C*-ALGEBRAIC BIEBERBACH, ROBERTSON, LEBEDEV-MILIN, ZALCMAN, KRZYZ AND CORONA CONJECTURES

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Abstract: We study C*-algebraic versions of following conjectures/theorems: (1) Bieberbach conjecture (de Branges theorem) (2) Robertson conjecture (3) Lebedev-Milin conjecture (4) Zalcman conjecture (5) Krzyz conjecture (6) Corona conjecture (Carleson theorem). We prove that the C*-algebraic Bieberbach Conjecture for invertible coefficients is true for second degree C*-algebraic polynomials.

Keywords: C*-algebra, Bieberbach conjecture, de Branges theorem, Robertson conjecture, Lebedev-Milin conjecture, Zalcman conjecture, Krzyz conjecture, Corona conjecture, Riemann mapping theorem.

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1. C*-algebraic Bieberbach, Robertson and Lebedev-Milin conjectures

Let $\mathbb{D}(0,1)$ be the open unit disc in $\mathbb{C}$ centered at 0 of radius 1. In 1916, Bieberbach made the following conjecture which became known as Bieberbach conjecture [9].

Conjecture 1.1. *[2,9,17,27,29,38,41,49,63,65,73,95,96,117]* ***(Bieberbach Conjecture/de Branges Theorem) If the power series***

$$f(z) \coloneqq z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}, \forall n \geq 2$$

converges for all $z \in \mathbb{D}(0,1)$ ***and the function*** f ***is injective on*** $\mathbb{D}(0,1)$***, then***

$$|a_n| \leq n, \quad \forall n \geq 2. \tag{1}$$

Inequality (1) is strict except for rotations of Koebe function on $\mathbb{D}(0,1)$ ***defined by***

$$k(z) \coloneqq \sum_{n=1}^{\infty} n z^n.$$

Bieberbach himself proved that $|a_2| \leq 2$ [9]. In 1923 Lowner proved that $|a_3| \leq 3$ [77]. In 1955 Garabedian and Schiffer gave a new proof for $|a_3| \leq 3$ [33]. The inequality $|a_4| \leq 4$ was proved by Garabedian and Schiffer in 1955 [34]. A simpler proof for $|a_4| \leq 4$ is later given by Charzynski and Schiffer in 1960 [15]. The inequality $|a_5| \leq 5$ was proved by Pederson and Schiffer in 1972 [92]. The inequality $|a_6| \leq 6$ was proved by Pederson in 1968 [93] as well as by Ozawa in 1969 [90,91]. On the other side, Littlewood in 1925 showed that $|a_n| \leq en$ for all n [75]. In 1957 Nehari showed that asymptotic Bieberbach conjecture implies Littlewood conjecture [86]. In 1982 Hamilton showed that Littlewood conjecture implies asymptotic Bieberbach conjecture [44].

In 1972 FitzGerald proved that $|a_n| < \sqrt{\frac{7}{6}}n$ for all n [26] which improved the bound obtained by Milin in 1965 [83]. In 1976 Horowitz proved that $|a_n| < \left(\frac{209}{140}\right)^{\frac{1}{6}} n$ for all n [52]. Horowitz improved his result in 1978 and obtained $|a_n| < \left(\frac{1659164137}{681080400}\right)^{\frac{1}{14}} n$ for all n [54] (this result was further improved by Hu in 1983 [55]). In 1955 Hayman proved that $\lim_{n\to\infty} \frac{a_n}{n}$ exists [45]. In 1986 Brown and Tsao showed that Zalcman conjecture implies Bieberbach conjecture [11] (also see [71, 72, 79, 80]) Further, Bieberbach conjecture has been proved for special classes of functions [6, 16, 22, 23, 37, 46, 53, 58, 76, 89, 98, 103]. Finally in 1985, de Branges proved the conjecture in full generality for all n [17]. In 1991 Weinstein gave another proof of Bieberbach conjecture [114] (also see [4, 12, 18, 19, 21, 24, 28, 47, 48, 59, 62, 88, 106–109, 115, 116]). In 1997 Xie proved a generalization of de Branges theorem [116]. It is interesting to note that Bieberbach conjecture for holomorphic mappings on several complex variables fails [73].

Let $\mathcal{A}$ be a unital C*-algebra. We define the C*-algebraic open unit disc centered at 0 and of radius 1, denoted as $\mathbb{D}^*(0,1)$ as the set of all strict contractions in $\mathcal{A}$, i.e., set of all elements of $\mathcal{A}$ having norm less than 1. Based on Conjecture 1.1 we set following conjectures.

Conjecture 1.2. ***(C*-algebraic Bieberbach Conjecture for general coefficients) Let*** $\mathcal{A}$ ***be a unital C*-algebra. If the C*-algebraic power series***

$$f(z) \coloneqq z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathcal{A}, \forall n \geq 2$$

converges (in norm of $\mathcal{A}$***) for all*** $z \in \mathbb{D}^*(0,1)$ ***and the function*** f ***is injective on*** $\mathbb{D}^*(0,1)$***, then***

$$a_n a_n^* \leq n^2, \quad a_n^* a_n \leq n^2, \quad \forall n \geq 2. \tag{2}$$

Inequality (2) is strict except for rotations of C*-algebraic Koebe function

$$k(z) \coloneqq \sum_{n=1}^{\infty} n z^n$$

on $\mathbb{D}^*(0,1)$***.***

Conjecture 1.3. ***(C*-algebraic Bieberbach Conjecture for invertible coefficients) Let*** $\mathcal{A}$ ***be a unital C*-algebra and let*** $I(\mathcal{A})$ ***be the set of all invertible elements in*** $\mathcal{A}$***. If the C*-algebraic power series***

$$f(z) \coloneqq z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in I(\mathcal{A}) \cup \{0\}, \forall n \geq 2$$

converges (in norm of $\mathcal{A}$***) for all*** $z \in \mathbb{D}^*(0,1)$ ***and the function*** f ***is injective on*** $\mathbb{D}^*(0,1)$***, then***

$$\frac{1}{\|a_n^{-1}\|} \leq n, \quad \forall a_n \in I(\mathcal{A}).$$

Here are strong forms of Conjecture 1.2 and Conjecture 1.3.

