)$  is nonsingular. Let  $H(\mu)$  be a real matrix function that satisfies  $H_0 = 1$ ,  $H_1 = 1$  and  $|H_2| < \infty$ , where  $I$  is the  $n \times n$  identity matrix. Let us also consider a multivariate matrix function  $G(\mu, \nu)$  such that  $G_0 = 1$ ,  $G_{11} = G_{12} = 0$ ,  $G_{2,1} = 0$ ,  $G_{2,2} = 2$ ,  $G_{2,3} = 0$  and  $|G_{3,i}| < \infty$  for  $i = 1, \dots, 4$ . Then, taking an initial guess  $x^{(0)}$  sufficiently close to the root  $\alpha$ , sequence  $\{x^{(k)}\}$  obtained by  $M_{7,\gamma}$  converges to  $\alpha$  with order 7.

*Proof.* We have already proven that

$$\begin{aligned} y^{(k)} - \alpha &= C_2(I + \gamma F'(\alpha))e_k^2 \\ &\quad - (2C_2^2 + 2\gamma C_2^2 F'(\alpha) + \gamma^2 (C_2 F'(\alpha))^2 - C_3(2I + 3\gamma F'(\alpha) + \gamma^2 F'(\alpha)F'(\alpha))) e_k^3 \\ &\quad + O(e_k^4) \end{aligned}$$

and

$$\begin{aligned} z^{(k)} - \alpha &= \left( -C_3(I + \gamma F'(\alpha)) + C_2 \left( \left( 3I - \frac{H_2}{2}I + \gamma F'(\alpha) \right) C_2 + \gamma C_2 F'(\alpha) \right) \right) \\ &\quad C_2 (I + \gamma F'(\alpha)) e_k^4 + O(e_k^5). \end{aligned}$$

We denote by  $Z_1$  the coefficient of  $e_k^4$  in the error equation.

Applying the Genochi-Hermite formula, we obtain

$$[z^{(k)}, y^{(k)}; F] = F'(\alpha) \left( I + C_2^2(I + \gamma F'(\alpha))e_k^2 + D_3e_k^3 \right) + O(e_k^4),$$

being  $D_3 = -(2C_2^2 + 2\gamma C_2^2 F'(\alpha) + \gamma^2((C_2 F'(\alpha))^2 - C_3(2I + 3\gamma F'(\alpha) + \gamma^2 F'(\alpha))^2) + C_2^4(I + \gamma F'(\alpha))$ .

Calculating the inverse of this divided difference operator as we have done above, we obtain

$$[z^{(k)}, y^{(k)}; F]^{-1} = \left( I - C_2^2(I + \gamma F'(\alpha))e_k^2 + J_3e_k^3 + O(e_k^4) \right) F'(\alpha)^{-1},$$

being  $J_3 = (C_2^2(I + \gamma F'(\alpha))^2 - D_3$ .

Now, we calculate  $v^{(k)}$  and obtain that

$$\begin{aligned} v^{(k)} &= I - [w^{(k)}, x^{(k)}; F]^{-1}[z^{(k)}, y^{(k)}; F]H(\mu^{(k)}) \\ &= I - (I + X_2e_k + X_3e_k^2)(I + C_2^2(I + \gamma F'(\alpha))e_k^2) \\ &\quad \left( I + C_2e_k + \left( M_3 + \frac{H_2}{2} \right) e_k^2 \right) + O(e_k^3) \\ &= -(X_2 + C_2)e_k - \left( X_3 - C_2(I + \gamma F'(\alpha)) + X_2C_2 + \left( M_3 + \frac{H_2}{2} \right) \right) e_k^2 + O(e_k^3) \\ &= I - (X_2 + C_2)e_k - V_2e_k^2 + O(e_k^3), \end{aligned}$$

being  $V_2 = X_3 - C_2(I + \gamma F'(\alpha)) + X_2C_2 + \left( M_3 + \frac{H_2}{2} \right)$ .

Then,

$$\begin{aligned} G(\mu^{(k)}, v^{(k)}) &= I + \mu^{(k)}v^{(k)} + \frac{1}{6}(G_{31}(\mu^{(k)})^3 + G_{32}(\mu^{(k)})^2v^{(k)} + G_{33}\mu^{(k)}(v^{(k)})^2 \\ &\quad + G_{34}(v^{(k)})^3) + O_4(\mu^{(k)}, v^{(k)}) \\ &= I + (C_2e_k + M_3e_k^2)(-(X_2 + C_2)e_k - V_2e_k^2) \\ &\quad + \frac{1}{6}(G_{31}C_2^3 - G_{32}C_2^2(X_2 + C_2) + G_{33}C_2(X_2 + C_2)^2 - G_{34}(X_2 + C_2)^3)e_k^3 \\ &= I - C_2(X_2 + C_2)e_k^2 + (-M_3(X_2 + C_2) - C_2V_2 \\ &\quad + \frac{1}{6}(G_{31}C_2^3 - G_{32}C_2^2(X_2 + C_2) + G_{33}C_2(X_2 + C_2)^2 - G_{34}(X_2 + C_2)^3))e_k^3. \end{aligned}$$

