A high-order predictor-corrector method for initial value problems with fractional derivative involving Mittag-Leffler kernel

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Abstract

In this paper, we propose a numerical scheme of the predictor-corrector type for solving nonlinear fractional initial value problems, the chosen fractional derivative is called the Atangana-Baleanu derivative defined in Caputo sense (ABC). This proposed method is based on Lagrangian quadratic polynomials to approximate the nonlinearity implied in the Volterra integral which is obtained by reducing the given fractional differential equation via the properties of the ABC-fractional derivative. Through this technique, we get corrector formula with high accuracy which is implicit as well as predictor formula which is explicit and has the same precision order as the corrective formula. On the other hand, the so-called memory term is computed only once for both prediction and correction phases, which indicates the low cost of the proposed method. Also, the error bound of the proposed numerical scheme is offered. Furthermore, numerical experiments are presented in order to assess the accuracy of the new method on two differential equations. Moreover, a case study is considered where the proposed predictor-corrector scheme is used to obtained approximate solutions of a coupled non-autonomous ABC-fractional differential system.

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In this paper, we propose a numerical scheme of the predictor-corrector type for solving nonlinear fractional initial value problems, the chosen fractional derivative is called the Atangana-Baleanu derivative defined in Caputo sense (ABC). This proposed method is based on Lagrangian quadratic polynomials to approximate the nonlinearity implied in the Volterra integral which is obtained by reducing the given fractional differential equation via the properties of the ABC-fractional derivative. Through this technique, we get corrector formula with high accuracy which is implicit as well as predictor formula which is explicit and has the same precision order as the corrective formula. On the other hand, the so-called memory term is computed only once for both prediction and correction phases, which indicates the low cost of the proposed method. Also, the error bound of the proposed numerical scheme is offered. Furthermore, numerical experiments are presented in order to assess the accuracy of the new method on two differential equations. Moreover, a case study is considered where the proposed predictor-corrector scheme is used to obtained approximate solutions of a coupled non-autonomous ABC-fractional differential system.

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1. Introduction

Despite the fact that fractional calculus has a lengthy history in mathematics, it has only recently seen a considerable number of real-world applications [1]. Fractional derivatives are important due to their attractive characteristics, (e.g. memory effect) [2, 3], including their wide dynamical range [4, 5]. Many definitions of fractional derivatives and integrals exist, including Riemann-Liouville, Caputo, Grunwald-Letnikov [6, 7]. In 2015, Caputo and Fabrizio (CF) have suggested a new idea of fractional derivatives based on the exponential decay [8]. Following that, Atangana and Baleanu proposed a novel concept of fractional derivative with non-singular and non-local kernel based on the Mittag-Leffler function [9].

Due to the difficulty or impossibility to obtain exact solutions of nonlinear differential equations involving fractional derivatives, numerical techniques are required to provide approximate solutions. Over the past few decades, numerical methods to solve initial value problems have attracted great interest from the research community, which support various disciplines, including medicine [10, 11], chemistry [12, 13], biology [14], and physics [9, 15], etc. Where computational algorithms are implemented. Moreover, numerical analysis techniques are the perfect tools to evaluate the behavior of real-life complicated models. Therefore, the implementation of accurate numerical methods is suitable in the problems representing real world phenomena.
Over the past twenty years, several numerical approaches have been developed by researchers for solving fractional ordinary differential equations with general nonlinearity and various definitions of fractional derivatives.

For fractional diffential equations with the Caputo fractional derivative, Diethelm et al. [16, 17, 18], Nguyen and Jang [21] used the linear or quadratic Lagrange interpolating polynomials to approximate the nonlinear term in Volterra integral equation, in order to solve it numerically with the so-called Adams-type predictor-corrector method, which is identical to the numerical solution

of the fractional differential equation. Li et al. [23] obtained the approximate solution for Volterra integral equation based on higher-order piecewise interpolation polynomial, and use the Simpson method to design a higher-order scheme for the fractional differential equation. The fractional Adams-Bashforth-Moulton method developed with Newton linear/quadratic interpolations, by Atangana and Araz, Douaifia et al. [19, 14]. For the fractional differential equations which involving variable-order (VO) fractional derivative (FD) in sense of Caputo (it is an extension of constant-order FD). Moghaddam et al. [24], Douaifia and Abdelmalek [25], got the numerical solutions of VO fractional differential equations with/without delay, using predictor-corrector approach. Also, the authors [27, 28] applied spectral collocation method to obtain solutions of the VO differential equations with/without integrals term.

Based on the previous mentioned approaches, several numerical techniques have developed to solve fractional differential equations with Atangana-Baleanu fractional derivative. Toufik and Atangana, Shah et al. [15, 29] used the two-step Lagrange polynomial interpolation to obtain solution of fractional differential equations. Baleanu et al. [30] constructed Adams-type predictor-corrector method. Sadeghi et al. [31] obtained the approximate solution based on Genocchi polynomials. Ganji et al. [26] applied the fifth-kind Chebyshev polynomials to obtain solution of the ABC-fractional multi-variable orders differential equations. Ganji and Jafari [32] used the spectral technique for solving integro-differential equations. Abdeljawad et al. [33] developed a predictor-corrector method to solve fractional differential equations involving a generalized Atangana-Baleanu derivative with a three-parameter Mittag-Leffler function in its kernel. Then, Baba et al. [34] used it to solve COVID-19 Awareness model in the setting of a generalized fractional Atangana-Baleanu derivative.

In the current paper, by following [21, 22] we suggest a high-order (i.e. lead to $\mathcal{O}(h^3)$ accuracy) numerical approach of the predictor-corrector type for solving nonlinear ABC-fractional differential equations by using quadratic lagrange interpolation to approximate the nonlinearity involving in the Volterra integral which is obtained by reducing the given fractional initial value problem via the properties of the ABC-fractional derivative. Thus, we obtain two formulas, an implicit corrector formula with high accuracy as well as an explicit predictor formula with the same precision order as the corrector formula. Furthermore, the memory term which appeared in each the previous formulas, is computed just once for both, which highlighting the low cost of the proposed approach.

The remainder of this paper is arranged as follows: Basic definitions and notations, including the Atangana-Baleanu fractional derivatives and integral, are introduced in Section 2. In Section 3, a new numerical method is proposed for solving fractional initial value problems involving the Atangana-Baleanu fractional derivative in Caputo sense. The schemes' error estimates are obtained in Section 4. Finally, in Section 5, we present numerical examples to demonstrate the efficacy of the proposed scheme, and we also compare the solutions with other methods [15, 30]. Also, numerical simulation of a coupled non-autonomous ABC-fractional differential system is discussed via the proposed scheme.

2. Preliminaries

The Atangana-Baleanu fractional derivative in the Caputo sense (ABC) is defined as (cf. [9]):

$$ {}^{ABC}_{0}D^{\alpha}_{t} f(t) = \frac{AB(\alpha)}{1-\alpha} \int_0^t f'(s) \mathbf{E}_{\alpha} \left(\frac{-\alpha}{1-\alpha} (t-s)^{\alpha} \right) ds, \tag{2.1} $$

where $\alpha \in (0,1)$, $AB(\alpha) > 0$ is a normalization function obeying $AB(0) = AB(1) = 1$ $\left(\text{e.g. } AB(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)} \right)$, $\mathbf{E}_{\alpha}(.)$ denotes the Mittag-Leffler function of order α defined by

$$ \mathbf{E}_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad z, \alpha \in \mathbb{C}, \quad \text{and } Re(\alpha) > 0, \tag{2.2} $$

and $\Gamma(.)$ denotes Gamma function, defined as

$$ \Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt, \qquad Re(\alpha) > 0. \tag{2.3} $$

The fractional integral in the sense of Atangana-Beleanu, is defined as follows (cf. [38]):

$$ {}^{ABC}_{0}I^{\alpha}_{t} f(t) = \frac{1-\alpha}{AB(\alpha)} f(t) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t f(s)(t-s)^{\alpha-1} ds. \tag{2.4} $$

We consider the initial-value problem with Atangana-Baleanu derivative

$$ \begin{cases} {}^{ABC}_{0}D^{\alpha}_{t} y(t) = f(t, y(t)), & 0 < t < T < \infty, \\ y(0) = y_0, \end{cases} \tag{2.5} $$

where f is a smooth nonlinear function that guarantees the existence of a unique solution for (2.5), with the fractional order $\alpha \in (0,1)$, and $y_0 \in \mathbb{R}$. A continuous function $y(t)$ is the solution of (2.5) if and only if it is the solution of following Voltera-integral equation:

$$y(t) \quad = y_0 + \frac{1-\alpha}{AB(\alpha)} f(t, y(t)) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t f(s, y(s)) (t-s)^{\alpha-1} \, ds. \tag{2.6}$$

At point $t_{n+1} = h(n+1)$, $n = 0, 1, \ldots, N \in \mathbb{N}$ with $h = \frac{T}{N}$, we get

$$\begin{aligned} y_{n+1} &= y_0 + \frac{1-\alpha}{AB(\alpha)} f_{n+1} + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^{t_{n+1}} f(s, y(s)) (t_{n+1}-s)^{\alpha-1} \, ds \\ &= y_0 + \frac{1-\alpha}{AB(\alpha)} f_{n+1} + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \left(\int_0^{t_n} f(s, y(s)) (t_{n+1}-s)^{\alpha-1} \, ds + \int_{t_n}^{t_{n+1}} f(s, y(s)) (t_{n+1}-s)^{\alpha-1} \, ds \right) \\ &= y_0 + \frac{1-\alpha}{AB(\alpha)} f_{n+1} + y_{n+1}^{lag} + y_{n+1}^{inc} \end{aligned} \tag{2.7}$$

where, y_{n+1}^{lag} and y_{n+1}^{inc} are called lag term and increment term respectively, which take the following forms:

$$y_{n+1}^{lag} := \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^{t_n} (t_{n+1}-s)^{\alpha-1} f(s, y(s)) ds, \tag{2.8}$$

and

$$y_{n+1}^{inc} := \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_{t_n}^{t_{n+1}} (t_{n+1}-s)^{\alpha-1} f(s, y(s)) ds. \tag{2.9}$$

3. Predictor-corrector scheme with quadratic interpolation

To start presenting the proposed numerical method to solve the initial-value problem (2.5), we need the following lemma:

Lemma 1 *([21]) Assume that $\psi \in \mathbb{P}_2([0,T])$, where $\mathbb{P}_2([0,T])$ is the space of all polynomials of degree less than or equal to two. Let ψ_k, $k = 0, ..., N$ be the restricted value of $\psi(t)$ on t_k ($0 \le k \le N$). Then there exist reals b_{n+1}^0, b_{n+1}^1 and b_{n+1}^2, such that*

$$\int_{t_n}^{t_{n+1}} (t_{n+1}-s)^{\alpha-1} \psi(s) ds = B \sum_{j=0}^{2} b_{n+1}^j \psi_{n+j-2}. \tag{3.1}$$

Where, $B = \dfrac{h^\alpha}{\alpha(\alpha+1)(\alpha+2)}$, $b_{n+1}^0 = \dfrac{\alpha+4}{2}$, $b_{n+1}^1 = -2(\alpha+3)$, $b_{n+1}^2 = \dfrac{2\alpha^2+9\alpha+12}{2}$.