Conjecture 1.4. ***(C*-algebraic Bieberbach Conjecture for general coefficients - strong form) Let*** $\mathcal{A}$ ***be a unital C*-algebra. If the C*-algebraic power series***

$$f(z) \coloneqq z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathcal{A}, \forall n \geq 2$$

converges (in norm of $\mathcal{A}$) for all $z \in \mathbb{D}^*(0,1)$, the C*-algebraic power series

$$f'(z) \coloneqq 1+\sum_{n=2}^{\infty}a_nnz^{n-1}$$

also converges (in norm of $\mathcal{A}$) for all $z \in \mathbb{D}^*(0,1)$ and the function f is injective on $\mathbb{D}^*(0,1)$, then

$$a_na_n^* \leq n^2, \quad a_n^*a_n \leq n^2, \quad \forall n \geq 2.$$

Conjecture 1.5. ***(C*-algebraic Bieberbach Conjecture invertible coefficients - strong form) Let $\mathcal{A}$ be a unital C*-algebra and let $I(\mathcal{A})$ be the set of all invertible elements in $\mathcal{A}$. If the C*-algebraic power series***

$$f(z) \coloneqq z+\sum_{n=2}^{\infty}a_nz^{n}, \quad a_n \in I(\mathcal{A})\cup\{0\}, \forall n \geq 2$$

converges (in norm of $\mathcal{A}$) for all $z \in \mathbb{D}^*(0,1)$, the C*-algebraic power series

$$f'(z) \coloneqq 1+\sum_{n=2}^{\infty}a_nnz^{n-1}$$

also converges (in norm of $\mathcal{A}$) for all $z \in \mathbb{D}^*(0,1)$ and the function f is injective on $\mathbb{D}^*(0,1)$, then

$$\frac{1}{\|a_n^{-1}\|} \leq n, \quad \forall a_n \in I(\mathcal{A}).$$

Since holomorphic functions (convergent power series inside the disc of convergence) are infinitely differentiable, strong form, namely Conjectures 1.4 and 1.5 reduce to Conjecture 1.3 for complex numbers. We don't know this for C*-algebraic convergent power series. If this is true, then the strong form is same as general form. Otherwise, note that we can even stronger form of Conjecture 1.4 by imposing second, third, ..., differentiable conditions. The same comment holds for other strong form of conjectures also. We next formulate two conjectures which are stronger than Conjecture 1.2 and Conjecture 1.3 for polynomials which is based on Proposition in Page 136 in [117].

Conjecture 1.6. ***(C*-algebraic Bieberbach Conjecture for polynomials - 1) Let $n \geq 2$. Let $\mathcal{A}$ be a unital C*-algebra. If the C*-algebraic polynomial***

$$p(z) \coloneqq z+a_2z^2+a_3z^3+\cdots+a_nz^n, \quad a_k \in \mathcal{A}, \forall 2 \leq k \leq n$$

is injective on $\mathbb{D}^*(0,1)$, then

$$a_na_n^* \leq \frac{1}{n^2}, \quad a_n^*a_n \leq \frac{1}{n^2}.$$

Conjecture 1.7. ***(C*-algebraic Bieberbach Conjecture for polynomials - 2) Let $n \geq 2$. Let $\mathcal{A}$ be a unital C*-algebra and let $I(\mathcal{A})$ be the set of all invertible elements in $\mathcal{A}$. If the C*-algebraic polynomial***

$$p(z) \coloneqq z+a_2z^2+a_3z^3+\cdots+a_nz^n, \quad a_k \in I(\mathcal{A})\cup\{0\}, \forall 2 \leq k \leq n-1, a_n \in I(\mathcal{A})$$

is injective on $\mathbb{D}^*(0,1)$, then

$$\frac{1}{\|a_n^{-1}\|} \leq \frac{1}{n}.$$

Theorem 1.8. *Conjecture 1.3 holds for C*-algebraic polynomials of degree 2.*

Proof. Let $\mathcal{A}$ be a unital C*-algebra, $a \in \mathcal{A}$ be invertible and $p(z) = z + az^2$ be a polynomial over $\mathcal{A}$ which is injective on $\mathbb{D}^*(0,1)$. Since p is injective and $p(-a^{-1}) = 0 = p(0)$, we must have $-a^{-1} \notin \mathbb{D}^*(0,1)$. Therefore $\|-a^{-1}\| \geq 1 > 1/2$. □

In 1936 Robertson formulated (after the failure of Littlewood-Paley conjecture [76] by Feketo and Szego in 1933 [25]) what came to known as Robertson conjecture [99]. This conjecture implies Bieberbach conjecture.

Conjecture 1.9. *[99, 117]* ***(Robertson conjecture) If the power series***

$$f(z) := z + \sum_{n=2}^{\infty} b_{2n-1} z^{2n-1}, \quad b_{2n-1} \in \mathbb{C}, \forall n \geq 2$$

converges for all $z \in \mathbb{D}(0,1)$ ***and the function*** f ***is injective on*** $\mathbb{D}(0,1)$***, then***

$$1 + \sum_{k=1}^{n} |b_{2k-1}|^2 \leq n, \quad \forall n \geq 2.$$

We now formulate C*-algebraic Robertson conjectures.

Conjecture 1.10. ***(C*-algebraic Robertson Conjecture for general coefficients) Let*** $\mathcal{A}$ ***be a unital C*-algebra. If the C*-algebraic power series***

$$f(z) := z + \sum_{n=2}^{\infty} b_{2n-1} z^{2n-1}, \quad b_{2n-1} \in \mathcal{A}, \forall n \geq 2$$

converges (in norm of $\mathcal{A}$***) for all*** $z \in \mathbb{D}^*(0,1)$ ***and the function*** f ***is injective on*** $\mathbb{D}^*(0,1)$***, then***

$$1 + \sum_{k=1}^{n} b_{2k-1} b_{2k-1}^* \leq n, \quad 1 + \sum_{k=1}^{n} b_{2k-1}^* b_{2k-1} \leq n, \quad \forall n \geq 2.$$

Conjecture 1.11. ***(C*-algebraic Robertson Conjecture for invertible coefficients) Let*** $\mathcal{A}$ ***be a unital C*-algebra and let*** $I(\mathcal{A})$ ***be the set of all invertible elements in*** $\mathcal{A}$***. If the C*-algebraic power series***

$$f(z) := z + \sum_{n=2}^{\infty} b_{2n-1} z^{2n-1}, \quad b_{2n-1} \in I(\mathcal{A}) \cup \{0\}, \forall n \geq 2$$

converges (in norm of $\mathcal{A}$***) for all*** $z \in \mathbb{D}^*(0,1)$ ***and the function*** f ***is injective on*** $\mathbb{D}^*(0,1)$***, then***

$$\frac{1}{\left\| \left(1 + \sum_{k=1}^{n} b_{2k-1} b_{2k-1}^*\right)^{-1} \right\|} \leq n, \quad \frac{1}{\left\| \left(1 + \sum_{k=1}^{n} b_{2k-1}^* b_{2k-1}\right)^{-1} \right\|} \leq n, \quad \forall n \geq 2.$$

