

p-ADIC WELCH BOUNDS AND p-ADIC ZAUNER CONJECTURE

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Abstract: Let p be a prime. For $d \in \mathbb{N}$, let $\mathbb{Q}_p^d$ be the standard d -dimensional p -adic Hilbert space. Let $m \in \mathbb{N}$ and $\text{Sym}^m(\mathbb{Q}_p^d)$ be the p -adic Hilbert space of symmetric m -tensors. We prove the following result. Let $\{\tau_j\}_{j=1}^n$ be a collection in $\mathbb{Q}_p^d$ satisfying (i) $\langle \tau_j, \tau_j \rangle = 1$ for all $1 \leq j \leq n$ and (ii) there exists $b \in \mathbb{Q}_p$ satisfying $\sum_{j=1}^n \langle x, \tau_j \rangle \tau_j = bx$ for all $x \in \mathbb{Q}_p^d$. Then

$$(1) \quad \max_{1 \leq j, k \leq n, j \neq k} \{ |n|, |\langle \tau_j, \tau_k \rangle|^{2m} \} \geq \frac{|n|^2}{\left| \binom{d+m-1}{m} \right|}.$$

We call Inequality (1) as the p -adic version of Welch bounds obtained by Welch [*IEEE Transactions on Information Theory*, 1974]. Inequality (1) differs from the non-Archimedean Welch bound obtained recently by M. Krishna as one can not derive one from another. We formulate p -adic Zauner conjecture.

Keywords: p -adic number field, p -adic Hilbert space, Welch bound, Zauner conjecture.

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1. INTRODUCTION

In 1974 Prof. L. Welch proved the following result [86].

Theorem 1.1. [86] (**Welch Bounds**) Let $n > d$. If $\{\tau_j\}_{j=1}^n$ is any collection of unit vectors in $\mathbb{C}^d$, then

$$\sum_{1 \leq j, k \leq n} |\langle \tau_j, \tau_k \rangle|^{2m} = \sum_{j=1}^n \sum_{k=1}^n |\langle \tau_j, \tau_k \rangle|^{2m} \geq \frac{n^2}{\binom{d+m-1}{m}}, \quad \forall m \in \mathbb{N}.$$

In particular,

$$\sum_{1 \leq j, k \leq n} |\langle \tau_j, \tau_k \rangle|^2 = \sum_{j=1}^n \sum_{k=1}^n |\langle \tau_j, \tau_k \rangle|^2 \geq \frac{n^2}{d}.$$

Further,

$$(\textbf{Higher order Welch bounds}) \quad \max_{1 \leq j, k \leq n, j \neq k} |\langle \tau_j, \tau_k \rangle|^{2m} \geq \frac{1}{n-1} \left[\frac{n}{\binom{d+m-1}{m}} - 1 \right], \quad \forall m \in \mathbb{N}.$$

In particular,

$$(\textbf{First order Welch bound}) \quad \max_{1 \leq j, k \leq n, j \neq k} |\langle \tau_j, \tau_k \rangle|^2 \geq \frac{n-d}{d(n-1)}.$$

It is impossible to list all applications of Theorem 1.1. A few are: in the study of root-mean-square (RMS) absolute cross relation of unit vectors [73], frame potential [10, 15, 19], correlations [72], codebooks [30],

numerical search algorithms [87, 88], quantum measurements [75], coding and communications [78, 82], code division multiple access (CDMA) systems [54, 55], wireless systems [70], compressed/compressive sensing [2, 7, 32, 35, 74, 80, 81, 83], ‘game of Sloanes’ [48], equiangular tight frames [79], equiangular lines [24, 34, 47, 63], digital fingerprinting [62] etc.

Theorem 1.1 has been improved/different proofs were given in [20, 25, 26, 31, 45, 71, 78, 84, 85]. In 2021 M. Krishna derived continuous version of Theorem 1.1 [56]. In 2022 M. Krishna obtained Theorem 1.1 for Hilbert C^* -modules [58], Banach spaces [57] and non-Archimedean Hilbert spaces [59].

In this paper we derive p -adic Welch bounds (Theorem 2.3). We formulate p -adic Zauner conjecture (Conjecture 3.3).

2. P -ADIC WELCH BOUNDS

Let p be a prime. For $d \in \mathbb{N}$, let $\mathbb{Q}_p^d$ be the standard p -adic Hilbert space equipped with the inner product

$$\langle (a_j)_{j=1}^d, (b_j)_{j=1}^d \rangle := \sum_{j=1}^d a_j b_j, \quad \forall (a_j)_{j=1}^d, (b_j)_{j=1}^d \in \mathbb{Q}_p^d.$$

Let $I_{\mathbb{Q}_p^d}$ be the identity operator on $\mathbb{Q}_p^d$. Note that $\mathbb{Q}_p^d$ is not a non-Archimedean Hilbert space as it does not satisfies Equation (2) in [59] (see Page 40, [67]). We refer [1, 28, 29, 51, 52] for more on p -adic Hilbert spaces.