By denoting  $R = -M_3(X_2 + C_2) - C_2V_2 + \frac{1}{6}(G_{31}C_2^3 - G_{32}C_2^2(X_2 + C_2) + G_{33}C_2(X_2 + C_2)^2 - G_{34}(X_2 + C_2)^3)$ , we have

$$G(\mu^{(k)}, v^{(k)}) = I - C_2(X_2 + C_2)e_k^2 + Re_k^3 + O(e_k^4).$$

From that

$$\begin{aligned} x^{(k+1)} - \alpha &= e_z - G(\mu^{(k)}, v^{(k)})[z^{(k)}, y^{(k)}; F]^{-1}F(z^{(k)}) \\ &= e_z - \left( I - C_2(X_2 + C_2)e_k^2 + Re_k^3 \right) \left( I - C_2^2(I + \gamma F'(\alpha))e_k^2 + J_3e_k^3 \right) \\ &\quad (e_z + C_2e_z^2) + O(e_k^8) \\ &= e_z - \left( I + (-C_2(X_2 + C_2) - C_2^2(I + \gamma F'(\alpha)))e_k^2 + (R + J_3)e_k^3 \right) \\ &\quad (e_z + C_2e_z^2) + O(e_k^8). \end{aligned}$$

As  $X_2 = -C_2(2I + \gamma F'(\alpha))$ , then  $X_2 + C_2 = -C_2(I + \gamma F'(\alpha))$ , so  $-C_2(X_2 + C_2) - C_2^2(I + \gamma F'(\alpha)) = 0$ . From this,

$$\begin{aligned} x^{(k+1)} - \alpha &= e_z - G(\mu^{(k)}, v^{(k)})[z^{(k)}, y^{(k)}; F]^{-1}F(z^{(k)}) \\ &= e_z - \left( I + (R + J_3)e_k^3 \right) (e_z + C_2e_z^2) + O(e_k^8) \\ &= -(R + J_3)e_k^3e_z + O(e_k^8) \\ &= -(R + J_3)Z_1e_k^7 + O(e_k^8). \end{aligned}$$

Thus, it is proven that parametric family  $M_{7,\gamma}$  has order of convergence 7.

In particular, if  $G_{31} = G_{32} = G_{34} = 0$  and  $G_{33} = 13$ , then  $e_{k+1} \sim (I + \gamma F'(\alpha))^4 e_k^7$ .

□

## 2.4 | Introducing memory to $M_{7,\gamma}$

As we did with class  $M_{4,\gamma}$ , in this section we introduce memory, in different ways, to family  $M_{7,\gamma}$ .

- If we choose  $\gamma_k = -[x^{(k)}, x^{(k-1)}, F]^{-1}$ , then replacing the parameter of family  $M_{7,\gamma}$  by this value, we obtain a method with memory, denoted by  $M_7D$ .
- Choosing  $\gamma = -[2x^{(k)} - x^{(k-1)}, x^{(k-1)}, F]^{-1}$  and substituting it in family  $M_{7,\gamma}$ , yields a method with memory denoted by  $M_7K$ .
- If we choose  $\gamma_k = -[x^{(k)}, y^{(k-1)}, F]^{-1}$  and substituting it in  $M_{7,\gamma}$ , a new scheme with memory  $M_7D_y$  is obtained.
- Finally, choosing  $\gamma = -[2x^{(k)} - y^{(k-1)}, y^{(k-1)}, F]^{-1}$  and substituting it in  $M_{7,\gamma}$ , a new scheme with memory,  $M_7K_y$ , is obtained.

The order of convergence of all these methods with memory is establishes in the next result, whose proof is similar to that of the previous results.

**Theorem 6.** Let us consider a sufficiently differentiable function  $F : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$  defined in a neighbourhood  $D$  of the roof  $\alpha$  of  $F$ . We assume that  $F'(\alpha)$  is nonsingular. Let  $H$  and  $G$  be real matrix functions that satisfy  $H_0 = 1$ ,  $H_1 = 1$  and  $H_2 = 2$ , and  $G_0 = 1$ ,  $G_{11} = G_{12} = 0$ ,  $G_{2,1} = 0$ ,  $G_{2,3} = 0$ ,  $G_{2,2} = 2$ ,  $G_{3,3} = 13$  and  $G_{3,i} = 0$  for  $i = 1, 2, 4$ . Therefore, considering an initial guess  $x^{(0)}$  sufficiently close to  $\alpha$ , we have:

- sequence  $\{x^{(k)}\}$  generated by  $M_7D$  converges to  $\alpha$  with order  $p = \frac{7 + \sqrt{65}}{2} \approx 7.5311$ .
- sequence  $\{x^{(k)}\}$  defined by  $M_7K$  converges to  $\alpha$  with order  $p = \frac{7 + \sqrt{78}}{2} \approx 7.9159$ .
- sequence  $\{x^{(k)}\}$  got by  $M_7D_y$  converges to  $\alpha$  with order  $p = 4 + \sqrt{17} \approx 8.1231$ .
- sequence  $\{x^{(k)}\}$  generated by  $M_7K_y$  converges to  $\alpha$  with order  $p = \frac{9 + \sqrt{89}}{2} \approx 9.21699$ .

In these methods with memory, we could also use variable  $z^{(k-1)}$  in order to obtain a better approximation of the parameter. Thus, if we choose  $\gamma_k = -[x^{(k)}, z^{(k-1)}, F]^{-1}$ , and replace the parameter of family  $M_{7,\gamma}$  by this expression we obtain a new method with memory denoted by  $M_7D_z$ .

