NON-ARCHIMEDEAN WELCH BOUNDS AND NON-ARCHIMEDEAN ZAUNER CONJECTU

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Abstract: Let $\mathbb{K}$ be a non-Archimedean (complete) valued field satisfying

$$\left|\sum_{j=1}^{n}\lambda_j^2\right|=\max_{1\leq j\leq n}|\lambda_j|^2,\quad \forall \lambda_j\in \mathbb{K}, 1\leq j\leq n, \forall n \in \mathbb{N}.$$

For $d\in \mathbb{N}$, let $\mathbb{K}^d$ be the standard d-dimensional non-Archimedean Hilbert space. Let $m\in \mathbb{N}$ and $\operatorname{Sym}^m(\mathbb{K}^d)$ be the non-Archimedean Hilbert space of symmetric m-tensors. We prove the following result. If $\{\tau_j\}_{j=1}^n$ is a collection in $\mathbb{K}^d$ satisfying $\langle \tau_j, \tau_j\rangle=1$ for all $1\leq j\leq n$ and the operator $\operatorname{Sym}^m(\mathbb{K}^d)\ni x\mapsto \sum_{j=1}^n\langle x, \tau_j^{\otimes m}\rangle \tau_j^{\otimes m}\in \operatorname{Sym}^m(\mathbb{K}^d)$ is diagonalizable, then

$$\max_{1\leq j,k\leq n, j\neq k}\{|n|, |\langle \tau_j, \tau_k\rangle|^{2m}\}\geq \frac{|n|^2}{\left|\binom{d+m-1}{m}\right|}. \tag{1}$$

We call Inequality (1) as the non-Archimedean version of Welch bounds obtained by Welch [*IEEE Transactions on Information Theory, 1974*]. We formulate non-Archimedean Zauner conjecture.

Keywords: Non-Archimedean valued field, non-Archimedean Hilbert space, Welch bound, Zauner conjecture.

Mathematics Subject Classification (2020): 12J25, 46S10, 47S10.

1. Introduction

Forty-eight years ago, Prof. L. Welch proved the following result [80].

Theorem 1.1. *[80] (**Welch Bounds**) Let $n>d$. If $\{\tau_j\}_{j=1}^n$ is any collection of unit vectors in $\mathbb{C}^d$, then*

$$\sum_{1\leq j,k\leq n}|\langle \tau_j, \tau_k\rangle|^{2m}=\sum_{j=1}^{n}\sum_{k=1}^{n}|\langle \tau_j, \tau_k\rangle|^{2m}\geq \frac{n^2}{\binom{d+m-1}{m}},\quad \forall m \in \mathbb{N}.$$

In particular,

$$\sum_{1\leq j,k\leq n}|\langle \tau_j, \tau_k\rangle|^{2}=\sum_{j=1}^{n}\sum_{k=1}^{n}|\langle \tau_j, \tau_k\rangle|^{2}\geq \frac{n^2}{d}.$$

Further,

$$\textit{(Higher order Welch bounds)}\quad \max_{1\leq j,k\leq n, j\neq k}|\langle \tau_j, \tau_k\rangle|^{2m}\geq \frac{1}{n-1}\left[\frac{n}{\binom{d+m-1}{m}}-1\right],\quad \forall m \in \mathbb{N}.$$

In particular,

$$\textbf{\textit{(First order Welch bound)}} \quad \max_{1\leq j,k\leq n, j\neq k} |\langle \tau_j, \tau_k\rangle|^2 \geq \frac{n-d}{d(n-1)}.$$

There are infinitely many applications of Theorem 1.1 such as in the study of root-mean-square (RMS) absolute cross relation of unit vectors [67], frame potential [9, 14, 18], correlations [66], codebooks [27], numerical search algorithms [81, 82], quantum measurements [69], coding and communications [72, 76], code division multiple access (CDMA) systems [49, 50], wireless systems [64], compressed/compressive sensing [1, 6, 29, 32, 68, 74, 75, 77], 'game of Sloanes' [45], equiangular tight frames [73], equiangular lines [23, 31, 44, 57], digital fingerprinting [56] etc.

Different proofs/improvements of Theorem 1.1 have been done in [19, 24, 25, 28, 42, 65, 72, 78, 79]. In 2021 M. Krishna derived continuous version of Theorem 1.1 [51]. In 2022 M. Krishna obtained Theorem 1.1 for Hilbert C*-modules [53] and Banach spaces [52].

In this paper we derive non-Archimedean Welch bounds (Theorem 2.3). We formulate non-Archimedean Zauner conjecture (Conjecture 3.2).

