# Existence of nonnegative nontrivial solutions for Kirchhoff type problems with variable exponent

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This paper is devoted to study a class of Kirchhoff type problems with variable exponent. By means of perturbation technique, variational methods and a priori estimation, the existence of nonnegative nontrivial solutions to this problem is obtain.

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**Abstract**

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**Keywords:** Kirchhoff type problems, variable exponent, variational methods, priori estimate, nonnegative nontrivial solution.

**Mathematics Subject Classification 2010:** 35J20, 35J62, 35Q55.

## 1 Introduction and main result

Let $0 \in \Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. In this paper, we consider the following Kirchhoff type problems with variable exponent

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2 dx)\Delta u = |u|^{q(x)-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $a \geq 0$, $b > 0$ are real numbers. Define $D_0 = \{0\}$, $D_\rho = \{x | x \in \overline{\Omega}, |x| < \rho\}$ and $\Omega_\rho = \Omega \setminus D_\rho$. We assume $q(x)$ satisfies the following conditions.

$(Q_1)$ $q \in C(\overline{\Omega})$, $q(0) = 4$ and $4 < q(x) \leq \max\limits_{x\in\overline{\Omega}}\{q(x)\} = q^+ < 6$ for $x \neq 0$;

$(Q_2)$ there exist $\alpha \in (0, \frac{5}{2})$ such that $q(x) \geq 4 + |x|^\alpha$ for $x \in \overline{\Omega}$.

Kirchhoff type problem on a smooth-bounded domain $\Omega \subset \mathbb{R}^3$ takes the form

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2 dx)\Delta u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

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which has been studied extensively. Indeed, such a class of problems is called a nonlocal because of the presence of the term $\int_\Omega |\nabla u|^2 dx$, which implies that equation in (1.2) is no longer a pointwise equation. Moreover, equation (1.2) is related to the stationary analogue equation, that is,

$$\begin{cases} u_{tt} - (a + b\int_\Omega |\nabla u|^2 dx)\Delta u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

which was first proposed by Kirchhoff (see [8]) in 1883 as an extension of the classical D'Alembert wave equation for free vibrations of elastic strings. The problem (1.2) has been studied by many authors, for example [2, 5, 6, 7, 10, 13, 14, 15, 16, 17, 19, 21, 22]. Many solvability conditions on the nonlinearity $f$ near zero and infinity for the problem (1.2) have been considered, such as the superlinear case [14] and asymptotical linear case [17]. In addition, the authors in [9] mentioned the following growth condition on $f$ is often assumed:

$(f)$ $\quad f(x,t)t \geq 4F(x,t)$ for $|t|$ large, where $F(x,t) = \int_0^t f(x,s)\,ds$,

which assures the boundedness of any Palais-Smale or Cerami sequence (see [20]). Indeed the condition $(f)$ may appear in different forms as follows:

$(f_0)$ $\quad$ there exists $\theta \geq 1$ such that $\theta G(x,t) \geq G(sx, st)$ for all $t \in R$ and $s \in [0,1]$, where $G(x,t) = tf(x,t) - 4F(x,t)$ for all $x \in \Omega$ (see [17]);

$(f_1)$ $\quad \lim\limits_{|t|\to\infty} G(x,t) = \infty$ for all $x \in \Omega$ (see [21]); or

$(f_2)$ $\quad \lim\limits_{|t|\to\infty} G(x,t) = \infty$ and there exists $\sigma > \max\{1, N/2\}$ such that $|f(x,t)|^\sigma \leq CG(x,t)|t|^\sigma$ for any $x \in \Omega$ and $|t|$ large enough (see [14]).

In the papers above, each of the conditions $(f_0) - (f_2)$ implies that the condition $(f)$ holds. Several researchers studied problem (1.2) with conditions weaker than $(f)$. For example, a more weaker super-quadratic condition is that

$(S)$ $\quad \frac{F(x,u)}{|u|^4} \to +\infty$ as $|u| \to +\infty$ uniformly in $x \in \Omega$.