Conjecture 1.12. ***(C*-algebraic Robertson Conjecture for general coefficients - strong form) Let*** $\mathcal{A}$ ***be a unital C*-algebra. If the C*-algebraic power series***

$$f(z) := z + \sum_{n=2}^{\infty} b_{2n-1} z^{2n-1}, \quad b_{2n-1} \in \mathcal{A}, \forall n \geq 2$$

converges (in norm of $\mathcal{A}$***) for all*** $z \in \mathbb{D}^*(0,1)$***, the C*-algebraic power series***

$$f'(z) := 1 + \sum_{n=2}^{\infty} (2n-1) b_{2n-1} z^{2n-2}$$

also converges (in norm of $\mathcal{A}$) for all $z \in \mathbb{D}^*(0,1)$ and the function f is injective on $\mathbb{D}^*(0,1)$, then

$$1+\sum_{k=1}^{n} b_{2k-1}b_{2k-1}^* \leq n, \quad 1+\sum_{k=1}^{n} b_{2k-1}^*b_{2k-1} \leq n, \quad \forall n \geq 2.$$

Conjecture 1.13. ***(C*-algebraic Robertson Conjecture for invertible coefficients - strong form) Let $\mathcal{A}$ be a unital C*-algebra and let $I(\mathcal{A})$ be the set of all invertible elements in $\mathcal{A}$. If the C*-algebraic power series***

$$f(z) \coloneqq z+\sum_{n=2}^{\infty} b_{2n-1}z^{2n-1}, \quad b_{2n-1} \in I(\mathcal{A})\cup\{0\}, \forall n \geq 2$$

converges (in norm of $\mathcal{A}$) for all $z \in \mathbb{D}^*(0,1)$, the C*-algebraic power series

$$f'(z) \coloneqq 1+\sum_{n=2}^{\infty} (2n-1)b_{2n-1}z^{2n-2}$$

also converges (in norm of $\mathcal{A}$) for all $z \in \mathbb{D}^*(0,1)$ and the function f is injective on $\mathbb{D}^*(0,1)$, then

$$\frac{1}{\left\|\left(1+\sum_{k=1}^{n} b_{2k-1}b_{2k-1}^*\right)^{-1}\right\|} \leq n, \quad \frac{1}{\left\|\left(1+\sum_{k=1}^{n} b_{2k-1}^*b_{2k-1}\right)^{-1}\right\|} \leq n, \quad \forall n \geq 2.$$

In 1970's Lebedev and Milin conjectured an inequality which became known as Lebedev-Milin conjecture [84]. This conjecture implies Robertson conjecture.

Conjecture 1.14. *[84, 117]* ***(Lebedev-Milin Conjecture) Let the power series***

$$f(z) \coloneqq z+\sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}, \forall n \geq 2$$

converges for all $z \in \mathbb{D}(0,1)$ and the function f is injective on $\mathbb{D}(0,1)$. Let the sequence $\{c_n\}_{n=1}^{\infty}$ of complex numbers be defined as the coefficients of the power series

$$\log\left(\frac{f(z)}{z}\right) = \sum_{n=1}^{\infty} c_n z^n, \quad \forall z \in \mathbb{D}(0,1).$$

Then for all $n = 1, 2, \dots$,

$$\sum_{m=1}^{n}\sum_{k=1}^{m}\left(k|c_k|^2-\frac{4}{k}\right) \leq 0. \tag{3}$$

Observe that, as is well-known, Equation (3)'s sum

$$\sum_{m=1}^{n}\sum_{k=1}^{m}\left(k|c_k|^2-\frac{4}{k}\right) = \sum_{k=1}^{n}(n+1-k)\left(k|c_k|^2-\frac{4}{k}\right), \quad \forall n = 1, 2, \dots$$

which can be proved by induction. We formulate C*-algebraic versions of Lebedev-Milin conjectures as follows.

Conjecture 1.15. ***(C*-algebraic Lebedev-Milin Conjecture for general coefficients) Let $\mathcal{A}$ be a unital C*-algebra. Assume the following.***

(i) ***The C*-algebraic power series***

$$f(z) \coloneqq z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathcal{A}, \forall n \geq 2$$

converges (in norm of $\mathcal{A}$***) for all*** $z \in \mathbb{D}^*(0,1)$***.***

(ii) ***The C*-algebraic power series***

$$f'(z) \coloneqq 1 + \sum_{n=2}^{\infty} a_n n z^{n-1}$$

converges (in norm of $\mathcal{A}$***) for all*** $z \in \mathbb{D}^*(0,1)$***.***

(iii) f ***is injective on*** $\mathbb{D}^*(0,1)$***.***

(iv) ***There exists a C*-algebraic power series***

$$g(z) \coloneqq \sum_{n=0}^{\infty} c_n z^n, \quad c_n \in \mathcal{A}, \forall n \geq 0$$

converging (in norm of $\mathcal{A}$***) for all*** $z \in \mathbb{D}^*(0,1)$ ***such that***

$$f(z) = e^{g(z)} z, \quad \forall z \in \mathbb{D}^*(0,1).$$

Then for all $n = 1, 2, \dots,$

$$\sum_{m=1}^{n} \sum_{k=1}^{m} \left(k c_k c_k^* - \frac{4}{k} \right) \leq 0, \quad \sum_{m=1}^{n} \sum_{k=1}^{m} \left(k c_k^* c_k - \frac{4}{k} \right) \leq 0.$$

Conjecture 1.16. ***(C*-algebraic Lebedev-Milin Conjecture for invertible coefficients) Let*** $\mathcal{A}$ ***be a unital C*-algebra and let*** $I(\mathcal{A})$ ***be the set of all invertible elements in*** $\mathcal{A}$***. Assume the following.***

(i) ***The C*-algebraic power series***

$$f(z) \coloneqq z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in I(\mathcal{A}) \cup \{0\}, \forall n \geq 2$$

converges (in norm of $\mathcal{A}$***) for all*** $z \in \mathbb{D}^*(0,1)$***.***

(ii) ***The C*-algebraic power series***

$$f'(z) \coloneqq 1 + \sum_{n=2}^{\infty} a_n n z^{n-1}$$

converges (in norm of $\mathcal{A}$***) for all*** $z \in \mathbb{D}^*(0,1)$***.***

(iii) f ***is injective on*** $\mathbb{D}^*(0,1)$***.***

(iv) ***There exists a C*-algebraic power series***

$$g(z) \coloneqq \sum_{n=0}^{\infty} c_n z^n, \quad c_n \in I(\mathcal{A}) \cup \{0\}, \forall n \geq 0$$

converging (in norm of $\mathcal{A}$***) for all*** $z \in \mathbb{D}^*(0,1)$ ***such that***

$$f(z) = e^{g(z)} z, \quad \forall z \in \mathbb{D}^*(0,1).$$

Then for all $n = 1, 2, \dots$***,***

$$\frac{1}{\left\|\left[\sum\limits_{m=1}^{n}\sum\limits_{k=1}^{m}\left(kc_kc_k^* - \frac{4}{k}\right)\right]^{-1}\right\|} \leq 0, \quad \frac{1}{\left\|\left[\sum\limits_{m=1}^{n}\sum\limits_{k=1}^{m}\left(kc_k^*c_k - \frac{4}{k}\right)\right]^{-1}\right\|} \leq 0.$$