2. Non-Archimedean Welch bounds

Let $\mathbb{K}$ be a non-Archimedean (complete) valued field satisfying

$$\left|\sum_{j=1}^{n}\lambda_j^2\right| = \max_{1\leq j\leq n}|\lambda_j|^2, \quad \forall \lambda_j\in\mathbb{K}, 1\leq j\leq n, \forall n\in\mathbb{N}. \tag{2}$$

Such non-Archimedean fields exist, see [61]. Throughout the paper, we assume that our non-Archimedean field satisfies Equation (2). For $d\in\mathbb{N}$, let $\mathbb{K}^d$ be the standard non-Archimedean Hilbert space equipped with the inner product

$$\langle (a_j)_{j=1}^d, (b_j)_{j=1}^d\rangle \coloneqq \sum_{j=1}^{d} a_j b_j, \quad \forall (a_j)_{j=1}^d, (b_j)_{j=1}^d \in \mathbb{K}^d.$$

Theorem 2.1. ***(First Order Non-Archimedean Welch Bound)*** *If $\{\tau_j\}_{j=1}^n$ is any collection in $\mathbb{K}^d$ such that the operator $S_\tau : \mathbb{K}^d \ni x \mapsto \sum_{j=1}^{n}\langle x, \tau_j\rangle\tau_j \in \mathbb{K}^d$ is diagonalizable, then*

$$\max_{1\leq j,k\leq n, j\neq k}\left\{\left|\sum_{l=1}^{n}\langle \tau_l, \tau_l\rangle^2\right|, |\langle \tau_j, \tau_k\rangle|^2\right\} \geq \frac{1}{|d|}\left|\sum_{j=1}^{n}\langle \tau_j, \tau_j\rangle\right|^2.$$

In particular, if $\langle \tau_j, \tau_j\rangle = 1$ for all $1\leq j\leq n$, then

$$\textbf{\textit{(First order non-Archimedean Welch bound)}} \quad \max_{1\leq j,k\leq n, j\neq k}\{|n|, |\langle \tau_j, \tau_k\rangle|^2\} \geq \frac{|n|^2}{|d|}.$$

Proof. We first note that

$$\operatorname{Tra}(S_\tau) = \sum_{j=1}^{n}\langle \tau_j, \tau_j\rangle,$$
$$\operatorname{Tra}(S_\tau^2) = \sum_{j=1}^{n}\sum_{k=1}^{n}\langle \tau_j, \tau_k\rangle\langle \tau_k, \tau_j\rangle.$$

Let $\lambda_1, \dots, \lambda_d$ be the diagonal entries in the diagonalization of S_τ. Then using the diagonalizability of S_τ and the non-Archimedean Cauchy-Schwarz inequality (Theorem 2.4.2 [61]), we get

$$\begin{aligned}\left|\sum_{j=1}^{n}\langle\tau_j,\tau_j\rangle\right|^2 &= |\operatorname{Tra}(S_\tau)|^2 = \left|\sum_{k=1}^{d}\lambda_k\right|^2 \leq |d|\left|\sum_{k=1}^{d}\lambda_k^2\right| = |d||\operatorname{Tra}(S_\tau^2)|\\ &= |d|\left|\sum_{j=1}^{n}\sum_{k=1}^{n}\langle\tau_j,\tau_k\rangle\langle\tau_k,\tau_j\rangle\right| = |d|\left|\sum_{l=1}^{n}\langle\tau_l,\tau_l\rangle^2 + \sum_{j,k=1,j\neq k}^{n}\langle\tau_j,\tau_k\rangle\langle\tau_k,\tau_j\rangle\right|\\ &\leq |d|\max_{1\leq j,k\leq n, j\neq k}\left\{\left|\sum_{l=1}^{n}\langle\tau_l,\tau_l\rangle^2\right|, |\langle\tau_j,\tau_k\rangle\langle\tau_k,\tau_j\rangle|\right\}\\ &= |d|\max_{1\leq j,k\leq n, j\neq k}\left\{\left|\sum_{l=1}^{n}\langle\tau_l,\tau_l\rangle^2\right|, |\langle\tau_j,\tau_k\rangle|^2\right\}.\end{aligned}$$

Whenever $\langle\tau_j,\tau_j\rangle = 1$ for all $1\leq j\leq n$,

$$|n|^2 \leq |d|\max_{1\leq j,k\leq n, j\neq k}\{|n|, |\langle\tau_j,\tau_k\rangle|^2\}.$$

$\square$

We next obtain higher order non-Archimedean Welch bounds. We use the following vector space result.