In the same way, the approximation by the Kurchatov divided difference  $\gamma = -[2x^{(k)} - z^{(k-1)}, z^{(k-1)}, F]^{-1}$ , gives us a scheme with memory,  $M_7K_z$ , whose convergence we are going establish.

**Theorem 7.** Let us consider a sufficiently differentiable function  $F : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$  defined in a neighbourhood  $D$  of the roof  $\alpha$  of  $F$ . We assume that  $F'(\alpha)$  is nonsingular. Let  $H$  and  $G$  be real matrix functions that satisfy  $H_0 = 1$ ,  $H_1 = 1$  and  $H_2 = 2$ , and that  $G_0 = 1$ ,  $G_{11} = G_{12} = 0$ ,  $G_{2,1} = 0$ ,  $G_{2,3} = 0$ ,  $G_{2,2} = 2$ ,  $G_{3,3} = 13$  and  $G_{3,i} = 0$  for  $i = 1, 2, 4$ . Therefore, choosing an initial guess  $x^{(0)}$  sufficiently close to the root  $\alpha$ , sequence  $\{x^{(k)}\}$  defined by  $M_7D_z$  converges to  $\alpha$  with order  $\frac{9 + \sqrt{89}}{2} \approx 9.21699$ , and sequence  $\{x^{(k)}\}$  defined by  $M_7K_z$  converges to  $\alpha$  with order 11.

*Proof.* Let us consider the Taylor development of  $F(z^{(k-1)})$ ,  $F'(z^{(k-1)})$  and  $F''(z^{(k-1)})$  around  $\alpha$  as was done in Theorem 3. Applying the Genochi-Hermite formula we obtain

$$[x^{(k)}, z^{(k-1)}; F] = F'(\alpha) (I + C_2(e_k + e_{z,k-1})) + O_2(e_k, e_{z,k-1}).$$

Then, we get

$$[x^{(k)}, z^{(k-1)}; F]^{-1} = (I - C_2(e_k + e_{z,k-1}))F'(\alpha)^{-1} + O_2(e_k, e_{z,k-1}).$$

Therefore

$$I + \gamma_k F'(\alpha) = C_2(e_k + e_{z,k-1}) + O_2(e_{z,k-1}, e_k).$$

Let us assume that sequence  $\{z^{(k)}\}$  has R-order  $p_1$  and the R-order of the scheme is  $p$ .

Then, it follows that

$$\frac{e_k}{e_{z,k-1}} = \frac{e_{k-1}^p}{e_{k-1}^{p_1}} = e_{k-1}^{p-p_1}.$$

Thus  $I + \gamma_k F'(\alpha) \sim e_{z,k-1}$  if it is verified that  $p > p_1$ . By the error equation (6) and the above relation we have

$$e_{k+1} \sim e_{z,k-1}^4 e_k^7. \quad (14)$$

Assuming that the  $R$ -order of the scheme is  $p$ , we have (10). In the same way that the relation (14) is obtained, and supposing that the sequence  $\{z^{(k)}\}$  has  $R$ -order at least  $p_1$  we obtain

$$e_{k+1} \sim e_{z,k-1}^4 e_k^7 \sim (e_{k-1}^{p_1})^4 (e_{k-1}^p)^7 \sim e_{k-1}^{7p+4p_1}. \quad (15)$$

By other way, from the error equation of  $e_{z,k}$  we have

$$\begin{aligned} e_{z,k} &\sim (-C_3(I + \gamma F'(\alpha)) + C_2((I + \gamma F'(\alpha)) C_2 + C_2(I + \gamma F'(\alpha)))) \\ &C_2(I + \gamma F'(\alpha)) e_k^4 \sim e_{z,k-1}^2 e_k^4 \sim e_{k-1}^{4p+2p_1}. \end{aligned} \quad (16)$$

Assuming that  $\{z^{(k)}\}$  has  $R$ -order  $p_1$ , we assure that

$$e_{z,k} \sim e_k^{p_1} \sim e_{k-1}^{pp_1}. \quad (17)$$

Then, by equaling the exponents of  $e_{k-1}$  of (10) and (15), and by equaling the exponents of  $e_{k-1}$  of (16) and (17), it follows that

$$\begin{aligned} p^2 &= 7p + 4p_1, \\ pp_1 &= 4p + 2p_1, \end{aligned}$$

whose only positive solution is the order of  $M_7D_z$  scheme, where  $p = \frac{9 + \sqrt{89}}{2} \approx 9.21699$ .

Now, we calculate  $[2x^{(k)} - z^{(k-1)}, z^{(k-1)}; F]$  by using the Genochi-Hermite formula:

$$\begin{aligned} [2x^{(k)} - z^{(k-1)}, z^{(k-1)}; F] &= F'(\alpha) \left( I + 2C_2 e_k - 2C_3 e_{z,k-1} e_k + C_3 e_{z,k-1}^2 + 4C_3 e_k^2 \right) \\ &+ O_3(e_k, e_{z,k-1}). \end{aligned}$$

Then, the inverse of this divided difference operator is:

$$\begin{aligned} [2x^{(k)} - z^{(k-1)}, z^{(k-1)}; F]^{-1} &= (I - 2C_2 e_k - C_3 e_{z,k-1}^2 + 2C_3 e_{z,k-1} e_k + 4(C_2^2 - C_3) e_k^2) \\ &F'(\alpha)^{-1} + O_3(e_k, e_{z,k-1}). \end{aligned}$$