Now, we use quadratic Lagrange polynomial of $f(s, y(s))$ over the intervals $[t_{i-1}, t_{i+1}]$, $1 \le i \le N-1$:

$$f(s, y(s)) \approx \sum_{l=i-1}^{i+1} f_l q_l^i(s), \tag{3.2}$$

where

$$q_l^i(s) = \prod_{\substack{k=i-1 \\ l \neq k}}^{i+1} \frac{s-t_k}{t_l-t_k}. \tag{3.3}$$

But, on $[t_0, t_1]$ we can interpolate $f(s, y(s))$ with the points $\left\{t_0, t_{\frac{1}{2}}, t_1\right\}$, then we get

$$f(s, y(s)) \approx f_0 q_0^0(s) + f_{\frac{1}{2}} q_{\frac{1}{2}}^0(s) + f_1 q_1^0(s), \tag{3.4}$$

where

$$q_0^0(s) = \frac{(s-t_0)(s-t_{\frac{1}{2}})}{(t_1-t_0)(t_1-t_{\frac{1}{2}})}, \; q_{\frac{1}{2}}^0(s) = \frac{(s-t_0)(s-t_1)}{(t_{\frac{1}{2}}-t_1)(t_{\frac{1}{2}}-t_1)}, \; q_1^0(s) = \frac{(s-t_0)(s-t_{\frac{1}{2}})}{(t_1-t_0)(t_1-t_{\frac{1}{2}})}. \tag{3.5}$$

The approximation of $y(t_{n+1})$ denoting by $\widetilde{y}_{n+1}$, according to (3.2)-(3.5), $\widetilde{y}_{n+1}$ takes the following form:

$$\widetilde{y}_{n+1} = y_0 + {}^{AB}\widetilde{f}_{n+1}^{P} + \widetilde{y}_{n+1}^{lag} + \widetilde{y}_{n+1}^{inc}, \tag{3.6}$$

where

$$ {}^{AB}\widetilde{f}^{P}_{n+1} = \frac{1-\alpha}{AB(\alpha)} f(t_{n+1}, \widetilde{y}^{P}_{n+1}), \tag{3.7} $$

$$ \widetilde{y}^{lag}_{n+1} = \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \left(\sum_{j=0}^{2} a^{j,0}_{n+1} \widetilde{f}_{\frac{j}{2}} + \sum_{j=0}^{2} \sum_{k=1}^{n-1} a^{j,k}_{n+1} \widetilde{f}_{k+j-1} \right) \tag{3.8} $$

and

$$ \widetilde{y}^{inc}_{n+1} = \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \left(\sum_{j=0}^{1} a^{j,n}_{n+1} \widetilde{f}_{n+j-1} + a^{2,n}_{n+1} \widetilde{f}^{P}_{n+1} \right) \tag{3.9} $$

with

$$ a^{j,k}_{n+1} = \int_{t_k}^{t_{k+1}} (t_{n+1} - s)^{\alpha-1} q^{k}_{\widehat{j}}(s) ds, \quad for \quad j \in \{0,1,2\}, \tag{3.10} $$

where,

$$ \widehat{j} = \begin{cases} \dfrac{j}{2} & \text{if } k = 0, \\ j + k - 1 & \text{if } 1 \le k \le n. \end{cases} \tag{3.11} $$

The predictor term can be approximated as follows:

$$ \widetilde{y}^{P}_{n+1} = y_0 + {}^{AB}\widetilde{f}^{apprx}_{n+1} + \widetilde{y}^{lag}_{n+1} + \frac{\alpha h^{\alpha}}{AB(\alpha)\Gamma(\alpha+3)} \sum_{j=0}^{2} b^{j}_{n+1} \widetilde{f}_{n+j-2}, \tag{3.12} $$

where

$$ {}^{AB}\widetilde{f}^{apprx}_{n+1} = \frac{1-\alpha}{AB(\alpha)} \left(f_{n-2} - 3f_{n-1} - 3f_n \right), \qquad n \ge 2. \tag{3.13} $$

4. Error analysis

Throughout this section, we need the following lemmas:

Lemma 2 *Let $f \in C^{n+1}([a,b])$ and $P_n \in \mathbb{P}_n([a,b])$ (where $\mathbb{P}_n([0,T])$ is the space of all polynomials of degree less than or equal to n) interpolate the function f at t_k, $0 \le k \le N$, with $t_0 = a, t_N = b,$, then there exists $\xi \in (a,b)$such that, for any $s \in [a,b]$*

$$ f(s) - P_n(s) = \frac{f^{n+1}(\xi)}{(n+1)!} \prod_{k=0}^{n} (s - t_k). \tag{4.1} $$

Lemma 3 *([21]) For $\delta > 0$, we have*

$$ \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_{n+1} - s)^{\delta-1} ds \le \frac{T^{\delta}}{\delta}. \tag{4.2} $$

Lemma 4 *([36]) (Discrete Gronwall's Inequality) Let $\{a_n\}_{n=0}^{N}, \{b_n\}_{n=0}^{N}$ be non-negative sequences with second one is monotonic increasing and satisfy that*

$$ a_n \le b_n + Mh^{\theta} \sum_{k=0}^{n-1} (n-k)^{\theta-1} a_k, \quad 0 \le n \le N, \tag{4.3} $$

where, $M > 0$ is independent of $h > 0$, and $0 < \theta \le 1$. Then,

$$ a_n \le b_n \mathbf{E}_{\theta}\left(M\Gamma(\theta)(nh)^{\theta} \right). \tag{4.4} $$

Lemma 5 *([21]) There exist $K_1, K_2 > 0$ such that for $\alpha \in (0,1)$, and $j = 0,1,2$, we have*

$$ \left| a^{j,k}_{n+1} \right| \le \begin{cases} K_1 (n-k)^{\alpha-1} h^{\alpha} & \text{if } 0 \le k \le n-1, \\ K_2 & \text{if } k = n. \end{cases} \tag{4.5} $$

Let T^{P}_{n+1} be the truncation error of prediction at point t_{n+1}, defined by

$$ \begin{aligned} T^{P}_{n+1} = \Bigg| & \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_{0}^{t_{n+1}} (t_{n+1} - s)^{\alpha-1} f(s, y(s)) ds - \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \sum_{j=0}^{2} a^{j,0}_{n+1} f_{\frac{j}{2}} \\ & - \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \sum_{j=0}^{2} \sum_{k=1}^{n-1} a^{j,k}_{n+1} f_{k+j-1} - \frac{\alpha h^{\alpha}}{AB(\alpha)\Gamma(\alpha+3)} \sum_{j=0}^{2} b^{j}_{n+1} f_{n+j-2} \Bigg|. \end{aligned} \tag{4.6} $$

Theorem 1 *Assume that $f(.,y(.)) \in C^3([0,T])$. Then, there exists $C>0$ (independent of all grid parameters) such that:*

$$T_{n+1}^P \leq Ch^3. \tag{4.7}$$

Proof: We set the notation, $AB_g = \dfrac{\alpha}{AB(\alpha)\Gamma(\alpha)}$, then

$$T_{n+1}^P \leq \sum_{j=1}^{3} \mathcal{I}_j, \tag{4.8}$$

where,

$$\mathcal{I}_1 := AB_g \int_0^{t_1} (t_{n+1}-s)^{\alpha-1} \Big| f(s,y(s)) - \sum_{j=0}^{2} f_{\frac{j}{2}}\, q_{\frac{j}{2}}^0(s) \Big| ds, \tag{4.9}$$

$$\mathcal{I}_2 := AB_g \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} (t_{n+1}-s)^{\alpha-1} \Big| f(s,y(s)) - \sum_{j=0}^{2} f_{k+j-1} q_{k+j-1}^k(s) \Big| ds, \tag{4.10}$$

$$\mathcal{I}_3 := \Big| AB_g \int_{t_n}^{t_{n+1}} (t_{n+1}-s)^{\alpha-1} f(s,y(s)) - \Gamma(\alpha) AB_g \frac{h^\alpha}{\Gamma(\alpha+3)} \sum_{j=0}^{2} b_{n+1}^j f_{n+j-2} \Big|. \tag{4.11}$$

Thanks to lemma 2, there exists $C_1>0$, such that

$$\mathcal{I}_j \leq C_1 h^3, \quad j=1,2. \tag{4.12}$$

The Taylor expansion of f around t_n gives:

$$f(t) = P_2(t) + \frac{1}{3!} f'''(\xi_n)(t-t_n)^3 + O(h^3), \quad \xi_n \in (t_n, t), \tag{4.13}$$

where,

$$P_2(t) = f_n + f'(t_n)(t-t_n) + \frac{1}{2} f''(t_n)(t-t_n)^2, \quad (P_2 \in \mathbb{P}_2([0,T])). \tag{4.14}$$

According to lemma 2 and (4.11), we have

$$\begin{aligned} \mathcal{I}_3 = & \Big| AB_g \int_{t_n}^{t_{n+1}} (t_{n+1}-s)^{\alpha-1} (P_2(s) + \frac{1}{3!} f'(\xi_n)(s-t_n)^3 + O(h^3)) ds \\ & - \Gamma(\alpha) AB_g \frac{h^\alpha}{\Gamma(\alpha+3)} \sum_{j=0}^{2} b_{n+1}^j f_{n+j-2} \Big|, \end{aligned} \tag{4.15}$$

it follows that

$$\begin{aligned} \mathcal{I}_3 \leq & \Gamma(\alpha) AB_g \frac{h^\alpha}{\Gamma(\alpha+3)} \Big(\mid b_{n+1}^0 \mid\mid P_2(t_{n-2}) - f_{n-2} - f_{n-2} \mid + \mid b_{n+1}^1 \mid\mid P_2(t_{n-1}) - f_{n-1} \mid \Big) \\ & + AB_g \left| \int_{t_n}^{t_{n+1}} (t_{n+1}-s)^{\alpha-1} \Big(\frac{1}{6} f'''(\xi_n)(s-t_n)^3 \Big) ds \right| + O(h^3). \end{aligned} \tag{4.16}$$

