NON-ARCHIMEDEAN AND P-ADIC FUNCTIONAL WELCH BOUNDS

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Abstract: We prove the non-Archimedean (resp. p-adic) Banach space version of non-Archimedean (resp. p-adic) Welch bounds recently obtained by M. Krishna. More precisely, we prove following results.

(i) Let $\mathbb{K}$ be a non-Archimedean (complete) valued field satisfying $\left|\sum_{j=1}^n \lambda_j^2\right| = \max_{1\leq j\leq n} |\lambda_j|^2$ for all $\lambda_j \in \mathbb{K}, 1 \leq j \leq n$, for all $n \in \mathbb{N}$. Let $\mathcal{X}$ be a d-dimensional non-Archimedean Banach space over $\mathbb{K}$. If $\{\tau_j\}_{j=1}^n$ is any collection in $\mathcal{X}$ and $\{f_j\}_{j=1}^n$ is any collection in $\mathcal{X}^*$ (dual of $\mathcal{X}$) satisfying $f_j(\tau_j) = 1$ for all $1 \leq j \leq n$ and the operator $S_{f,\tau} : \text{Sym}^m(\mathcal{X}) \ni x \mapsto \sum_{j=1}^n f_j^{\otimes m}(x)\tau_j^{\otimes m} \in \text{Sym}^m(\mathcal{X})$, is diagonalizable, then

$$\max_{1\leq j,k\leq n, j\neq k}\{|n|, |f_j(\tau_k)f_k(\tau_j)|^m\} \geq \frac{|n|^2}{\left|\binom{d+m-1}{m}\right|}. \tag{1}$$

We call Inequality (1) as non-Archimedean functional Welch bounds.

(ii) For a prime p, let $\mathbb{Q}_p$ be the p-adic number field. Let $\mathcal{X}$ be a d-dimensional p-adic Banach space over $\mathbb{Q}_p$. If $\{\tau_j\}_{j=1}^n$ is any collection in $\mathcal{X}$ and $\{f_j\}_{j=1}^n$ is any collection in $\mathcal{X}^*$ (dual of $\mathcal{X}$) satisfying $f_j(\tau_j) = 1$ for all $1 \leq j \leq n$ and there exists $b \in \mathbb{Q}_p$ such that $\sum_{j=1}^n f_j^{\otimes m}(x)\tau_j^{\otimes m} = bx$ for all $x \in \text{Sym}^m(\mathcal{X})$, then

$$\max_{1\leq j,k\leq n, j\neq k}\{|n|, |f_j(\tau_k)f_k(\tau_j)|^m\} \geq \frac{|n|^2}{\left|\binom{d+m-1}{m}\right|}. \tag{2}$$

We call Inequality (2) as p-adic functional Welch bounds.

We formulate non-Archimedean functional and p-adic functional Zauner conjectures.

Keywords: Non-Archimedean valued field, Non-Archimedean Banach space, p-adic number field, p-adic Banach space, Welch bound, Zauner conjecture.

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1. Introduction

Everything starts from the result Prof. L. Welch, obtained in 1974 [82].

Theorem 1.1. *[82] (**Welch Bounds**) Let $n > d$. If $\{\tau_j\}_{j=1}^n$ is any collection of unit vectors in $\mathbb{C}^d$, then*

$$\sum_{1\leq j,k\leq n} |\langle \tau_j, \tau_k\rangle|^{2m} = \sum_{j=1}^n\sum_{k=1}^n |\langle \tau_j, \tau_k\rangle|^{2m} \geq \frac{n^2}{\binom{d+m-1}{m}}, \quad \forall m \in \mathbb{N}.$$

In particular,

$$\sum_{1\leq j,k\leq n}|\langle \tau_j, \tau_k\rangle|^2=\sum_{j=1}^{n}\sum_{k=1}^{n}|\langle \tau_j, \tau_k\rangle|^2\geq \frac{n^2}{d}.$$

Further,

$$\textbf{\textit{(Higher order Welch bounds)}}\quad \max_{1\leq j,k\leq n, j\neq k}|\langle \tau_j, \tau_k\rangle|^{2m}\geq \frac{1}{n-1}\left[\frac{n}{\binom{d+m-1}{m}}-1\right],\quad \forall m \in \mathbb{N}.$$

In particular,

$$\textbf{\textit{(First order Welch bound)}}\quad \max_{1\leq j,k\leq n, j\neq k}|\langle \tau_j, \tau_k\rangle|^{2}\geq \frac{n-d}{d(n-1)}.$$

Theorem 1.1 is a powerful tool in many areas such as in the study of root-mean-square (RMS) absolute cross relation of unit vectors [69], frame potential [9, 14, 18], correlations [68], codebooks [27], numerical search algorithms [83, 84], quantum measurements [71], coding and communications [74, 78], code division multiple access (CDMA) systems [49, 50], wireless systems [66], compressed/compressive sensing [1, 6, 29, 32, 70, 76, 77, 79], 'game of Sloanes' [45], equiangular tight frames [75], equiangular lines [23, 31, 44, 59], digital fingerprinting [58] etc.