Therefore,

$$\begin{aligned} I + \gamma_k F'(\alpha) &= 2C_2 e_k + C_3 e_{z,k-1}^2 - 2C_3 e_{z,k-1} e_k - 4(C_2^2 - C_3) e_k^2 F'(\alpha)^{-1} \\ &+ O_3(e_k, e_{z,k-1}). \end{aligned}$$

Thus,  $I + \gamma_k F'(\alpha)$  can have the behaviour of  $e_k$  or  $e_{z,k-1}^2$ , since the factors  $e_k e_{z,k-1}$  and  $e_k^2$  tend to have higher speed at 0 than  $e_k$ , so we have to see whether  $e_k$  or  $e_{z,k-1}^2$  converges faster. Assume the  $R$ -order of the scheme is  $p$ . On the other hand, as sequence  $z^{(k)}$  has  $R$ -order  $p_1$ , we have

$$\frac{e_k}{e_{z,k-1}^2} \sim \frac{D_{k-1,p} e_{k-1}^p}{e_{k-1}^{2p_1}}.$$

Then, if we assume that  $p > 2p_1$ , we have that the behaviour will be like that of  $e_{z,k-1}^2$ , that is,  $I + \gamma_k F'(\alpha) \sim e_{z,k-1}^2$ . From the error equation and the above relationship the following relation is obtained

$$e_{k+1} \sim e_{z,k-1}^8 e_k^7. \quad (18)$$

In addition, relation (10) holds since the  $R$ -order of the procedure is  $p$ .

In the same way that the relation (18) is obtained, and taking into account that sequence  $z^{(k)}$  has  $R$ -order  $p_1$ , we obtain

$$e_{k+1} \sim e_{z,k-1}^8 (e_{k-1}^p)^7 \sim e_{k-1}^{8p_1} e_{k-1}^{7p} \sim e_{k-1}^{7p+8p_1}. \quad (19)$$

On the other hand, by the error equation of  $e_{z,k}$  it is obtained that

$$e_{z,k} \sim e_{z,k-1}^4 e_k^4. \quad (20)$$

Then by equaling the exponents of  $e_{k-1}$  of (10) and (19), and by equaling the exponents of  $e_{k-1}$  of (20) and (17) it is obtained that

$$\begin{aligned} p^2 &= 7p + 8p_1, \\ pp_1 &= 4p + 4p_1, \end{aligned}$$

whose only positive solution is  $p \approx 11.3523$  and  $p_1 \approx 6.17$ , therefore it does not satisfy the property for which  $I + \gamma_k F'(\alpha) \sim e_{z,k-1}^2$ , thus  $I + \gamma_k F'(\alpha) \sim e_k$ , and therefore

$$e_{k+1} \sim e_k^4 e_k^7 \sim e_k^{11}. \quad (21)$$

Thus, we conclude that the order of method  $M_7 K_z$  is  $p = 11$ .  $\square$

As we can see, by introducing memory to families  $M_{4,\gamma}$  and  $M_{7,\gamma}$  we have managed to increase the order up to 2 and 4 units, thus obtaining methods with memory up to order 6 and 11, respectively. Let us note that  $p = 11$  is the maximum order that can be reached by introducing memory in family  $M_{7,\gamma}$ .

### 3 | NUMERICAL EXPERIMENTS

Now, we perform several numerical experiments in order to see the behaviour of our  $M_{4,\gamma}$ ,  $M_{7,\gamma}$  families and the methods derived from them when introducing memory. We present two numerical experiments, one of them applied to the Hammerstein equation and other applied to academical nonlinear system, in which we also make a comparison with two known methods of order 8.

The first thing we would like to point out is that in this case Matlab 2020b has been used to carry out the numerical experiments, with an arithmetical precision variable of 1000 digits. As stopping criterion we choose that  $\|x^{(k+1)} - x^{(k)}\|_2 + \|F(x^{(k+1)})\|_2$  is less than a chosen tolerance. We use also a maximum of 100 iterations.

For all methods and all numerical experiments the following matrix functions have been selected as weight functions:

- $H(\mu) = \mu^2 + \mu + I$ ,
- $G(\mu, \nu) = I + \mu\nu + \frac{13}{6}\mu\nu^2$ ,

being  $I$  the identity matrix.

In the different tables we show :

- the norm  $\|F(x^{(k+1)})\|_2$ ,
- the norm  $\|x^{(k+1)} - x^{(k)}\|_2$ ,
- the number of iterations necessary to satisfy the required tolerance,
- and the ACOC, approximated computational order of convergence, defined by Cordero and Torregrosa in<sup>17</sup>, which has the following expression:

$$p \approx ACOC = \frac{\ln(\|x^{(k+1)} - x^{(k)}\|_2 / \|x^{(k)} - x^{(k-1)}\|_2)}{\ln(\|x^{(k)} - x^{(k-1)}\|_2 / \|x^{(k-1)} - x^{(k-2)}\|_2)}.$$

#### 3.1 | Hammerstein equation

In this test, we consider the Hammerstein integral equation, appearing for example in<sup>15</sup>,

$$x(s) = 1 + \frac{1}{5} \int_0^1 F(s, t)x(t)^3 dt, \quad (22)$$

being

$$F(s, t) = \begin{cases} (1-s)t, & t \leq s, \\ s(1-t), & s \leq t \end{cases}$$

and  $s, t \in [0, 1]$ ,  $x \in C[0, 1]$ .