According to (4.8) and (4.13), we have

$$\Big| P_2(t_{n-2}) - f_{n-2} \Big| \leq \frac{4Mh^3}{3}, \quad \text{where } M := \max \{ \mid f'''(\xi_k) \mid \; : 0 \leq k \leq N \}, \tag{4.17}$$

and

$$\Big| P_2(t_{n-1}) - f_{n-1} \Big| \leq \frac{M}{6} h^3. \tag{4.18}$$

We thus obtain the estimate

$$\mathcal{I}_3 \leq \Gamma(\alpha) AB_g \frac{h^\alpha}{\Gamma(\alpha+3)} \Big(b_{n+1}^0 \frac{4Mh^3}{3} + \big| b_{n+1}^1 \big| \frac{M}{6} h^3 \Big) + \frac{Mh^{3+\alpha}}{6AB(\alpha)\Gamma(\alpha)} + O(h^3). \tag{4.19}$$

Consequently, there exists $C_2>0$ such that

$$\mathcal{I}_3 \leq C_2 h^3. \tag{4.20}$$

According to (4.8), (4.12) and (4.20), proof of theorem is achieved. □

Theorem 2 *(Global Predictor Error) Assume that $f(.,y(.)) \in C^3([0,T])$ and is Lipschitz continuous in its second argument, i.e.*

$$\exists L > 0, \text{ such that } |f(t,y_1) - f(t,y_2)| \leq L\,|y_1 - y_2|\,, \quad \forall y_1, y_2 \in \mathbb{R}. \tag{4.21}$$

Then, there exist $K_1, K_2 > 0$ such that, the global predictor error satisfies

$$E^P_{n+1} := \left|y_{n+1} - \widetilde{y}^P_{n+1}\right| \leq T^P_{n+1} + K_1 AB_g L h^\alpha E_{\frac{1}{2}} + K_2 AB_g L h^\alpha \sum_{k=1}^{n} (n-k+1)^{\alpha-1} E_k + O(h^3). \tag{4.22}$$

Where, $AB_g := \dfrac{\alpha}{AB(\alpha)\Gamma(\alpha)}$.

Proof: We set ${}^{AB}f_{n+1} := \dfrac{1-\alpha}{AB(\alpha)} f(t_{n+1}, y_{n+1})$. Thus

$$E^P_{n+1} = \left| {}^{AB}f_{n+1} + AB_g \int_0^{n+1} (t_{n+1}-s)^{\alpha-1} f(s,y(s))ds - {}^{AB}\widetilde{f}^{apprx}_{n+1} - \widetilde{y}^{lag}_{n+1} - \frac{\Gamma(\alpha) AB_g h^\alpha}{\Gamma(\alpha+3)} \sum_{j=0}^{2} b^j_{n+1} \widetilde{f}_{n+j-2} \right|, \tag{4.23}$$

therefore

$$\begin{aligned} E^P_{n+1} = \Bigg| & {}^{AB}f_{n+1} + AB_g \int_0^{t_{n+1}} (t_{n+1}-s)^{\alpha-1} f(s,y(s))ds - AB_g \sum_{j=0}^{2} a^{j,0}_{n+1} f_{\frac{j}{2}} \\ & - AB_g \sum_{j=0}^{2} \sum_{k=1}^{n-1} a^{j,k}_{n+1} f_{k+j-1} - AB_g h^\alpha \sum_{j=0}^{2} b^j_{n+1} f_{n+j-2} + AB_g \sum_{j=0}^{2} a^{j,0}_{n+1} f_{\frac{j}{2}} \\ & + AB_g \sum_{j=0}^{2} \sum_{k=1}^{n-1} a^{j,k}_{n+1} f_{k+j-1} - \frac{\Gamma(\alpha) AB_g h^\alpha}{\Gamma(\alpha+3)} \sum_{j=0}^{2} b^j_{n+1} f_{n+j-2} - {}^{AB}\widetilde{f}^{apprx}_{n+1} - \widetilde{y}^{lag}_{n+1} \\ & - \frac{\Gamma(\alpha) AB_g h^\alpha}{\Gamma(\alpha+3)} \sum_{j=0}^{2} b^j_{n+1} \widetilde{f}_{n+j-2} \Bigg|, \end{aligned} \tag{4.24}$$

it follows that

$$\begin{aligned} E^P_{n+1} \leq T^P_{n+1} + & \left| {}^{AB}f_{n+1} - {}^{AB}\widetilde{f}^{apprx}_{n+1} \right| \\ & + AB_g \sum_{j=0}^{2} \left|a^{j,0}_{n+1}\right| \left| f_{\frac{j}{2}} - \widetilde{f}_{\frac{j}{2}} \right| + AB_g \sum_{j=0}^{2} \sum_{k=1}^{n-1} \left|a^{j,k}_{n+1}\right| \left| f_{k+j-1} - \widetilde{f}_{k+j-1} \right| \\ & + \frac{\Gamma(\alpha) AB_g h^\alpha}{\Gamma(\alpha+3)} \sum_{j=0}^{2} \left|b^j_{n+1}\right| \left| f_{n+j-2} - \widetilde{f}_{n+j-2} \right|. \end{aligned} \tag{4.25}$$

Hence, according to the lemma 3 there exist $C_1, K_1 > 0$, such that

$$\begin{aligned} E^P_{n+1} \leq T^P_{n+1} + & C_1 \sum_{j=0}^{2} E_{n+j-2} + K_1 AB_g L h^\alpha n^{\alpha-1} (E_{\frac{1}{2}} + E_1) \\ & + K_1 AB_g L h^\alpha \sum_{j=0}^{2} \sum_{k=1}^{n-1} (n-k)^{\alpha-1} E_{k+j-1} + \frac{\Gamma(\alpha) AB_g L h^\alpha}{\Gamma(\alpha+3)} \sum_{j=0}^{2} \left|b^j_{n+1}\right| E_{n+j-2} + O(h^3). \end{aligned} \tag{4.26}$$

On the other hand, for $\alpha \in (0,1)$ we have

$$(n+1-i)^{\alpha-1} + (n-1)^{\alpha-1} + (n-i-1)^{\alpha-1} \leq 6(n+1-i)^{\alpha-1}, \quad 1 \leq i \leq n-2. \tag{4.27}$$

Consequently, there exists $K_2 > 0$ such that the estimate (4.26), becomes

$$E^P_{n+1} \leq T^P_{n+1} + K_1 AB_g L h^\alpha E_{\frac{1}{2}} + K_2 AB_g L h^\alpha \sum_{k=1}^{n} (n+1-k)^{\alpha-1} E_k + O(h^3). \tag{4.28}$$

□

Theorem 3 *Assume that $f(.,y(.)) \in C^3([0,T])$. Then there exists positive constants C_1 and C_2 (independent of grid parameters) such that:*

$$T_{n+1}^C \leq C_1 h^3 + C_2 E_{n+1}^P, \tag{4.29}$$

where,

$$\begin{aligned} T_{n+1}^C = \Bigg| & AB_g \int_0^{t_{n+1}} (t_{n+1}-s)^{\alpha-1} f(s,y(s))ds - AB_g \sum_{j=0}^{2} a_{n+1}^{j,0} f_{\frac{j}{2}} \\ & - AB_g \left(\sum_{j=0}^{2} \sum_{k=1}^{n-1} a_{n+1}^{j,k} f_{k+j-1} + \sum_{j=0}^{1} a_{n+1}^{j,n} f_{n+j-1} + a_{n+1}^{2,n} f_{n+1}^P \right) \Bigg|, \end{aligned} \tag{4.30}$$

with, $AB_g = \dfrac{\alpha}{AB(\alpha)\Gamma(\alpha)}$.

Proof: It follows immediately that,

$$\begin{aligned} T_{n+1}^C \leq & AB_g \int_{t_0}^{t_1} (t_{n+1}-s)^{\alpha-1} \left| f(s,y(s)) - \sum_{j=0}^{2} q_{\frac{j}{2}}^0(s) f_{\frac{j}{2}} \right| ds \\ & + AB_g \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} (t_{n+1}-s)^{\alpha-1} \left| f(s,y(s)) - \sum_{j=0}^{2} q_{k+j-1}^k(s) f_{k+j-1} \right| ds \\ & + AB_g \int_{t_n}^{t_{n+1}} (t_{n+1}-s)^{\alpha-1} \left| f(s,y(s)) - \sum_{j=0}^{1} q_{n+j-1}^n(s) f_{n+j-1} - q_{n+1}^n(s) f_{n+1}^P \right| ds. \end{aligned} \tag{4.31}$$

Thus, according to the lemma 2, there exist $\widehat{C}, \widetilde{C} > 0$ such that

$$T_{n+1}^C \leq \frac{\widetilde{C} T^\alpha}{6AB(\alpha)\Gamma(\alpha)} h^3 + \frac{\widehat{C} h^\alpha}{AB(\alpha)\Gamma(\alpha)} E_{n+1}^P + O(h^3). \tag{4.32}$$

The desired estimate (4.30) can be derived by direct consideration of the last inequality (4.32). □

Theorem 4 *(Global Error of the proposed method) With the same assumptions as those of Theorem 2. Then, we have*

$$E_{n+1} := |y_{n+1} - \widetilde{y}_{n+1}| \leq C h^3, \tag{4.33}$$

where $C > 0$ (independent of grid parameters), given $E_1, E_2 \leq \widehat{C}_1 h^3$, and $E_{\frac{1}{2}} \leq \widehat{C}_2 h^{3-\alpha}$, with $\widehat{C}_1, \widehat{C}_2 > 0$.