In [3], the authors obtained the existence of one least energy sign-changing solution for problem (1.2) by condition $(S)$ and some extra assumptions. It is worth mentioning that the above conditions are usually for the non-degenerate case, i.e. the case $a > 0$. In fact, for the degenerate case $a = 0$, some conditions need to be strengthened(see [4]). For example, the condition $(f)$ should be replaced by

$(f')$ $\quad f(x,t)t > 4F(x,t)$ for $|t|$ large.

Indeed, if there exists $x_0 \in \Omega$ such that $q(x_0) = \inf_{x\in\Omega}\{q(x)\} = 4$, then the conditions $(f^{'})$ and $(S)$ do not hold. This phenomenon does not exist for the constant exponent. Therefore, the problem we intend to study is a new phenomenon. If $a = 0$, it is difficult to verify the boundedness of Palais-Smale sequence of the corresponding functional to equation (1.1). In recent years, some literature has used perturbation method to overcome this difficulty(see [11, 12, 18, 23]). Inspired by the above literature, we first modify the nonlinear term to guarantee the boundedness of Palais-Smale sequence of the corresponding functional and obtain a nonnegative nontrivial solution of perturbation problem by the method of mountain pass lemma. Subsequently, we use the Moser iteration to prove that the nonnegative nontrivial solution of auxiliary problem is indeed a nonnegative nontrivial solution of original problem (1.1).

The main result of this paper reads as follows.

**Theorem 1.1.** *Suppose that $a \geq 0$, $b > 0$, the conditions $(Q_1)$ and $(Q_2)$ hold. Then, problem (1.1) has at least a nonnegative nontrivial solution.*

**Remark 1.2.** *We use the perturbation method and Moser iteration mainly to deal with the degenerate cases. Our results include both degenerate and non-degenerate cases.*

Throughout this paper, We use $\|\cdot\|$ to denote the usual norms of $H_0^1(\Omega)$. The letter $C$ stands for positive constant which may take different values at different places.

# 2 The modified Kirchhoff problem

According to $q(0) = 4$, it seems to be difficult to confirm whether the energy function $I$ corresponding to (1.1) satisfies the Palais-Smale condition or not. To apply variational methods, the first step in proving Theorem 1.1 is modifying the nonlinear term to obtain the perturbation equation. Since $q(x)$ is a continuous function and $q^+ < 6$, we can choose $r > 0$ such that

$$r < \min\left\{6 - q^+,\ \frac{1}{12}\right\}. \tag{2.1}$$

Let $\psi(t) \in C_0^\infty(\mathbb{R}, [0,1])$ be a smooth even function with the following properties: $\psi(t) = 1$ for $|t| \leq 1$, $\psi(t) = 0$ for $|t| \geq 2$ and $\psi(t)$ is monotonically decreasing on the interval $(0, +\infty)$. Define

$$b_\mu(t) = \psi(\mu t), \qquad m_\mu(t) = \int_0^t b_\mu(\tau)d\tau,$$

for $\mu \in (0, 1]$. We will deal with the modified problem

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2 dx)\Delta u = (\frac{u}{m_\mu(u)})^r |u|^{q(x)-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

**Theorem 2.1.** *Suppose that $a \geq 0$, $b > 0$, the conditions $(Q_1)$ and $(Q_2)$ hold. Then, for any $\mu \in (0,1]$, there exists $L > 0$ independent of $\mu$ such that problem (2.2) has at least a nonnegative nontrivial solution $u_\mu$ satisfying $0 < I_\mu(u_\mu) < L$.*

The formal energy functional $I_\mu : H_0^1(\Omega) \to \mathbb{R}$ associated with (2.2) is defined by

$$I_\mu(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_\Omega K_\mu(x, u^+)\, dx,$$

where $k_\mu(x,t) = \left(\frac{t}{m_\mu(t)}\right)^r |t|^{q(x)-2}t$, $K_\mu(x,t) = \displaystyle\int_0^t k_\mu(x,\tau)\, d\tau$.