Theorem 2.1. (First Order p -adic Welch Bound) *Let p be a prime and $n, d \in \mathbb{N}$. If $\{\tau_j\}_{j=1}^n$ is any collection in $\mathbb{Q}_p^d$ such that there exists $b \in \mathbb{Q}_p$ satisfying*

$$\sum_{j=1}^n \langle x, \tau_j \rangle \tau_j = bx, \quad \forall x \in \mathbb{Q}_p^d,$$

then

$$\max_{1 \leq j, k \leq n, j \neq k} \left\{ \left| \sum_{l=1}^n \langle \tau_l, \tau_l \rangle^2 \right|, |\langle \tau_j, \tau_k \rangle|^2 \right\} \geq \frac{1}{|d|} \left| \sum_{j=1}^n \langle \tau_j, \tau_j \rangle \right|^2.$$

In particular, if $\langle \tau_j, \tau_j \rangle = 1$ for all $1 \leq j \leq n$, then

$$\textbf{(First order } p\text{-adic Welch bound)} \quad \max_{1 \leq j, k \leq n, j \neq k} \{ |n|, |\langle \tau_j, \tau_k \rangle|^2 \} \geq \frac{|n|^2}{|d|}.$$

Proof. Define $S_\tau : \mathbb{Q}_p^d \ni x \mapsto \sum_{j=1}^n \langle x, \tau_j \rangle \tau_j \in \mathbb{Q}_p^d$. Then

$$\begin{aligned} bd &= \text{Tra}(bI_{\mathbb{Q}_p^d}) = \text{Tra}(S_\tau) = \sum_{j=1}^n \langle \tau_j, \tau_j \rangle, \\ b^2d &= \text{Tra}(b^2I_{\mathbb{Q}_p^d}) = \text{Tra}(S_\tau^2) = \sum_{j=1}^n \sum_{k=1}^n \langle \tau_j, \tau_k \rangle \langle \tau_k, \tau_j \rangle. \end{aligned}$$

Therefore

$$\begin{aligned}
\left| \sum_{j=1}^n \langle \tau_j, \tau_j \rangle \right|^2 &= |\text{Tra}(S_\tau)|^2 = |bd|^2 = |d||b^2d| = |d| \left| \sum_{j=1}^n \sum_{k=1}^n \langle \tau_j, \tau_k \rangle \langle \tau_k, \tau_j \rangle \right| \\
&= |d| \left| \sum_{l=1}^n \langle \tau_l, \tau_l \rangle^2 + \sum_{j,k=1, j \neq k}^n \langle \tau_j, \tau_k \rangle \langle \tau_k, \tau_j \rangle \right| \\
&\leq |d| \max_{1 \leq j, k \leq n, j \neq k} \left\{ \left| \sum_{l=1}^n \langle \tau_l, \tau_l \rangle^2 \right|, |\langle \tau_j, \tau_k \rangle \langle \tau_k, \tau_j \rangle| \right\} \\
&= |d| \max_{1 \leq j, k \leq n, j \neq k} \left\{ \left| \sum_{l=1}^n \langle \tau_l, \tau_l \rangle^2 \right|, |\langle \tau_j, \tau_k \rangle|^2 \right\}.
\end{aligned}$$

Whenever $\langle \tau_j, \tau_j \rangle = 1$ for all $1 \leq j \leq n$,

$$|n|^2 \leq |d| \max_{1 \leq j, k \leq n, j \neq k} \{ |n|, |\langle \tau_j, \tau_k \rangle|^2 \}.$$

□

We next derive higher order p-adic Welch bounds. For this, we need the following result.