Conjecture 1.17. ***(C*-algebraic Lebedev-Milin Conjecture for general coefficients - strong form) Let*** $\mathcal{A}$ ***be a unital C*-algebra. Assume the following.***

(i) ***The C*-algebraic power series***

$$f(z) \coloneqq z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathcal{A}, \forall n \geq 2$$

converges (in norm of $\mathcal{A}$***) for all*** $z \in \mathbb{D}^*(0,1)$***.***

(ii) ***The C*-algebraic power series***

$$f'(z) \coloneqq 1 + \sum_{n=2}^{\infty} a_n n z^{n-1}$$

converges (in norm of $\mathcal{A}$***) for all*** $z \in \mathbb{D}^*(0,1)$***.***

(iii) f ***is injective on*** $\mathbb{D}^*(0,1)$***.***

(iv) ***There exists a C*-algebraic power series***

$$g(z) \coloneqq \sum_{n=0}^{\infty} c_n z^n, \quad c_n \in \mathcal{A}, \forall n \geq 0$$

converging (in norm of $\mathcal{A}$***) for all*** $z \in \mathbb{D}^*(0,1)$ ***such that***

$$f(z) = e^{g(z)} z, \quad \forall z \in \mathbb{D}^*(0,1).$$

(v) ***The C*-algebraic power series***

$$g'(z) \coloneqq \sum_{n=1}^{\infty} c_n n z^{n-1}$$

converges (in norm of $\mathcal{A}$***) for all*** $z \in \mathbb{D}^*(0,1)$***.***

Then for all $n = 1, 2, \dots$***,***

$$\sum_{m=1}^{n}\sum_{k=1}^{m}\left(kc_kc_k^* - \frac{4}{k}\right) \leq 0, \quad \sum_{m=1}^{n}\sum_{k=1}^{m}\left(kc_k^*c_k - \frac{4}{k}\right) \leq 0.$$

Conjecture 1.18. ***(C*-algebraic Lebedev-Milin Conjecture for invertible coefficients - strong form) Let*** $\mathcal{A}$ ***be a unital C*-algebra and let*** $I(\mathcal{A})$ ***be the set of all invertible elements in*** $\mathcal{A}$***. Assume the following.***

(i) ***The C*-algebraic power series***

$$f(z) \coloneqq z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in I(\mathcal{A}) \cup \{0\}, \forall n \geq 2$$

converges (in norm of $\mathcal{A}$***) for all*** $z \in \mathbb{D}^*(0,1)$***.***

(ii) ***The C*-algebraic power series***

$$f'(z) \coloneqq 1 + \sum_{n=2}^{\infty} a_n n z^{n-1}$$

converges (in norm of $\mathcal{A}$***) for all*** $z \in \mathbb{D}^*(0,1)$***.***

(iii) f ***is injective on*** $\mathbb{D}^*(0,1)$***.***

(iv) ***There exists a C*-algebraic power series***

$$g(z) \coloneqq \sum_{n=0}^{\infty} c_n z^n, \quad c_n \in I(\mathcal{A}) \cup \{0\}, \forall n \geq 0$$

converging (in norm of $\mathcal{A}$***) for all*** $z \in \mathbb{D}^*(0,1)$ ***such that***

$$f(z) = e^{g(z)} z, \quad \forall z \in \mathbb{D}^*(0,1).$$

(v) ***The C*-algebraic power series***

$$g'(z) \coloneqq \sum_{n=1}^{\infty} c_n n z^{n-1}$$

converges (in norm of $\mathcal{A}$***) for all*** $z \in \mathbb{D}^*(0,1)$***.***

Then for all $n = 1, 2, \dots$***,***

$$\frac{1}{\left\| \left[\sum\limits_{m=1}^{n} \sum\limits_{k=1}^{m} \left(k c_k c_k^* - \frac{4}{k} \right) \right]^{-1} \right\|} \leq 0, \quad \frac{1}{\left\| \left[\sum\limits_{m=1}^{n} \sum\limits_{k=1}^{m} \left(k c_k^* c_k - \frac{4}{k} \right) \right]^{-1} \right\|} \leq 0.$$

By assuming Lebedev-Milin conjecture one proves the Robertson conjecture using Lebedev-Milin inequalities stated below.

Theorem 1.19. *[63, 73]* ***(Lebedev-Milin Inequalities) Let the power series***

$$f(z) \coloneqq 1 + \sum_{n=1}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}, \forall n \geq 1$$

converges for all $z \in \mathbb{D}(0,1)$ ***and let*** g ***be the power series defined by***

$$e^{f(z)} = \sum_{n=0}^{\infty} b_n z^n =: g(z), \quad \forall z \in \mathbb{D}(0,1).$$

Then

$$\sum_{n=0}^{\infty} |b_n|^2 \leq e^{\sum\limits_{n=0}^{\infty} n|a_n|^2},$$

$$\frac{1}{n+1} \sum_{k=0}^{n} |b_k|^2 \leq e^{\frac{1}{n+1} \sum\limits_{k=1}^{n} (n+1-k)\left(k|a_k|^2 - \frac{4}{k}\right)}, \quad \forall n = 1, 2, \dots,$$

$$|b_n|^2 \leq e^{\sum\limits_{k=1}^{n} \left(k|a_k|^2 - \frac{4}{k}\right)}, \quad \forall n = 1, 2, \dots.$$

Based on Theorem 1.19 we set the following conjecture.

Conjecture 1.20. ***Let*** $\mathcal{A}$ ***be a unital C*-algebra. Let the C*-algebraic power series***

$$f(z) \coloneqq 1 + \sum_{n=1}^{\infty} a_n z^n, \quad a_n \in \mathcal{A}, \forall n \geq 1$$

converges for all $z \in \mathbb{D}^*(0,1)$ ***and assume that there exists a C*-algebraic power series*** g ***such that***

$$e^{f(z)} = \sum_{n=0}^{\infty} b_n z^n =: g(z), \quad \forall z \in \mathbb{D}^*(0,1).$$

Then

$$\sum_{n=0}^{\infty} b_n b_n^* \leq e^{\sum\limits_{n=0}^{\infty} n a_n a_n^*},$$

$$\frac{1}{n+1}\sum_{k=0}^{n} b_k b_k^* \leq e^{\frac{1}{n+1}\sum\limits_{k=1}^{n}(n+1-k)\left(k a_k a_k^* - \frac{4}{k}\right)}, \quad \forall n = 1, 2, \dots,$$

$$b_n b_n^* \leq e^{\sum\limits_{k=1}^{n}\left(k|a_k a_k^* - \frac{4}{k}\right)}, \quad \forall n = 1, 2, \dots.$$

and

$$\sum_{n=0}^{\infty} b_n^* b_n \leq e^{\sum\limits_{n=0}^{\infty} n a_n^* a_n},$$

$$\frac{1}{n+1}\sum_{k=0}^{n} b_k^* b_k \leq e^{\frac{1}{n+1}\sum\limits_{k=1}^{n}(n+1-k)\left(k a_k^* a_k - \frac{4}{k}\right)}, \quad \forall n = 1, 2, \dots,$$

$$b_n^* b_n \leq e^{\sum\limits_{k=1}^{n}\left(k|a_k^* a_k - \frac{4}{k}\right)}, \quad \forall n = 1, 2, \dots.$$

We next formulate C*-algebraic versions of Gronwall area theorem [42].