**Lemma 2.2.** *The function $K_\mu(x,t)$ defined above satisfies the following inequalities:*

$$K_\mu(x,t) \leq \frac{1}{q(x)} t k_\mu(x,t), \qquad K_\mu(x,t) \leq \frac{1}{q(x)+r} t k_\mu(x,t) + C_\mu,$$

*for $t > 0$, where $C_\mu > 0$ is a positive constant.*

**Proof.** Since $b_\mu(t)$ is a monotonically decreasing on the interval $(0,+\infty)$, we have

$$\frac{d}{dt}\left(\frac{t}{m_\mu(t)}\right) = \frac{m_\mu(t) - t b_\mu(t)}{m_\mu^2(t)} = \frac{t(b_\mu(\xi) - b_\mu(t))}{m_\mu^2(t)} \geq 0,$$

for $t > 0$, where $\xi \in (0,t)$. Therefore, $\frac{t}{m_\mu(t)}$ is monotonically increasing on the interval $(0,+\infty)$. Hence, $\frac{k_\mu(x,t)}{t^{q(x)-1}} = \left(\frac{t}{m_\mu(t)}\right)^r$ is also monotonically increasing on the interval $(0,+\infty)$. It follows that

$$K_\mu(x,t) = \int_0^t k_\mu(x,\tau) d\tau \leq \int_0^t \frac{k_\mu(x,t)}{t^{q(x)-1}} \tau^{q(x)-1} d\tau = \frac{1}{q(x)} t k_\mu(x,t), \tag{2.3}$$

for $t > 0$.

By definition of the function $m_\mu$, we have $m_\mu(t) = \frac{A}{\mu}$ for $t \geq \frac{2}{\mu}$, where $A = 1 + \int_1^2 \psi(\tau) d\tau$. For $t > \frac{2}{\mu}$, one has

$$\begin{aligned}
K_\mu(x,t) &= \int_0^{\frac{2}{\mu}} k_\mu(x,\tau)\, d\tau + \int_{\frac{2}{\mu}}^t \left(\frac{\mu}{A}\right)^r \tau^{q(x)+r-1}\, d\tau \\
&= \int_0^{\frac{2}{\mu}} \left(k_\mu(x,\tau) - \left(\frac{\mu}{A}\right)^r \tau^{q(x)+r-1}\right) d\tau + \int_0^t \left(\frac{\mu}{A}\right)^r \tau^{q(x)+r-1}\, d\tau \\
&\leq C_\mu + \frac{t k_\mu(x,t)}{q(x)+r}.
\end{aligned} \tag{2.4}$$

It implies from (2.3) and (2.4) that

$$K_\mu(x,t) \leq \frac{1}{q(x)+r} t k_\mu(x,t) + C_\mu$$

for $t > 0$. $\square$

**Lemma 2.3.** *Suppose that $a \geq 0$, $b > 0$, the conditions $(Q_1)$ and $(Q_2)$ hold. Then, for any $\mu \in (0, 1]$, $I_\mu$ satisfies the $(PS)$ condition.*

**Proof.** Let $\{u_n\}$ be a $(PS)$ sequence of $I_\mu$ in $H_0^1(\Omega)$. This means that there exists $C > 0$ such that

$$|I_\mu(u_n)| \leq C, \qquad I_\mu'(u_n) \to 0 \text{ as } n \to \infty. \tag{2.5}$$

From (2.5) and Lemma 2.2, we derive that

$$\begin{aligned} & I_\mu(u_n) - \frac{1}{4+r}\langle I_\mu'(u_n), u_n\rangle \\ \geq \; & \frac{(2+r)a}{2(4+r)}\|u_n\|^2 + \frac{br}{4(4+r)}\|u_n\|^4 + \int_\Omega \left(\frac{1}{4+r} - \frac{1}{q(x)+r}\right) k_\mu(x, u_n^+)u_n^+ dx - C_\mu \\ \geq \; & \frac{br}{4(4+r)}\|u_n\|^4 - C_\mu, \end{aligned}$$

which implies that $\frac{br}{4(4+r)}\|u_n\|^4 \leq C + C_\mu + o(\|u_n\|)$. We obtain $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Since the functional $I_\mu(u)$ is of subcritical growth, by a standard argument $I_\mu(u)$ satisfies the $(PS)$ condition.