Theorem 2.2. *[13, 21] If $\mathcal{V}$ is a vector space of dimension d and $Sym^m(\mathcal{V})$ denotes the vector space of symmetric m-tensors, then*

$$dim(Sym^m(\mathcal{V})) = \binom{d+m-1}{m}, \quad \forall m\in\mathbb{N}.$$

Theorem 2.3. ***(Higher Order Non-Archimedean Welch Bounds)*** *Let $m\in\mathbb{N}$. If $\{\tau_j\}_{j=1}^n$ is any collection in $\mathbb{K}^d$ such that the operator $S_\tau : Sym^m(\mathbb{K}^d) \ni x \mapsto \sum_{j=1}^{n}\langle x,\tau_j^{\otimes m}\rangle\tau_j^{\otimes m} \in Sym^m(\mathbb{K}^d)$ is diagonalizable, then*

$$\max_{1\leq j,k\leq n, j\neq k}\left\{\left|\sum_{l=1}^{n}\langle\tau_l,\tau_l\rangle^{2m}\right|, |\langle\tau_j,\tau_k\rangle|^{2m}\right\} \geq \frac{1}{\left|\binom{d+m-1}{m}\right|}\left|\sum_{j=1}^{n}\langle\tau_j,\tau_j\rangle^m\right|^2.$$

In particular, if $\langle\tau_j,\tau_j\rangle = 1$ for all $1\leq j\leq n$, then

$$\textbf{\textit{(Higher order non-Archimedean Welch bound)}}\quad \max_{1\leq j,k\leq n, j\neq k}\{|n|, |\langle\tau_j,\tau_k\rangle|^{2m}\} \geq \frac{|n|^2}{\left|\binom{d+m-1}{m}\right|}.$$

Proof. Let $\lambda_1,\dots,\lambda_{\dim(\mathrm{Sym}^m(\mathbb{K}^d))}$ be the diagonal entries in the diagonalization of S_τ. Then

$$\begin{aligned}
\left|\sum_{j=1}^{n}\langle\tau_j,\tau_j\rangle^m\right|^2 &= \left|\sum_{j=1}^{n}\langle\tau_j^{\otimes m},\tau_j^{\otimes m}\rangle\right|^2 = |\operatorname{Tra}(S_\tau)|^2 = \left|\sum_{k=1}^{\dim(\operatorname{Sym}^m(\mathbb{K}^d))}\lambda_k\right|^2\\
&\leq |\dim(\operatorname{Sym}^m(\mathbb{K}^d))|\left|\sum_{k=1}^{\dim(\operatorname{Sym}^m(\mathbb{K}^d))}\lambda_k^2\right| = |\dim(\operatorname{Sym}^m(\mathbb{K}^d))||\operatorname{Tra}(S_\tau^2)|\\
&= \left|\binom{d+m-1}{m}\right||\operatorname{Tra}(S_\tau^2)| = \left|\binom{d+m-1}{m}\right|\left|\sum_{j=1}^{n}\sum_{k=1}^{n}\langle\tau_j^{\otimes m},\tau_k^{\otimes m}\rangle\langle\tau_k^{\otimes m},\tau_j^{\otimes m}\rangle\right|\\
&= \left|\binom{d+m-1}{m}\right|\left|\sum_{j=1}^{n}\sum_{k=1}^{n}\langle\tau_j,\tau_k\rangle^m\langle\tau_k,\tau_j\rangle^m\right|\\
&= \left|\binom{d+m-1}{m}\right|\left|\sum_{l=1}^{n}\langle\tau_l,\tau_l\rangle^{2m}+\sum_{j,k=1,j\neq k}^{n}\langle\tau_j,\tau_k\rangle^m\langle\tau_k,\tau_j\rangle^m\right|\\
&\leq \left|\binom{d+m-1}{m}\right|\max_{1\leq j,k\leq n,j\neq k}\left\{\left|\sum_{l=1}^{n}\langle\tau_l,\tau_l\rangle^{2m}\right|,|\langle\tau_j,\tau_k\rangle^m\langle\tau_k,\tau_j\rangle^m|\right\}\\
&= \left|\binom{d+m-1}{m}\right|\max_{1\leq j,k\leq n,j\neq k}\left\{\left|\sum_{l=1}^{n}\langle\tau_l,\tau_l\rangle^{2m}\right|,|\langle\tau_j,\tau_k\rangle|^{2m}\right\}.
\end{aligned}$$

Whenever $\langle\tau_j,\tau_j\rangle=1$ for all $1\leq j\leq n$,

$$|n|^2\leq\left|\binom{d+m-1}{m}\right|\max_{1\leq j,k\leq n,j\neq k}\{|n|,|\langle\tau_j,\tau_k\rangle|^{2m}\}.$$

□

Remark 2.4. *Theorems 2.1 and 2.3 hold by replacing $\mathbb{K}^d$ by a d-dimensional non-Archimedean Hilbert space over $\mathbb{K}$.*

3. Non-Archimedean Zauner Conjecture and open problems

Theorem 2.1 straight away brings the following question.