Proof: By taking into account the previous results, there exist $C_j > 0$ (with $j = 1, \ldots, 10$), such that

$$\begin{aligned} E_{n+1} \leq & T_{n+1}^C + AB_g \left| f_{n+1}^P - \widetilde{f}_{n+1}^P \right| + \frac{L(1-\alpha)}{AB(\alpha)} E_{n+1}^P + AB_g \sum_{j=0}^{2} \left| a_{n+1}^{j,0} \right| \left| f_{\frac{j}{2}} - \widetilde{f}_{\frac{j}{2}} \right| \\ & + AB_g \sum_{j=0}^{2} \sum_{k=1}^{n-1} \left| a_{n+1}^{j,k} \right| \left| f_{k+j-1} - \widetilde{f}_{k+j-1} \right| + AB_g \sum_{j=1}^{1} \left| a_{n+1}^{j,n} \right| \left| f_{n+j-1} - \widetilde{f}_{n+j-1} \right| \\ \leq & T_{n+1}^C + \frac{L(1-\alpha)}{AB(\alpha)} E_{n+1}^P + AB_g L h^\alpha E_{\frac{1}{2}} + \frac{\widetilde{C}_1 h^\alpha}{AB(\alpha)\Gamma(\alpha)} E_{n+1}^P \\ & + AB_g \widetilde{C}_2 L h^\alpha \sum_{k=1}^{n} (n-k+1)^{\alpha-1} E_k + O(h^3) \\ \leq & \widetilde{C}_3 h^3 + \widetilde{C}_4 E_{n+1}^P + \frac{L(1-\alpha)}{AB(\alpha)} E_{n+1}^P + AB_g L h^\alpha E_{\frac{1}{2}} \\ & + \frac{\widetilde{C}_5 h^\alpha}{AB(\alpha)} E_{n+1}^p + AB_g \widetilde{C}_6 L h^\alpha \sum_{k=1}^{n} (n-k+1)^{\alpha-1} E_k \\ \leq & \widetilde{C}_7 h^3 + \widetilde{C}_8 T_{n+1}^P + \widetilde{C}_9 h^\alpha E_{\frac{1}{2}} + \widetilde{C}_{10} h^\alpha \sum_{k=1}^{n} (n-k+1)^{\alpha-1} E_k. \end{aligned} \tag{4.34}$$

Consequently, by the discrete Gronwall's inequality (i.e. lemma 4) the desired result holds. □

Remark 1 *Since the global error indicated in the Theorem 4 is dependent on that of the start-up ($E_{\frac{1}{2}}$, E_1, and E_2). We suggest employing the start-up scheme described in* **Appendix A** *to generate approximate solutions for the first stages (i.e. $\widetilde{y}_{\frac{1}{2}}$, $\widetilde{y}_1$, and $\widetilde{y}_2$).*

Table 1. The absolute error of various numerical methods for problem (5.3) with $n = 2$, N=40, and various values of α.

Methods	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 0.99$
PPC (3.6)-(3.12)	5.3e−15	1.8e−15	3.6e−15	9.0e−16
BJH-PC [30]	4.1e−4	6.2e−4	7.7e−4	8.3e−4
TAE [15]	1.3e−1	7.4e−2	2.5e−2	6.1e−3

5. Numerical Illustrations and Simulation

In this section, we give some numerical experiments through the proposed predictor-corrector scheme (PPC) (3.6)-(3.12), the predictor-corrector method introduced by Baleanu-Jajarmi-Hajipour (BJH-PC) [30], and the explicit numerical scheme introduced by Toufik-Atangana (TAE) [15], to show the efficiency and accuracy of our new method. The experimental order of convergence (EOC) is computed by

$$EOC = log_2 \frac{AE\left(\frac{N}{2}\right)}{AE\left(N\right)}, \tag{5.1}$$

where AE is the absolute error, which takes the following form:

$$AE := AE(N) = \max_{1 \leqslant k \leqslant N} \left| y\left(t_k\right) - \widetilde{y}_k \right|. \tag{5.2}$$

Example 1 *Consider the following fractional Initial value problem (IVP):*

$$\begin{cases} {}^{ABC}_{0}D^{\alpha}_{t} y\left(t\right) = t^n, \quad 0 < t \leq 2, \\ \\ y\left(0\right) = 1, \end{cases} \tag{5.3}$$

where $\alpha \in (0,1)$, and $n \in \mathbb{N}$. On account of [30], the exact solution of (5.3) is given by

$$y\left(t\right) = 1 + \frac{1-\alpha}{AB\left(\alpha\right)} t^n + \frac{\alpha \Gamma\left(n+1\right)}{AB\left(\alpha\right)\Gamma\left(\alpha+n+1\right)} t^{\alpha+n}. \tag{5.4}$$

In addition, the problem (5.3) is numerically solved. Table 1 and table 2 show the comparison of absolute error of different numerical methods ((PPC) (3.6)-(3.12), BJH-PC [30], and TAE [15]) for (5.3) with $n = 2, 3$, $t \in [0,2]$ and various values of fractional order $\alpha \in \{0.5, 0.7, 0.9, 0.99\}$, where we notice that our method is superior to them in terms of accuracy (i.e. PPC (3.6)-(3.12) achieves a lower error than TAE and BJH-PC). Moreover, table 1 and table 2 offer that EOC for TAE are roughly 1, for BJH-PC are roughly 2, and for PPC (3.6)-(3.12) are roughly 3. Also, we note that the approximate solutions obtained by our proposed scheme get closer to the exact solutions by the increase in α. From Figure 1 and Figure 2, we note that the approximate solution obtained by PPC (3.6)-(3.12) almost matches with the exact solution with small step size compared to its counterparts. These figures and tables, indicate the efficacy of the current predictor-corrector numerical method (PPC) (3.6)-(3.12).

Example 2 *Consider the following Atangana-Beleanu-fractional differential equation:*

$$\begin{cases} {}^{ABC}_{0}D^{\alpha}_{t} y\left(t\right) = t - y(t), \quad 0 < t \leq 1, \\ \\ y\left(0\right) = 0, \end{cases} \tag{5.5}$$

where $\alpha \in (0,1)$. According to [30], the exact solution of (5.5) is given by

$$y\left(t\right) = \frac{1}{AB\left(\alpha\right) + 1 - \alpha} \left(\left(1-\alpha\right) t \mathbf{E}_{\alpha,2} \left(-\frac{\alpha}{AB\left(\alpha\right) + 1 - \alpha} t^{\alpha} \right) + \alpha t^{\alpha+1} \mathbf{E}_{\alpha,\alpha+2} \left(-\frac{\alpha}{AB\left(\alpha\right) + 1 - \alpha} t^{\alpha} \right) \right), \tag{5.6}$$

with $\mathbf{E}_{\alpha,\beta}(.)$ denotes the Mittag-Leffler function of two parameters α and β, which defined by

$$\mathbf{E}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad z, \alpha, \beta \in \mathbb{C}, \quad \text{and } Re(\alpha), Re(\beta) > 0, \tag{5.7}$$

and the exact solution (5.6) is calculated using the algorithm $mlf.m$ (see [39]) evaluated with accuracy 10^{-12}. Furthermore, the problem (5.5) is numerically solved. Table 3 and table 4 show the comparison of absolute error of different numerical methods ((PPC) (3.6)-(3.12), BJH-PC [30], and TAE [15]) for (5.5) with $AB(\alpha) = 1$, $AB(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}$, $t \in [0,1]$ and various values of fractional order $\alpha \in \{0.5, 0.55, 0.7, 0.9, 0.95\}$, where we notice that our method is superior to them in terms of accuracy (i.e. PPC (3.6)-(3.12) achieves a lower error than TAE and BJH-PC). From Figure 3 and Figure 4, we note that the approximate solution obtained by PPC (3.6)-(3.12) almost matches with the exact solution with small step size compared to its counterparts. These figures and tables, again confirm the efficacy of the current predictor-corrector numerical method (PPC) (3.6)-(3.12).

Table 2. The absolute error, experimental order of convergence, and CPU time in seconds (CTs) of various numerical methods for problem (5.3) with $n = 3$.

Methods		$\alpha = 0.5$			$\alpha = 0.7$			$\alpha = 0.9$		
	N	AE	EOC	CTs	AE	EOC	CTs	AE	EOC	CTs
PPC (3.6)-(3.12)	10	1.8e−3	–	3.8e−2	2.7e−3	–	2.4e−3	3.4e−3	–	2.2e−3
	20	2.4e−4	2.94	7.7e−3	3.6e−4	2.94	7.3e−3	4.4e−4	2.93	6.4e−3
	40	3.1e−5	2.97	2.0e−2	4.6e−5	2.97	2.1e−2	5.7e−5	2.97	2.1e−2
	80	3.9e−6	2.98	7.5e−2	5.8e−6	2.99	7.6e−2	7.2e−6	2.98	7.5e−2
	160	4.9e−7	2.99	0.31	7.2e−7	2.99	0.29	9.0e−7	2.99	0.29
	320	6.2e−8	2.99	1.12	9.1e−8	3.00	1.14	1.1e−7	3.00	1.15
BJH-PC [30]	10	2.4e−2	–	2.7e−2	3.4e−2	–	1.8e−3	3.9e−2	–	9.0e−4
	20	6.3e−3	1.95	3.5e−3	8.6e−3	1.98	1.7e−3	9.7e−3	2.00	1.9e−3
	40	1.6e−3	1.97	3.7e−3	2.2e−3	1.99	3.9e−3	2.4e−3	2.00	4.0e−3
	80	4.1e−4	1.98	9.3e−3	5.4e−4	1.99	1.0e−2	6.1e−4	2.00	1.0e−2
	160	1.0e−4	1.99	2.6e−2	1.4e−4	2.00	3.1e−2	1.5e−4	2.00	3.1e−2
	320	2.6e−5	1.99	8.4e−2	3.4e−5	2.00	0.10	3.8e−5	2.00	0.11
TAE [15]	10	1.5e−0	–	9.6e−3	9.5e−1	–	2.6e−3	4.1e−1	–	1.3e−3
	20	7.7e−1	1.01	4.7e−3	4.5e−1	1.07	4.6e−3	1.7e−1	1.28	4.8e−3
	40	3.9e−1	1.00	1.7e−2	2.2e−1	1.04	1.8e−2	7.4e−2	1.18	1.8e−2
	80	1.9e−1	1.00	8.4e−2	1.1e−1	1.02	7.0e−2	3.4e−2	1.10	7.0e−2
	160	9.6e−2	1.00	0.27	5.4e−2	1.01	0.33	1.7e−2	1.06	0.29
	320	4.8e−2	1.00	1.11	2.7e−2	1.00	1.13	8.1e−3	1.03	1.15

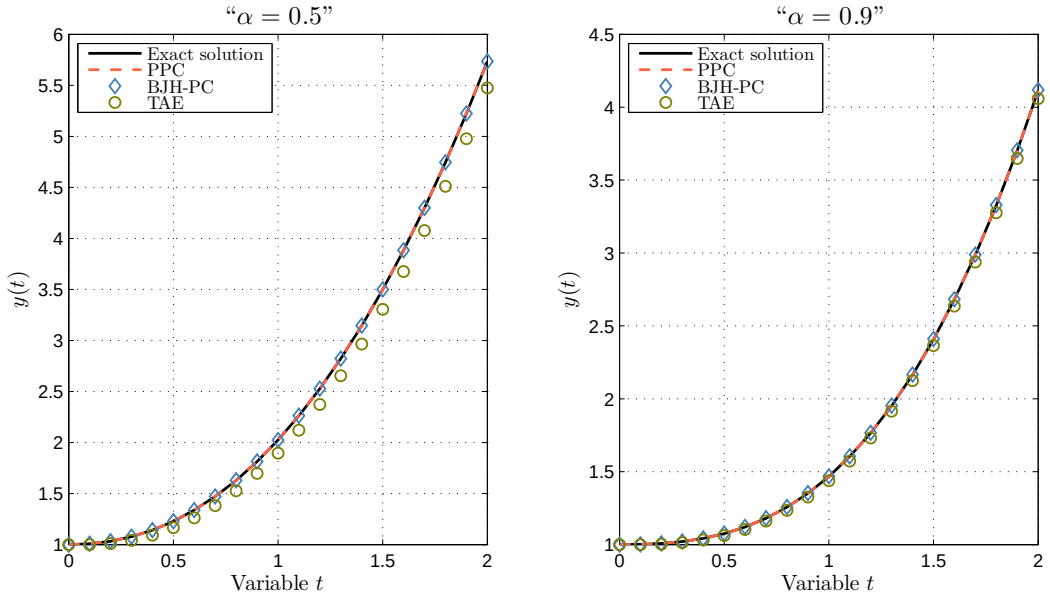

Figure 1. Comparison of the exact and the numerical solutions of problem (5.3) with $n = 2$, and $N = 10$.