□

In the following lemma, we will verify that $I_\mu$ possesses the mountain pass geometry.

**Lemma 2.4.** *Suppose that $a \geq 0$, $b > 0$, the conditions $(Q_1)$ and $(Q_2)$ hold. Then, the functional $I_\mu$ possesses the mountain pass geometry, namely,*
*(1) there exist $m,\ \rho > 0$ such that $I_\mu(u) > m$ for any $u \in H_0^1(\Omega)$ with $\|u\| = \rho$;*
*(2) there exists $w \in H_0^1(\Omega)$ such that $\|w\| > \rho$ and $I_\mu(w) < 0$.*

**Proof.** By definition of the function $k_\mu$, we have

$$|k_\mu(x, t)| \leq |t|^{q(x)-1} + \left(\frac{\mu}{A}\right)^r |t|^{q(x)+r-1}.$$

It follows that

$$|K_\mu(x, t)| \leq \frac{|t|^{q(x)}}{q(x)} + \left(\frac{\mu}{A}\right)^r \frac{|t|^{q(x)+r}}{q(x)+r}.$$

Therefore, there exists $C_\mu > 0$ such that

$$\left|\int_\Omega K_\mu(x, u^+)\, dx\right| \leq C_\mu \int_\Omega (|u|^{q(x)} + |u|^{q(x)+r})\, dx. \tag{2.6}$$

By the Sobolev imbedding theorem, it implies from $4 \leq q(x) < q(x) + r < 6$ that

$$\int_\Omega |u|^{q(x)+r}\, dx \leq \int_\Omega (|u|^{4+r} + |u|^6)\, dx \leq C(\|u\|^{4+r} + \|u\|^6). \tag{2.7}$$

Set $V_\varepsilon = \{x \in \Omega | 4 \le q(x) < 4+\varepsilon\}$. By the Hölder inequality and the Sobolev imbedding theorem, we have

$$\begin{aligned}\int_\Omega |u|^{q(x)}\,dx &= \int_{V_\varepsilon} |u|^{q(x)}\,dx + \int_{\Omega\setminus V_\varepsilon} |u|^{q(x)}\,dx \\ &\le \int_{V_\varepsilon} (|u|^4 + |u|^{4+\varepsilon})\,dx + \int_{\Omega\setminus V_\varepsilon} (|u|^{4+\varepsilon} + |u|^6)\,dx \\ &\le \int_{V_\varepsilon} |u|^4\,dx + \int_\Omega (|u|^{4+\varepsilon} + |u|^6)\,dx \\ &\le C|V_\varepsilon|^{\frac{1}{3}}\|u\|^4 + C(\|u\|^{4+\varepsilon} + \|u\|^6). \end{aligned} \tag{2.8}$$

Since $V_0 = \{0\}$, we obtain $|V_\varepsilon| \to 0$ as $\varepsilon \to 0$. Fix $\mu \in (0,1]$, it implies that there exists $\varepsilon_0 > 0$ such that

$$|V_\varepsilon|^{\frac{1}{3}} < \frac{b}{8CC_\mu}, \tag{2.9}$$

for any $\varepsilon \in (0, \varepsilon_0)$. From (2.6)-(2.9), we obtain

$$I_\mu(u) \ge \frac{b}{8}\|u\|^4 - C_\mu(\|u\|^{4+\varepsilon} + \|u\|^{4+r} + \|u\|^6).$$

Therefore, there exist $m,\ \rho > 0$ such that $I_\mu(u) > m$ for any $u \in H_0^1(\Omega)$ with $\|u\| = \rho$.

By definition of the function $k_\mu$, we obtain $k_\mu(x,t) \ge t^{q(x)-1}$. According to $(Q_1)$, we know that there exists a positive measurable set $U_0 \subset \Omega$ such that

$$q(x) \ge \frac{4+q^+}{2} \quad \text{for any} \quad x \in U_0. \tag{2.10}$$

Fix a nonnegative function $v_0 \in H_0^1(U_0)\setminus\{0\}$. Then, for $t > 0$ sufficiently large, we obtain

$$I_\mu(tv_0) \le \frac{at^2}{2}\|v_0\|^2 + \frac{bt^4}{4}\|v_0\|^4 - t^{\frac{4+q^+}{2}} \int_\Omega \frac{|v_0|^{q(x)}}{q(x)}\,dx < 0.$$

Choosing $w = t_0 v_0$ with $t_0 > 0$ large enough, we have $\|w\| > \rho$ and $I_\mu(w) < 0$. $\square$

Now we are in a position to prove the main result of this section.

