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On the asymptotic behaviour of difference equations: piecewise-linear analysis

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Abstract

In this paper, general one-dimensional difference equations (DE) are studied. In fact, the asymptotic behaviour of such DE's will be studied via an algebraic closed-form condition plus a linear DE. Some examples are presented including the Collatz sequence, proving the existence of at most one invariant set surrounding the number 1: $\{1, 2, 4\}$. Some conclusions will be also depicted.

Keywords: Difference equations, asymptotic solutions, piecewise-linear

1 Introduction

In this short note, the asymptotic behaviour of the following general one-dimensional difference equations (DE) will be studied:

$$x(k+1) = g(x(k)), \quad k \in \mathbb{N} \cup 0, \quad g : \Omega \subset \Re \to \Omega \subset \Re \tag{1}$$

A sufficient condition for the asymptotic analysis of the given DE will be provided based on a closed-form formula plus a linear DE.

1.1 Asymptotic solutions of Difference Equations

The asymptotic behaviour of one-dimensional DE's given in (1) can be studied via the following main theorem:

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Theorem 1.1 *Given a one-dimensional DE (1), if the following condition is satisfied:*

$$\Omega_1 = \left\{ x : \frac{1}{a} \cdot ln\left(\frac{g(x)+\frac{1}{a}}{x+\frac{1}{a}}\right) > 0, a = 1 \right\},$$
$$\Omega_2 = \left\{ x : \frac{1}{a} \cdot ln\left(\frac{g(x)+\frac{1}{a}}{x+\frac{1}{a}}\right) > 0, a = -1 \right\}$$
$$\Omega_1 \cup \Omega_2 = \Omega$$

Then, the asymptotic behaviour: $lim_{k\to\infty} x(k)$ *is captured by:*

- *Bounded trajectories:*

$$x(k) < \infty \sim \textit{Fixed point} \quad (k \to \infty)$$
$$\textit{or/and}$$
$$x(k) < \infty \sim \textit{An invariant set containing } \pm 1 \quad (k \to \infty)$$

- *Unbounded trajectories:*

$$x(k) \to \infty \Leftrightarrow z(k+1) = b \cdot z(k), \quad z(k) \to \infty, \quad b = lim_{x\to\infty} \frac{g(x)}{x}$$

Proof 1.2 *Let's consider a piecewise linear 2-D system as follows:*

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = a \cdot x_2 + 1, \quad a = \pm 1 \end{cases} \tag{2}$$

Then, defining an impact curve or a curve where parameter a changes from $a = +1$ *to* $a = -1$ *and vice-versa (piecewise linear changing rule):* $x_1 = \phi(x_2, k)$*, with the initial condition:*

$$x_1(0) = \phi(x_2(0), 0)$$

It is possible to write the consecutive impact's location of system's (2) over the curve $x_1 = \phi(x_2, k)$ *as time goes on:*

$$\begin{cases} x_1(k+1) - \frac{x_2(k+1)}{a} = -\frac{1}{a} \cdot T_k + \left(x_1(k) - \frac{x_2(k)}{a}\right) \\ x_2(k+1) = e^{a \cdot T_k} \cdot x_2(k) + \left(e^{a \cdot T_k} - 1\right) \cdot \frac{1}{a} \end{cases} \tag{3}$$

Where $\{x_1(k), x_2(k)\}$ *denotes the points:* $x_1(T_k) = \phi(x_2(T_k), k)$ *with* T_k *the successive flying times from* $x_1(T_{k-1}) = \phi(x_2(T_{k-1}), k-1)$ *to the next* $x_1(T_k) = \phi(x_2(T_k, k))$ *and a alternating from* $a = +1$ *to* $a = -1$*.*

Rewriting (3) as follows:

$$\begin{cases} T_k = -a \cdot (x_1(k+1) - x_1(k)) + (x_2(k+1) - x_2(k)) \\ x_2(k+1) = e^{a \cdot T_k} \cdot x_2(k) + \left(e^{a \cdot T_k} - 1\right) \cdot \frac{1}{a} \end{cases}$$