Example 3 *In the previous examples, we have considered some differential equations with known exact solutions. Let us now analyze numerically the following coupled non-autonomous ABC-fractional differential system:*

$$\begin{cases} {}^{ABC}_{0}D^{\alpha}_{t}u(t) = -\sin^2(t)u(t) - \sin(t)\cos(t)v(t), & t > 0, \\ {}^{ABC}_{0}D^{\alpha}_{t}v(t) = -\sin(t)\cos(t)u(t) - \cos^2(t)v(t), & t > 0, \\ u(0) = u_0 > 0, \;\; v(0) = v_0 = 0, \end{cases} \tag{5.8}$$

where $\alpha \in (0, 1)$. The Caputo-fractional version of (5.8) has been previously investigated in [35]. It is classical task to prove the existence of unique solution for system (5.8) (see e.g. [29, 40, 41]).

Next, we present some numerical simulations of the coupled non-autonomous ABC-fractional differential system (5.8) through several sets of parameters to examine the effectiveness of the proposed predictor-corrector scheme in the current work (i.e. PPC (3.6)-(3.12)).

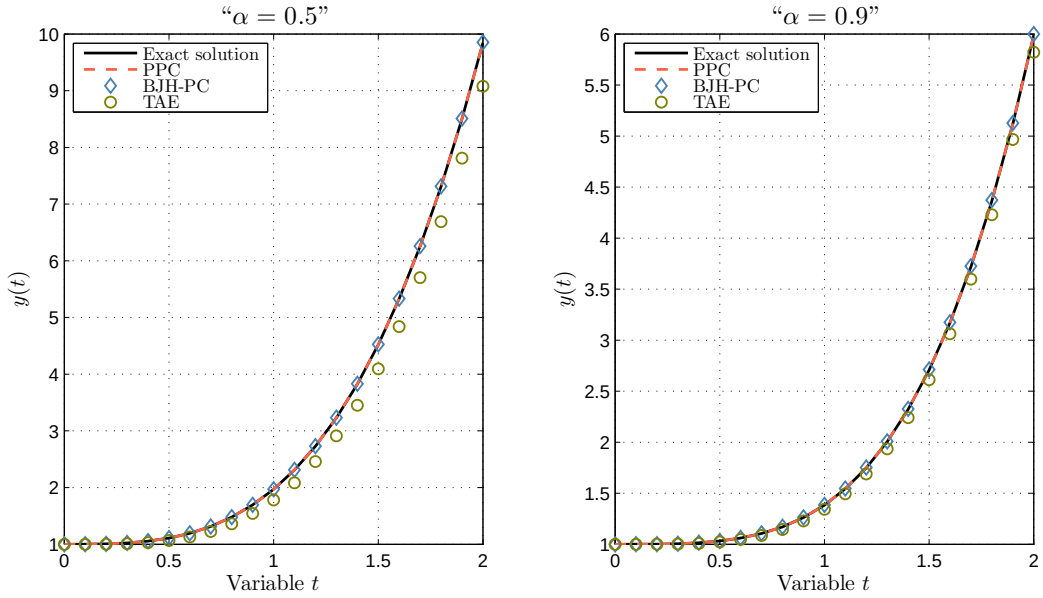

Figure 2. Comparison of the exact and the numerical solutions of problem (5.3) with $n = 3$, and $N = 10$.

Table 3. The absolute error, experimental order of convergence, and CPU time in seconds (CTs) of various numerical methods for problem (5.5) with $AB(\alpha) = 1$.

Methods		$\alpha = 0.55$			$\alpha = 0.75$			$\alpha = 0.95$		
	N	AE	EOC	CTs	AE	EOC	CTs	AE	EOC	CTs
PPC (3.6)-(3.12)	10	1.5e−3	–	3.4e−2	3.5e−4	–	1.1e−3	2.7e−5	–	6.0e−4
	20	7.4e−4	1.03	2.9e−3	8.8e−5	1.99	1.1e−3	3.2e−6	3.05	1.1e−3
	40	3.7e−4	0.99	3.5e−3	2.3e−5	1.93	3.0e−3	4.4e−7	2.89	3.0e−3
	80	1.8e−4	1.03	1.0e−2	6.1e−6	1.91	1.0e−2	6.7e−8	2.70	1.0e−2
	160	9.0e−5	1.03	3.8e−2	2.7e−6	1.21	3.7e−2	1.2e−8	2.51	3.7e−2
	320	4.4e−5	1.03	0.14	1.3e−6	1.03	0.14	2.3e−9	2.33	0.14
BJH-PC [30]	10	2.6e−2	–	1.7e−2	8.9e−3	–	8.0e−4	1.8e−3	–	1.0e−3
	20	1.2e−2	1.18	3.3e−3	3.7e−3	1.26	1.5e−3	6.7e−4	1.41	1.7e−3
	40	5.3e−3	1.13	3.4e−3	1.7e−3	1.16	3.3e−3	2.8e−4	1.25	3.3e−3
	80	2.5e−3	1.09	8.2e−3	7.8e−4	1.09	8.3e−3	1.3e−4	1.15	8.2e−3
	160	1.2e−3	1.06	2.5e−2	3.8e−4	1.06	2.4e−2	6.0e−5	1.08	2.4e−2
	320	5.8e−4	1.04	7.9e−2	1.8e−4	1.03	7.9e−2	2.9e−5	1.04	7.7e−2
TAE [15]	10	3.6e−2	–	5.6e−3	2.5e−2	–	7.0e−4	9.6e−3	–	6.0e−4
	20	1.7e−2	1.06	2.5e−3	1.2e−2	1.12	2.2e−3	3.7e−3	1.40	2.2e−3
	40	8.4e−3	1.05	8.8e−3	5.5e−3	1.08	8.6e−3	1.5e−3	1.26	8.4e−3
	80	4.1e−3	1.03	3.6e−2	2.6e−3	1.05	3.4e−2	6.8e−4	1.16	3.4e−2
	160	2.0e−3	1.02	0.14	1.3e−3	1.03	0.13	3.2e−4	1.09	0.14
	320	9.9e−4	1.02	0.55	6.4e−4	1.02	0.54	1.5e−4	1.05	0.55

The following is a description of the results:

- *Figures 5 and 6 provide numerical comparisons of solutions for system (5.8), with the ordinary derivative, Caputo-fractional ($\alpha = 0.8$), and ABC-fractional ($\alpha = 0.8$) derivatives.*
- *Figures 7 and 8 depict simulations of system (5.8) solutions for $\alpha = 0.5, 0.7, 0.9$, and $AB(\alpha) = 1$. The results shown in these figures show that the dynamical behavior of the system (5.8) is highly dependent on the fractional order α.*
- *Figures 9 and 10 depict simulations of system (5.8) solutions for $\alpha = 0.5, 0.7, 0.9$, and $AB(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}$. The results shown in these figures show that the dynamical behavior of the system (5.8) is highly dependent on the fractional order α and $AB(\alpha)$.*

Table 4. The absolute error, experimental order of convergence, and CPU time in seconds (CTs) of various numerical methods for problem (5.5) with $AB(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}$.

Methods		$\alpha = 0.55$			$\alpha = 0.75$			$\alpha = 0.95$		
	N	AE	EOC	CTs	AE	EOC	CTs	AE	EOC	CTs
PPC (3.6)-(3.12)	10	2.3e−2	–	3.4e−2	5.5e−4	–	1.1e−3	2.9e−5	–	6.0e−4
	20	9.7e−3	1.25	2.7e−3	1.3e−4	2.03	1.2e−3	3.5e−6	3.05	1.1e−3
	40	4.3e−3	1.17	3.5e−3	3.4e−5	1.99	3.0e−3	4.7e−7	2.88	3.1e−3
	80	2.0e−3	1.12	1.1e−2	1.2e−5	1.45	1.0e−2	7.3e−8	2.70	1.0e−2
	160	9.3e−4	1.09	3.8e−2	6.1e−6	1.01	3.7e−2	1.3e−8	2.51	3.7e−2
	320	4.5e−4	1.06	0.14	3.0e−6	1.01	0.14	2.6e−9	2.33	0.14
BJH-PC [30]	10	4.9e−2	–	1.6e−2	1.2e−2	–	9.0e−4	1.9e−3	–	8.0e−4
	20	2.1e−2	1.23	3.2e−3	4.9e−3	1.26	1.5e−3	6.9e−4	1.41	1.5e−3
	40	9.5e−3	1.15	3.3e−3	2.2e−3	1.17	3.4e−3	2.9e−4	1.25	3.2e−3
	80	4.4e−3	1.10	8.2e−3	1.0e−3	1.11	8.3e−3	1.3e−4	1.15	8.3e−3
	160	2.1e−3	1.07	2.4e−2	4.8e−4	1.07	2.4e−2	6.2e−5	1.08	2.4e−2
	320	1.0e−3	1.05	7.8e−2	2.4e−4	1.04	7.8e−2	3.0e−5	1.04	8.5e−2
TAE [15]	10	4.2e−2	–	5.5e−3	2.8e−2	–	6.0e−4	9.9e−3	–	6.0e−4
	20	2.0e−2	1.06	2.5e−3	1.3e−2	1.11	2.2e−3	3.8e−3	1.39	2.3e−3
	40	9.7e−3	1.04	8.8e−3	6.1e−3	1.08	8.5e−3	1.6e−3	1.26	8.7e−3
	80	4.8e−3	1.03	3.5e−2	3.0e−3	1.05	3.4e−2	7.0e−4	1.16	3.4e−2
	160	2.3e−3	1.02	0.14	1.4e−3	1.03	0.14	3.3e−4	1.09	0.14
	320	1.2e−3	1.02	0.56	7.2e−4	1.02	0.55	1.6e−4	1.05	0.57