**Proof of Theorem 2.1.** From Lemmas 2.3 and 2.4 we see that the functional $I_\mu$ satisfies the $(PS)$ condition and has the mountain pass geometry. Define

$$\Gamma = \{\gamma \in C([0,1], H_0^1(\Omega)) |\ \gamma(0) = 0,\ \gamma(1) = w\}, \quad c_\mu = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\mu(\gamma(t)).$$

By the mountain pass theorem(see [1]), we obtain that problem (2.2) has a solution $u_\mu$. After a direct calculation, we derive that $a\|u_\mu^-\|^2 + b\|u_\mu^-\|^4 = \langle I'_\mu(u_\mu), u_\mu^-\rangle = 0$, which implies that $u_\mu^- = 0$. Hence, $u_\mu \ge 0$. Since $I_\mu(u_\mu) = c_\mu > 0 = I_\mu(0)$, we have $u_\mu \ne 0$.

So we obtain $u_\mu$ is a nonnegative nontrivial solution of problem (2.2). It follows from (2.10) that

$$I_\mu(u_\mu) = c_\mu \leq \max_{t\in[0,1]} I_\mu(tw) \leq \max_{t\in[0,1]} \left( \frac{at^2}{2}\|w\|^2 + \frac{bt^4}{4}\|w\|^4 - t^{\frac{4+q^+}{2}} \int_\Omega \frac{|w|^{q(x)}}{q(x)}\, dx \right) = L,$$

where $L$ independent of $\mu$. The proof is complete. □

# 3 A priori estimate and proof of Theorem 1.1

In this section, we will show that solutions of auxiliary problem (2.2) are indeed solutions of original problem (1.1). For this purpose, we need the following uniform $L^\infty$-estimate for critical points of the functional $I_\mu$.

**Lemma 3.1.** *Suppose that $a \geq 0$, $b > 0$, the conditions $(Q_1)$ and $(Q_2)$ hold. If $I_\mu(v) \leq L$ and $I'_\mu(v) = 0$, then there exists $C = C(L) > 0$ independent of $\mu$ such that $\int_\Omega |\nabla v|^2 dx \leq C$.*

**Proof.** By Lemma 2.2, $I_\mu(v) \leq L$ and $I'_\mu(v) = 0$, we have

$$\begin{aligned} L &\geq I_\mu(v) - \frac{1}{4}\langle I'_\mu(v), v\rangle \\ &\geq \frac{a}{2}\|v\|^2 + \int_\Omega \left(\frac{1}{4} - \frac{1}{q(x)}\right) k_\mu(x,v) v dx. \end{aligned} \tag{3.1}$$

Case 1. If $a > 0$, then according to $(Q_1)$ and (3.1), we obtain $\|v\|^2 \leq \frac{2L}{a}$, then there exist $C = C(L) > 0$ independent of $\mu$ such that $\int_\Omega |\nabla v|^2 dx \leq C$.
Case 2. If $a = 0$, then according to $(Q_2)$ and (3.1), there exists $C_0 > 0$ such that

$$\begin{aligned} L &\geq \int_\Omega \left(\frac{1}{4} - \frac{1}{q(x)}\right) k_\mu(x,v) v dx \\ &\geq C_0 \int_\Omega |x|^\alpha k_\mu(x,v) v dx. \end{aligned} \tag{3.2}$$

Therefore, for any $\rho > 0$, we know that there exists $m_\rho > 0$ such that

$$\int_{\Omega_\rho} k_\mu(x,v) v dx \leq \frac{1}{\rho^\alpha} \int_{\Omega_\rho} |x|^\alpha k_\mu(x,v) v dx \leq m_\rho, \tag{3.3}$$

and

$$\int_{D_\rho} |x|^\alpha |v|^{q(x)} dx \leq \int_{D_\rho} |x|^\alpha k_\mu(x,v) v dx \leq L/C_0. \tag{3.4}$$