Conjecture 1.21. ***(C*-algebraic Gronwall Area Conjecture for general coefficients) Let $\mathcal{A}$ be a unital C*-algebra. Assume the following.***

(i) ***The C*-algebraic power series***

$$f(z) \coloneqq z + \sum_{n=0}^{\infty} b_n \frac{1}{z^n}, \quad b_n \in \mathcal{A}, \forall n \geq 0$$

converges (in norm of $\mathcal{A}$) for all $z \in \Delta \coloneqq \{x \in \mathcal{A} : \|x\| > 1\}$.

(ii) ***The C*-algebraic power series***

$$f'(z) \coloneqq 1 + \sum_{n=1}^{\infty} b_n \frac{-n}{z^{n-1}}$$

converges (in norm of $\mathcal{A}$) for all $z \in \Delta$.

(iii) f ***is injective on*** Δ.

(iv) $\lim_{z\to\infty} f(z) = \infty$.

(v) $\lim_{z\to\infty} f'(z) = 1$.

Then

$$\sum_{n=1}^{\infty} n b_n b_n^* \leq 1, \quad \sum_{n=1}^{\infty} n b_n^* b_n \leq 1.$$

Conjecture 1.22. ***(C*-algebraic Gronwall Area Conjecture for invertible coefficients) Let $\mathcal{A}$ be a unital C*-algebra and let $I(\mathcal{A})$ be the set of all invertible elements in $\mathcal{A}$. Assume the following.***

(i) ***The C*-algebraic power series***

$$f(z) \coloneqq z + \sum_{n=0}^{\infty} b_n \frac{1}{z^n}, \quad b_n \in I(\mathcal{A}) \cup \{0\}, \forall n \geq 0$$

converges (in norm of $\mathcal{A}$) for all $z \in \Delta \coloneqq \{x \in \mathcal{A} : \|x\| > 1\}$.

(ii) ***The C*-algebraic power series***

$$f'(z) \coloneqq 1 + \sum_{n=1}^{\infty} b_n \frac{-n}{z^{n-1}}$$

converges (in norm of $\mathcal{A}$***) for all*** $z \in \Delta$.

(iii) f ***is injective on*** Δ.

(iv) $\lim_{z\to\infty} f(z) = \infty$.

(v) $\lim_{z\to\infty} f'(z) = 1$.

Then

$$\frac{1}{\left\|\left(\sum\limits_{n=1}^{\infty} nb_nb_n^*\right)^{-1}\right\|} \leq 1, \quad \frac{1}{\left\|\left(\sum\limits_{n=1}^{\infty} nb_n^*b_n\right)^{-1}\right\|} \leq 1.$$

We now recall Zalcman conjecture which, as mentioned earlier, implies Bieberbach conjecture.

Conjecture 1.23. *[11]* ***(Zalcman Conjecture) If the power series***

$$f(z) \coloneqq z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}, \forall n \geq 2$$

converges for all $z \in \mathbb{D}(0,1)$ ***and the function*** f ***is injective on*** $\mathbb{D}(0,1)$***, then***

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2, \quad \forall n \geq 2.$$

In 1998 Ma proposed a generalization of Conjecture 1.23 [80].

Conjecture 1.24. *[80]* ***(Ma Conjecture) If the power series***

$$f(z) \coloneqq z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}, \forall n \geq 2$$

converges for all $z \in \mathbb{D}(0,1)$ ***and the function*** f ***is injective on*** $\mathbb{D}(0,1)$***, then***

$$|a_n a_m - a_{n+m-1}| \leq (n-1)(m-1), \quad \forall n, m \geq 2.$$

We next formulate C*-algebraic versions of Zalcman conjecture and Ma conjecture as follows.

Conjecture 1.25. ***(C*-algebraic Zalcman Conjecture) Let*** $\mathcal{A}$ ***be a unital C*-algebra. If the C*-algebraic power series***

$$f(z) \coloneqq z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathcal{A}, \forall n \geq 2$$

converges (in norm of $\mathcal{A}$***) for all*** $z \in \mathbb{D}^*(0,1)$ ***and the function*** f ***is injective on*** $\mathbb{D}^*(0,1)$***, then***

$$(a_n^2 - a_{2n-1})(a_n^2 - a_{2n-1})^* \leq (n-1)^4, \quad (a_n^2 - a_{2n-1})^*(a_n^2 - a_{2n-1}) \leq (n-1)^4, \quad \forall n \geq 2.$$

Conjecture 1.26. ***(C*-algebraic Zalcman Conjecture - strong form) Let*** $\mathcal{A}$ ***be a unital C*-algebra. If the C*-algebraic power series***

$$f(z) \coloneqq z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathcal{A}, \forall n \geq 2$$

converges (in norm of $\mathcal{A}$) for all $z \in \mathbb{D}^*(0,1)$, the C*-algebraic power series

$$f'(z) \coloneqq 1 + \sum_{n=2}^{\infty} a_n n z^{n-1}$$

also converges (in norm of $\mathcal{A}$) for all $z \in \mathbb{D}^*(0,1)$ and the function f is injective on $\mathbb{D}^*(0,1)$, then

$$(a_n^2 - a_{2n-1})(a_n^2 - a_{2n-1})^* \leq (n-1)^4, \quad (a_n^2 - a_{2n-1})^*(a_n^2 - a_{2n-1}) \leq (n-1)^4, \quad \forall n \geq 2.$$

Conjecture 1.27. ***(C*-algebraic Ma Conjecture) Let $\mathcal{A}$ be a unital C*-algebra. If the C*-algebraic power series***

$$f(z) \coloneqq z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathcal{A}, \forall n \geq 2$$

converges (in norm of $\mathcal{A}$) for all $z \in \mathbb{D}^*(0,1)$ and the function f is injective on $\mathbb{D}^*(0,1)$, then

$$(a_n a_m - a_{n+m-1})(a_n a_m - a_{n+m-1})^* \leq (n-1)^2(m-1)^2,$$
$$(a_n a_m - a_{n+m-1})^*(a_n a_m - a_{n+m-1}) \leq (n-1)^2(m-1)^2, \quad \forall n, m \geq 2.$$