Question 3.1. ***Given non-Archimedean field $\mathbb{K}$ satisfying Equation (2), for which $(d,n)\in\mathbb{N}\times\mathbb{N}$, there exist vectors $\tau_1,\dots,\tau_n\in\mathbb{K}^d$ satisfying the following.***

- **(i)** $\langle\tau_j,\tau_j\rangle=1$ ***for all*** $1\leq j\leq n$.
- **(ii)** ***The operator*** $S_\tau:\mathbb{K}^d\ni x\mapsto\sum_{j=1}^{n}\langle x,\tau_j\rangle\tau_j\in\mathbb{K}^d$ ***is diagonalizable.***
- **(iii)**

$$\max_{1\leq j,k\leq n,j\neq k}\{|n|,|\langle\tau_j,\tau_k\rangle|^2\}=\frac{|n|^2}{|d|}.$$

A particular case of Question 3.1 is the following non-Archimedean version of Zauner conjecture (see [2–5,10–12,33,36,43,48,51,55,63,70,84] for Zauner conjecture in Hilbert spaces, [53] for Zauner conjecture in Hilbert C*-modules and [52] for Zauner conjecture in Banach spaces).

Conjecture 3.2. ***(Non-Archimedean Zauner Conjecture) Let $\mathbb{K}$ be a non-Archimedean field satisfying Equation (2). For each $d\in\mathbb{N}$, there exist vectors $\tau_1,\dots,\tau_{d^2}\in\mathbb{K}^d$ satisfying the following.***

(i) $\langle \tau_j, \tau_j\rangle = 1$ ***for all*** $1 \leq j \leq d^2$***.***
(ii) ***The operator*** $S_\tau : \mathbb{K}^d \ni x \mapsto \sum_{j=1}^{d^2}\langle x, \tau_j\rangle \tau_j \in \mathbb{K}^d$ ***is diagonalizable.***
(iii)

$$|\langle \tau_j, \tau_k\rangle|^2 = |d|, \quad \forall 1 \leq j, k \leq d^2, j \neq k.$$

We remember the definition of Gerzon's bound which allows us to recall the bounds which are in the same way to Welch bounds in Hilbert spaces.

Definition 3.3. *[45] Given $d \in \mathbb{N}$, define* ***Gerzon's bound***

$$\mathcal{Z}(d, \mathbb{K}) := \begin{cases} d^2 & \text{if} \quad \mathbb{K} = \mathbb{C} \\ \frac{d(d+1)}{2} & \text{if} \quad \mathbb{K} = \mathbb{R}. \end{cases}$$

Theorem 3.4. *[16, 22, 41, 45, 58, 62, 71, 81] Define $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $m := \dim_{\mathbb{R}}(\mathbb{K})/2$. If $\{\tau_j\}_{j=1}^n$ is any collection of unit vectors in $\mathbb{K}^d$, then*

(i) ***(Bukh-Cox bound)***

$$\max_{1\leq j,k\leq n, j\neq k} |\langle \tau_j, \tau_k\rangle| \geq \frac{\mathcal{Z}(n-d, \mathbb{K})}{n(1 + m(n-d-1)\sqrt{m^{-1} + n - d}) - \mathcal{Z}(n-d, \mathbb{K})} \quad \text{if} \quad n > d.$$

(ii) ***(Orthoplex/Rankin bound)***

$$\max_{1\leq j,k\leq n, j\neq k} |\langle \tau_j, \tau_k\rangle| \geq \frac{1}{\sqrt{d}} \quad \text{if} \quad n > \mathcal{Z}(d, \mathbb{K}).$$

(iii) ***(Levenstein bound)***

$$\max_{1\leq j,k\leq n, j\neq k} |\langle \tau_j, \tau_k\rangle| \geq \sqrt{\frac{n(m+1) - d(md+1)}{(n-d)(md+1)}} \quad \text{if} \quad n > \mathcal{Z}(d, \mathbb{K}).$$

(iv) ***(Exponential bound)***

$$\max_{1\leq j,k\leq n, j\neq k} |\langle \tau_j, \tau_k\rangle| \geq 1 - 2n^{\frac{-1}{d-1}}.$$

Theorem 3.4 and Theorem 2.1 give the following problem.