From $(Q_2)$ and (2.1), we have

$$0 < \frac{\alpha(1+r)}{2-(1+r)} = \frac{\alpha(1+r)}{1-r} < \frac{13\alpha}{11} < 3.$$

Therefore, we can choose $p \in \left(1, \frac{3}{2}\right)$ satisfying

$$\frac{p(1-r)}{p-1} > 2 \quad \text{and} \quad 0 < \frac{p\alpha(1+r)}{2+r-p(1+r)} < \frac{23}{8} < 3. \tag{3.5}$$

Let $q_\rho^+ = \sup\{q(x)|x \in D_\rho\}$. It follows from $(Q_1)$ and (3.5) that

$$q_\rho^+ \leq \frac{p(1-r)}{p-1} \quad \text{and} \quad 0 < \frac{p\alpha(q_\rho^+ - 1 + r)}{q_\rho^+ - p(q_\rho^+ - 1 + r)} < 3 \tag{3.6}$$

Using the Young inequality, we deduce from (3.4) and (3.6) that

$$\begin{aligned}
\int_{D_\rho} |v|^{p(q(x)+r-1)}dx &= \int_{D_\rho} \left(|x|^\alpha |v|^{q(x)}\right)^{\frac{p(q(x)+r-1)}{q(x)}} \cdot |x|^{-\frac{p\alpha(q(x)+r-1)}{q(x)}} dx \\
&\leq C\int_{D_\rho} |x|^\alpha |v|^{q(x)}dx + C\int_{D_\rho} |x|^{-\frac{p\alpha(q(x)+r-1)}{q(x)-p(q(x)+r-1)}}dx \\
&\leq C\int_{D_\rho} |x|^\alpha |v|^{q(x)}dx + C\int_{D_\rho} |x|^{-\frac{p\alpha(q_\rho^+ +r-1)}{q_\rho^+ -p(q_\rho^+ +r-1)}}dx \\
&\leq \frac{CL}{C_0} + C\rho^{3-\frac{p\alpha(q_\rho^+ +r-1)}{q_\rho^+ -p(q_\rho^+ +r-1)}} \\
&\leq C.
\end{aligned} \tag{3.7}$$

Choose $r$ and $\rho$ such that $6(q(x) + r - 4) \leq p(q(x) + r - 1)$. By the Hölder inequality, we deduce from (3.7) that

$$\begin{aligned}
\int_{D_\rho} |v|^{q(x)+r}dx &\leq \left(\int_{D_\rho} |v|^6 dx\right)^{\frac{4}{6}} \left(\int_{D_\rho} |v|^{6(q(x)+r-4)}dx\right)^{\frac{1}{6}} \left(\int_{D_\rho} dx\right)^{\frac{1}{6}} \\
&\leq C\|v\|_6^4 \left(\int_{D_\rho} (1 + |v|^{p(q(x)+r-1)})dx\right)^{\frac{1}{6}} |D_\rho|^{\frac{1}{6}} \\
&\leq C\rho^{\frac{1}{2}}\|v\|_6^4.
\end{aligned} \tag{3.8}$$

Since $v \in H_0^1(\Omega)$ is a solution of problem (2.2), when $a = 0$, we have

$$-(b\int_\Omega |\nabla v|^2 dx)\triangle v = k_\mu(x, v), \text{ in } \Omega. \tag{3.9}$$