Theorem 1.1 has been upgraded/different proofs were given in [19, 24, 25, 28, 42, 67, 74, 80, 81]. In 2021 M. Krishna derived continuous version of Theorem 1.1 [51]. In 2022 M. Krishna obtained Theorem 1.1 for Hilbert C*-modules [53], Banach spaces [52], non-Archimedean Hilbert spaces [54] and p-adic Hilbert spaces [55].

In this paper we derive non-Archimedean (resp. p-adic) Banach space version of non-Archimedean (resp. p-adic) Welch bounds in Theorem 2.3 (resp. Theorem 3.2). We formulate non-Archimedean functional Zauner conjecture (Conjecture 2.5) and p-adic functional Zauner conjecture (Conjecture 3.4). We also formulate non-Archimedean functional equiangular line problem (Question 2.5) and p-adic functional equiangular line problem (Question 3.4).

2. Non-Archimedean Functional Welch bounds

In this section we derive non-Archimedean Banach space version of results derived in [54]. Let $\mathbb{K}$ be a non-Archimedean (complete) valued field satisfying

$$\left|\sum_{j=1}^{n}\lambda_j^2\right|=\max_{1\leq j\leq n}|\lambda_j|^2,\quad \forall \lambda_j \in \mathbb{K}, 1\leq j\leq n, \forall n \in \mathbb{N}. \tag{3}$$

For examples of such fields, we refer [63]. Throughout this section, we assume that our non-Archimedean field satisfies Equation (3). Letter $\mathcal{X}$ stands for a d-dimensional non-Archimedean Banach space over $\mathbb{K}$. Identity operator on $\mathcal{X}$ is denoted by $I_{\mathcal{X}}$. The dual of $\mathcal{X}$ is denoted by $\mathcal{X}^*$.

Theorem 2.1. ***(First Order Non-Archimedean Functional Welch Bound)*** *Let $\mathcal{X}$ be a d-dimensional non-Archimedean Banach space over $\mathbb{K}$. If $\{\tau_j\}_{j=1}^n$ is any collection in $\mathcal{X}$ and $\{f_j\}_{j=1}^n$ is any collection in $\mathcal{X}^*$ such that the operator $S_{f,\tau}:\mathcal{X}\ni x\mapsto \sum_{j=1}^{n}f_j(x)\tau_j\in \mathcal{X}$ is diagonalizable, then*

$$\max_{1\leq j,k\leq n, j\neq k}\left\{\left|\sum_{l=1}^{n}f_l(\tau_l)^2\right|, |f_j(\tau_k)f_k(\tau_j)|\right\}\geq \frac{1}{|d|}\left|\sum_{j=1}^{n}f_j(\tau_j)\right|^2.$$

In particular, if $f_j(\tau_j)=1$ for all $1\leq j\leq n$, then

(First order non-Archimedean functional Welch bound) $\displaystyle\max_{1\leq j,k\leq n, j\neq k}\{|n|,|f_j(\tau_k)f_k(\tau_j)|\}\geq \frac{|n|^2}{|d|}.$

Proof. We first note that

$$\operatorname{Tra}(S_{f,\tau})=\sum_{j=1}^{n}f_j(\tau_j),$$
$$\operatorname{Tra}(S_{f,\tau}^2)=\sum_{j=1}^{n}\sum_{k=1}^{n}f_j(\tau_k)f_k(\tau_j).$$