Theorem 2.2. [14, 22] *If $\mathcal{V}$ is a vector space of dimension d and $\text{Sym}^m(\mathcal{V})$ denotes the vector space of symmetric m -tensors, then*

$$\dim(\text{Sym}^m(\mathcal{V})) = \binom{d+m-1}{m}, \quad \forall m \in \mathbb{N}.$$

Theorem 2.3. (Higher Order p-adic Welch Bounds) *Let p be a prime and $n, d, m \in \mathbb{N}$. If $\{\tau_j\}_{j=1}^n$ is any collection in $\mathbb{Q}_p^d$ such that there exists $b \in \mathbb{Q}_p$ satisfying*

$$\sum_{j=1}^n \langle x, \tau_j^{\otimes m} \rangle \tau_j^{\otimes m} = bx, \quad \forall x \in \text{Sym}^m(\mathbb{Q}_p^d),$$

then

$$\max_{1 \leq j, k \leq n, j \neq k} \left\{ \left| \sum_{l=1}^n \langle \tau_l, \tau_l \rangle^{2m} \right|, |\langle \tau_j, \tau_k \rangle|^{2m} \right\} \geq \frac{1}{\left| \binom{d+m-1}{m} \right|} \left| \sum_{j=1}^n \langle \tau_j, \tau_j \rangle^m \right|^2.$$

In particular, if $\langle \tau_j, \tau_j \rangle = 1$ for all $1 \leq j \leq n$, then

$$(\text{Higher order p-adic Welch bound}) \quad \max_{1 \leq j, k \leq n, j \neq k} \{ |n|, |\langle \tau_j, \tau_k \rangle|^{2m} \} \geq \frac{|n|^2}{\left| \binom{d+m-1}{m} \right|}.$$

Proof. Define $S_\tau : \text{Sym}^m(\mathbb{Q}_p^d) \ni x \mapsto \sum_{j=1}^n \langle x, \tau_j^{\otimes m} \rangle \tau_j^{\otimes m} \in \text{Sym}^m(\mathbb{Q}_p^d)$. Then

$$b \dim(\text{Sym}^m(\mathbb{Q}_p^d)) = \text{Tra}(bI_{\text{Sym}^m(\mathbb{Q}_p^d)}) = \text{Tra}(S_\tau) = \sum_{j=1}^n \langle \tau_j^{\otimes m}, \tau_j^{\otimes m} \rangle,$$

$$b^2 \dim(\text{Sym}^m(\mathbb{Q}_p^d)) = \text{Tra}(b^2 I_{\text{Sym}^m(\mathbb{Q}_p^d)}) = \text{Tra}(S_\tau^2) = \sum_{j=1}^n \sum_{k=1}^n \langle \tau_j^{\otimes m}, \tau_k^{\otimes m} \rangle \langle \tau_k^{\otimes m}, \tau_j^{\otimes m} \rangle.$$

Therefore

$$\begin{aligned}
 & \left| \sum_{j=1}^n \langle \tau_j, \tau_j \rangle^m \right|^2 = \left| \sum_{j=1}^n \langle \tau_j^{\otimes m}, \tau_j^{\otimes m} \rangle \right|^2 = |\text{Tra}(S_\tau)|^2 = |b \dim(\text{Sym}^m(\mathbb{Q}_p^d))|^2 \\
 & = |\dim(\text{Sym}^m(\mathbb{Q}_p^d))| |b^2 \dim(\text{Sym}^m(\mathbb{Q}_p^d))| \\
 & = |\dim(\text{Sym}^m(\mathbb{Q}_p^d))| \left| \sum_{j=1}^n \sum_{k=1}^n \langle \tau_j^{\otimes m}, \tau_k^{\otimes m} \rangle \langle \tau_k^{\otimes m}, \tau_j^{\otimes m} \rangle \right| \\
 & = \left| \binom{d+m-1}{m} \right| \left| \sum_{j=1}^n \sum_{k=1}^n \langle \tau_j^{\otimes m}, \tau_k^{\otimes m} \rangle \langle \tau_k^{\otimes m}, \tau_j^{\otimes m} \rangle \right| \\
 & = \left| \binom{d+m-1}{m} \right| \left| \sum_{j=1}^n \sum_{k=1}^n \langle \tau_j, \tau_k \rangle^m \langle \tau_k, \tau_j \rangle^m \right| \\
 & = \left| \binom{d+m-1}{m} \right| \left| \sum_{l=1}^n \langle \tau_l, \tau_l \rangle^{2m} + \sum_{j,k=1, j \neq k}^n \langle \tau_j, \tau_k \rangle^m \langle \tau_k, \tau_j \rangle^m \right| \\
 & \leq \left| \binom{d+m-1}{m} \right| \max_{1 \leq j, k \leq n, j \neq k} \left\{ \left| \sum_{l=1}^n \langle \tau_l, \tau_l \rangle^{2m} \right|, |\langle \tau_j, \tau_k \rangle^m \langle \tau_k, \tau_j \rangle^m| \right\} \\
 & = \left| \binom{d+m-1}{m} \right| \max_{1 \leq j, k \leq n, j \neq k} \left\{ \left| \sum_{l=1}^n \langle \tau_l, \tau_l \rangle^{2m} \right|, |\langle \tau_j, \tau_k \rangle|^{2m} \right\}.
 \end{aligned}$$