Multiply problem (3.9) by $v$ and integrate, it implies from (3.3) and (3.8) that

$$\begin{aligned} b\|v\|^4 &= \int_\Omega k_\mu(x,v)vdx \\ &= \int_{\Omega\setminus\Omega_\rho} k_\mu(x,v)vdx + \int_{\Omega_\rho} k_\mu(x,v)vdx \\ &\le \int_{D_\rho} |v|^{q(x)+r}dx + m_\rho \\ &\le C\rho^{\frac{1}{2}}\|v\|_6^4 + m_\rho \\ &\le C\rho^{\frac{1}{2}}(S_3^{-1})^2\|v\|^4 + m_\rho \\ &\le C\rho^{\frac{1}{2}}\|v\|^4 + m_\rho, \end{aligned} \tag{3.10}$$

where $S_3$ is the best embedding constant for $H_0^1(\Omega) \to L^6(\Omega)$. We can choose $\rho_0 > 0$ sufficiently small such that

$$b - C\rho_0^{\frac{1}{2}} > 0.$$

It follows from (3.10) that

$$\int_\Omega |\nabla v|^2 dx \le C.$$

□

**Lemma 3.2.** *Suppose that $a \ge 0$, $b > 0$, the conditions $(Q_1)$ and $(Q_2)$ hold. If $v$ is a critical point of $I_\mu$ with $I_\mu(v) \le L$, then there exists a positive constant $M = M(L)$ independent of $\mu$ such that $\|V\|_\infty \le M$.*

**Proof.** Using the Sobolev embedding theorem, we have

$$\int_\Omega |v|^6 dx \le C\left(\int_\Omega |\nabla v|^2 dx\right)^3 \le C. \tag{3.11}$$

Let $s > 0$ and $t = q^+ + r$. We have

$$\int_\Omega \nabla\varphi\nabla\varphi^{2s+1}dx = (2s+1)\int_\Omega |\nabla\varphi|^2\varphi^{2s}dx > 0,$$

for any $\varphi \in H_0^1(\Omega)$. Since $v$ is a solution of problem (2.2), multiply problem (2.2) by $v^{2s+1}$ and integrate to obtain

$$\begin{aligned} b\|v\|^2\int_\Omega \nabla v\nabla v^{2s+1}dx &\le (a+b\|v\|^2)\int_\Omega \nabla v\nabla v^{2s+1}dx \\ &= \int_\Omega k_\mu(x,v)v^{2s+1}dx \\ &\le C\left(\int_\Omega |v|^2dx + \int_\Omega |v|^{2s+t}dx\right). \end{aligned}$$

It implies that

$$
\begin{aligned}
b\|v\|^2 \int_\Omega |\nabla v|^2 v^{2s} dx &= \frac{b\|v\|^2}{2s+1} \int_\Omega \nabla v \nabla v^{2s+1} dx \\
&\leq C\left( \int_\Omega |v|^2 dx + \int_\Omega |v|^{2s+t} dx \right). \qquad (3.12)
\end{aligned}
$$

On the one hand, by the Sobolev embedding theorem, we have

$$
\begin{aligned}
b\|v\|^2 \int_\Omega |\nabla v|^2 v^{2s} dx &= \frac{b\|v\|^2}{(1+s)^2} \int_\Omega |\nabla v^{1+s}|^2 dx \\
&\geq \frac{b\|v\|^2}{(1+s)^2} C \left( \int_\Omega |v|^{6(1+s)} dx \right)^{\frac{1}{3}}. \qquad (3.13)
\end{aligned}
$$

On the other hand, by the Hölder inequality and (3.11), we have

$$
\begin{aligned}
\int_\Omega |v|^{2s+t} dx &\leq \left( \int_\Omega |v|^6 dx \right)^{\frac{t-2}{6}} \left( \int_\Omega |v|^{2(1+s)\frac{6}{8-t}} dx \right)^{\frac{8-t}{6}} \\
&\leq C\|v\|^{t-2} \left( \int_\Omega |v|^{(1+s)\frac{6}{d}} dx \right)^{\frac{d}{3}}, \qquad (3.14)
\end{aligned}
$$