Let $\lambda_1,\dots,\lambda_d$ be the diagonal entries in the diagonalization of $S_{f,\tau}$. Then using the diagonalizability of $S_{f,\tau}$ and the non-Archimedean Cauchy-Schwarz inequality (Theorem 2.4.2 [63]), we get

$$\begin{aligned}\left|\sum_{j=1}^{n}f_j(\tau_j)\right|^2&=|\operatorname{Tra}(S_{f,\tau})|^2=\left|\sum_{k=1}^{d}\lambda_k\right|^2\leq|d|\left|\sum_{k=1}^{d}\lambda_k^2\right|=|d||\operatorname{Tra}(S_{f,\tau}^2)|\\&=|d|\left|\sum_{j=1}^{n}\sum_{k=1}^{n}f_j(\tau_k)f_k(\tau_j)\right|=|d|\left|\sum_{l=1}^{n}f_l(\tau_l)^2+\sum_{j,k=1,j\neq k}^{n}f_j(\tau_k)f_k(\tau_j)\right|\\&\leq|d|\max_{1\leq j,k\leq n,j\neq k}\left\{\left|\sum_{l=1}^{n}f_l(\tau_l)^2\right|,|f_j(\tau_k)f_k(\tau_j)|\right\}.\end{aligned}$$

Whenever $f_j(\tau_j)=1$ for all $1\leq j\leq n$,

$$|n|^2\leq|d|\max_{1\leq j,k\leq n,j\neq k}\{|n|,|f_j(\tau_k)f_k(\tau_j)|\}.$$

□

Next we obtain higher order non-Archimedean functional Welch bounds. We use the following vector space result.

Theorem 2.2. *[13, 21] If $\mathcal{V}$ is a vector space of dimension d and $Sym^m(\mathcal{V})$ denotes the vector space of symmetric m-tensors, then*

$$dim(Sym^m(\mathcal{V}))=\binom{d+m-1}{m},\quad\forall m\in\mathbb{N}.$$

Theorem 2.3. *(**Higher Order Non-Archimedean Functional Welch Bounds**) Let $\mathcal{X}$ be a d-dimensional non-Archimedean Banach space over $\mathbb{K}$. Let $m\in\mathbb{N}$. If $\{\tau_j\}_{j=1}^n$ is any collection in $\mathcal{X}$ and $\{f_j\}_{j=1}^n$ is any collection in $\mathcal{X}^*$ such that the operator $S_{f,\tau}: Sym^m(\mathcal{X})\ni x\mapsto\sum_{j=1}^n f_j^{\otimes m}(x)\tau_j^{\otimes m}\in Sym^m(\mathcal{X})$ is diagonalizable, then*

$$\max_{1\leq j,k\leq n,j\neq k}\left\{\left|\sum_{l=1}^{n}f_l(\tau_l)^{2m}\right|,|f_j(\tau_k)f_k(\tau_j)|^m\right\}\geq\frac{1}{\left|\binom{d+m-1}{m}\right|}\left|\sum_{j=1}^{n}f_j(\tau_j)^m\right|^2.$$

In particular, if $f_j(\tau_j)=1$ for all $1\leq j\leq n$, then

(Higher order non-Archimedean functional Welch bounds)

$$\max_{1\leq j,k\leq n,j\neq k}\{|n|,|f_j(\tau_k)f_k(\tau_j)|^m\}\geq\frac{|n|^2}{\left|\binom{d+m-1}{m}\right|}.$$

Proof. Let $\lambda_1, \dots, \lambda_{\dim(\mathrm{Sym}^m(\mathcal{X}))}$ be the diagonal entries in the diagonalization of $S_{f,\tau}$. We note that

$$b\dim(\mathrm{Sym^m}(\mathcal{X})) = \mathrm{Tra}(bI_{\mathrm{Sym}^m(\mathcal{X})}) = \mathrm{Tra}(S_{f,\tau}) = \sum_{j=1}^{n} f_j^{\otimes m}(\tau_j^{\otimes m}),$$

$$b^2\dim(\mathrm{Sym^m}(\mathcal{X})) = \mathrm{Tra}(b^2I_{\mathrm{Sym}^m(\mathcal{X})}) = \mathrm{Tra}(S_{f,\tau}^2) = \sum_{j=1}^{n}\sum_{k=1}^{n} f_j^{\otimes m}(\tau_k^{\otimes m})f_k^{\otimes m}(\tau_j^{\otimes m}).$$