Conjecture 1.28. ***(C*-algebraic Ma Conjecture - strong form) Let $\mathcal{A}$ be a unital C*-algebra. If the C*-algebraic power series***

$$f(z) \coloneqq z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathcal{A}, \forall n \geq 2$$

converges (in norm of $\mathcal{A}$) for all $z \in \mathbb{D}^*(0,1)$, the C*-algebraic power series

$$f'(z) \coloneqq 1 + \sum_{n=2}^{\infty} a_n n z^{n-1}$$

also converges (in norm of $\mathcal{A}$) for all $z \in \mathbb{D}^*(0,1)$ and the function f is injective on $\mathbb{D}^*(0,1)$, then

$$(a_n a_m - a_{n+m-1})(a_n a_m - a_{n+m-1})^* \leq (n-1)^2(m-1)^2,$$
$$(a_n a_m - a_{n+m-1})^*(a_n a_m - a_{n+m-1}) \leq (n-1)^2(m-1)^2, \quad \forall n, m \geq 2.$$

In 1948 Goodman formulated the version of Bieberbach conjecture for p-valent functions [35].

Conjecture 1.29. *[35, 36, 43, 78]* ***(Goodman Conjecture) Let $p \in \mathbb{N}$. If the power series***

$$f(z) \coloneqq \sum_{n=1}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}, \forall n \geq 1$$

converges for all $z \in \mathbb{D}(0,1)$ and the function f takes each value almost p times on $\mathbb{D}(0,1)$, then

$$|a_n| \leq \sum_{k=1}^{p} \frac{2k(n+p)!}{(p-k)!(p+k)!(n-p-1)(n^2-k^2)} |a_k|, \quad \forall n > p.$$

We state C*-algebraic versions of Conjecture 1.29 as follows.

Conjecture 1.30. ***(C*-algebraic Goodman Conjecture for general coefficients) Let*** $p \in \mathbb{N}$ $\mathcal{A}$ ***be a unital C*-algebra. If the C*-algebraic power series***

$$f(z) \coloneqq \sum_{n=1}^{\infty} a_n z^n, \quad a_n \in \mathcal{A}, \forall n \geq 1$$

converges (in norm of $\mathcal{A}$***) for all*** $z \in \mathbb{D}^*(0,1)$ ***and the function*** f ***takes each value at most*** p ***times on*** $\mathbb{D}^*(0,1)$***, then***

$$a_n a_n^* \leq \left(\sum_{k=1}^{p} \frac{2k(n+p)!}{(p-k)!(p+k)!(n-p-1)(n^2-k^2)} a_k\right)\left(\sum_{k=1}^{p} \frac{2k(n+p)!}{(p-k)!(p+k)!(n-p-1)(n^2-k^2)} a_k\right)^*, \quad \forall n > p$$

and

$$a_n^* a_n \leq \left(\sum_{k=1}^{p} \frac{2k(n+p)!}{(p-k)!(p+k)!(n-p-1)(n^2-k^2)} a_k\right)^* \left(\sum_{k=1}^{p} \frac{2k(n+p)!}{(p-k)!(p+k)!(n-p-1)(n^2-k^2)} a_k\right), \quad \forall n > p.$$

Conjecture 1.31. ***(C*-algebraic Goodman Conjecture for invertible coefficients) Let*** $p \in \mathbb{N}$***,*** $\mathcal{A}$ ***be a unital C*-algebra and let*** $I(\mathcal{A})$ ***be the set of all invertible elements in*** $\mathcal{A}$***. If the C*-algebraic power series***

$$f(z) \coloneqq \sum_{n=1}^{\infty} a_n z^n, \quad a_n \in I(\mathcal{A}) \cup \{0\}, \forall n \geq 1$$

converges (in norm of $\mathcal{A}$***) for all*** $z \in \mathbb{D}^*(0,1)$ ***and the function*** f ***takes each value at most*** p ***times on*** $\mathbb{D}^*(0,1)$***, then***

$$\frac{1}{\|a_n^{-1}\|} \leq \frac{1}{\left\|\left(\sum_{k=1}^{p} \frac{2k(n+p)!}{(p-k)!(p+k)!(n-p-1)(n^2-k^2)} a_k\right)^{-1}\right\|}, \quad \forall n > p.$$

Conjecture 1.32. ***(C*-algebraic Goodman Conjecture for general coefficients - strong form) Let*** $p \in \mathbb{N}$ $\mathcal{A}$ ***be a unital C*-algebra. If the C*-algebraic power series***

$$f(z) \coloneqq \sum_{n=1}^{\infty} a_n z^n, \quad a_n \in \mathcal{A}, \forall n \geq 1$$

converges (in norm of $\mathcal{A}$***) for all*** $z \in \mathbb{D}^*(0,1)$***, the C*-algebraic power series***

$$f'(z) \coloneqq \sum_{n=1}^{\infty} n a_n z^{n-1}, \quad a_n \in I(\mathcal{A}) \cup \{0\}, \forall n \geq 1$$

also converges (in norm of $\mathcal{A}$***) for all*** $z \in \mathbb{D}^*(0,1)$ ***and the function*** f ***takes each value at most*** p ***times on*** $\mathbb{D}^*(0,1)$***, then***

$$a_n a_n^* \leq \left(\sum_{k=1}^{p} \frac{2k(n+p)!}{(p-k)!(p+k)!(n-p-1)(n^2-k^2)} a_k\right)\left(\sum_{k=1}^{p} \frac{2k(n+p)!}{(p-k)!(p+k)!(n-p-1)(n^2-k^2)} a_k\right)^*, \quad \forall n > p$$

and

$$a_n^*a_n \leq \left(\sum_{k=1}^{p}\frac{2k(n+p)!}{(p-k)!(p+k)!(n-p-1)(n^2-k^2)}a_k\right)^*\left(\sum_{k=1}^{p}\frac{2k(n+p)!}{(p-k)!(p+k)!(n-p-1)(n^2-k^2)}a_k\right),$$
$$\forall n > p.$$

Conjecture 1.33. ***(C*-algebraic Goodman Conjecture for invertible coefficients - strong form) Let*** $p \in \mathbb{N}$***,*** $\mathcal{A}$ ***be a unital C*-algebra and let*** $I(\mathcal{A})$ ***be the set of all invertible elements in*** $\mathcal{A}$***. If the C*-algebraic power series***

$$f(z) \coloneqq \sum_{n=1}^{\infty}a_nz^n, \quad a_n \in I(\mathcal{A}) \cup \{0\}, \forall n \geq 1$$

converges (in norm of $\mathcal{A}$***) for all*** $z \in \mathbb{D}^*(0,1)$***, the C*-algebraic power series***

$$f'(z) \coloneqq \sum_{n=1}^{\infty}na_nz^{n-1}, \quad a_n \in I(\mathcal{A}) \cup \{0\}, \forall n \geq 1$$

also converges (in norm of $\mathcal{A}$***) for all*** $z \in \mathbb{D}^*(0,1)$ ***and the function*** f ***takes each value at most*** p ***times on*** $\mathbb{D}^*(0,1)$***, then***

$$\frac{1}{\|a_n^{-1}\|} \leq \frac{1}{\left\|\left(\sum\limits_{k=1}^{p}\frac{2k(n+p)!}{(p-k)!(p+k)!(n-p-1)(n^2-k^2)}a_k\right)^{-1}\right\|}, \quad \forall n > p.$$

Next we wish to state C*-algebraic version of Koebe distortion theorem. First we recall the result.