By means of Gauss–Legendre quadrature, Hammerstein equation is transformed in

$$\int_0^1 f(t)dt \approx \sum_{i=1}^7 \omega_i f(t_i),$$

appearing the abscissas  $t_i$  and the weights  $\omega_i$  for  $n = 7$  in the following table.

$i$	Weight $\omega_i$	Nodes $t_i$
1	0.0647424831	0.0254460438
2	0.1398526957	0.1292344072
3	0.1909150252	0.2970774243
4	0.2089799185	0.5
5	0.1909150252	0.7029225757
6	0.1398526955	0.8707655928
7	0.0647424831	0.9745539561

Let us also denote by  $x_i, i = 1, \dots, 7$  the approximations of  $x(t_i)$ . Then,

$$5x_i - 5 - \sum_{j=1}^7 a_{ij}x_j^3 = 0,$$

where  $i = 1, \dots, 7$  and

$$a_{ij} = \begin{cases} w_j t_j (1 - t_i) & j \leq i, \\ w_j t_i (1 - t_j) & i < j. \end{cases}$$

We initialize the process with  $x^{(0)} = (0.5, \dots, 0.5)^T$ , we choose as initial approximations for  $x^{(-1)}, y^{(-1)}$  and  $z^{(-1)}$  vector  $(0.4, \dots, 0.4)^T$ , and we use a tolerance of  $10^{-50}$ . In Table 1, we can see the results obtained by each method for the Hammerstein’s equation.

**Table 1** Numerical results of Hammerstein’s equation

Method	$\ x^{(k+1)} - x^{(k)}\ _2$	$\ F(x^{(k+1)})\ _2$	Iteration	ACOC
$M_{4,-1}$	1.01573e-166	2.41351e-666	5	3.99986
$M_4D$	2.05575e-154	4.58506e-688	4	4.4952
$M_4K$	7.31331e-177	2.37182e-853	4	4.9600
$M_4D_y$	7.12038e-204	1.62192e-1021	4	4.9971
$M_4K_y$	1.31159e-295	3.12493e-1776	4	5.9975
$M_{7,-1}$	4.53896e-171	9.75609e-1027	4	6.9996
$M_7D$	2.42252e-79	3.32054e-516	3	7.5291
$M_7K$	2.45812e-82	3.43156e-571	3	7.8613
$M_7D_y$	2.35271e-88	4.7518e-622	3	8.1898
$M_7K_y$	3.16092e-101	7.79194e-813	3	9.1692
$M_7D_z$	3.17456e-99	2.21039e-796	3	9.2162
$M_7K_z$	2.85847e-114	1.65181e-1032	3	10.9981

We can see that in all cases the ACOC is close to the theoretical convergence order demonstrated in Section 2 and that the number of iterations required is similar for the methods of the same family, being one unit higher in the case of the methods without memory.

We notice that in all cases, the required tolerance is well below the results obtained. It can be seen that the best results for these numerical experiments are given by the memory methods that use the Kurchatov divided difference operator to approximate the family parameter. These methods give the closest approximations to the solution and the biggest ACOC.

### 3.2 | Academic problems

We also approximate the solution of the following academic system of nonlinear equations. In this case, we compare the results obtained with the different methods proposed with those provided by two known schemes, both of order 8. These schemes are the *CCGT1* method which can be found in<sup>18</sup> and the *NM8* method which can be found in<sup>19</sup>.

The system that we use in our experiment, denoted by System F, is:

$$F_i(x) = x_i^2 x_{i+1} - 1,$$

$$F_{200}(x) = x_{200}^2 x_1 - 1,$$

a system with 200 unknown and 200 equations.

For this example we use a tolerance of  $10^{-50}$ , an initial estimation  $x^{(0)} = (0.9, \dots, 0.9)^T$ , and as initial approximations for  $x^{(-1)}$ ,  $y^{(-1)}$  and  $z^{(-1)}$  vector  $(0.7, \dots, 0.7)^T$ .

**Table 2** Numerical results for System F

Method	$\ x^{(k+1)} - x^{(k)}\ _2$	$\ F(x^{(k+1)})\ _2$	Iteration	ACOC
$M_{4,-1}$	2.61975e-71	2.275e-102	5	3.92262
$M_4D$	1.84221e-51	6.63527e-83	4	4.36222
$M_4K$	5.58564e-57	5.32079e-89	4	4.71234
$M_4D_y$	3.72943e-54	1.45955e-85	4	5.1739
$M_4K_y$	1.51937e-63	1.46077e-95	4	5.9701
$M_{7,-1}$	4.32096e-77	1.40986e-123	4	6.93731
$M_7D$	1.10549e-89	2.70395e-137	4	7.53147
$M_7K$	8.76112e-51	6.9915e-401	3	7.85679
$M_7D_y$	1.52107e-94	1.2392e-139	4	8.19609
$M_7K_y$	8.76112e-51	6.9915e-401	3	9.18679
$M_7D_z$	5.21322e-97	8.1106e-144	4	9.22566
$M_7K_z$	8.76112e-51	6.9915e-401	3	10.9754
<i>CCGT1</i>	2.64372e-64	1.59087e-516	4	8.09479
<i>NM8</i>	2.81063e-292	3.0869e-2337	4	8.0

The results obtained for the System F and for each method are shown in Table 2. We can see from the table that the number of iterations change for the family  $M_{7,y}$  and their memory methods. In this case, the iterations range between 3 and 4, making the methods that perform 4 iterations have the ACOC closer to the theoretical convergence order and those that perform 3 iterations not so close since the tolerance is small.