Then

$$\begin{aligned}
\left|\sum_{j=1}^{n} f_j(\tau_j)^m\right|^2 &= \left|\sum_{j=1}^{n} f_j^{\otimes m}(\tau_j^{\otimes m})\right|^2 = |\operatorname{Tra}(S_{f,\tau})|^2 = \left|\sum_{k=1}^{\dim(\mathrm{Sym}^m(\mathcal{X}))}\lambda_k\right|^2\\
&\leq |\dim(\mathrm{Sym}^m(\mathcal{X}))|\left|\sum_{k=1}^{\dim(\mathrm{Sym}^m(\mathcal{X}))}\lambda_k^2\right| = |\dim(\mathrm{Sym}^m(\mathcal{X}))||\operatorname{Tra}(S_{f,\tau}^2)|\\
&= \left|\binom{d+m-1}{m}\right||\operatorname{Tra}(S_{f,\tau}^2)| = \left|\binom{d+m-1}{m}\right|\left|\sum_{j=1}^{n}\sum_{k=1}^{n} f_j^{\otimes m}(\tau_k^{\otimes m})f_k^{\otimes m}(\tau_j^{\otimes m})\right|\\
&= \left|\binom{d+m-1}{m}\right|\left|\sum_{j=1}^{n}\sum_{k=1}^{n} f_j(\tau_k)^m f_k(\tau_j)^m\right|\\
&= \left|\binom{d+m-1}{m}\right|\left|\sum_{l=1}^{n} f_l(\tau_l)^{2m} + \sum_{j,k=1, j\neq k}^{n} f_j(\tau_k)^m f_k(\tau_j)^m\right|\\
&\leq \left|\binom{d+m-1}{m}\right|\max_{1\leq j,k\leq n, j\neq k}\left\{\left|\sum_{l=1}^{n} f_l(\tau_l)^{2m}\right|, |f_j(\tau_k)^m f_k(\tau_j)^m|\right\}\\
&= \left|\binom{d+m-1}{m}\right|\max_{1\leq j,k\leq n, j\neq k}\left\{\left|\sum_{l=1}^{n} f_l(\tau_l)^{2m}\right|, |f_j(\tau_k) f_k(\tau_j)|^m\right\}.
\end{aligned}$$

Whenever $f_j(\tau_j) = 1$ for all $1 \leq j \leq n$,

$$|n|^2 \leq \left|\binom{d+m-1}{m}\right|\max_{1\leq j,k\leq n, j\neq k}\{|n|, |f_j(\tau_k)f_k(\tau_j)|^m\}.$$

$\square$

Motivated from 2.1 we formulate the following question.

Question 2.4. ***Let $\mathbb{K}$ non-Archimedean field satisfying Equation (3) and $\mathcal{X}$ be a d-dimensional non-Archimedean Banach space over $\mathbb{K}$. For which $n \in \mathbb{N}$, there exist vectors $\tau_1, \dots, \tau_n \in \mathcal{X}$ and functionals $f_1, \dots, f_n \in \mathcal{X}^*$ satisfying the following.***

- **(i)** $f_j(\tau_j) = 1$ ***for all*** $1 \leq j \leq n$.
- **(ii)** ***The operator*** $S_{f,\tau} : \mathcal{X} \ni x \mapsto \sum_{j=1}^{n} f_j(x)\tau_j \in \mathcal{X}$ ***is diagonalizable.***
- **(iii)**
$$\max_{1\leq j,k\leq n, j\neq k}\{|n|, |f_j(\tau_k)f_k(\tau_j)|\} = \frac{|n|^2}{|d|}.$$
- **(iv)** $\|f_j\| = 1$ ***for all*** $1 \leq j \leq n$, $\|\tau_j\| = 1$ ***for all*** $1 \leq j \leq n$.

A particular case of Question 2.4 is the following non-Archimedean functional version of Zauner conjecture which comes by taking $n = d^2$ (see [2–5, 10–12, 33, 36, 43, 48, 51, 57, 65, 72, 86] for Zauner conjecture in

Hilbert spaces, [53] for Zauner conjecture in Hilbert C*-modules, [52] for Zauner conjecture in Banach spaces, [54] for Zauner conjecture in non-Archimedean Hilbert spaces and [55] for Zauner conjecture in p-adic Hilbert spaces).

Conjecture 2.5. ***(Non-Archimedean Functional Zauner Conjecture) Let $\mathbb{K}$ non-Archimedean field satisfying Equation (3). For each $d \in \mathbb{N}$, there exist vectors $\tau_1, \dots, \tau_{d^2} \in \mathbb{K}^d$ (w.r.t. any non-Archimedean norm) and functionals $f_1, \dots, f_{d^2} \in (\mathbb{K}^d)^*$ satisfying the following.***

- **(i)** $f_j(\tau_j) = 1$ ***for all*** $1 \leq j \leq d^2$.
- **(ii)** ***The operator*** $S_{f,\tau} : \mathbb{K}^d \ni x \mapsto \sum_{j=1}^{d^2} f_j(x)\tau_j \in \mathbb{K}^d$ ***is diagonalizable.***
- **(iii)**
$$|f_j(\tau_k)f_k(\tau_j)| = |d|, \quad \forall 1 \leq j, k \leq d^2, j \neq k.$$
- **(iv)** $\|f_j\| = 1$ ***for all*** $1 \leq j \leq d^2$, $\|\tau_j\| = 1$ ***for all*** $1 \leq j \leq d^2$.