Whenever $\langle \tau_j, \tau_j \rangle = 1$ for all $1 \leq j \leq n$,

$$|n|^2 \leq \left| \binom{d+m-1}{m} \right| \max_{1 \leq j, k \leq n, j \neq k} \{ |n|, |\langle \tau_j, \tau_k \rangle|^{2m} \}.$$

□

Remark 2.4. Conditions given in the Theorem 2.3 says that the operator S_τ in the proof of Theorem 2.3 is diagonalizable. Thus Theorem 2.3 is restrictive as the hypothesis is stronger than that of Theorem 2.3 in [59]. However, note that the field $\mathbb{Q}_p$ does not satisfies the Equation (2) in [59] (see [67]) and hence neither the results in this paper can be derived from the results in [59] nor the results in [59] can be derived from the results in this paper.

Remark 2.5. Theorems 2.1 and 2.3 hold by replacing $\mathbb{Q}_p^d$ by a d -dimensional p -adic Hilbert space over any non-Archimedean (complete) valued field (such as $\mathbb{C}_p$).

3. P-ADIC ZAUNER CONJECTURE AND OPEN PROBLEMS

Using Theorem 2.1 we ask the following question.

Question 3.1. *Given a prime p , for which $(d, n) \in \mathbb{N} \times \mathbb{N}$, there exist vectors $\tau_1, \dots, \tau_n \in \mathbb{Q}_p^d$ satisfying the following.*

- (i) $\langle \tau_j, \tau_j \rangle = 1$ for all $1 \leq j \leq n$.
- (ii) There exists $b \in \mathbb{Q}_p$ satisfying

$$\sum_{j=1}^n \langle x, \tau_j \rangle \tau_j = bx, \quad \forall x \in \mathbb{Q}_p^d.$$

(iii)

$$\max_{1 \leq j, k \leq n, j \neq k} \{ |n|, |\langle \tau_j, \tau_k \rangle|^2 \} = \frac{|n|^2}{|d|}.$$

We can formulate a strong form of Question 3.1 as follows.

Question 3.2. *Given a prime p , for which $(d, n) \in \mathbb{N} \times \mathbb{N}$, there exist vectors $\tau_1, \dots, \tau_n \in \mathbb{Q}_p^d$ satisfying the following.*

(i) $\langle \tau_j, \tau_j \rangle = 1$ for all $1 \leq j \leq n$.

(ii) There exists $b \in \mathbb{Q}_p$ satisfying

$$\sum_{j=1}^n \langle x, \tau_j \rangle \tau_j = bx, \quad \forall x \in \mathbb{Q}_p^d.$$

(iii)

$$\max_{1 \leq j, k \leq n, j \neq k} \{ |n|, |\langle \tau_j, \tau_k \rangle|^2 \} = \frac{|n|^2}{|d|}.$$

(iv) $\|\tau_j\| = 1$ for all $1 \leq j \leq n$.

Why Question 3.2 is different than Question 3.1? Reason is that unlike non-Archimedean Hilbert spaces, in p-adic Hilbert spaces, norm is not defined as $\sqrt{|\langle \cdot, \cdot \rangle|}$. A particular case of Question 3.1 is the following p-adic version of Zauner conjecture (see [3–6, 11–13, 36, 39, 46, 53, 56, 61, 69, 76, 90] for Zauner conjecture in Hilbert spaces, [58] for Zauner conjecture in Hilbert C*-modules, [57] for Zauner conjecture in Banach spaces and [59] for Zauner conjecture in non-Archimedean Hilbert spaces).

Conjecture 3.3. (*p*-adic Zauner Conjecture) *Let p be a prime. For each $d \in \mathbb{N}$, there exist vectors $\tau_1, \dots, \tau_{d^2} \in \mathbb{Q}_p^d$ satisfying the following.*

(i) $\langle \tau_j, \tau_j \rangle = 1$ for all $1 \leq j \leq d^2$.

(ii) There exists $b \in \mathbb{Q}_p$ satisfying

$$\sum_{j=1}^{d^2} \langle x, \tau_j \rangle \tau_j = bx, \quad \forall x \in \mathbb{Q}_p^d.$$

(iii)

$$|\langle \tau_j, \tau_k \rangle|^2 = |n|, \quad \forall 1 \leq j, k \leq d^2, j \neq k.$$

Question 3.2 gives the following Zauner conjecture.