It can be seen that the best results for these numerical experiments are given by the memory methods that use the Kurchatov's divided difference operator to approximate the family parameter, although these are also the ones that perform the fewest iterations, which means that they are still closer to the solution than the rest, although their ACOC is not adequate for 3 iterations.

As we can see in the tables, our methods  $M_7K$ ,  $M_7K_y$  and  $M_7K_z$  are quite similar to the results obtained by the *CCGT1* method, and that the *NM8* method performs one more iteration than them to verify the tolerance, so it would be more advisable to use the methods derived from the parametric family.

### 3.3 | An example of real dynamics.

In the previous section, we have introduced memory to two parametric families and studied the order of convergence of the resulting methods. These are important concepts of iterative methods, but not the only ones. Another important concept is the behaviour of the method according to the initial estimation chosen, since we would like to know a priori if the method converge to any of the solutions according to the estimate taken. This analysis is the study of the stability of the method, also called dynamical study.

In this section, we will only show the dynamical planes associated with each of the methods when they are applied on a simple system of quadratic polynomials. In this case, the mentioned polynomial system is

$$\begin{aligned}x_1^2 - 1 &= 0, \\x_2^2 - 1 &= 0,\end{aligned}$$

where  $(x_1, x_2)^T \in \mathbb{R}^2$ .

The appearance of an iterative scheme with memory using only two previous iterations is:

$$x^{(k+1)} = \phi(x^{(k-1)}, x^{(k)}), \quad k \geq 1,$$

being the initial estimations  $x^{(0)}$  and  $x^{(1)}$ . It is clear that no fixed point can be defined in a function from  $\mathbb{R}^n \times \mathbb{R}^n$  to  $\mathbb{R}^n$ . Therefore, we define an auxiliary function  $O$  defined in vectorial form as

$$O(x^{(k-1)}, x^{(k)}) = (x^{(k)}, \phi(x^{(k-1)}, x^{(k)})), \quad k = 1, 2, \dots$$

Whether  $(x^{(k-1)}, x^{(k)})$  is a fixed point of  $O$ , then  $O(x^{(k-1)}, x^{(k)}) = (x^{(k-1)}, x^{(k)})$ , and from the definition of  $O$ , we have that  $(x^{(k-1)}, x^{(k)}) = (x^{(k)}, x^{(k+1)})$ . Thus, the discrete dynamical system  $O : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n$  is defined as

$$O(z, x) = (x, \phi(z, x)),$$

being  $\phi$  is the operator of the iterative scheme with memory. Then, a point  $(z, x)$  is a fixed pint of  $O$  if  $z = x$  and  $x = \phi(z, x)$ . If a fixed point  $(z, x)$  of the operator  $O$  does not verify that  $F(x) = 0$ , it is called strange fixed point.

The basin of attraction of a fixed point  $x^*$ , is defined as

$$\mathcal{A}(x^*) = \{y \in \mathbb{R}^n : O^m(y) \rightarrow x^*, m \rightarrow \infty\}.$$

We know that the roots of the polynomial system are:

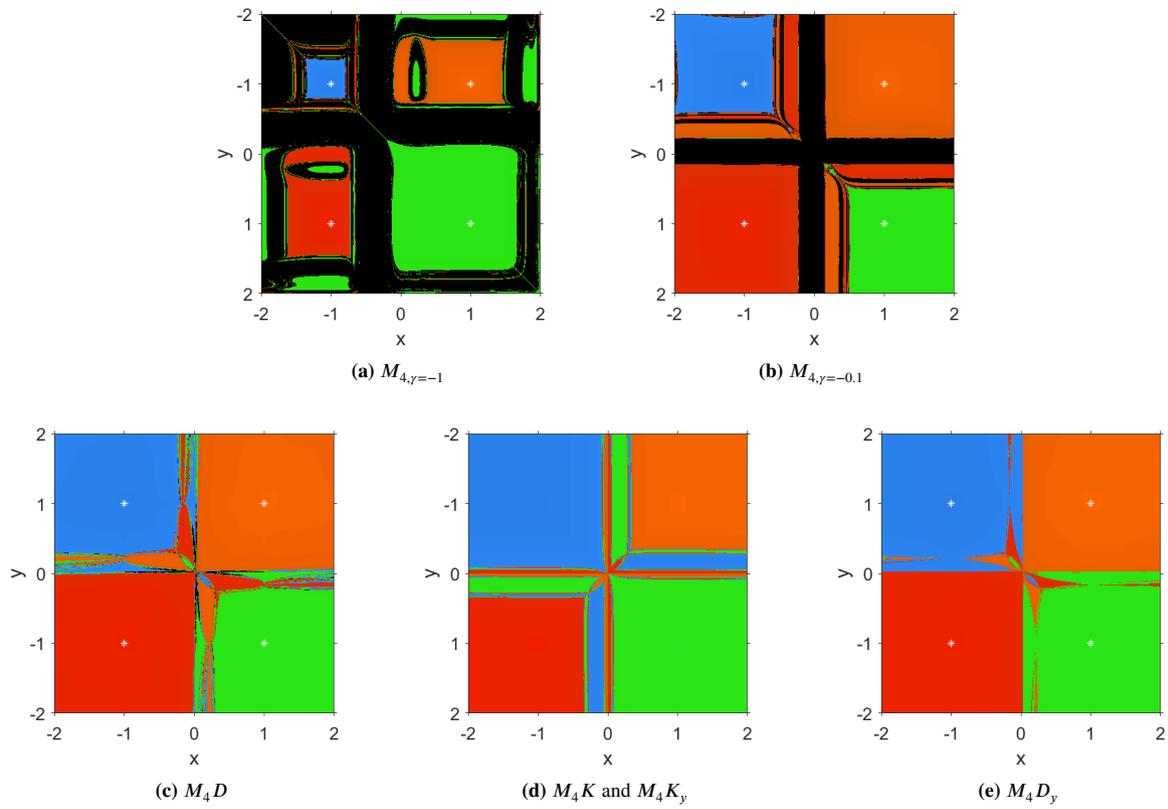
- $(1, -1)^T$ ,
- $(-1, 1)^T$ ,
- $(-1, -1)^T$ ,
- $(1, 1)^T$ .