Question 3.5. ***Whether there is a non-Archimedean version of Theorem 3.4? In particular, does there exists a version of***

(i) ***non-Archimedean Bukh-Cox bound?***
(ii) ***non-Archimedean Orthoplex/Rankin bound?***
(iii) ***non-Archimedean Levenstein bound?***
(iv) ***non-Archimedean Exponential bound?***

As written already, Welch bounds have applications in study of equiangular lines. Therefore we wish to formulate equiangular line problem for non-Archimedean Hilbert spaces. For the study of equiangular lines in Hilbert spaces we refer [7, 8, 15, 17, 26, 34, 35, 37, 40, 46, 47, 54, 59, 60, 83], quaternion Hilbert space we refer [30], octonion Hilbert space we refer [20], finite dimensional vector spaces over finite fields we refer [38, 39] and for Banach spaces we refer [52] (there equiangular line problem for Banach spaces is not mentioned explicitly but Zauner conjecture for Banach spaces is formulated. One can easily formulate equiangular line problem using that).

Question 3.6. ***(Non-Archimedean Equiangular Line Problem) Let*** $\mathbb{K}$ ***be a non-Archimedean field. Given*** $a \in \mathbb{K}$***,*** $d \in \mathbb{N}$ ***and*** $\gamma > 0$***, what is the maximum*** $n = n(\mathbb{K}, a, d, \gamma) \in \mathbb{N}$ ***such that there exist vectors*** $\tau_1, \dots, \tau_n \in \mathbb{K}^d$ ***satisfying the following.***

(i) $\langle \tau_j, \tau_j \rangle = a$ ***for all*** $1 \leq j \leq n$.

(ii) $|\langle \tau_j, \tau_k \rangle|^2 = \gamma$ ***for all*** $1 \leq j, k \leq n, j \neq k$.

In particular, whether there is a non-Archimedean Gerzon bound?

Question 3.6 can be easily modified to formulate question of non-Archimedean regular s-distance sets.

References

[1] W. O. Alltop. Complex sequences with low periodic correlations. *IEEE Trans. Inform. Theory*, 26(3):350–354, 1980.

[2] D. M. Appleby. Symmetric informationally complete-positive operator valued measures and the extended Clifford group. *J. Math. Phys.*, 46(5):052107, 29, 2005.

[3] Marcus Appleby, Ingemar Bengtsson, Steven Flammia, and Dardo Goyeneche. Tight frames, Hadamard matrices and Zauner's conjecture. *J. Phys. A*, 52(29):295301, 26, 2019.

[4] Marcus Appleby, Steven Flammia, Gary McConnell, and Jon Yard. SICs and algebraic number theory. *Found. Phys.*, 47(8):1042–1059, 2017.

[5] Marcus Appleby, Steven Flammia, Gary McConnell, and Jon Yard. Generating ray class fields of real quadratic fields via complex equiangular lines. *Acta Arith.*, 192(3):211–233, 2020.

[6] Waheed U. Bajwa, Robert Calderbank, and Dustin G. Mixon. Two are better than one: fundamental parameters of frame coherence. *Appl. Comput. Harmon. Anal.*, 33(1):58–78, 2012.

[7] Igor Balla, Felix Draxler, Peter Keevash, and Benny Sudakov. Equiangular lines and spherical codes in Euclidean space. *Invent. Math.*, 211(1):179–212, 2018.

[8] Alexander Barg and Wei-Hsuan Yu. New bounds for equiangular lines. In *Discrete geometry and algebraic combinatorics*, volume 625 of *Contemp. Math.*, pages 111–121. Amer. Math. Soc., Providence, RI, 2014.

[9] John J. Benedetto and Matthew Fickus. Finite normalized tight frames. *Adv. Comput. Math.*, 18(2-4):357–385, 2003.

[10] Ingemar Bengtsson. The number behind the simplest SIC-POVM. *Found. Phys.*, 47(8):1031–1041, 2017.

[11] Ingemar Bengtsson. SICs: some explanations. *Found. Phys.*, 50(12):1794–1808, 2020.

[12] Ingemar Bengtsson and Karol Zyczkowski. On discrete structures in finite Hilbert spaces. *arXiv:1701.07902v1 [quant-ph]*, 26 January 2017.

[13] Cristiano Bocci and Luca Chiantini. *An introduction to algebraic statistics with tensors*, volume 118 of *Unitext*. Springer, Cham, 2019.