There are four bounds which are companions of Welch bounds in Hilbert spaces. To recall them we need the notion of Gerzon's bound.

Definition 2.6. *[45] Given $d \in \mathbb{N}$, define* ***Gerzon's bound***
$$\mathcal{Z}(d, \mathbb{K}) := \begin{cases} d^2 & \text{if} \quad \mathbb{K} = \mathbb{C} \\ \frac{d(d+1)}{2} & \text{if} \quad \mathbb{K} = \mathbb{R}. \end{cases}$$

Theorem 2.7. *[16, 22, 41, 45, 60, 64, 73, 83] Define $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $m := \dim_{\mathbb{R}}(\mathbb{K})/2$. If $\{\tau_j\}_{j=1}^n$ is any collection of unit vectors in $\mathbb{K}^d$, then*

(i) ***(Bukh-Cox bound)***
$$\max_{1 \leq j,k \leq n, j \neq k} |\langle \tau_j, \tau_k \rangle| \geq \frac{\mathcal{Z}(n-d, \mathbb{K})}{n(1 + m(n-d-1)\sqrt{m^{-1} + n - d}) - \mathcal{Z}(n-d, \mathbb{K})} \quad \text{if} \quad n > d.$$

(ii) ***(Orthoplex/Rankin bound)***
$$\max_{1 \leq j,k \leq n, j \neq k} |\langle \tau_j, \tau_k \rangle| \geq \frac{1}{\sqrt{d}} \quad \text{if} \quad n > \mathcal{Z}(d, \mathbb{K}).$$

(iii) ***(Levenstein bound)***
$$\max_{1 \leq j,k \leq n, j \neq k} |\langle \tau_j, \tau_k \rangle| \geq \sqrt{\frac{n(m+1) - d(md+1)}{(n-d)(md+1)}} \quad \text{if} \quad n > \mathcal{Z}(d, \mathbb{K}).$$

(iv) ***(Exponential bound)***
$$\max_{1 \leq j,k \leq n, j \neq k} |\langle \tau_j, \tau_k \rangle| \geq 1 - 2n^{\frac{-1}{d-1}}.$$

Motivated from Theorem 2.7 and Theorem 2.1 we ask the following problem.

Question 2.8. ***Whether there is a non-Archimedean functional version of Theorem 2.7? In particular, does there exists a version of***

(i) ***non-Archimedean functional Bukh-Cox bound?***
(ii) ***non-Archimedean functional Orthoplex/Rankin bound?***
(iii) ***non-Archimedean functional Levenstein bound?***
(iv) ***non-Archimedean functional Exponential bound?***

In the introduction we wrote that Welch bounds have applications in study of equiangular lines. Therefore wish to formulate equiangular line problem for non-Archimedean Banach spaces. For the study of equiangular lines in Hilbert spaces we refer [7, 8, 15, 17, 26, 34, 35, 37, 40, 46, 47, 56, 61, 62, 85], quaternion Hilbert space we refer [30], octonion Hilbert space we refer [20], finite dimensional vector spaces over finite fields we refer [38, 39], for Banach spaces we refer [52], for non-Archimedean Hilbert spaces we refer [54] and for p-adic Hilbert spaces we refer [55].

Question 2.9. ***(Non-Archimedean Functional Equiangular Line Problem) Let*** $\mathbb{K}$ ***be a non-Archimedean field satisfying Equation (3). Given*** $a \in \mathbb{K}$, $d \in \mathbb{N}$ ***and*** $\gamma > 0$, ***what is the maximum*** $n = n(\mathbb{K}, a, d, \gamma) \in \mathbb{N}$ ***such that there exist vectors*** $\tau_1, \dots, \tau_n \in \mathbb{K}^d$ ***(w.r.t. any non-Archimedean norm) and functionals*** $f_1, \dots, f_n \in (\mathbb{K}^d)^*$ ***satisfying the following.***

(i) $f_j(\tau_j) = a$ ***for all*** $1 \leq j \leq n$.
(ii) $|f_j(\tau_k) f_k(\tau_j)| = \gamma$ ***for all*** $1 \leq j, k \leq n, j \neq k$.
(iii) $\|f_j\| = 1$ ***for all*** $1 \leq j \leq n$, $\|\tau_j\| = 1$ ***for all*** $1 \leq j \leq n$.