where $d = \frac{8-t}{2} > 1$. And by the Sobolev embedding theorem, we have

$$
\int_\Omega |v|^2 dx \leq C\|v\|^2. \qquad (3.15)
$$

According to (3.12), (3.13), (3.14) and (3.15), we obtain

$$
\begin{aligned}
b\|v\|^2 \left( \int_\Omega |v|^{6(1+s)} dx \right)^{\frac{1}{3}} &\leq \frac{(1+s)^2}{C} b\|v\|^2 \int_\Omega |\nabla v|^2 v^{2s} dx \\
&\leq \frac{(1+s)^2}{C} \cdot C \left( \int_\Omega |v|^2 dx + \int_\Omega |v|^{2s+t} dx \right) \\
&\leq \frac{(1+s)^2}{C} \cdot C \left( C\|v\|^2 + C\|v\|^{t-2} \left( \int_\Omega |v|^{(1+s)\frac{6}{d}} dx \right)^{\frac{d}{3}} \right),
\end{aligned}
$$

then, by Lemma 3.1, when $b > 0$, we have

$$
\begin{aligned}
\left( \int_\Omega |v|^{6(1+s)} dx \right)^{\frac{1}{3}} &\leq (C(1+s))^2 \left( 1 + \|v\|^{t-4} \left( \int_\Omega |v|^{(1+s)\frac{6}{d}} dx \right)^{\frac{d}{3}} \right) \\
&\leq (C(1+s))^2 \left( 1 + (\int_\Omega |v|^{(1+s)\frac{6}{d}} dx)^{\frac{d}{3}} \right).
\end{aligned}
$$

Notice that $\frac{1}{2(1+s)} < 1$, we have

$$
\begin{aligned}
&\left(\int_\Omega |v|^{6(1+s)}dx\right)^{\frac{1}{6(1+s)}} \\
\leq\ & (C(1+s))^{\frac{1}{1+s}}\left(1+\left(\int_\Omega |v|^{(1+s)\frac{6}{d}}dx\right)^{\frac{d}{6(1+s)}}\right) \\
\leq\ & (2C(1+s))^{\frac{1}{1+s}}\max\left\{1,\left(\int_\Omega |v|^{(1+s)\frac{6}{d}}dx\right)^{\frac{d}{6(1+s)}}\right\}.
\end{aligned} \tag{3.16}
$$

Now we carry out an iteration process. Set $s_k = d^k - 1$ for $k = 1, 2, \cdots$. By (3.16), we have

$$
\begin{aligned}
\left(\int_\Omega |v|^{6d^k}dx\right)^{\frac{1}{6d^k}} &\leq (2Cd^k)^{\frac{1}{d^k}}\max\left\{1,\left(\int_\Omega |v|^{6d^{k-1}}dx\right)^{\frac{1}{6d^{k-1}}}\right\} \\
&\leq \Pi_{j=1}^k (2Cd^j)^{\frac{1}{d^j}}\max\left\{1,\left(\int_\Omega |v|^6dx\right)^{\frac{1}{6}}\right\} \\
&\leq (2C)^{\sum_{j=1}^k d^{-j}}\cdot d^{\sum_{j=1}^k jd^{-j}}\max\left\{1,\left(\int_\Omega |v|^6dx\right)^{\frac{1}{6}}\right\}.
\end{aligned} \tag{3.17}
$$

Since $d > 1$, the series $\sum\limits_{j=1}^{\infty} d^{-j}$ and $\sum\limits_{j=1}^{\infty} jd^{-j}$ are convergent. Letting $k \to \infty$, we conclude from (3.11) and (3.17) that $\|v\|_\infty \leq M$. The proof is complete. □

**Proof of Theorem 1.1.** By Theorem 2.1, we know that problem (2.2) has at least a nonnegative nontrivial solution $u_\mu$ satisfying $0 < I_\mu(u_\mu) < L$. By definition of the function $m_\mu$, we have $m_\mu(t) = t$ for $t \leq \frac{1}{\mu}$. Hence, problem (2.2) reduce to problem (1.1) for $|u| \leq \frac{1}{\mu}$. Let $\mu < \frac{1}{2M}$. By Lemma 3.2, we have

$$
|u_\mu| \leq \frac{1}{\mu}, \tag{3.18}
$$

then $u_\mu$ is indeed a nonnegative nontrivial solution of problem (1.1). The proof of Theorem 1.1 is now complete. □

# Availability of data and material

The authors declare that our manuscript has no associated date.

# Competing interests

The authors declare that they have no competing interests.

# Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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