[14] Bernhard G. Bodmann and John Haas. Frame potentials and the geometry of frames. *J. Fourier Anal. Appl.*, 21(6):1344–1383, 2015.

[15] Boris Bukh. Bounds on equiangular lines and on related spherical codes. *SIAM J. Discrete Math.*, 30(1):549–554, 2016.

[16] Boris Bukh and Christopher Cox. Nearly orthogonal vectors and small antipodal spherical codes. *Israel J. Math.*, 238(1):359–388, 2020.

[17] A. R. Calderbank, P. J. Cameron, W. M. Kantor, and J. J. Seidel. Z_4-Kerdock codes, orthogonal spreads, and extremal Euclidean line-sets. *Proc. London Math. Soc. (3)*, 75(2):436–480, 1997.

[18] Peter G. Casazza, Matthew Fickus, Jelena Kovačević, Manuel T. Leon, and Janet C. Tremain. A physical interpretation of tight frames. In *Harmonic analysis and applications*, Appl. Numer. Harmon. Anal., pages 51–76. Birkhäuser Boston, Boston, MA, 2006.

[19] Ole Christensen, Somantika Datta, and Rae Young Kim. Equiangular frames and generalizations of the Welch bound to dual pairs of frames. *Linear Multilinear Algebra*, 68(12):2495–2505, 2020.

[20] Henry Cohn, Abhinav Kumar, and Gregory Minton. Optimal simplices and codes in projective spaces. *Geom. Topol.*, 20(3):1289–1357, 2016.

[21] Pierre Comon, Gene Golub, Lek-Heng Lim, and Bernard Mourrain. Symmetric tensors and symmetric tensor rank. *SIAM J. Matrix Anal. Appl.*, 30(3):1254–1279, 2008.

[22] John H. Conway, Ronald H. Hardin, and Neil J. A. Sloane. Packing lines, planes, etc.: packings in Grassmannian spaces. *Experiment. Math.*, 5(2):139–159, 1996.

[23] G. Coutinho, C. Godsil, H. Shirazi, and H. Zhan. Equiangular lines and covers of the complete graph. *Linear Algebra Appl.*, 488:264–283, 2016.

[24] S. Datta, S. Howard, and D. Cochran. Geometry of the Welch bounds. *Linear Algebra Appl.*, 437(10):2455–2470, 2012.

[25] Somantika Datta. Welch bounds for cross correlation of subspaces and generalizations. *Linear Multilinear Algebra*, 64(8):1484–1497, 2016.

[26] D. de Caen. Large equiangular sets of lines in Euclidean space. *Electron. J. Combin.*, 7:Research Paper 55, 3, 2000.

[27] Cunsheng Ding and Tao Feng. Codebooks from almost difference sets. *Des. Codes Cryptogr.*, 46(1):113–126, 2008.

[28] M. Ehler and K. A. Okoudjou. Minimization of the probabilistic p-frame potential. *J. Statist. Plann. Inference*, 142(3):645–659, 2012.

[29] Yonina C. Eldar and Gitta Kutyniok, editors. *Compressed sensing : Theory and application*. Cambridge University Press, Cambridge, 2012.

[30] Boumediene Et-Taoui. Quaternionic equiangular lines. *Adv. Geom.*, 20(2):273–284, 2020.

[31] Matthew Fickus, John Jasper, and Dustin G. Mixon. Packings in real projective spaces. *SIAM J. Appl. Algebra Geom.*, 2(3):377–409, 2018.

[32] Simon Foucart and Holger Rauhut. *A mathematical introduction to compressive sensing*. Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, New York, 2013.

[33] Hoang M.C. Fuchs, C.A. and B.C Stacey. The SIC question: History and state of play. *Axioms*, 6(3), 2017.

[34] Alexey Glazyrin and Wei-Hsuan Yu. Upper bounds for s-distance sets and equiangular lines. *Adv. Math.*, 330:810–833, 2018.

[35] Chris Godsil and Aidan Roy. Equiangular lines, mutually unbiased bases, and spin models. *European J. Combin.*, 30(1):246–262, 2009.

[36] Gilad Gour and Amir Kalev. Construction of all general symmetric informationally complete measurements. *J. Phys. A*, 47(33):335302, 14, 2014.

[37] Gary Greaves, Jacobus H. Koolen, Akihiro Munemasa, and Ferenc Szollosi. Equiangular lines in Euclidean spaces. *J. Combin. Theory Ser. A*, 138:208–235, 2016.