In particular, whether there is a non-Archimedean functional Gerzon bound?

Question 2.9 can be easily generalized to formulate question of non-Archimedean functional regular s-distance sets.

3. p-adic Functional Welch bounds

In this section we derive p-adic Banach space version of results done in [55]. Let p be a prime and $\mathbb{Q}_p$ be the filed of p-adic numbers. In this section, $\mathcal{X}$ is a d-dimensional p-adic Banach space over $\mathbb{Q}_p$.

Theorem 3.1. ***(First Order p-adic Functional Welch Bound)*** *Let p be a prime and $\mathcal{X}$ be a d-dimensional p-adic Banach space over $\mathbb{Q}_p$. If $\{\tau_j\}_{j=1}^n$ is any collection in $\mathcal{X}$ and $\{f_j\}_{j=1}^n$ is any collection in $\mathcal{X}^*$ such that there exists $b \in \mathbb{Q}_p$ satisfying*

$$\sum_{j=1}^{n} f_j(x)\tau_j = bx, \quad \forall x \in \mathcal{X},$$

then

$$\max_{1 \leq j,k \leq n, j \neq k} \left\{ \left| \sum_{l=1}^{n} f_l(\tau_l)^2 \right|, |f_j(\tau_k) f_k(\tau_j)| \right\} \geq \frac{1}{|d|} \left| \sum_{j=1}^{n} f_j(\tau_j) \right|^2.$$

In particular, if $f_j(\tau_j) = 1$ for all $1 \leq j \leq n$, then

$$\textbf{(First order p-adic functional Welch bound)} \quad \max_{1 \leq j,k \leq n, j \neq k} \{|n|, |f_j(\tau_k) f_k(\tau_j)|\} \geq \frac{|n|^2}{|d|}.$$

Proof. Define $S_{f,\tau} : \mathcal{X} \ni x \mapsto \sum_{j=1}^{n} f_j(x)\tau_j \in \mathcal{X}$. Then

$$bd = \text{Tra}(bI_{\mathcal{X}}) = \text{Tra}(S_{f,\tau}) = \sum_{j=1}^{n} f_j(\tau_j),$$

$$b^2 d = \text{Tra}(b^2 I_{\mathcal{X}}) = \text{Tra}(S_{f,\tau}^2) = \sum_{j=1}^{n} \sum_{k=1}^{n} f_j(\tau_k) f_k(\tau_j).$$

Therefore

$$\begin{aligned}
\left|\sum_{j=1}^{n}f_j(\tau_j)\right|^2 &= |\operatorname{Tra}(S_{f,\tau})|^2=|bd|^2=|d||b^2d|=|d|\left|\sum_{j=1}^{n}\sum_{k=1}^{n}f_j(\tau_k)f_k(\tau_j)\right|\\
&=|d|\left|\sum_{l=1}^{n}f_l(\tau_l)^2+\sum_{j,k=1,j\neq k}^{n}f_j(\tau_k)f_k(\tau_j)\right|\\
&\leq |d|\max_{1\leq j,k\leq n,j\neq k}\left\{\left|\sum_{l=1}^{n}f_l(\tau_l)^2\right|,|f_j(\tau_k)f_k(\tau_j)|\right\}.
\end{aligned}$$

Whenever $f_j(\tau_j)=1$ for all $1\leq j\leq n$,

$$|n|^2\leq |d|\max_{1\leq j,k\leq n,j\neq k}\{|n|,|f_j(\tau_k)f_k(\tau_j)|\}.$$

□

We derive higher order version of Theorem 3.1.