Theorem 1.34. *[73, 117]* ***(Koebe Distortion Theorem) If the power series***

$$f(z) \coloneqq z + \sum_{n=2}^{\infty}a_nz^n, \quad a_n \in \mathbb{C}, \forall n \geq 2$$

converges for all $z \in \mathbb{D}(0,1)$ ***and the function*** f ***is injective on*** $\mathbb{D}(0,1)$***, then***

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}, \quad \forall z \in \mathbb{D}(0,1) \setminus \{0\}$$

and

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}, \quad \forall z \in \mathbb{D}(0,1) \setminus \{0\}.$$

Conjecture 1.35. ***(C*-algebraic Koebe Distortion Conjecture) Let*** $\mathcal{A}$ ***be a unital C*-algebra. If the C*-algebraic power series***

$$f(z) \coloneqq z + \sum_{n=2}^{\infty}a_nz^n, \quad a_n \in \mathcal{A}, \forall n \geq 2$$

converges (in norm of $\mathcal{A}$***) for all*** $z \in \mathbb{D}^*(0,1)$***, the C*-algebraic power series***

$$f'(z) \coloneqq 1 + \sum_{n=2}^{\infty}a_nnz^{n-1}$$

also converges (in norm of $\mathcal{A}$***) for all*** $z \in \mathbb{D}^*(0,1)$ ***and the function*** f ***is injective on*** $\mathbb{D}^*(0,1)$***, then***

$$\frac{\|z\|}{(1+\|z\|)^2} \leq \|f(z)\| \leq \frac{\|z\|}{(1-\|z\|)^2}, \quad \forall z \in \mathbb{D}^*(0,1) \setminus \{0\}$$

and

$$\frac{1-\|z\|}{(1+\|z\|)^3} \leq \|f'(z)\| \leq \frac{1+\|z\|}{(1-\|z\|)^3}, \quad \forall z \in \mathbb{D}^*(0,1) \setminus \{0\}.$$

2. C*-algebraic Krzyz conjecture

A conjecture similar to that of Bieberbach conjecture is Krzyz conjecture.

Conjecture 2.1. *[5, 7, 81, 94]* ***(Krzyz Conjecture) If the power series***

$$f(z) \coloneqq \sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}, \forall n \geq 0$$

converges for all $z \in \mathbb{D}(0,1)$ ***and*** $0 < |f(z)| < 1$ ***on*** $\mathbb{D}(0,1)$***, then***

$$|a_n| \leq \frac{2}{e}, \quad \forall n \geq 1. \tag{4}$$

Inequality (4) is strict except for rotations of functions

$$f_n(z) \coloneqq e^{\frac{z^n+1}{z^n-1}}.$$

on $\mathbb{D}(0,1)$***,*** $n \in \mathbb{N}$***.***

By taking $z = 0$, we see that $|a_0| \leq 1$. In 1953 while answering a problem by Shapiro in **The American Mathematical Monthly**, Robertson proved that $|a_1| \leq \frac{2}{e}$ [104] (also see [97]). In 1977 Hummel, Scheinberg, and Zalcman proved that $|a_2| \leq \frac{2}{e}$ and $|a_3| \leq \frac{2}{e}$ [56]. In 1987 Brown proved that $|a_4| \leq \frac{2}{e}$ [10]. In 2003 Samaris proved that $|a_5| \leq \frac{2}{e}$ [102]. In 1978 Horowitz proved that there exists $0 < c < 1$ such that $|a_n| \leq c$ for all n [51]. In fact, $c = 1 - \frac{1}{3\pi} + \frac{4}{\pi}\sin\left(\frac{1}{12}\right)$. In 2021, Agler and McCarthy obtained a connection between Conjecture 2.1 and the entropy conjecture for polynomials with zeros on the standard unit circle group [1]. An extension of Krzyz conjecture has been formulated by Samaris in 2001 [101]. We state C*-algebraic versions of Krzyz conjecture as follows.

Conjecture 2.2. ***(C*-algebraic Krzyz Conjecture for general coefficients) Let*** $\mathcal{A}$ ***be a unital C*-algebra. If the C*-algebraic power series***

$$f(z) \coloneqq \sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathcal{A}, \forall n \geq 0$$

converges (in norm of $\mathcal{A}$***) for all*** $z \in \mathbb{D}^*(0,1)$ ***and*** $0 < \|f(z)\| < 1$ ***on*** $\mathbb{D}^*(0,1)$***, then***

$$a_n a_n^* \leq \frac{4}{e^2}, \quad a_n^* a_n \leq \frac{4}{e^2}, \quad \forall n \geq 1. \tag{5}$$

Inequality (5) is strict except for rotations of C*-algebraic functions

$$f_n(z) \coloneqq e^{\frac{z^n+1}{z^n-1}}.$$

on $\mathbb{D}^*(0,1)$***,*** $n \in \mathbb{N}$***.***

Conjecture 2.3. ***(C*-algebraic Krzyz Conjecture for invertible coefficients) Let*** $\mathcal{A}$ ***be a unital C*-algebra and let*** $I(\mathcal{A})$ ***be the set of all invertible elements in*** $\mathcal{A}$***. If the C*-algebraic power series***

$$f(z) \coloneqq \sum_{n=0}^{\infty} a_n z^n, \quad a_n \in I(\mathcal{A}) \cup \{0\}, \forall n \geq 0$$

converges (in norm of $\mathcal{A}$) for all $z \in \mathbb{D}^*(0,1)$ and $0 < \|f(z)\| < 1$ on $\mathbb{D}^*(0,1)$, then

$$\frac{1}{\|a_n^{-1}\|} \leq \frac{2}{e}, \quad \forall a_n \in I(\mathcal{A}), n \geq 1.$$