[38] Gary R. W. Greaves, Joseph W. Iverson, John Jasper, and Dustin G. Mixon. Frames over finite fields: basic theory and equiangular lines in unitary geometry. *Finite Fields Appl.*, 77:Paper No. 101954, 41, 2022.

[39] Gary R. W. Greaves, Joseph W. Iverson, John Jasper, and Dustin G. Mixon. Frames over finite fields: equiangular lines in orthogonal geometry. *Linear Algebra Appl.*, 639:50–80, 2022.

[40] Gary R. W. Greaves, Jeven Syatriadi, and Pavlo Yatsyna. Equiangular lines in low dimensional Euclidean spaces. *Combinatorica*, 41(6):839–872, 2021.

[41] John I. Haas, Nathaniel Hammen, and Dustin G. Mixon. The Levenstein bound for packings in projective spaces. *Proceedings, volume 10394, Wavelets and Sparsity XVII, SPIE Optical Engineering+Applications, San Diego, California, United States of America*, 2017.

[42] Marina Haikin, Ram Zamir, and Matan Gavish. Frame moments and Welch bound with erasures. *2018 IEEE International Symposium on Information Theory (ISIT)*, pages 2057–2061, 2018.

[43] Pawel Horodecki, Lukasz Rudnicki, and Karol Zyczkowski. Five open problems in theory of quantum information. *arXiv:2002.03233v2 [quant-ph]*, 21 December 2020.

[44] Joseph W. Iverson and Dustin G. Mixon. Doubly transitive lines I: Higman pairs and roux. *J. Combin. Theory Ser. A*, 185:Paper No. 105540, 47, 2022.

[45] John Jasper, Emily J. King, and Dustin G. Mixon. Game of Sloanes: best known packings in complex projective space. *Proc. SPIE 11138, Wavelets and Sparsity XVIII*, 2019.

[46] Zilin Jiang and Alexandr Polyanskii. Forbidden subgraphs for graphs of bounded spectral radius, with applications to equiangular lines. *Israel J. Math.*, 236(1):393–421, 2020.

[47] Zilin Jiang, Jonathan Tidor, Yuan Yao, Shengtong Zhang, and Yufei Zhao. Equiangular lines with a fixed angle. *Ann. of Math. (2)*, 194(3):729–743, 2021.

[48] Gene S. Kopp. SIC-POVMs and the Stark Conjectures. *Int. Math. Res. Not. IMRN*, (18):13812–13838, 2021.

[49] J. Kovacevic and A. Chebira. Life beyond bases: The advent of frames (part I). *IEEE Signal Processing Magazine*, 24(4):86–104, 2007.

[50] J. Kovacevic and A. Chebira. Life beyond bases: The advent of frames (part II). *IEEE Signal Processing Magazine*, 24(5):115–125, 2007.

[51] K. Mahesh Krishna. Continuous Welch bounds with applications. *arXiv:2109.09296v1 [FA]*, 20 September 2021.

[52] K. Mahesh Krishna. Discrete and continuous Welch bounds for Banach spaces with applications. *arXiv:2201.00980v1 [math.FA] 4 January*, 2022.

[53] K. Mahesh Krishna. Modular Welch bounds with applications. *arXiv:2201.00319v1 [OA] 2 January*, 2022.

[54] P. W. H. Lemmens and J. J. Seidel. Equiangular lines. *J. Algebra*, 24:494–512, 1973.

[55] M. Maxino and Dustin G. Mixon. Biangular Gabor frames and Zauner's conjecture. In *Wavelets and Sparsity XVIII*. 2019.

[56] Dustin G. Mixon, Christopher J. Quinn, Negar Kiyavash, and Matthew Fickus. Fingerprinting with equiangular tight frames. *IEEE Trans. Inform. Theory*, 59(3):1855–1865, 2013.

[57] Dustin G. Mixon and James Solazzo. A short introduction to optimal line packings. *College Math. J.*, 49(2):82–91, 2018.

[58] K.K. Mukkavilli, A. Sabharwal, E. Erkip, and B. Aazhang. On beamforming with finite rate feedback in multiple-antenna systems. *IEEE Transactions on Information Theory*, 49(10):2562–2579, 2003.

[59] A. Neumaier. Graph representations, two-distance sets, and equiangular lines. *Linear Algebra Appl.*, 114/115:141–156, 1989.

[60] Takayuki Okuda and Wei-Hsuan Yu. A new relative bound for equiangular lines and nonexistence of tight spherical designs of harmonic index 4. *European J. Combin.*, 53:96–103, 2016.