Theorem 3.2. *(**Higher Order p-adic Functional Welch Bounds**) Let p be a prime and $\mathcal{X}$ be a d-dimensional p-adic Banach space over $\mathbb{Q}_p$. If $\{\tau_j\}_{j=1}^n$ is any collection in $\mathcal{X}$ and $\{f_j\}_{j=1}^n$ is any collection in $\mathcal{X}^*$ such that there exists $b\in\mathbb{Q}_p$ satisfying*

$$\sum_{j=1}^{n}f_j^{\otimes m}(x)\tau_j^{\otimes m}=bx,\quad \forall x\in \mathit{Sym}^m(\mathcal{X}),$$

then

$$\max_{1\leq j,k\leq n,j\neq k}\left\{\left|\sum_{l=1}^{n}f_l(\tau_l)^{2m}\right|,|f_j(\tau_k)f_k(\tau_j)|^m\right\}\geq \frac{1}{\left|\binom{d+m-1}{m}\right|}\left|\sum_{j=1}^{n}f_j(\tau_j)^m\right|^2.$$

In particular, if $f_j(\tau_j)=1$ for all $1\leq j\leq n$, then

$$\textbf{\textit{(Higher order p-adic functional Welch bound)}}\quad \max_{1\leq j,k\leq n,j\neq k}\{|n|,|f_j(\tau_k)f_k(\tau_j)|^m\}\geq\frac{|n|^2}{\left|\binom{d+m-1}{m}\right|}.$$

Proof. Define $S_{f,\tau}:\mathrm{Sym}^m(\mathcal{X})\ni x\mapsto\sum_{j=1}^{n}f_j^{\otimes m}(x)\tau_j^{\otimes m}\in\mathrm{Sym}^m(\mathcal{X})$. Then

$$b\dim(\mathrm{Sym^m}(\mathcal{X}))=\operatorname{Tra}(bI_{\mathrm{Sym}^m(\mathcal{X})})=\operatorname{Tra}(S_{f,\tau})=\sum_{j=1}^{n}f_j^{\otimes m}(\tau_j^{\otimes m}),$$

$$b^2\dim(\mathrm{Sym^m}(\mathcal{X}))=\operatorname{Tra}(b^2I_{\mathrm{Sym}^m(\mathcal{X})})=\operatorname{Tra}(S_{f,\tau}^2)=\sum_{j=1}^{n}\sum_{k=1}^{n}f_j^{\otimes m}(\tau_k^{\otimes m})f_k^{\otimes m}(\tau_j^{\otimes m}).$$

Therefore

$$\begin{aligned}
\left|\sum_{j=1}^{n} f_j(\tau_j)^m\right|^2 &= \left|\sum_{j=1}^{n} f_j^{\otimes m}(\tau_j^{\otimes m})\right|^2 = |\operatorname{Tra}(S_{f,\tau})|^2 = |b\dim(\mathrm{Sym^m}(\mathcal{X}))|^2\\
&= |\dim(\mathrm{Sym^m}(\mathcal{X}))|\left|b^2\dim(\mathrm{Sym^m}(\mathcal{X}))\right|\\
&= |\dim(\mathrm{Sym^m}(\mathcal{X}))|\left|\sum_{j=1}^{n}\sum_{k=1}^{n} f_j^{\otimes m}(\tau_k^{\otimes m})f_k^{\otimes m}(\tau_j^{\otimes m})\right|\\
&= \left|\binom{d+m-1}{m}\right|\left|\sum_{j=1}^{n}\sum_{k=1}^{n} f_j^{\otimes m}(\tau_k^{\otimes m})f_k^{\otimes m}(\tau_j^{\otimes m})\right|\\
&= \left|\binom{d+m-1}{m}\right|\left|\sum_{j=1}^{n}\sum_{k=1}^{n} f_j(\tau_k)^m f_k(\tau_j)^m\right|\\
&= \left|\binom{d+m-1}{m}\right|\left|\sum_{l=1}^{n} f_l(\tau_l)^{2m} + \sum_{j,k=1, j\neq k}^{n} f_j(\tau_k)^m f_k(\tau_j)^m\right|\\
&\leq \left|\binom{d+m-1}{m}\right|\max_{1\leq j,k\leq n, j\neq k}\left\{\left|\sum_{l=1}^{n} f_l(\tau_l)^{2m}\right|, |f_j(\tau_k)^m f_k(\tau_j)^m|\right\}\\
&= \left|\binom{d+m-1}{m}\right|\max_{1\leq j,k\leq n, j\neq k}\left\{\left|\sum_{l=1}^{n} f_l(\tau_l)^{2m}\right|, |f_j(\tau_k)f_k(\tau_j)|^m\right\}.
\end{aligned}$$

Whenever $f_j(\tau_j)=1$ for all $1\leq j\leq n$,

$$|n|^2 \leq \left|\binom{d+m-1}{m}\right|\max_{1\leq j,k\leq n, j\neq k}\{|n|, |f_j(\tau_k)f_k(\tau_j)|^m\}.$$

□

Difference between Theorem 2.3 and Theorem 3.2 should be clearly emphasized. Assumption in (at vector space level) Theorem 3.2 is more than the assumption in Theorem 2.3 (as any scalar times identity is already diagonal) but the field in Theorem 2.3 is much more restrictive than the field in Theorem 3.2. Theorem 3.2 works on any non-Archimedean field not just $\mathbb{Q}_p$. Using Theorem 3.1 we ask the following question.