Conjecture 2.4. ***(C*-algebraic Krzyz Conjecture for general coefficients - strong form) Let $\mathcal{A}$ be a unital C*-algebra. If the C*-algebraic power series***

$$f(z) \coloneqq \sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathcal{A}, \forall n \geq 0$$

converges (in norm of $\mathcal{A}$) for all $z \in \mathbb{D}^*(0,1)$, $0 < \|f(z)\| < 1$ on $\mathbb{D}^*(0,1)$ and the C*-algebraic power series

$$f'(z) \coloneqq \sum_{n=1}^{\infty} a_n n z^{n-1}$$

also converges (in norm of $\mathcal{A}$) for all $z \in \mathbb{D}^*(0,1)$, then

$$a_n a_n^* \leq \frac{4}{e^2}, \quad a_n^* a_n \leq \frac{4}{e^2}, \quad \forall n \geq 0.$$

Conjecture 2.5. ***(C*-algebraic Krzyz Conjecture for invertible coefficients - strong form) Let $\mathcal{A}$ be a unital C*-algebra and let $I(\mathcal{A})$ be the set of all invertible elements in $\mathcal{A}$. If the C*-algebraic power series***

$$f(z) \coloneqq \sum_{n=0}^{\infty} a_n z^n, \quad a_n \in I(\mathcal{A}) \cup \{0\}, \forall n \geq 0$$

converges (in norm of $\mathcal{A}$) for all $z \in \mathbb{D}^*(0,1)$, $0 < \|f(z)\| < 1$ on $\mathbb{D}^*(0,1)$ and the C*-algebraic power series

$$f'(z) \coloneqq \sum_{n=1}^{\infty} a_n n z^{n-1}$$

also converges (in norm of $\mathcal{A}$) for all $z \in \mathbb{D}^*(0,1)$, then

$$\frac{1}{\|a_n^{-1}\|} \leq \frac{2}{e}, \quad \forall a_n \in I(\mathcal{A}), n \geq 1.$$

We end this section by asking a problem which corresponds to **Riemann Mapping Theorem** in one complex variable [31, 32, 39, 40, 61, 74, 82, 85, 112, 113]. We set the following notion of analyticity and conformality.

Definition 2.6. ***Let $\mathcal{A}$ be a unital C*-algebra and Ω be an open set in $\mathcal{A}$ (in the norm topology). A map $f : \Omega \to \mathcal{A}$ is said to be C*-algebraic holomorphic or analytic if for each $a \in \Omega$, there exists a C*-algebraic power series which converges in the norm around a, i.e., there exists a real $r > 0$, a C*-algebraic disc $\mathbb{D}^*(a,r) \coloneqq \{z \in \mathcal{A} : \|z - a\| < r\} \subseteq \Omega$ and a sequence $\{a_n\}_{n=0}^{\infty} \subseteq \mathcal{A}$ such that***

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n, \quad \forall z \in \mathbb{D}^*(a,r)$$

where the series converges in the norm of $\mathcal{A}$.

Definition 2.7. ***Let $\mathcal{A}$ be a unital C*-algebra and Ω_1, Ω_2 be open sets in $\mathcal{A}$. We say that $\Omega_!$ and Ω_2 are C*-algebraic conformal or C*-algebraic biholomorphic to each other if there***

is a bijective C*-algebraic holomorphic function $f : \Omega_1 \to \Omega_2$ ***such that*** $f^{-1} : \Omega_2 \to \Omega_1$ ***is C*-algebraic holomorphic.***

Problem 2.8. ***(C*-algebraic Riemann Mapping Problem) Let*** $\mathcal{A}$ ***be a unital C*-algebra. Classify open subsets of*** $\mathcal{A}$ ***which is C*-algebraic biholomorphic to*** $\mathbb{D}^*(0,1)$***.***

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3. C*-algebraic corona conjecture

Everything started from the paper of Carleson published in 1962 [14].

Theorem 3.1. *[13,14,66,87]* ***(Corona Conjecture/Carleson Theorem) Let*** $n \in \mathbb{N}$ ***and*** $f_1, \dots, f_n : \mathbb{D}(0,1) \to \mathbb{C}$ ***be bounded analytic functions such that***

$$\sum_{j=1}^{n} |f_j(z)|^2 \geq \delta, \quad \forall z \in \mathbb{D}(0,1)$$

for some $\delta > 0$***. Then there are bounded analytic functions*** $g_1, \dots, g_n : \mathbb{D}(0,1) \to \mathbb{C}$ ***such that***

$$\sum_{j=1}^{n} f_j(z) g_j(z) = 1, \quad \forall z \in \mathbb{D}(0,1).$$

Other proofs of Theorem 3.1 were given by Wolff [3,30,64], Slodkowski [105], Berndtsson and Ransford [8], Hormander [50], Kelleher and Taylor [60] and Jones [57]. In 1980 Rosenblum [100] and Tolokonnikov [110] proved Theorem 3.1 for countably many functions. History of Theorem 3.1 is beautifully presented in [20]. In 2007 Trent gave an algorithm to produce $g_1, \dots, g_n$ in Theorem 3.1 whenever $f_1, \dots, f_n$ are polynomials [111].
We state C*-algebraic version of Conjecture 3.1 as follows.

Conjecture 3.2. ***(C*-algebraic Corona Conjecture) Let*** $\mathcal{A}$ ***be a unital C*-algebra. Let*** $n \in \mathbb{N}$ ***and*** $f_1, \dots, f_n : \mathbb{D}^*(0,1) \to \mathcal{A}$ ***be bounded C*-algebraic holomorphic functions such that***

$$\sum_{j=1}^{n} f_j(z) f_j(z)^* \geq \delta, \quad \forall z \in \mathbb{D}^*(0,1) \quad \textbf{\textit{and}} \quad \sum_{j=1}^{n} f_j(z)^* f_j(z) \geq \delta, \quad \forall z \in \mathbb{D}^*(0,1).$$

for some real $\delta > 0$***. Then there are bounded C*-algebraic holomorphic functions*** $g_1, \dots, g_n$, $h_1, \dots, h_n : \mathbb{D}^*(0,1) \to \mathcal{A}$ ***such that***

$$\sum_{j=1}^{n} f_j(z) g_j(z) = 1, \quad \forall z \in \mathbb{D}^*(0,1)$$

and

$$\sum_{j=1}^{n} h_j(z) f_j(z) = 1, \quad \forall z \in \mathbb{D}^*(0,1).$$

Remark 3.3. *Some of the above conjectures can be formulated for unital Banach algebras as well as for Banach *-algebras.*

Remark 3.4.
(i) ***C*-algebraic Sendov Conjecture*** *has been formulated in [68].*
(ii) ***C*-algebraic Schoenberg Conjecture*** *has been formulated in [69].*
(iii) ***C*-algebraic Smale Mean Value Conjecture and Dubinin-Sugawa Dual Mean Value Conjecture*** *have been formulated in [70].*
(iv) ***C*-algebraic Casas-Alvero Conjecture*** *has been formulated in [67].*

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