[61] C. Perez-Garcia and W. H. Schikhof. *Locally convex spaces over non-Archimedean valued fields*, volume 119 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010.

[62] R. A. Rankin. The closest packing of spherical caps in n dimensions. *Proc. Glasgow Math. Assoc.*, 2:139–144, 1955.

[63] Joseph M. Renes, Robin Blume-Kohout, A. J. Scott, and Carlton M. Caves. Symmetric informationally complete quantum measurements. *J. Math. Phys.*, 45(6):2171–2180, 2004.

[64] C. Rose, S. Ulukus, and R. D. Yates. Wireless systems and interference avoidance. *EEE Transactions on Wireless Communications*, 1(3):415–428, 2002.

[65] Moshe Rosenfeld. In praise of the Gram matrix. In *The mathematics of Paul Erdős, II*, volume 14 of *Algorithms Combin.*, pages 318–323. Springer, Berlin, 1997.

[66] Dilip V. Sarwate. Bounds on crosscorrelation and autocorrelation of sequences. *IEEE Trans. Inform. Theory*, 25(6):720–724, 1979.

[67] Dilip V. Sarwate. Meeting the Welch bound with equality. In *Sequences and their applications (Singapore, 1998)*, Springer Ser. Discrete Math. Theor. Comput. Sci., pages 79–102. Springer, London, 1999.

[68] Karin Schnass and Pierre Vandergheynst. Dictionary preconditioning for greedy algorithms. *IEEE Trans. Signal Process.*, 56(5):1994–2002, 2008.

[69] A. J. Scott. Tight informationally complete quantum measurements. *J. Phys. A*, 39(43):13507–13530, 2006.

[70] A. J. Scott and M. Grassl. Symmetric informationally complete positive-operator-valued measures: a new computer study. *J. Math. Phys.*, 51(4):042203, 16, 2010.

[71] Mojtaba Soltanalian, Mohammad Mahdi Naghsh, and Petre Stoica. On meeting the peak correlation bounds. *IEEE Transactions on Signal Processing*, 62(5):1210–1220, 2014.

[72] Thomas Strohmer and Robert W. Heath, Jr. Grassmannian frames with applications to coding and communication. *Appl. Comput. Harmon. Anal.*, 14(3):257–275, 2003.

[73] Mátyás A. Sustik, Joel A. Tropp, Inderjit S. Dhillon, and Robert W. Heath, Jr. On the existence of equiangular tight frames. *Linear Algebra Appl.*, 426(2-3):619–635, 2007.

[74] Yan Shuo Tan. Energy optimization for distributions on the sphere and improvement to the Welch bounds. *Electron. Commun. Probab.*, 22:Paper No. 43, 12, 2017.

[75] Joel A. Tropp. Greed is good: algorithmic results for sparse approximation. *IEEE Trans. Inform. Theory*, 50(10):2231–2242, 2004.

[76] Joel A. Tropp, Inderjit S. Dhillon, Robert W. Heath, Jr., and Thomas Strohmer. Designing structured tight frames via an alternating projection method. *IEEE Trans. Inform. Theory*, 51(1):188–209, 2005.

[77] M. Vidyasagar. *An introduction to compressed sensing*, volume 22 of *Computational Science & Engineering*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2020.

[78] Shayne Waldron. Generalized Welch bound equality sequences are tight frames. *IEEE Trans. Inform. Theory*, 49(9):2307–2309, 2003.

[79] Shayne Waldron. A sharpening of the Welch bounds and the existence of real and complex spherical t-designs. *IEEE Trans. Inform. Theory*, 63(11):6849–6857, 2017.

[80] L. Welch. Lower bounds on the maximum cross correlation of signals. *IEEE Transactions on Information Theory*, 20(3):397–399, 1974.

[81] Pengfei Xia, Shengli Zhou, and Georgios B. Giannakis. Achieving the Welch bound with difference sets. *IEEE Trans. Inform. Theory*, 51(5):1900–1907, 2005.

[82] Pengfei Xia, Shengli Zhou, and Georgios B. Giannakis. Correction to: "Achieving the Welch bound with difference sets" [IEEE Trans. Inform. Theory 51 (2005), no. 5, 1900–1907]. *IEEE Trans. Inform. Theory*, 52(7):3359, 2006.

[83] Wei-Hsuan Yu. New bounds for equiangular lines and spherical two-distance sets. *SIAM J. Discrete Math.*, 31(2):908–917, 2017.

[84] Gerhard Zauner. Quantum designs: foundations of a noncommutative design theory. *Int. J. Quantum Inf.*, 9(1):445–507, 2011.