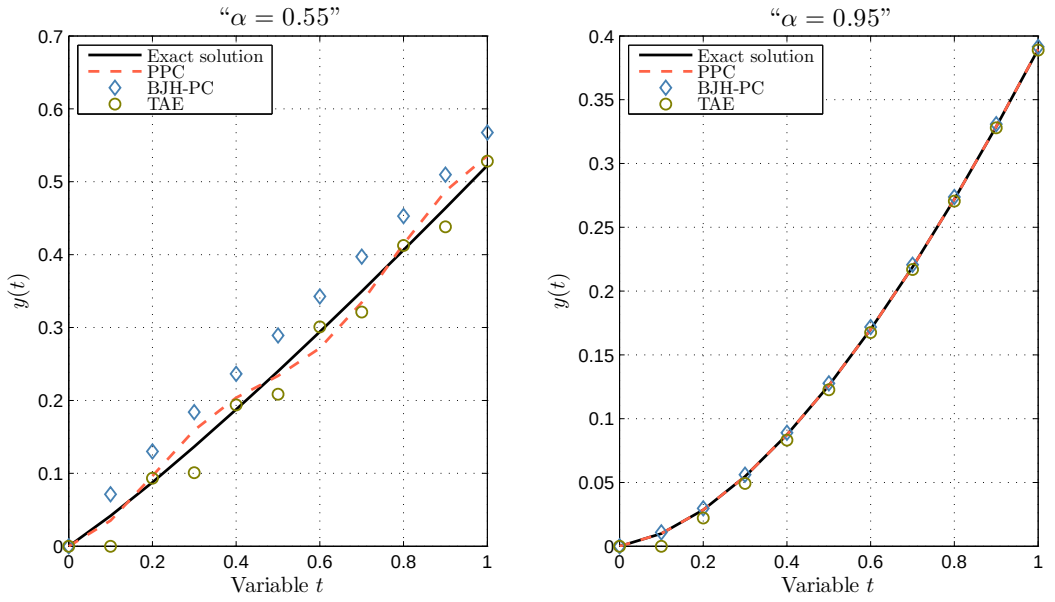

Figure 3. Comparison of the exact and the numerical solutions of problem (5.5) with $N = 10$, and $AB(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}$.

- *Figure 11 depicts simulations of the solution $u(t)$ of system (5.8) for $\alpha = 0.8$, and $AB(\alpha) = 1$, considering different values of initial data $(u_0, v_0) \in \{(-1, 0); (1, 0); (2, 0)\}$.*

- *Figure 12 depicts simulations of phase portrait of the solutions of system (5.8) for $\alpha = 0.55, 0.75, 0.95$, and $AB(\alpha) = 1$. The results shown in these figure shows that the dynamical behavior of the system (5.8) is highly dependent on the fractional order α.*

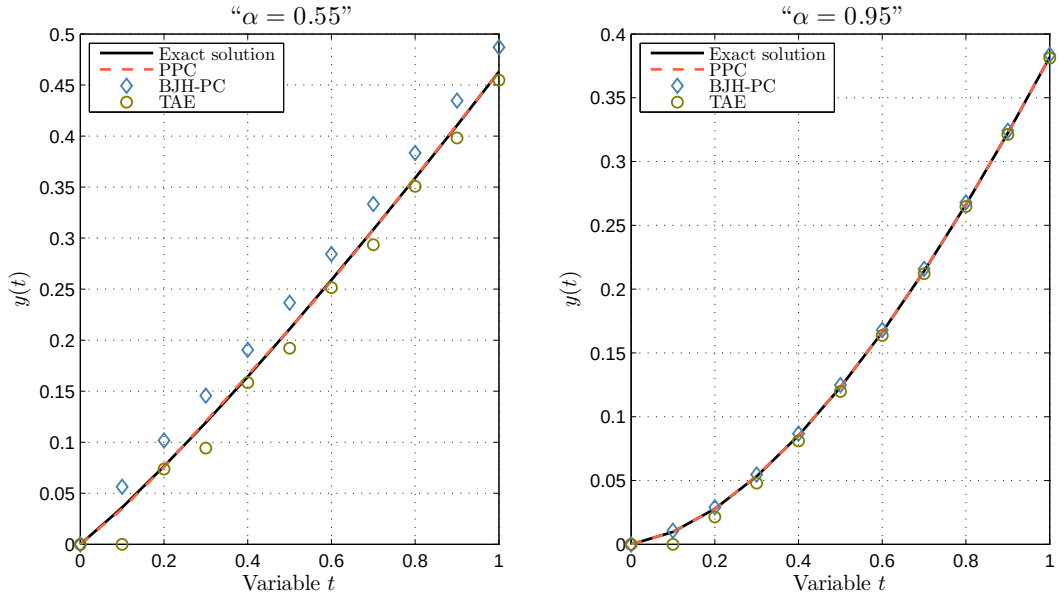

Figure 4. Comparison of the exact and the numerical solutions of problem (5.5) with $N = 10$, and $AB(\alpha) = 1$.

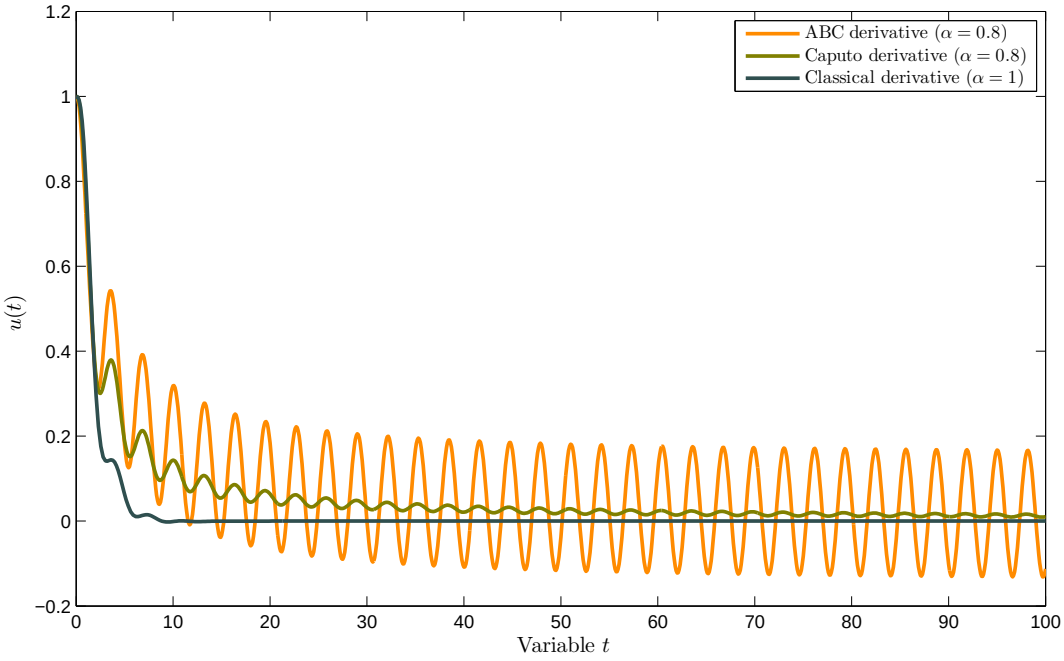

Figure 5. The approximate solution $u(t)$ of system (5.8) by PPC (3.6)-(3.12), ordinary derivative, and Caputo-fractional, subject to $(u_0, v_0) = (1, 0)$, and $\alpha = 0.8$.

6. Conclusion

In this article, an efficient predictor-corrector method to solve ABC-fractional order (for the fractional order $0 < \alpha < 1$) nonlinear diferential equations is established by using Lagrange quadratic interpolation. We decreased total processing costs by using the same discretization approach for the lag term in both the prediction and correction stages. This is the algorithm's most notable benefit, as the convergent order of this technique is shown to be $\mathcal{O}(h^3)$. This confirms the accuracy which has been numerically obtained, of the proposed approach through numerous numerical tests and simulations. The same approach with or without changes can be used to solve ABC-fractional differential equations with multi-variable orders, delay, and a generalized

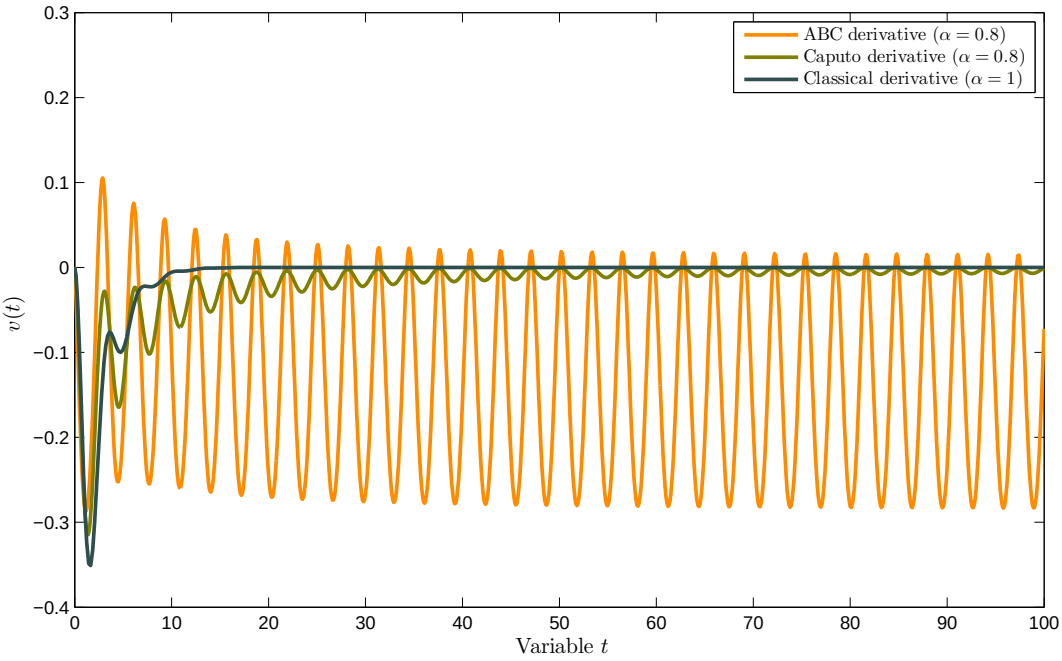

Figure 6. The approximate solution $v(t)$ of system (5.8) by PPC (3.6)-(3.12), ordinary derivative, and Caputo-fractional, subject to $(u_0, v_0) = (1, 0)$, and $\alpha = 0.8$.