Taking into account that $x_1(k+1) = \phi(x_2(k+1), k+1)$*,* $x_1(k) = \phi(x_2(k), k)$*:*

$$\begin{cases} T_k = -a \cdot \underbrace{(\phi(x_2(k+1), k+1) - \phi(x_2(k), k)}_{\Delta\phi} + \underbrace{(x_2(k+1) - x_2(k))}_{\Delta x_2} \\ x_2(k+1) = e^{a \cdot T_k} \cdot x_2(k) + \left(e^{a \cdot T_k} - 1\right) \cdot \frac{1}{a} \end{cases} \tag{4}$$

In other words:

$$T_k = -a \cdot \Delta\phi + \Delta x_2 \geq 0, \quad \forall x \in \Omega \subset \Re \tag{5}$$

$$ln\left(\frac{x_2(k+1) + \frac{1}{a}}{x_2(k) + \frac{1}{a}}\right) = a \cdot T_k \tag{6}$$

Where $ln(.)$ *is the natural logarithm function. Eliminating* T_k *from the equation:*

$$\Delta\phi = \frac{\Delta x_2}{a} - ln\left(\frac{x_2(k+1) + \frac{1}{a}}{x_2(k) + \frac{1}{a}}\right)$$

These DE can be made to exactly match the given: $x(k+1) = g(x(k))$ *by replacing* $x \to x_2$*, if and only if, a solution to this DE exists for the function* $\phi(x_2, k)$*. Moreover, such a solution can be obtained in closed form to be:*

$$\phi(x_2, k) = \frac{x_2}{a} - ln\left(x_2 + \frac{1}{a}\right) + r(k)$$

Where $r(k) \in \mathbb{C}$ *is a (possible) complex function matching the given DE* $x(k+1) = g(x(k))$*.*

Once the existence solution is proved, continuous impacts over the curve $x_1 = \phi(x_2, k)$ *must be also ensured, so replacing into (6), the flying times* T_k *must be positive:*

$$T_k = \frac{1}{a} \cdot ln\left(\frac{\underbrace{x(k+1)}_{g(x)} + \frac{1}{a}}{x + \frac{1}{a}}\right) \geq 0, \quad \forall x \in \Omega \subset \Re \tag{7}$$

This is not more than the sufficient condition to satisfy. On the other hand, the possible scenarios for the impact curve $x_1(k) = \phi(x_2(k))$ *as* $k \to \infty$ *are as follows:*

- $x_1(k) \to \infty, \quad x_2(k) < \infty, \quad k \to \infty$

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The case $x_1(k) \to \infty, \quad x_2(k) < \infty, \quad k \to \infty$

$$x_1(k) = \phi(x_2, k) = \frac{x_2}{a} - ln\left(x_2 + \frac{1}{a}\right) + r(k) \sim r(k) \quad (k \to \infty)$$

That means:

$$\Delta\phi(x_2) \sim 0 \quad (k \to \infty)$$

Recalling (5):

$$T_k \sim \Delta x_2$$

On the other hand and from (2):

$$\Delta x_2 = \int_0^{T_k} (a \cdot x_2 + 1) \cdot dt = \rho(T_k) - \rho(0)$$

Where $\rho(t) = \int_0^t (a \cdot x_2 + 1) \cdot d\sigma$. *Finally:*

$$T_k \sim \Delta x_2 = \rho(T_k) - \rho(0)$$

That means:

$$\rho(t) \sim t \Rightarrow \frac{d\rho(t)}{dt} = 1 \sim a \cdot x_2 + 1$$

In other words: $x_2(k) \sim 0 \quad (k \to \infty)$.

The case $x_1(k) < \infty, x_2(k) < \infty, \quad k \to \infty$*:*

On the other hand: $\dot{x_1} = x_2$*, looking for invariant sets (*x_1 *and* x_2 *bounded), three possible scenarios come across:*

- $x_2 < -1$
- $x_2 \in [-1, 1]$
- $x_2 > 1$

The first and third cases are not possible for bounded orbits: $\dot{x_1} = x_2 < 0$ *or* $\dot{x_1} = x_2 > 0$ *respectively, so:* $x_1 \to \infty$ *(unbounded).*