For all methods the following matrix functions have been selected as weight functions:

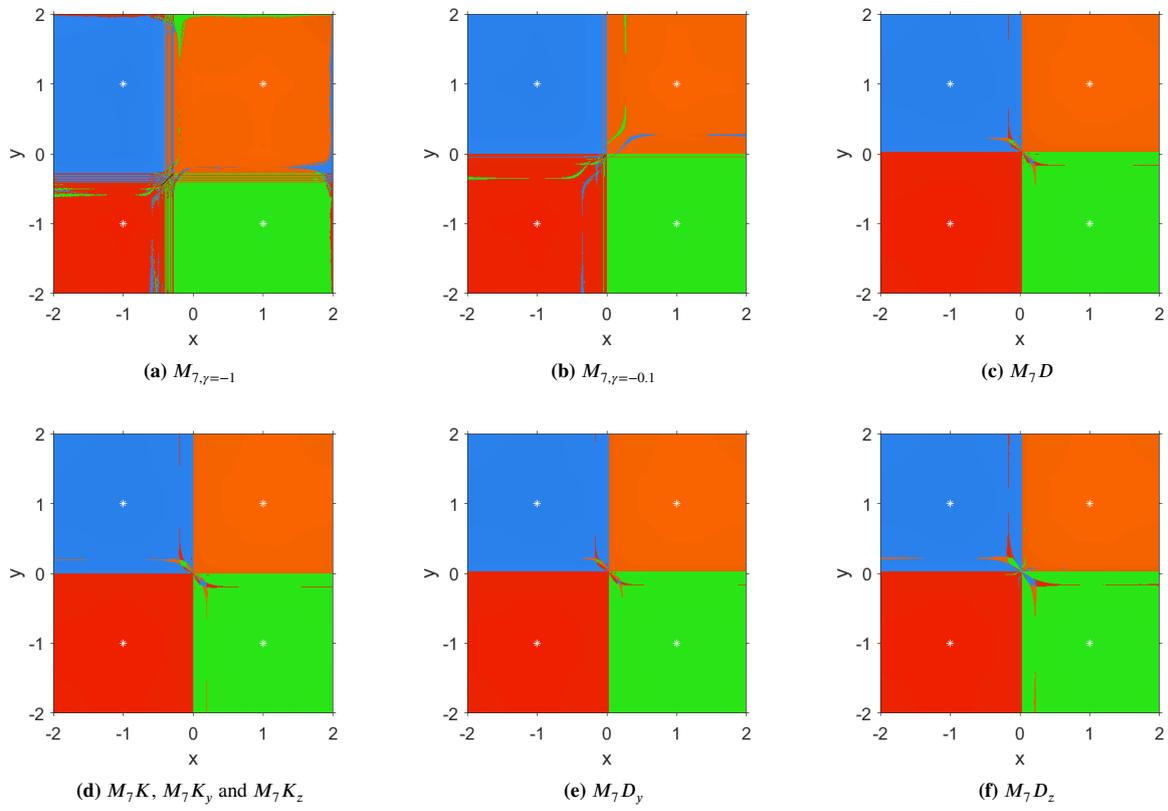
- $H(\mu) = \mu^2 + \mu + I$ ,
- $G(\mu, \nu) = I + \mu\nu + \frac{13}{6}\mu\nu^2$ ,

being  $I$  is the  $2 \times 2$  identity matrix.

To generate the dynamical planes, we have chosen a mesh of  $400 \times 400$  points, and what we will do is apply our methods to each of these points, taking the point as the initial estimate. We have also defined that the maximum number of iterations that each initial estimate must do is 80, and that we will determine that the initial point converges to one of the solutions if the distance to that solution is less than  $10^{-3}$ . We paint in orange the initial points converging to  $(1, 1)^T$ , in green the initial points converging to  $(1, -1)^T$ , in blue the initial points converging to  $(-1, 1)^T$ , in red the initial points converging to  $(-1, -1)^T$  and in black the initial guesses not converging to any of the roots.