Conjecture 3.4. (*p*-adic Zauner Conjecture - strong form) *Let p be a prime. For each $d \in \mathbb{N}$, there exist vectors $\tau_1, \dots, \tau_{d^2} \in \mathbb{Q}_p^d$ satisfying the following.*

(i) $\langle \tau_j, \tau_j \rangle = 1$ for all $1 \leq j \leq d^2$.

(ii) There exists $b \in \mathbb{Q}_p$ satisfying

$$\sum_{j=1}^{d^2} \langle x, \tau_j \rangle \tau_j = bx, \quad \forall x \in \mathbb{Q}_p^d.$$

(iii)

$$|\langle \tau_j, \tau_k \rangle|^2 = |n|, \quad \forall 1 \leq j, k \leq d^2, j \neq k.$$

(iv) $\|\tau_j\| = 1$ for all $1 \leq j \leq d^2$.

We recall the definition of Gerzon's bound which allows us to remember companions to Welch bounds in Hilbert spaces.

Definition 3.5. [48] Given $d \in \mathbb{N}$, define **Gerzon's bound**

$$\mathcal{Z}(d, \mathbb{K}) := \begin{cases} d^2 & \text{if } \mathbb{K} = \mathbb{C} \\ \frac{d(d+1)}{2} & \text{if } \mathbb{K} = \mathbb{R}. \end{cases}$$

Theorem 3.6. [17, 23, 44, 48, 64, 68, 77, 87] Define $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $m := \dim_{\mathbb{R}}(\mathbb{K})/2$. If $\{\tau_j\}_{j=1}^n$ is any collection of unit vectors in $\mathbb{K}^d$, then

(i) (**Bukh-Cox bound**)

$$\max_{1 \leq j, k \leq n, j \neq k} |\langle \tau_j, \tau_k \rangle| \geq \frac{\mathcal{Z}(n-d, \mathbb{K})}{n(1+m(n-d-1)\sqrt{m^{-1}+n-d}) - \mathcal{Z}(n-d, \mathbb{K})} \quad \text{if } n > d.$$

(ii) (**Orthoplex/Rankin bound**)

$$\max_{1 \leq j, k \leq n, j \neq k} |\langle \tau_j, \tau_k \rangle| \geq \frac{1}{\sqrt{d}} \quad \text{if } n > \mathcal{Z}(d, \mathbb{K}).$$

(iii) (**Levenstein bound**)

$$\max_{1 \leq j, k \leq n, j \neq k} |\langle \tau_j, \tau_k \rangle| \geq \sqrt{\frac{n(m+1) - d(md+1)}{(n-d)(md+1)}} \quad \text{if } n > \mathcal{Z}(d, \mathbb{K}).$$

(iv) (**Exponential bound**)

$$\max_{1 \leq j, k \leq n, j \neq k} |\langle \tau_j, \tau_k \rangle| \geq 1 - 2n^{\frac{-1}{d-1}}.$$

Theorem 3.6 and Theorem 2.1 give the following problem.

Question 3.7. *Whether there is a p -adic version of Theorem 3.6? In particular, does there exists a version of*

- (i) *p -adic Bukh-Cox bound?*
- (ii) *p -adic Orthoplex/Rankin bound?*
- (iii) *p -adic Levenstein bound?*
- (iv) *p -adic Exponential bound?*

We already wrote that Welch bounds have applications in study of equiangular lines. We wish to formulate equiangular line problem for p -adic Hilbert spaces. For the study of equiangular lines in Hilbert spaces we refer [8, 9, 16, 18, 27, 37, 38, 40, 43, 49, 50, 60, 65, 66, 89], quaternion Hilbert spaces we refer [33], octonion Hilbert spaces we refer [21], finite dimensional vector spaces over finite fields we refer [41, 42], for Banach spaces we refer [57] and for non-Archimedean Hilbert spaces we refer [59].

Question 3.8. (p -adic Equiangular Line Problem) *Let p be a prime. Given $a \in \mathbb{Q}_p$, $d \in \mathbb{N}$ and $\gamma > 0$, what is the maximum $n = n(p, a, d, \gamma) \in \mathbb{N}$ such that there exist vectors $\tau_1, \dots, \tau_n \in \mathbb{Q}_p^d$ satisfying the following.*

- (i) $\langle \tau_j, \tau_j \rangle = a$ for all $1 \leq j \leq n$.
- (ii) $|\langle \tau_j, \tau_k \rangle|^2 = \gamma$ for all $1 \leq j, k \leq n, j \neq k$.

In particular, whether there is a p -adic Gerzon bound?

Question 3.8 can be easily lifted to formulate question of p -adic regular s -distance sets.

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