The second possibility above implies the dynamics $\dot{x_2} = a \cdot x_2 + 1$ *increases* $x_2(t)$ *until reaching the border* $x_2 = +1$*. In conclusion, fixed points (*$+1$*) or invariant sets containing the points* ± 1 *are possible (under appropriate switching conditions).*

The case $x_2(k) \to \infty, \quad k \to \infty$*:*
In this case, from (4):

$$x_2(k+1) = e^{a \cdot T_k} \cdot x_2(k) + \frac{\left(e^{a \cdot T_k} - 1\right)}{a} \Rightarrow \frac{x_2(k+1)}{x_2(k)} = e^{a \cdot T_k} + \frac{\left(e^{a \cdot T_k} - 1\right)}{a \cdot x_2(k)}$$

Then, two sub cases are in order:

$$T_k < \infty \Rightarrow \frac{x_2(k+1)}{x_2(k)} \sim e^{a \cdot T_k} < \infty$$

Taking into account $x_2(k+1) = g(x_2(k))$*:* $\frac{g(x_2(k))}{x_2(k)} \sim e^{a \cdot T_k} < \infty$*, if unbounded orbits do exists, these orbits are captured by:*

$$x_2(k+1) \sim b \cdot x_2(k) \quad (x_2 \to \infty)$$
$$b = lim_{x_2 \to \infty} \frac{g(x_2)}{x_2}$$

It turns out that in cases where this asymptotic DE posses no unbounded orbits, then the analysis return to previous cases.
The second possibility leads:

$$T_k \to \infty \Rightarrow \frac{x_2(k+1)}{x_2(k)} \sim 0, \quad a = -1$$
$$T_k \to \infty \Rightarrow \frac{x_2(k+1)}{x_2(k)} \sim e^{T_k} \cdot \left(1 + \frac{1}{x_2(k)}\right), \quad (x_2 \to \infty), \quad a = 1$$

Recalling (7), then:

$$\frac{x_2(k+1)}{x_2(k)} \sim \frac{\left(\frac{x_2(k+1)+1}{x_2(k)+1}\right)^{\frac{1}{a=1}}}{x_2(k)} \quad (x_2 \to \infty)$$

Finally:

$$\frac{x_2(k+1)}{x_2(k)} \sim \frac{x_2(k+1)}{x_2(k)^2} \Leftrightarrow x_2(k) \sim 1$$

In other words: bounded trajectories or a contradiction to the unbounded hypothesis. This completes the proof.

2 Examples

2.1 An example from [1]

Considering the Example 6.1 in [1]:

$$x(k+1) = x(k)^2, \quad x(k) \in \Re$$

The sufficient condition in Theorem 1.1 reads as follows:

$$\frac{1}{a} \cdot ln\left(\frac{x^2 + \frac{1}{a}}{x + \frac{1}{a}}\right) > 0, \quad a = \pm 1 \quad \forall x \in \Omega$$

In fact, for $a = +1$:

$$ln\left(\frac{x^2+1}{x+1}\right) > 0 \Leftrightarrow \frac{x^2+1}{x+1} > 1, \forall x > -1$$

Notice that, for $x \leq -1$ nothing can be said with this theorem. Once this condition is satisfied, unbounded trajectories are concluded:

$$x(k) \to \infty \Leftrightarrow z(k+1) = \infty \cdot z(k), \quad z(k) \to \infty, \quad lim_{x \to \infty} \frac{x^2}{x} = \infty$$

This result agrees with the conclusions in [1]

2.2 The Collatz sequence

The Collatz sequence can be recast as a DE as follows (see for instance [4] and [3]):

$$x(k+1) = (3 \cdot x(k) + 1) \cdot \phi(x(k)) + \frac{x(k)}{2} \cdot (1 - \phi(x(k))), \quad x \in \mathbb{N}$$

where $\phi(x) = \begin{cases} 1, & x = odd \\ 0, & x = even \end{cases}$. Then, the sufficient condition in Theorem 1.1 leads:

$$\frac{1}{a} \cdot ln\left(\frac{(3 \cdot x + 1) \cdot \phi(x) + \frac{x}{2} \cdot (1 - \phi(x)) + \frac{1}{a}}{x + \frac{1}{a}}\right) > 0, \quad a = \pm 1 \quad \forall x \in \mathbb{N}$$