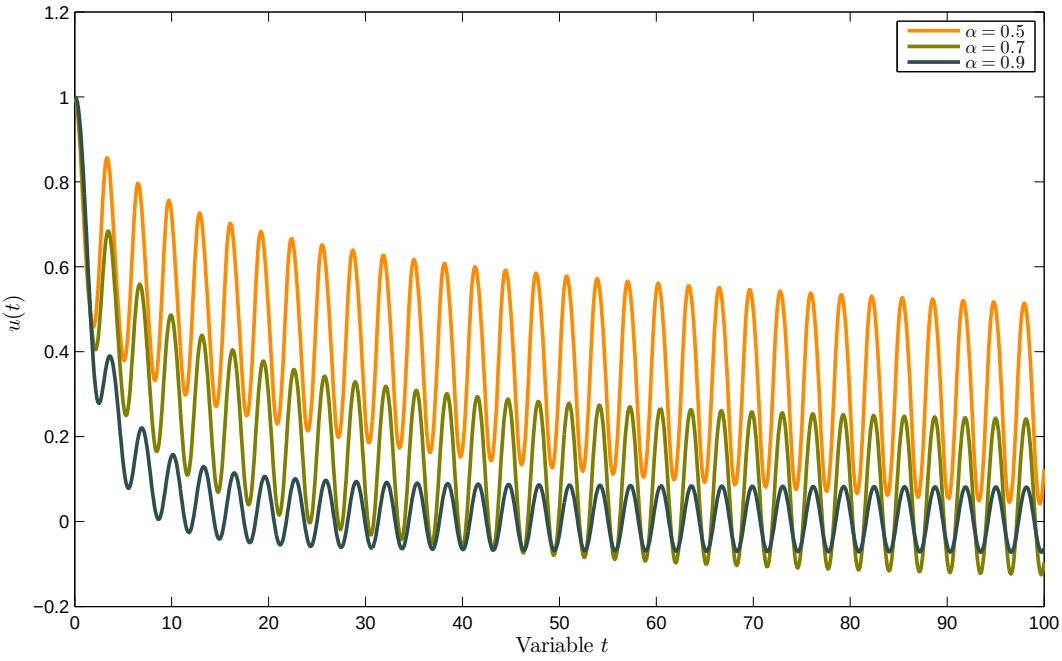

Figure 7. The approximate solution $u(t)$ of system (5.8) by PPC (3.6)-(3.12), subject to $(u_0, v_0) = (1, 0)$, $AB(\alpha) = 1$, and different values of fractional order α.

ABC-fractional derivative with a three-parameter Mittag-Leffler function in its kernel, as future works.

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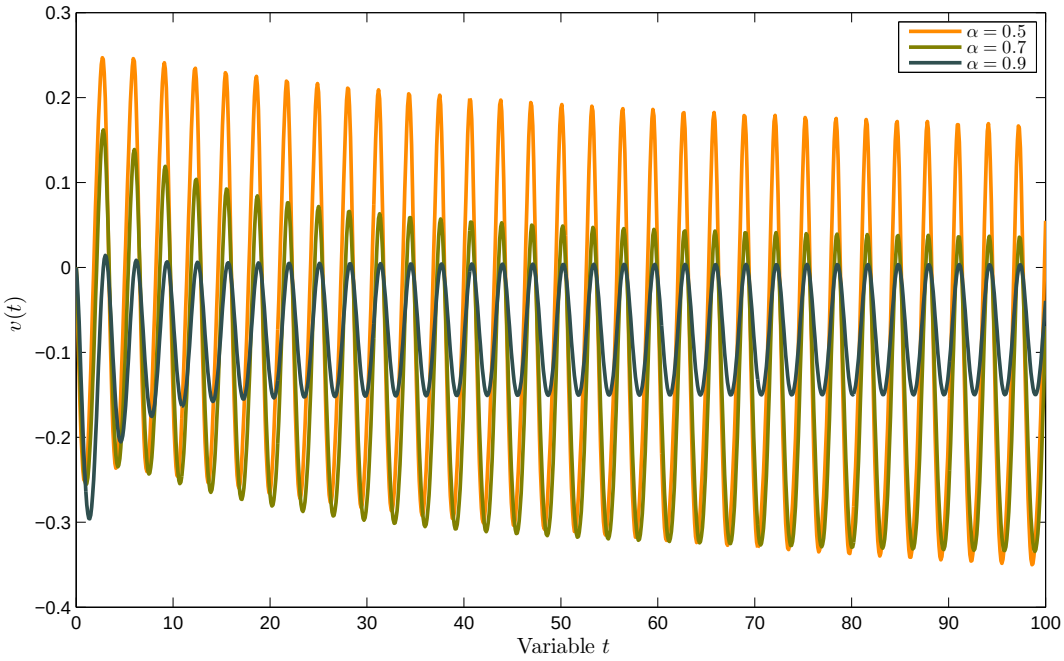

Figure 8. The approximate solution $v(t)$ of system (5.8) by PPC (3.6)-(3.12), subject to $(u_0, v_0) = (1, 0)$, $AB(\alpha) = 1$, and different values of fractional order α.

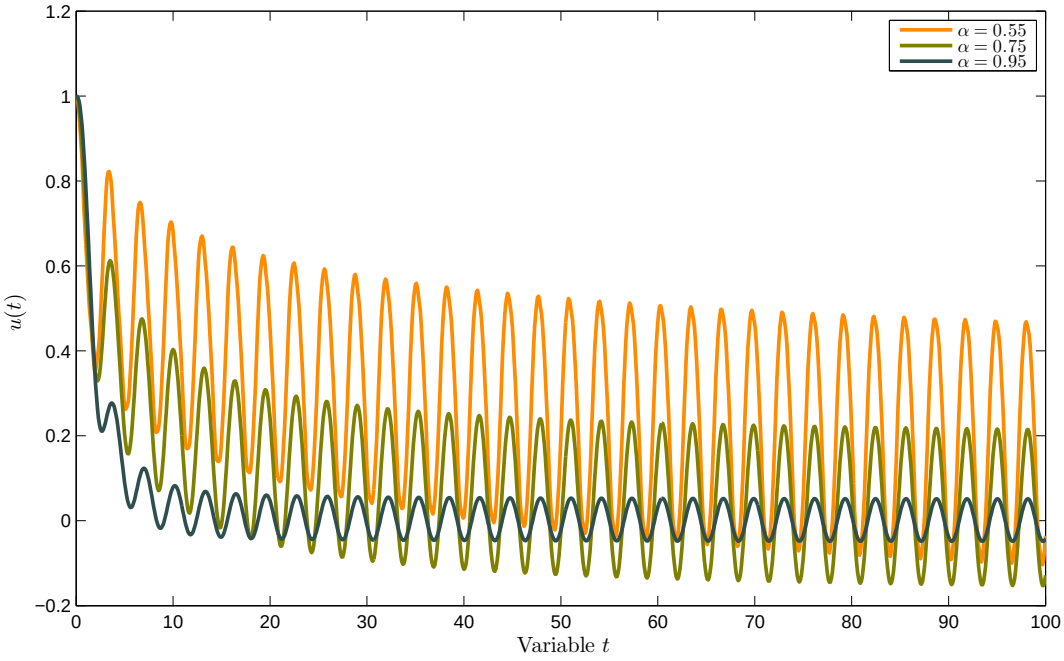

Figure 9. The approximate solution $u(t)$ of system (5.8) by PPC (3.6)-(3.12), subject to $(u_0, v_0) = (1, 0)$, $AB(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}$, and different values of fractional order α.

the algorithms file related to the methods developed in [21]. The work of S. Aljhani is supported by Taibah University - Saudi Arabia. This work does not have any conflicts of interest.

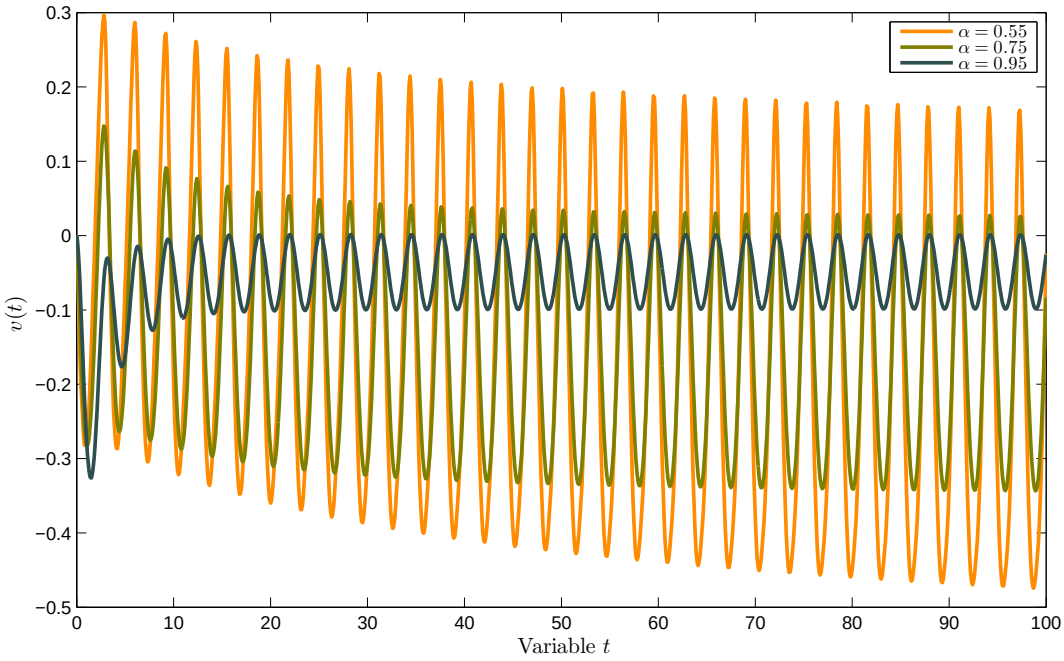

Figure 10. The approximate solution $v(t)$ of system (5.8) by PPC (3.6)-(3.12), subject to $(u_0, v_0) = (1, 0)$, $AB(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}$, and different values of fractional order α.