Question 3.3. ***Given a prime*** p***, for which*** d***-dimensional p-adic Banach space*** $\mathcal{X}$ ***over*** $\mathbb{Q}_p$ ***and*** $n\in\mathbb{N}$***, there exist vectors*** $\tau_1,\dots,\tau_n\in\mathcal{X}$ ***and functionals*** $f_1,\dots,f_n\in\mathcal{X}^*$ ***satisfying the following.***

- **(i)** $f_j(\tau_j)=1$ ***for all*** $1\leq j\leq n$***.***
- **(ii)** ***There exists*** $b\in\mathbb{Q}_p$ ***satisfying***

$$\sum_{j=1}^{n} f_j(x)\tau_j = bx, \quad \forall x\in\mathcal{X},$$

- **(iii)**

$$\max_{1\leq j,k\leq n, j\neq k}\{|n|, |f_j(\tau_k)f_k(\tau_j)|\} = \frac{|n|^2}{|d|}.$$

- **(iv)** $\|f_j\|=1$ ***for all*** $1\leq j\leq n$***,*** $\|\tau_j\|=1$ ***for all*** $1\leq j\leq n$***.***

A particular case of Question 3.3 is the following p-adic functional Zauner conjecture.

Conjecture 3.4. ***(p-adic Functional Zauner Conjecture) Let*** p ***be a prime. For each*** $d \in \mathbb{N}$***, there exist vectors*** $\tau_1, \dots, \tau_{d^2} \in \mathbb{Q}_p^d$ ***(w.r.t. any non-Archimedean norm) and functionals*** $f_1, \dots, f_n \in (\mathbb{Q}_p^d)^*$ ***satisfying the following.***

- **(i)** $f_j(\tau_j) = 1$ ***for all*** $1 \leq j \leq d^2$***.***
- **(ii)** ***There exists*** $b \in \mathbb{Q}_p$ ***satisfying***

$$\sum_{j=1}^{d^2} f_j(x)\tau_j = bx, \quad \forall x \in \mathcal{X},$$

- **(iii)**

$$|f_j(\tau_k)f_k(\tau_j)| = |d|, \quad \forall 1 \leq j,k \leq d^2, j \neq k.$$

- **(iv)** $\|f_j\| = 1$ ***for all*** $1 \leq j \leq d^2$***,*** $\|\tau_j\| = 1$ ***for all*** $1 \leq j \leq d^2$***.***

Theorem 2.7 and Theorem 3.1 give the following problem.

Question 3.5. ***Whether there is a p-adic functional version of Theorem 2.7? In particular, does there exists a version of***

- (i) ***p-adic functional Bukh-Cox bound?***
- (ii) ***p-adic functional Orthoplex/Rankin bound?***
- (iii) ***p-adic functional Levenstein bound?***
- (iv) ***p-adic functional Exponential bound?***

We end by formulating p-adic functional equiangular line problem.

Question 3.6. ***(p-adic Functional Equiangular Line Problem) Let*** p ***be a prime. Given*** $a \in \mathbb{Q}_p$***,*** $d \in \mathbb{N}$ ***and*** $\gamma > 0$***, what is the maximum*** $n = n(p, a, d, \gamma) \in \mathbb{N}$ ***such that there exist vectors*** $\tau_1, \dots, \tau_n \in \mathbb{Q}_p^d$ ***(w.r.t. any non-Archimedean norm) and functionals*** $f_1, \dots, f_n \in (\mathbb{Q}_p^d)^*$ ***satisfying the following.***

- **(i)** $f_j(\tau_j) = a$ ***for all*** $1 \leq j \leq n$***.***
- **(ii)** $|f_j(\tau_k)f_k(\tau_j)| = \gamma$ ***for all*** $1 \leq j,k \leq n, j \neq k$***.***
- **(iii)** $\|f_j\| = 1$ ***for all*** $1 \leq j \leq n$***,*** $\|\tau_j\| = 1$ ***for all*** $1 \leq j \leq n$***.***

In particular, whether there is a p-adic functional Gerzon bound?

Question 3.6 can be easily reformulated to formulate question of p-adic functional regular s-distance sets.

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