That is:

$$ln\left(\frac{(3 \cdot x + 1) + 1)}{x + 1}\right) > 0, \quad a = +1 \Leftrightarrow \frac{3 \cdot x + 2}{x + 1} > 1 \quad \forall x(odd) \in \mathbb{N}$$
$$-ln\left(\frac{\frac{x}{2} - 1}{x - 1}\right) > 0, \quad a = -1 \Leftrightarrow \frac{\frac{x}{2} - 1}{x - 1} < 1 \quad \forall x(even) \geq 2 \in \mathbb{N}$$

Once this condition is satisfied $\forall x \in \mathbb{N}$, the asymptotic behaviour can be examined:

Bounded trajectories:

Since no equilibrium points are possible in this DE:

$$x(k) < \infty \sim \text{An invariant set containing } \pm 1 \quad (k \to \infty)$$

The only invariant set for this DE is in fact: $\{1, 4, 2\}$
Unbounded trajectories:
In this case:

$$lim_{x\to\infty} \frac{3 \cdot x + 1}{x} = 3$$
$$lim_{x\to\infty} \frac{\frac{x}{2}}{x} = \frac{1}{2}$$

The asymptotic equivalent DE looks like: $z(k+1) = \{3, \frac{1}{2}\} \cdot z(k), \quad (k \to \infty)$. On the other hand, given $x = 2 \cdot p + 1, \quad p \in \mathbb{N}$ (odd number):

$$x(k+1) = 3 \cdot (2 \cdot p + 1) + 1 = 6 \cdot p + 4 = 2 \cdot (3 \cdot p + 2) \Leftrightarrow (even)$$

This conclusion means that, given an odd number $x(k)$, the next number $x(k+1)$ will be even, so the sequence $\{3, \frac{1}{2}\}$, is out of at least one number $\frac{1}{2}$ for each number 3, so the only possible case asymptotically equivalent to infinity is the sequence: $\{3, \frac{1}{2}, 3, \frac{1}{2}, \ldots\}$ (otherwise, the product $3 \cdot \frac{1}{2} \cdot \frac{1}{2} \ldots < 1$ and the equivalent DE tends to zero, see for instance [5], Theorem 6).

That is:

$$x(k+1) = \frac{3 \cdot x(k) + 1}{2}, \quad \forall x(0) \quad \text{Odd}$$

This linear DE can be solved in closed-form as follows (see for instance [2], pp. 452):

$$x(k) = \left(\frac{3}{2}\right)^k \cdot x(0) + \sum_{i=0}^{k-1} \left(\prod_{j=i}^{k-1} \right) \cdot \frac{1}{2}$$

Equivalently:

$$x(k) = \frac{3^k \cdot (x(0) + 1) - 2^k}{2^k}$$

It is not difficult to prove that these sequence obtaining only odd numbers can not continue for ever. Moreover, let's denote the number iteration from $x(0)$ odd to reach an even number by L, then:

$$x(L) = 2^s \cdot w = \frac{3^L \cdot (x(0) + 1) - 2^L}{2^L}$$

Where $2^s \cdot w$ is the primer decomposition of the even number $x(L)$ and w is an odd number. That is:

$$2^s \cdot 2^L \cdot w = 3^L \cdot (x(0) + 1) - 2^L \Leftrightarrow 2^L \cdot (2^s \cdot w + 1) = 3^L \cdot (x(0) + 1)$$

However, 3^L is an odd number, so: $2^L = x(0) + 1$, then L is a finite number given by the initial number $x(0)$. This completes the proof that the Collatz DE is asymptotically equivalent (with $k \to \infty$) to the invariant set $\{1, 4, 2\}$.

3 Conclusions

In this short paper, a new asymptotic analysis method has been presented base upon impacts over a non-linear border collision curve using a 2-D piecewise linear continuous ODE for DE equations.

The main theorem provides a ready to use sufficient condition, thus checking for asymptotic invariant sets or unbounded trajectories.

Some example were examined including the well-known Collatz sequence written as DE, proving the existence of only invariant set: $\{1, 4, 2\}$.

The methodology can be used to check bounded/unbound trajectories existence for any non-linear DE on the basis of a simple algebraic condition and a linear asymptotic DE.

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