**Figure 1** Dynamical planes of  $M_{4,\gamma}$  and their methods with memory.



**Figure 2** Dynamical planes of  $M_{7,\gamma}$  and their methods with memory.

In addition, to having increased the convergence order by introducing memory, it has been seen in the above dynamical planes that for the selected system, the introduction of memory has also helped to obtain a more stable behaviour and an easier to predict behaviour as the convergence zones of the roots are simpler.

## 4 | CONCLUSIONS

In this work, two parametric classes of iterative methods with orders of convergence 4 and 7, respectively, for solving nonlinear systems, have been designed

Memory has been introduced, in different ways, to these two families in order to obtain iterative methods with higher convergence order without the need to increase the number of functional evaluations per iteration. These methods with memory have managed to increase the order by up to 2 units for the family of order 4 and up to 4 units for the family of order 7.

But not only does the introduction of memory improve the order of convergence, but as we have seen in the dynamical planes that have been carried out, it has also improved the behaviour of the method, since we obtain that more points converge when it comes to the methods with memory, or else the attraction zones of the roots are simpler.

In the numerical experiments, the theoretical results are confirmed, and when comparing our methods with other known ones of high order (order of convergence 8) it can be seen that most of the proposed methods obtain a closer approximation to the solution than the known methods.

**Conflicts of interest:** This work does not have any conflicts of interest. **Acknowledgement:** This research was partially supported Grant PGC2018-095896-B-C22 funded by MCIN/AEI/10.13039/501100011000113033 by "ERDF A way of making Europe", European Union and was partially supported by Universitat Politècnica de València Contrato Predoctoral PAID-01-20-17 (UPV).

## References

1. Steffensen J.F.. Remarks on iteration.. *Skand. Aktuarietidskr.*. 1933;16:64–72.
2. Cordero A., Villalba E. G., Torregrosa J. R., Triguero-Navarro P. Convergence and Stability of a Parametric Class of Iterative Schemes for Solving Nonlinear Systems. *Mathematics.* ;9 (1)(2021).
3. Amiri A., Cordero A., Darvishi M. T., Torregrosa J. R.. A fast algorithm to solve systems of nonlinear equations. *Journal of Comput. Appl. Math.*. 2019;354:242-258.
4. Chicharro F.I., Cordero A., Garrido N., Torregrosa J.R.. A new efficient parametric family of iterative methods for solving nonlinear systems. *Journal of Difference Equations and Applications.* 2019;25(9-10):1454-1467.
5. Singh A. An efficient fifth-order Steffensen-type method for solving systems of nonlinear equations. *Int. J. Comput. Sci. Math.* ;9(2021).
6. Sharma J. R., Arora H.. Efficient derivative-free numerical methods for solving systems of nonlinear equations.. *Comput. Appl. Math.* ;35 (1)(2016).
7. Wang X., Zhang T., Qian W., Teng M.. Seventh-order derivative-free iterative method for solving nonlinear systems. *Numer. Algor.*. 2015;70(3):545–558.
8. Artidiello S., Cordero A., Torregrosa J. R., Vassileva M. P.. Multidimensional generalization of iterative methods for solving nonlinear problems by means of weight-function procedure. *Appl. Math. Comput.*. 2015;268:1064-1071.
9. Sharma R.K., Sharma R.. An efficient fourth order weighted-Newton method for systems of nonlinear equations. *Numer. Algor.*. 2013;62:307-323.
10. Wang X., Chen X.. Derivative-Free Kurchatov-Type Accelerating Iterative Method for Solving Nonlinear Systems: Dynamics and Applications.. *Fractal Fract.*. 2022;6(19).
11. Cordero A., Maimó J. G., Torregrosa J. R., Vassileva M.P.. Iterative Methods with Memory for Solving Systems of Nonlinear Equations Using a Second Order Approximation. *Mathematics.* ;7 (11)(2019).

12. Chicharro F.I., Cordero A., Garrido N., Torregrosa J.R.. On the improvement of the order of convergence of iterative methods for solving nonlinear systems by means of memory. *Appl. Math. Letters*. 2020;104:106277.
13. Wang X., Jin Y., Zhao Y.. Derivative-Free Iterative Methods with Some Kurchatov-Type Accelerating Parameters for Solving Nonlinear Systems. *Symmetry*. ;13(2021).
14. Traub J.F.. *Iterative Methods for the Solution of Equations*. Prentice-Hall, New York.; 1964.
15. Ortega J.M., Rheinboldt W.C.. *Iterative Solution of Nonlinear Equations in Several Variables*. Society for Industrial and Applied Mathematics, New York.; 1970.
16. Cordero A., Torregrosa J. R.. A modified Newton-Jarratt's composition.. *Numer. Algor.*. 2010;55:87–99.
17. Cordero A., Torregrosa J. R.. Variants of Newton's method using fifth-order quadrature formulas.. *Appl. Math. Comput.*. 2007;190:686–698.
18. Cordero A., Gómez E., Torregrosa J.R.. Efficient High-Order Iterative Methods for Solving Nonlinear Systems and Their Application on Heat Conduction Problems. *Complexity*. ;(2017).
19. Wang X.. Fixed-Point Iterative Method with Eighth-Order Constructed by Undetermined Parameter Technique for Solving Nonlinear Systems. *Symmetry*. ;13 (5)(2021).