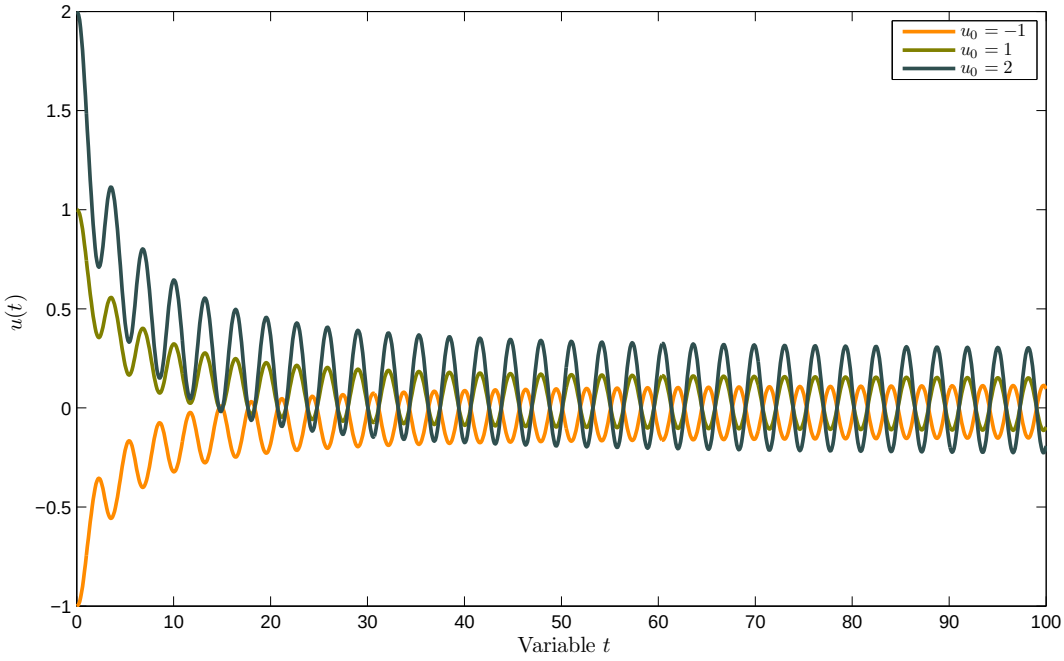

Figure 11. The approximate solution $u(t)$ of system (5.8) by PPC (3.6)-(3.12), subject to $\alpha = 0.8$, $AB(\alpha) = 1$, and different values of initial data $(u_0, v_0) \in \{(-1, 0); (1, 0); (2, 0)\}$.

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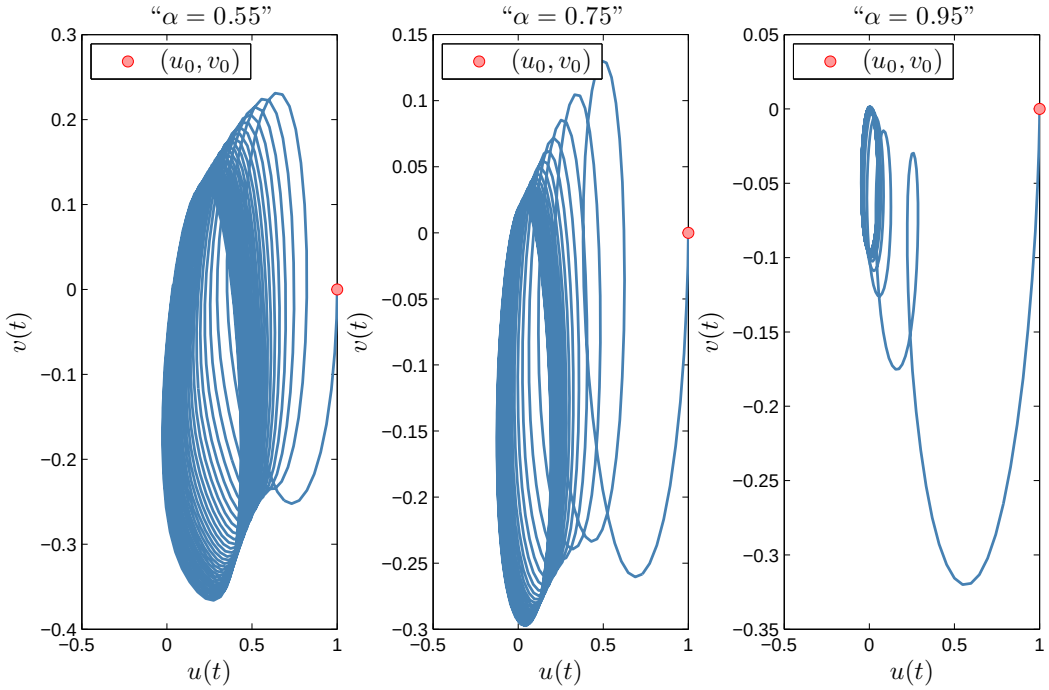

Figure 12. Phase portrait of the approximate solutions of system (5.8) by PPC (3.6)-(3.12), subject to $(u_0, v_0) = (1, 0)$, $AB(\alpha) = 1$, and different values of fractional order α.

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Appendix A. Start-up of the Scheme

To find a desired accuracy for $\widetilde{y}_1$ and $\widetilde{y}_2$, we find the approximate solutions at points $t_{\frac{1}{4}}$ and $t_{\frac{1}{2}}$ using the constant, linear and quadratic interpolation. Let $I^0(f_a)$ be the constant interpolation of f at point $t = a$ that is $I^0(f_a) = f(a)$. Let $I^1(f_a, f_b)$ and $I^2(f_a, f_b, f_c)$ be linear and quadratic interpolation of f with grids (a, b) and (a, b, c) respectively. In the algorithm below we describe how to find the approximate solution of $y(t)$ at points $t_{\frac{1}{4}}$, $t_{\frac{1}{2}}$, t_1, and t_2 (i.e. $\widetilde{y}_{\frac{1}{4}}$, $\widetilde{y}_{\frac{1}{2}}$, $\widetilde{y}_1$, and $\widetilde{y}_2$) using a predictor-corrector scheme:

- Approximate solution of $y(t)$ at point $t_{\frac{1}{4}}$:

$$\widetilde{y}^P_{\frac{1}{4}} = y_0 + AB_f \widetilde{f}(t_{\frac{1}{4}}, \widetilde{y}_0) + AB_y \int_0^{\frac{1}{4}} (t_{\frac{1}{4}} - s)^{\alpha-1} I^0(\widetilde{f}_0) ds,$$
$$\widetilde{y}_{\frac{1}{4}} = y_0 + AB_f \widetilde{f}^P_{\frac{1}{4}} + AB_g \int_0^{t_{\frac{1}{4}}} I^1(\widetilde{f}_0, \widetilde{f}^P_{\frac{1}{4}})(t_{\frac{1}{4}} - s)^{\alpha-1} ds.$$

- Approximate solution of $y(t)$ at point $t_{\frac{1}{2}}$:

$$\widetilde{y}^{P_1}_{\frac{1}{2}} = y_0 + AB_f(2\widetilde{f}_{\frac{1}{4}} - \widetilde{f}_0) + AB_g \int_0^{t_{\frac{1}{2}}} I^0(\widetilde{f}_{\frac{1}{4}})(t_{\frac{1}{2}} - s)^{\alpha-1} ds,$$
$$\widetilde{y}^{P_2}_{\frac{1}{2}} = y_0 + AB_f \widetilde{f}^{P_1}_{\frac{1}{2}} + AB_g \int_0^{t_{\frac{1}{2}}} I^1(\widetilde{f}_{\frac{1}{4}}, \widetilde{f}^{P_1}_{\frac{1}{2}})(t_{\frac{1}{2}} - s)^{\alpha-1} ds,$$
$$\widetilde{y}_{\frac{1}{2}} = y_0 + AB_f \widetilde{f}^{P_2}_{\frac{1}{2}} + AB_g \int_0^{t_{\frac{1}{2}}} I^2(\widetilde{f}_0, \widetilde{f}_{\frac{1}{4}}, \widetilde{f}^{P_2}_{\frac{1}{2}})(t_{\frac{1}{2}} - s)^{\alpha-1} ds.$$

- Approximate solution of $y(t)$ at point t_1:

$$\widetilde{y}^{P_1}_1 = y_0 + AB_f(6\widetilde{f}_{\frac{1}{2}} - 8\widetilde{f}_{\frac{1}{4}} - 3\widetilde{f}_0) + AB_g \int_0^{t_n} I^0(\widetilde{f}_{\frac{1}{2}})(t_n - s)^{\alpha-1} ds,$$
$$\widetilde{y}^{P_2}_1 = y_0 + AB_f \widetilde{f}^{P_1}_1 + AB_g \int_0^{t_1} I^1(\widetilde{f}_{\frac{1}{2}}, \widetilde{f}^{P_1}_1)(t_1 - s)^{\alpha-1} ds,$$
$$\widetilde{y}_1 = y_0 + AB_f \widetilde{f}^{P_2}_1 + AB_g \int_0^{t_1} I^2(\widetilde{f}_0, \widetilde{f}_{\frac{1}{2}}, \widetilde{f}^{P_2}_1)(t_1 - s)^{\alpha-1} ds.$$

- Approximate solution of $y(t)$ at point t_2:

$$\widetilde{y}^{P_1}_2 = y_0 + AB_f(6\widetilde{f}_1 - 8\widetilde{f}_{\frac{1}{2}} - 3\widetilde{f}_0) + \int_0^{t_2} I^0(\widetilde{f}_1)(t_2 - s)^{\alpha-1} ds,$$
$$\widetilde{y}^{P_2}_2 = y_0 + AB_f \widetilde{f}^{P_1}_2 + \int_0^{t_2} I^1(\widetilde{f}_1, \widetilde{f}^{P_1}_2)(t_2 - s)^{\alpha-1} ds,$$
$$\widetilde{y}_2 = y_0 + AB_f \widetilde{f}^{P_2}_2 + \int_0^{t_2} I^2(\widetilde{f}_0, \widetilde{f}_1, \widetilde{f}^{P_2}_2)(t_2 - s)^{\alpha-1} ds.$$

where

$$\left\{ AB_f = \frac{1-\alpha}{AB(\alpha)} \quad \textit{and} \quad AB_g = \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \right\}.$$